Equations rétrogrades avec singularités et autres contributions au calcul stochastique

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Équations rétrogrades avec singularités et autres contributions au calcul stochastique

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Contents

Introduction

This document is a synthesis of the research that I have been conducting, along with my co–authors, since the defense of my PhD thesis. This includes several topics, which are connected by the stochastic calculus (and also by the stochastic control). I have chosen to regroup them along two different main parts:

I. Monotone backward stochastic differential equations (BSDEs in short) with singularities.

II. Some contributions in stochastic analysis.

The first part forms a coherent whole. The main questions are: how can a terminal singularity be handled to obtain a solution ? And which applications are related to these singular equations ? The second is more disparate, which leads to this vague title. Here are aggregated different problems on measure solutions for BSDEs, on optimal switching, on parameter estimation for fractional diffusion and on homogenization of random partial differential equations (PDEs). If the first two works still deal with BSDE theory, estimation and homogenization are away from this topic. Nonetheless the initial motivation of the statistical work was an optimal control problem: how to design the parameters to obtain the more efficient estimation ? And the study of some random PDEs with small parameters is a very nice application of the BSDE theory, although we used completely different tools in our papers.

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The first part encompasses several works on singularity for BSDEs and is a direct pursuit of my PhD thesis. The starting question of Étienne Pardoux comes from the following observation. The function $t \mapsto (T-t)^{-1}$ is the solution of the ordinary differential equation (ODE) $y' = y^2$ with terminal value $+\infty$ at time T. This solution is finite (even bounded) except at time T. Similar results have been already proved for reaction-diffusion PDE:

$$\frac{\partial u}{\partial t} + \Delta u(t,x) - u(t,x)^2 = 0$$

with a prescribed singular terminal value $u(T,\cdot)$. Quantity $u(T,\cdot)$ is a measure, called the trace of u. This measure is singular in the sense that $u(T,\cdot)$ can be equal to $+\infty$ on some closed subset of $\mathbb{R}^d$. Since BSDE are a way to extend the Feynman–Kac representation to semi-linear PDE, this trace theory explains why we consider a singular terminal condition also for BSDEs. However if we consider an ODE of the form

$y' = -f(y)$ where f is Lipschitz continuous, then it is not possible to obtain a solution equal to $+\infty$ at time T and finite on $[0,T[$. Hence generator f has to be super-linear. This leads to the following two-steps program:

1. Solving a BSDE with a so-called monotone (or one-sided Lipschitz continuous) generator:

   (1) $$\langle x - x', f(x) - f(x') \rangle \leq \chi |x - x'|^2,$$

   but with integrable terminal value. The map $x \mapsto -x|x|$ is a typical example (with $\chi = 0$).

2. Adding a singularity at the final time: the terminal value can be equal to $+\infty$ or at least is not integrable.

These two items are the first two sections of the first part.

⋆

For BSDEs the underlying filtration $\mathbb{F} = (\mathcal{F}_t,\ t \geq 0)$ plays a key role. Let us simply evoke that the terminal value is supposed to be $\mathcal{F}_T$-measurable, whereas the solution is assumed to be adapted to the filtration, that is Y_t depends only on the information available at any time t, or is $\mathcal{F}_t$-measurable. When there is no generator, the solution at time $t \in [0,T]$ is the conditional expectation knowing $\mathcal{F}_t$ of the terminal value. Thus it is a martingale[1]. If $\mathbb{F}$ supports a Brownian motion W and a Poisson random measure π, this martingale can be decomposed into three parts

$$\int_0^{\cdot} Z_s dW_s + \int_0^{\cdot} \int_{\mathcal{E}} U_s(e) \widetilde{\pi}(de, ds) + M,$$

M being an additional orthogonal martingale. Combining this representation with the ODE part yields to the next form for a BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{E}} U_s(e) \widetilde{\pi}(de, ds) - (M_T - M_t).$$

When ξ belongs to some L^p-space and f is continuous and monotone w.r.t. y (Equation (1)) and Lipschitz continuous w.r.t. (z,u), the existence and uniqueness of the solution, together with a comparison result, are proved in the papers [X] and [XV], with Thomas Kruse. Note that for these BSDEs, we can replace the deterministic terminal time T by a stopping time τ. For $p \geq 2$, the results are obtained using standard technics. However for $p < 2$, when the generator f depends on U, there is a real issue. We needed some new properties for f and new arguments to get the existence of the solution (see [XV]).

We also extend these results:

[1] Provided some integrability condition is assumed.

- For backward doubly stochastic differential equations (BDSDEs) without jumps:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - \int_t^T g(s, Y_s, Z_s)\overleftarrow{dB_s}$$

  in [XIII], with Anis Matoussi and Lambert Piozin.

- For reflected backward stochastic differential equations (RBSDEs): $Y_t \geq L_t$ and

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s - (M_T - M_t) + (K_T - K_t),$$

  in [VII] with Saïd Hamadène, and in [XVII] with Chao Zhou.

- For second order backward stochastic differential equations (2BSDEs):

$$Y_t = \xi + \int_t^T \widehat{f}_u^{\mathbb{P}}(Y_u, \widehat{a}_u^{\frac{1}{2}} Z_u)du - \left(\int_t^T Z_u dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dM_u^{\mathbb{P}} + \int_t^T dK_u^{\mathbb{P}}$$

  also in [XVII].

In all these papers, the generator f is supposed to be monotone w.r.t. y and the terminal value has a r-moment for $r > 1$. If for the BSDE, we can deal with a general growth condition for f, some restrictions on the growth of f w.r.t. y are introduced for RBSDEs, BDSDEs and 2BSDEs. Essentially f should growth at most polynomially w.r.t. y. But in any case, if g is Lipschitz continuous w.r.t. (y, z, u), and if η is a positive process, generators of the form

$$f(y) = -\frac{1}{\eta_t} y|y|^{q-1} + g(t, y, z, u), \quad q > 1,$$

are allowed.

$\star$

Singularity is the most original topic of my research. The terminal value ξ doesn't belong to some $\mathbb{L}^p(\Omega)$; for example ξ can be equal to $+\infty$ with a positive probability. Loosing integrability[2] should be compensated by additional properties for generator f. We suppose[3] that f is decreasing sufficiently fast when y is large, namely there exist $q > 1$ and a positive process η such that

$$\forall y \geq 0,\ \forall (t, z, u), \quad f(t, y, z, u) - f(t, 0, z, u) \leq -\frac{1}{\eta_t} y|y|^{q-1}. \tag{2}$$

The existence of a minimal (super)solution is obtained by a truncation procedure in [I, II] and in [XI] with Thomas Kruse. Replacing ξ by $\xi \wedge L$ leads to a unique solution (Y^L, Z^L, U^L, M^L) to the BSDE with generator f. The comparison principle implies that

[2] On the positive part of ξ. We still suppose that ξ^- is integrable, even if it can be relaxed.
[3] The other conditions are more technical and less important.

the sequence $(Y^L,\ L \geq 0)$ is non-decreasing. Hence there exists a process $Y^{\min}$ which is the limit of Y^L. The key point is the existence of a priori estimate on Y^L. Indeed in classical estimates for BSDEs, Y^L at time t is controlled by the conditional expectation of $\xi \wedge L$ at time t. But this expectation tends to $+\infty$ when L goes to ∞, even if ξ is not a.s. equal to $+\infty$. For example, if $\xi = +\infty \mathbf{1}_{W_T \geq 0}$, then for any $0 \leq t < T$

$$\mathbb{E}(\xi \wedge L \big| \mathcal{F}_t) = L\mathbb{P}(W_T \geq 0 \big| \mathcal{F}_t) = L\Phi\left(\frac{W_t}{\sqrt{T-t}}\right) > 0$$

where Φ is the cumulative distribution function of the standard Gaussian law.

The previous growth condition (2) of f is sufficient to get a suitable a priori estimate[4] of the form:

$$(3) \qquad Y_t^L \leq \frac{C}{(T-t)^p}\left[\mathbb{E}\left(\int_t^T \left((\eta_s)^{p-1} + (T-s)^p (f(s,0,0,\mathbf{0}))^+\right)^\ell ds \bigg| \mathcal{F}_t\right)\right]^{1/\ell}.$$

Constant p is the Hölder conjugate of q, C and $\ell > 1$ come from the dependence of f w.r.t. z and u. This estimate shows that before time T, Y^L is finite and belongs to some $\mathbb{L}^\ell$-integrability space. Thus on any time interval $[0, T-\varepsilon]$, it is possible to pass to the limit on (Z^L, U^L, M^L). The limit process $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ solves the BSDE on $[0, T[$. Moreover if the filtration is left-continuous at time T (T is not a thin time), then a.s.

$$(4) \qquad \liminf_{t \to T} Y_t^{\min} \geq \xi = Y_T^{\min}.$$

This is the reason why we call $Y^{\min}$ a supersolution. Finally this solution is minimal, that is every other supersolution dominates it. Note that it implies that any other truncated approximation converges to the same supersolution $Y^{\min}$. This construction has been:

- Done if the terminal time is a stopping time τ in [XI]. However an a priori estimate has been proved only if τ is the exit time of a continuous diffusion process (Keller-Osserman inequality).
- Extended to BDSDEs in [XIII] and to 2BSDEs in [XVII].

The first immediate question concerns the uniqueness of this supersolution. It is still an open question, except if $\xi = +\infty$ a.s. In this particular case, a.s.

$$\lim_{t \to T} Y_t^{\min} = \xi = +\infty.$$

Another construction has been developed first by Horst and Graewe, based on the exact asymptotic of solution $Y^{\min}$. In [XXI] with Paulwin Graewe, if $f(t,y,z,u) = g(y)/\eta_t + \gamma_t$, we prove that

$$Y_t^{\min} = \phi(A_t) + \psi(A_t) H_t,$$

[4] If T is a stopping time, the a priori upper bound is very different and more delicate.

where ϕ and ψ are given functions depending of g, A is a non-negative process, such that a.s.

$$\lim_{t\to T} \phi(A_t) = \lim_{t\to T} \psi(A_t) = +\infty.$$

The remainder H solves a BSDE with terminal condition equal to zero, but with a *singular generator* F^H: a.s.

$$\int_0^T F^H(t,h,0)dt = +\infty.$$

Nonetheless the advantage of this method is that uniqueness of H and thus of $Y^{\min}$ can be proved if g is concave. And if $g(y) = -y|y|^{q-1}$, solution H can be obtained using Picard's iterations (in the right weighted space), and not only as the increasing limit of the approximating sequence.

From the definition of a supersolution, a second question concerns the behavior at time T of solution $Y^{\min}$:

- Does the limit at time T exist ?
- Is the limit (or at least the liminf) equal a.s. to ξ ?

We call this question the *continuity problem*. Remark that this question is related to the uniqueness of the solution. Indeed (4) is too weak to ensure the uniqueness. Suppose $Y^{(1)}$ and $Y^{(2)}$ are minimal supersolutions of the BSDE for two distinct terminal conditions $\xi^{(1)}$ and $\xi^{(2)}$. With (4), they may be equal on $[0,T[$! Assume now that $Y^{(i)}$ are solutions to the BSDE with these terminal conditions, such that $Y^{(i)}$ are both continuous at time T. This and $\xi^{(1)} \neq \xi^{(2)}$ imply that $Y^{(1)}$ and $Y^{(2)}$ are distinct processes.

In [XII], we show that the existence of the limit depends on generator f. The minimal solution $Y^{\min}$ is càdlàg on $[0,T]$ if there exists a lower bound on generator f:

$$\forall y \geq 0,\ \forall (z,u), \quad -b_t g(y) \leq f(t,y,z,u) - f(t,0,z,u).$$

Then $Y^{\min}$ is equal to a function of the difference of two non-negative càdlàg supermartingales ψ^+ and ψ^-:

$$Y_t = \Theta\left(\mathbb{E}^{\mathcal{F}_t}\left[\Theta^{-1}(\xi)\right] + \psi_t^- - \psi_t^+\right).$$

It is known that non-negative supermartingales have a limit and here a.s. $\lim_{t\to T} \psi_t^- = 0$. Thereby from the properties of Θ, $Y^{\min}$ has a limit at time T and

$$\lim_{t\to T} Y_t^{\min} \geq \xi.$$

Unfortunately we are not able to prove that ψ^+ tends to zero, which would lead to the a.s. equality of the limit and the terminal value ξ. A similar result has been obtained in [XIII] for singular BDSDEs.

Conversely equality in (4) only depends on ξ. In [I, II, XII], the (half-)Markovian[5] setting is considered, that is

$$\xi = \Phi(X_T)$$

[5] Half-Markovian because there is no similar assumption on generator f.

is a function of the terminal value of a diffusion process X. Then equality holds[6]. In [XIX], Dmytro Marushkevich and I extend this result to smooth functionals of X, using the functional Itô calculus developed by Dupire, Cont and Fournié. Non smooth functionals have been considered in [XVI, XXIV] with Thomas Kruse, Devin Sezer and Mahdi Ahmadi.

The probabilistic representation of a semi-linear PDE is one motivation of my research. For some reaction-diffusion PDE:

$$\frac{\partial u}{\partial t} + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^{q-1} = 0$$

Marcus and Véron developed the notion of the trace of a non-negative solution u. The operator $\mathcal{L}$ is the infinitesimal generator of a continuous Markov diffusion process. They show that this trace is in general singular, that is u can explode at time T (or on the boundary for an elliptic PDE). Legall, Dynkin and Kuznetsov obtain very similar results with the Brownian snake or superdiffusions. The minimal supersolution $Y^{\min}$ of the BSDE with singular terminal value provides another probabilistic representation of the solution of this PDE (see [I] for the parabolic case and [II] for the elliptic case). Moreover this method can be adapted to other types of PDE:

- In [XIV] for integro-partial differential equations: $u(T,\cdot) = \Phi$ and
$$\frac{\partial u}{\partial t} + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u(t,x),\nabla u(t,x),\mathcal{B}(t,x,u)) = 0$$
where $\mathcal{I}$ and $\mathcal{B}$ are non-local integro-differential operators.
- In [XIII] for SPDEs:
$$\begin{aligned} u(t,x) = \Phi(x) &+ \int_t^T \left[\mathcal{L}u(s,x) + f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))\right] ds \\ &+ \int_t^T g(s,x,u(s,x),(\nabla u \sigma)(s,x))\overleftarrow{dB_r}. \end{aligned}$$

In all cases, Φ takes values in $[0,+\infty]$. Even if we didn't check the details, the results of [XVII] can be used to deal with fully non-linear PDEs with singular terminal conditions. As far as I know, these types of PDE (IPDE, SPDE or fully non-linear) with singularities have never been studied before.

$\star$

Let us now tackle an interesting application of BSDEs with terminal singularity. The basic problem of the calculus of variations consists in minimizing an integral functional over a set of functions satisfying an initial condition and a terminal constraint:

$$\begin{aligned} &\text{Minimize } J(X) = \mathbb{E}\left[\int_0^T j(t,X_t,\dot{X}_t)dt\right] \text{ over all absolutely continuous} \\ &\text{and progr. mb. processes } X \text{ satisfying } X_0 = x_0 \in \mathbb{R} \text{ and } X_T = 0. \end{aligned}$$

[6] In [XIII], we prove that it also holds for BDSDEs.

We interpret t as time, X_t as the state and $\dot{X}_t$ as the velocity at time t:

$$X_t = x + \int_0^t \dot{X}_s ds.$$

If instead of the strong constraint on X_T at time T we add a terminal cost $g(X_T)$ in $J(X)$, the Pontryagin maximum principle characterizes the optimal control as the solution of a forward backward SDE:

$$\begin{aligned} X_t &= x - \int_0^t j_y^*(s, X_s, Y_s) ds, \\ Y_t &= g'(X_T) + \int_t^T j_x(s, X_s, j_y^*(s, X_s, Y_s)) ds - \int_t^T Z_s dW_s, \end{aligned} \tag{5}$$

where $j^*(t,x,\cdot)$ denotes the convex conjugate of the function $a \mapsto j(t,x,a)$, j_y^* its derivative w.r.t. y and j_x the derivative of j w.r.t. x. If the terminal cost is considered as a penalty on X_T, like $L(X_T)^2$, letting L go to $+\infty$ should give an optimal control for the constrained control problem ($X_T = 0$). Solving an FBSDE is challenging. One method is based on the existence of a decoupling field u: $Y_t = u(t, X_t)$. Roughly speaking the aim is to separate the forward equation from the backward equation.

One focus of the literature is set on cost functions j that are *additive* and *homogeneous*: $j(t,x,a) = \gamma_t |x|^p + \eta_t |a|^p$, where $p > 1$ and (η, γ) is a pair of non-negative progressively measurable processes. The terminal constraint is of the form $L|x|^p$. The particular form allows to decouple the FBSDE, after a variable change: $Y_t = pX_t^{p-1}\theta_t$, where θ solves the BSDE

$$\theta_t = L + \int_t^T \left[(p-1)\frac{\theta_s^q}{\eta_s^{q-1}} + \gamma_s \right] ds - \int_t^T \zeta_s dW_s.$$

This decoupling method can be also justified as follows: by homogeneity of the running and terminal costs, the value fonction

$$v(t,x) = \inf_{\dot{X}, X_s = x + \int_t^s \dot{X}_u du} \mathbb{E}\left[\int_t^T \left(\gamma_s |X_s|^p + \eta_s |\dot{X}_s|^p \right) ds + L|X_T|^p \middle| \mathcal{F}_t \right]$$

should be $v(t,x) = |x|^p \theta_t$. As the penalty L of any deviation of X_T from 0 increases to infinity, the backward part of the decoupled FBSDE converges to a solution of a BSDE with singular terminal condition. This additive–homogeneous case has been studied first by Ankirchner, Kruse and Jeanblanc. In [XI] Thomas Kruse and I extend the result to:

- General filtration $\mathbb{F}$;
- Poisson random measure as an additional source of randomness;
- Random terminal time;
- General terminal condition: we relax the mandatory constraint $X_T = 0$, adding in $J(X)$ a terminal cost $\xi |X_T|^p$ where $\xi \geq 0$ can be equal to $+\infty$. Finite costs require that $X_T = 0$ when $\xi = +\infty$.

The analysis of optimal control problems with state constraints on the terminal value is also motivated by models of optimal portfolio liquidation under stochastic price impact. The traditional assumption that all trades can be settled without impact on market dynamics is not always appropriate when investors need to close large positions over short time periods. In this framework the state process X denotes the agent's position in the financial market. She has two means to control her position. At each point in time t she can trade in the primary venue at a rate $\dot{X}_t$ which generates costs $\eta_t|\dot{X}_t|^p$ incurred by the stochastic price impact parameter η_t. Moreover, she can submit passive orders to a secondary venue ("dark pool"). These orders get executed at the jump times of the Poisson random measure π and generate so called slippage costs $\int_{\mathcal{E}} \lambda_t(e)|\beta_t(e)|^p\mu(de)$. The term $\gamma_t|X_t|^p$ can be understood as a measure of risk associated to the open position. Thus

$$J(X) = \mathbb{E}\left[\int_0^\tau \left(\eta_s|\dot{X}_s|^p + \gamma_s|X_s|^p + \int_{\mathcal{E}} \lambda_s(e)|\beta_s(e)|^p\mu(de)\right) ds + \xi|X_\tau|^p\right]$$

represents the overall expected costs for closing an initial position x over the time period $[0,\tau]$ using strategy X:

$$X_s = x + \int_0^s \dot{X}_u du + \int_0^s \int_{\mathcal{E}} \beta_u(e)\pi(de, du).$$

Working with general filtration means that the noise is not necessarily generated by a Brownian motion. Moreover, the liquidation constraint is relaxed in the following way. If $\xi = +\infty$ a.s., the position has to be closed imperatively (binding liquidation). Our model is flexible enough to allow for a specification of a set of market scenarios $\mathcal{S} \subset \mathcal{F}_\tau$ where liquidation is mandatory: $X_\tau \mathbb{1}_{\mathcal{S}} = 0$. On the complement $\mathcal{S}^c$ a penalization depending on the remaining position size can be implemented. This terminal constraint is described by the $\mathcal{F}_\tau$-measurable non negative random variable ξ such that $\mathcal{S} = \{\xi = +\infty\}$. For excepted scenarios, we can consider $\xi = \infty \mathbb{1}_{\mathcal{S}}$ with for example $\mathcal{S} = \{\max_{t\in[0,T]} \eta_t \leq H\}$ or $\mathcal{S} = \{\int_0^T \eta_t dt \leq H\}$ for a given threshold $H > 0$. This means that liquidation is only mandatory if the maximal price impact (or the average price impact) is small enough throughout the liquidation period. If the illiquidity of the market is too high, the trader has not obligatorily to close his position. Since the terminal time is a random time horizon τ, one can consider price-sensitive liquidation periods where the position has to be closed before the first time when the unaffected market price S (a diffusion) falls below some threshold level $K > 0$, i.e. $\tau = \inf\{t \geq 0 | S_t \leq K\}$.

In the work [XVII], Chao Zhou and I also consider *Knightian uncertainty* for this problem. Instead of a fixed probability measure $\mathbb{P}$, we suppose that there exists a family of probability measures such that we minimize

$$\sup_{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^T \left(\gamma_t|X_t|^p + \eta_t|\dot{X}_t|^p\right) dt + \xi|X_T|^p\right].$$

It corresponds for an agent to compute the worst case scenario for the liquidation of her portfolio. An optimal control is given by the solution of the related second order BSDE:

$$
\begin{aligned}
Y_t = \xi + \int_t^T \left[-\frac{(Y_u)^q}{(q-1)(\eta_u)^{q-1}} + \gamma_u \right] du \\
- \left(\int_t^T Z_s dX_s^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_t^T dM_s^{\mathbb{P}} + (K_T^{\mathbb{P}} - K_t^{\mathbb{P}}), \quad \mathbb{P} - a.s.
\end{aligned}
$$

If there are a large number N of players, each of them minimizes its own cost

$$
J^{N,i} = \mathbb{E}\left[\int_0^T \left(\frac{\kappa_t^i}{N} \sum_{j=1}^N \xi_t^j X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \right) dt \right]
$$

where $\xi^i = \dot{X}_t^i$. But now the agent has to take into account the control of the other players. Using a mean-field approach, Guanxing Fu, Ulrich Horst, Paulwin Graewe and I show in [XX] that

- The mean-field equilibrium is attained by the unique solution of a mean-field type FBSDE
$$
\begin{cases}
dX_t = -\dfrac{Y_t}{2\eta_t} dt, \\
-dY_t = \left(\kappa_t \mathbb{E}\left[\dfrac{Y_t}{2\eta_t} \middle| \mathcal{F}_t^0 \right] + 2\lambda_t X_t \right) dt - Z_t \, d\widetilde{W}_t, \\
X_0 = x, \; X_T = 0.
\end{cases}
$$
- The mean-field equilibrium ξ^* leads to an approximate Nash equilibrium for the N-players game of order $1/\sqrt{N}$: for each $1 \leq i \leq N$ and each admissible ξ^i,
$$
J^{N,i}(\xi^*) \leq J^{N,i}(\xi^i, \xi^{*,-i}) + O\left(\frac{1}{\sqrt{N}} \right),
$$
where $(\xi^i, \xi^{*,-i}) = (\xi^{*,1}, \cdots, \xi^{*,i-1}, \xi^i, \xi^{*,i+1}, \cdots, \xi^{*,N})$.

Surprisingly the solution of the mean-field type FBSDE cannot be directly obtained by a penalized scheme. We solve it using an adapted version of the continuation method and then we prove that the penalized scheme converges to the solution with the terminal constraint.

In the non-homogeneous case, we have to solve FBSDE (5), which is in general a difficult issue. We already mention the continuation method. The decoupling field theory is another way to tackle the problem. Roughly speaking we search Y of the form $Y_t = u(\omega, t, X_t)$, where u is the decoupling field. Stefan Ankirchner, Alexander Fromm, Thomas Kruse and I use this method in [XVIII] to solve the initial control problem: minimize $J(x_0) = \mathbb{E}\left[\int_0^T j(t, X_t, \dot{X}_t) dt \right]$ with the terminal constraint $X_T = 0$. Indeed we prove that the FBSDE

$$
\begin{aligned}
X_t &= x - \int_0^t j_y^*(s, X_s, Y_s)ds, \\
Y_t &= LX_T^2 + \int_t^T j_x(s, X_s, j_y^*(s, X_s, Y_s))ds - \int_t^T Z_s dW_s,
\end{aligned}
$$

has a solution with a unique decoupling field u^L. The sequence u^L is non decreasing w.r.t. L. Together with an appropriate upper bound[7], u^L converges to u^∞. If X^∞ is the unique solution to the ODE

$$
X_t^\infty := x_0 - \int_0^t j_y^*(s, X_s^\infty, u^\infty(s, X_s^\infty))ds, \quad t \in [0, T),
$$

$X_T^\infty = 0$ and if we define $\alpha_s^\infty := j_y^*(s, X_s^\infty, u^\infty(s, X_s^\infty))$, for $s \in [0, T)$, while setting $\alpha_T^\infty := 0$, then strategy α^∞ minimizes $J(x_0)$. Moreover if $Y^\infty = u^\infty(\cdot, X^\infty)$, we obtain the solution of the FBSDE: for $0 \leq t \leq r < T$

$$
\begin{aligned}
X_t^\infty &= x_0 - \int_0^t j_y^*(s, X_s^\infty, Y_s^\infty)ds, \quad X_T^\infty = 0, \\
Y_t^\infty &= Y_r^\infty + \int_t^r j_x(s, X_s^\infty, j_y^*(s, X_s^\infty, Y_s^\infty))ds - \int_t^r Z_s^\infty dW_s.
\end{aligned}
$$

Similar FBSDEs have been already obtained by Mikami and Thieullen or Tan and Touzi for optimal transportation problem, but with a different setting. Note that the singularity is in u^∞, which explodes at time T.

⋆⋆

The second part of this document is less coherent and encompasses different works, which are not directly related with each other. We start with two papers on BSDEs. In the first one [IV], Stefan Ankirchner, Peter Imkeller and I introduce a new notion of solution. Up to now in this document, the solution of a BSDE is a strong solution; the probability space is a priori given and the solution is constructed on this space. Buckdahn, Engelbert and Rascanu in 2004 already developed the concept of weak solutions of BSDEs: here the probability space is a part of the solution. Our definition of solution is somehow in between. Indeed a BSDE can be written as follows

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Z_s)ds - \int_t^T Z_s dW_s \\
&= \xi - \int_t^T Z_s \left(dW_s - \frac{f(s, Z_s)}{Z_s} ds \right) = \xi - \int_t^T Z_s d\widetilde{W}_s.
\end{aligned}
$$

$\widetilde{W}$ is a Brownian motion under a new probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$. Since we only change the probability measure, we call this solution *measure solution.* More

[7] Adapted version of estimate (3). However the bound is on the decoupling field, not on Y here.

precisely a triplet $(Y, Z, \mathbb{Q})$ is called a measure solution of the BSDE, if $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$, (Y, Z) a pair of $(\mathcal{F}_t)$–predictable stochastic processes such that $\int_0^T Z_s^2 ds < \infty$, $\mathbb{Q}$-a.s. and the following conditions are satisfied:

$$
\begin{aligned}
&\widetilde{W} = W - \int_0^{\cdot} g(s, Z_s)ds \quad \text{is a} \quad \mathbb{Q}- \text{ Brownian motion,}\\
&\xi \in L^1(\Omega, \mathcal{F}, \mathbb{Q}),\\
&Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t) = \xi - \int_t^T Z_s d\widetilde{W}_s, \quad t \in [0, T].
\end{aligned}
$$

Here $g(s, z) = f(s, z)/z$. In [IV], we show that if f is Lipschitz continuous w.r.t. z, then we can construct a measure solution from scratch. The quadratic case, $|f(s, z)| \leq c(1 + z^2)$, is more challenging. For bounded terminal value ξ, classical solution and measure solution are equivalent. But for unbounded terminal condition, under some integrability conditions on ξ, a measure solution exists whereas uniqueness is lost (we even construct a continuum of solutions).

⋆

Optimal switching problem is a class of control problems. In broad terms, a planner can choose between different modes of production and when she switches from a mode to another one. The goal is to determine optimal times and optimal modes to maximize the benefits of the firm. If there are only two modes (start and stop), the problem reduces to the choice of optimal times and has been solved by Hamadène and Jeanblanc in 2007. Suppose that the two modes are denoted 1 and 2, that there is a running profit ψ_i in mode i and that changing from the mode i to j has a sunk cost ℓ_{ij}. Solving a reflected BSDE with two barriers:

$$
Y_t = \int_t^T (\psi_1(s, X_s) - \psi_2(s, X_s))ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dW_s,
$$

with

$$
-\ell_{12} \leq Y_t \leq \ell_{21},
$$

and defining

$$
\begin{aligned}
Y_t^1 &= \mathbb{E}\left[\int_t^T \psi_1(s, X_s)ds + (K_T^+ - K_t^+)\middle|\mathcal{F}_t\right],\\
Y_t^2 &= \mathbb{E}\left[\int_t^T \psi_2(s, X_s)ds + (K_T^- - K_t^-)\middle|\mathcal{F}_t\right],
\end{aligned}
$$

Hamadène and Jeanblanc proved that

$$
\begin{aligned}
Y_t^1 = \ &\operatorname*{ess\,sup}_{\tau} \mathbb{E}\left[\int_t^\tau \psi_1(s, X_s)ds + (-\ell_{12} + Y_\tau^2)\mathbf{1}_{\tau<T}\middle|\mathcal{F}_t\right],\\
Y_t^2 = \ &\operatorname*{ess\,sup}_{\tau} \mathbb{E}\left[\int_t^\tau \psi_2(s, X_s)ds + (-\ell_{21} + Y_\tau^1)\mathbf{1}_{\tau<T}\middle|\mathcal{F}_t\right]
\end{aligned}
$$

give a sequence of optimal stopping times. These times are maximizers of the previous essential suprema.

In [V], Boualem Djehiche, Saïd Hamadène and I consider the case where there are more than three choices $i \in \{1, \cdots, q\}$. At a switching time, a mode should be optimally chosen. If we can define

$$Y_t^i = \operatorname*{ess\,sup}_{\tau} \mathbb{E}\left[\int_t^\tau \psi_i(s, X_s)ds + \max_{j \neq i}(-\ell_{ij} + Y_\tau^j)\mathbf{1}_{\tau < T}\middle|\mathcal{F}_t\right],$$

as in the two modes case, we can construct an optimal sequence of stopping times and modes. Instead of a reflected BSDE with two barriers, the Y^i are obtained as solution of a system of reflected BSDEs with inter-connected obstacles. We also relate this system to a system of variational inequalities: the Y^i are equal to $v^i(\cdot, X)$, v^i being viscosity solutions of this system of PDEs with obstacles.

⋆

The next topic is a collaboration with Alexandre Brouste and Marina Kleptsyna. In [VI] and [VIII], we study the large sample asymptotic properties of the Maximum Likelihood Estimator (MLE) for the signal drift parameter ϑ in a partially observed and possibly controlled fractional diffusion system.

Our model is the following. The signal and the observation are represented by real-valued processes $X = (X_t, t \geq 0)$ and $Y = (Y_t, t \geq 0)$ and they are governed by the following linear system of stochastic differential equation interpreted as integral equation:

$$\begin{cases} dX_t &= -\vartheta X_t dt + (1-\varepsilon)u(t)dt + \varepsilon dV_t^H, \quad X_0 = 0, \\ dY_t &= \mu X_t dt + dW_t^H, \quad Y_0 = 0. \end{cases}$$

Here, $V^H = (V_t^H, t \geq 0)$ and $W^H = (W_t^H, t \geq 0)$ are independent normalized fractional Brownian motions (fBm in short) with the same Hurst parameter $H \in (0,1)$ and the coefficients $\vartheta \in \mathbb{R}_+^*$ and $\mu \neq 0$ are real constants. Depending on $\varepsilon = 1$ or $\varepsilon = 0$, the unobserved signal process $X = (X_t, t \geq 0)$ is respectively stochastic or controlled by the real-valued function $u = (u(t), t \geq 0)$.

For the first statement $\varepsilon = 0$ (controlled, deterministic and partially observable signal) we establish the asymptotic (for large observation time) design problem of the input signal which gives an efficient estimator of the drift parameter. We separate the initial problem in two subproblems, when the first subproblem is equivalent to the explicit computations of the first eigenvalue of a certain self-adjoint operator and the second one is devoted to the analysis of the asymptotic properties of the MLE. In contrast with the previous works, we propose to use (for the both subproblems) Laplace transform computations, in particular, the Cameron-Martin formula and the link between the Laplace transform and the eigenvalues of a covariance operator.

For the second statement $\varepsilon = 1$, we work with a linear Gaussian system, perturbed by fBm noises. We suppose that the Hurst parameter H is known and it is the same for the signal X and for the observations Y, which means that the initial observation model is not Markovian. Again, our goal is to establish the large sample asymptotic properties of the MLE for the signal drift parameter ϑ. To analyze the large sample asymptotic properties of the MLE, we use the program proposed by Ibragimov and Khasminski. The main idea of this approach is to deduce strong properties of MLE from the weak convergence of scaled likelihoods in appropriate functional spaces, especially the convergence of moments which was not addressed even for discrete time hidden Markov models.

Although both models are different, the methodology of the proofs is very similar:

1. transform the system using the tools developed by Kleptsyna and Lebreton and compute the likelihood function and the Fisher information to deduce the MLE;

2. reduce the optimization problem to the resolvent estimation problem;

3. apply the Ibragimov-Khasminskii program.

In both cases the use of the Laplace transform is powerful to prove that the conditions of this program are satisfied.

In the first model, we maximize the Fisher information

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T\left(\vartheta, u\right) \,,$$

over a set of controls u. We prove that the asymptotical optimal input in the class of controls $\mathcal{U}_T$ is $u(t) = u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$, where the constants λ and κ_H only depend on H. The key point is that u_{opt} does not depend on the parameter ϑ. Then we demonstrate that when T goes to $+\infty$, the MLE is uniformly consistent on compacts $\mathbb{K} \subset \mathbb{R}_+^*$, is uniformly on compacts asymptotically normal with a limit variance equal to $\frac{\vartheta^4}{\mu^2}$ **which does not depend on** H. Moreover we have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments. Finally, the MLE is efficient in the sense that, for any compact $\mathbb{K} \subset \mathbb{R}^+$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbb{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 = 1 + o(1) \,.$$

For the second model the MLE $\hat{\vartheta}_T$ is also uniformly on compacts $\mathbb{K} \subset \mathbb{R}_+^*$ consistent, uniformly on compacts asymptotically normal where $\mathcal{I}(\vartheta)$ stands for the Fisher information **which does not depend on** H:

$$\mathcal{I}(\vartheta) = \frac{1}{2\vartheta} - \frac{2\vartheta}{\alpha(\alpha+\vartheta)} + \frac{\vartheta^2}{2\alpha^3}$$

and $\alpha = \sqrt{\mu^2 + \vartheta^2}$. We have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments.

To finish this part, let me mention that in [A] we started to study a similar problem of the form:

$$\left\{ \begin{array}{lcl} dX_t & = & -\vartheta X_t dt + dV_t^H \,, \quad X_0 = 0 \,, \\ dY_t & = & \mu X_t dt + dW_t \,, \quad Y_0 = 0. \end{array} \right.$$

Here, $V^H = (V_t^H,\, t \geq 0)$ is a normalized fBm with the Hurst parameter H in $(0,1)$ and $W = (W_t, t \geq 0)$ is independent Wiener process and the coefficients ϑ and $\mu \neq 0$ are real constants. Observing Y when T goes to $+\infty$, we wanted to estimate the parameter ϑ. Up to now there are still some issues we cannot overcome. Nonetheless some new results and technics developed by Chigansky and Kleptsyna could be useful to succeed in this estimation program.

$\star$

To conclude the second part of this thesis, we come to the homogenization problem studied in [IX, XXIII, B]. Marina Kleptsyna, Andrey Piatnitski and I deal with a Cauchy problem that takes the form

$$\partial_t u^\varepsilon = \mathrm{div}\Big(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)\nabla u^\varepsilon\Big),$$

with some fixed parameter $\alpha > 0$, and with the initial condition $u^\varepsilon(x,0) = \imath(x)$. We assume that the matrix $a(z,s) = \{a^{ij}(z,s)\}$ is uniformly elliptic, $(0,1)^d$-periodic in z variable, and random stationary ergodic in s. When ε tends to zero, it is known that u^ε converges to the solution u^0 of:

$$\partial_t u^0 = \mathrm{div}\big(\mathrm{a}^{\mathrm{eff}} \nabla u^0\big)$$

with a constant (non-random) positive definite matrix $\mathrm{a}^{\mathrm{eff}}$. The value of $\mathrm{a}^{\mathrm{eff}}$ depends on the value of α. In the diffusive case, $\alpha = 2$, there is an equilibrium between the periodic component and the random part. For $\alpha > 2$, the random part is dominating and vice versa for $\alpha < 2$.

Our goal was to obtain the asymptotic development of $u^\varepsilon - u^0$, up to a stochastic remaining term. Namely for suitable positive constants $\mathfrak{c}_j$ and correctors $\mathfrak{C}_j$

$$U^\varepsilon(x,t) = \frac{1}{\varepsilon^{\alpha/2}}\left[u^\varepsilon(x,t) - u^0(x,t) - \sum_{j\geq 1} \varepsilon^{\mathfrak{c}_j}\mathfrak{C}_j\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)\right]$$

converges to a solution U^0 of a SPDE with constants coefficients and an additive noise:

$$dU^0 = \mathrm{div}\Big(\mathrm{a}^{\mathrm{eff}}\nabla U^0 + \mu \frac{\partial^3}{\partial x^3}u^0\Big)dt + \Lambda^{1/2}\frac{\partial^2}{\partial x^2}u^0\, dW_t.$$

Our results can be summarized as follows:

- For $\alpha = 2$ (diffusive case, [IX]), the convergence holds with only one corrector, for any dimension d. We only suppose uniform ellipticity for a and some mixing condition for the stationary dynamical system.

- For $\alpha < 2$ and if the dynamical system is smooth in time, the convergence result remains valid (see [B]). The number of correctors increases when α goes to 2. However this smooth case excludes the diffusion example: $a(y,s) = \mathrm{a}(y,\xi_s)$ where ξ is the solution of a SDE driven by some Brownian motion.

- For $\alpha \neq 2$ and for the diffusion example, in [XXIII] we prove the convergence result only if the dimension d is equal to one.

To circumvent the issue, we thought about the properties of the fundamental solution of the heat equation. This is the reason why we investigate this topic in [C]. If we found interesting new properties of the fundamental solution and of the solution of the stochastic heat equation, these features are not sufficient to obtain the convergence for $d > 1$.

⋆⋆

To finish this introduction, I want to mention two other papers, which are not presented in this document.

In [III] Stefan Ankirchner, Peter Imkeller and I focus on the idea of *cross hedging*, i.e. using the potentials that are due to the negative correlation of risk exposure of different agents on a finance and insurance market. We try to develop a basic understanding of the impact of the correlation of risk exposure of different agents on prices and hedges of weather or climate derivatives. On the other hand, we try to keep it simple enough to obtain explicit solutions for pricing and hedging of financial products designed for climate or weather risk, which we simply call external risk henceforth. The key ingredient is the *utility indifference price* p for the derivative $F(X_T)$:

$$\sup_\pi \mathbb{E}(U(V_T^\pi + F(X_T) - p) = \sup_\pi \mathbb{E}(U(V_T^\pi)).$$

Paper [XXII] is a by-product of a work in progress with Dmytro Marushkevych and Thomas Kruse. Evoke that for singular BSDE we suppose that the growth condition (2) holds. Nonetheless for ODE, it can be weaken: instead of $-y|y|^{q-1}$, we could consider any non-increasing function g such that $-1/g$ is integrable on some interval $[c, +\infty[$ (see also [XXI]). Then the issue is to obtain an a priori estimate. The homogeneity of the power function allows us to deal only with BSDEs. But for general functions g, the upper bound has a more complex dynamics, given by a backward stochastic Volterra integral equation:

$$Y(t) = \Phi(t) + \int_t^T f(t,s,Y(s),Z(t,s))ds - \int_t^T Z(t,s)dW_s.$$

The terminal value and the generator also depend on the current time t. To solve our problem, we encountered some issues that have not been studied before. In [XXII], we extend known results to a widely class of BSVIEs:

$$\begin{aligned} Y(t) = \Phi(t) &+ \int_t^T f(t,s,Y(s),Z(t,s),Z(s,t),U(t,s),U(s,t))dB_s \\ &- \int_t^T Z(t,s)dX^\circ_s - \int_t^T \int_{\mathbb{R}^m} U(t,s,x)\widetilde{\pi}^\natural(dx,ds) - \int_t^T dM(t,s). \end{aligned}$$

containing the class of general BSDEs studied in [267]. In the Itô setting, $B_s = s$, $X^\circ = W$ and $\widetilde{\pi}^\natural$ is a compensated Poisson random measure.

Outline

In Chapter 1, we fix some notations used in the rest of this report. Monotone BSDEs (and BDSDE, RBSDE, 2BSDE) are studied in Chapter 2. The results of this chapter are extension of known results for BSDEs.

The most original part is Chapter 3. Here we develop some new notions and results when the terminal value of the BSDE is not in some L^p-space; in particular if the terminal value can be equal to $+\infty$ with positive probability. BSDEs with singular terminal conditions are applied to

- the probabilistic representation of the solution of some PDEs, IPDEs and SPDEs in Chapter 4;
- some stochastic control problems with terminal constraint in Chapter 5.

In the second part, we bring together two problems on BSDEs in Chapter 6. Chapters 7 and 8 are concerned respectively estimation problems and homogenization. Some additional technical results are in the appendix (Chapter 9)

Conditions' index

Chapter 1

Notations

This chapter contains all notations, spaces, etc. used in the rest of this report. No result is presented here.

Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and let $\mathbb{R}_+^*$ be the set of real positive numbers. For every $d-$dimensional vector b with $d \in \mathbb{N}^*$, we denote by $b^1, \dots, b^d$ its coordinates and for $\alpha, \beta \in \mathbb{R}^d$ we denote by $\alpha \cdot \beta$ the usual inner product, with associated norm $\|\cdot\|$, which we simplify to $|\cdot|$ when d is equal to 1. Moreover when the context is clear, $\|\cdot\|$ is sometimes denoted by $|\cdot|$. $\mathbb{R}^{d\times k}$ is identified with the space of real matrices with d rows and k columns. If $z \in \mathbb{R}^{d\times k}$, $|z|^2 = \text{Trace}(zz^*)$. $\mathcal{B}(.)$ is the Borelian sigma-field of the indicated Banach space. We also let $\mathbf{1}_d$ be the vector whose coordinates are all equal to 1. For any $(l, c) \in \mathbb{N}^* \times \mathbb{N}^*$, $\mathcal{M}_{l,c}(\mathbb{R})$ denotes the space of $l \times c$ matrices with real entries. Elements of the matrix $M \in \mathcal{M}_{l,c}$ are denoted by $(M^{i,j})_{1\leq i\leq l,\ 1\leq j\leq c}$, and the transpose of M by $M^\top$. When $l = c$, we let $\mathcal{M}_l(\mathbb{R}) := \mathcal{M}_{l,l}(\mathbb{R})$. We also identify $\mathcal{M}_{l,1}(\mathbb{R})$ and $\mathbb{R}^l$.

Let $\mathbb{S}_d^{\geq 0}$ denote the set of all symmetric positive semi-definite $d \times d$ matrices. We fix a map $\psi : \mathbb{S}_d^{\geq 0} \longrightarrow \mathcal{M}_d(\mathbb{R})$ which is (Borel) measurable and satisfies $\psi(a)(\psi(a))^\top = a$ for all $a \in \mathbb{S}_d^{\geq 0}$, and denote $a^{\frac{1}{2}} := \psi(a)$.

1.1 Under a single probability measure

Quadruplet $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ always denotes a filtered probability space. Filtration $\mathbb{F}$ is assumed to be complete and right continuous. Without loss of generality we suppose that all semimartingales have right continuous paths with left limits or càdlàg paths[1]. If M is a $\mathbb{R}^d$-valued martingale in $\mathcal{M}$, the bracket process $[M]_t$ is

$$[M]_t = \sum_{i=1}^{d} [M^i]_t,$$

where M^i is the i-th component of vector M.

On $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$, W denotes a k-dimensional Brownian motion and π a Poisson random measure with intensity $\mu(de)dt$ on space $\mathcal{E} \subset \mathbb{R}^m \setminus \{0\}$. Measure μ is

[1]French acronym for right continuous with left limits.

σ-finite on $\mathcal{E}$ such that

$$\int_{\mathcal{E}} (1 \wedge |e|^2)\mu(de) < +\infty.$$

The compensated Poisson random measure $\widetilde{\pi}(de, dt) = \pi(de, dt) - \mu(de)dt$ is a martingale w.r.t. filtration $\mathbb{F}$.

Remark 1.1 *All results can be generalized to the case where the compensator of π is random and equivalent to measure $dt \otimes \mu(\omega, de)$ with a bounded density for example (see the introduction of [34]).*

If X is an adapted process, $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ is the augmented filtration generated by X, namely

$$\mathbb{F}_t^X = \sigma(X_s,\ s \leq t) \vee \mathcal{N}$$

where $\mathcal{N}$ is the set of negligible subsets of Ω (w.r.t. $\mathbb{P}$).

Furthermore for a given $T \geq 0$, we denote:

- $Prog$ denotes the sigma-field of progressive subsets of $\Omega \times [0, T]$.
- $\mathcal{P}$: the predictable σ-field on $\Omega \times [0, T]$ and

  $$\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}(\mathcal{E})$$

  where $\mathcal{B}(\mathcal{E})$ is the Borelian σ-field on $\mathcal{E}$.
- On $\widetilde{\Omega} = \Omega \times [0, T] \times \mathcal{E}$, a function that is $\widetilde{\mathcal{P}}$-measurable, is called predictable. $G_{loc}(\mu)$ is the set of $\widetilde{\mathcal{P}}$-measurable functions ψ on $\widetilde{\Omega}$ such that for any $t \geq 0$ a.s.

  $$\int_0^t \int_{\mathcal{E}} (|\psi_s(e)|^2 \wedge |\psi_s(e)|)\mu(de) < +\infty.$$
- $\mathcal{D}$ (resp. $\mathcal{D}(0, T)$): the set of all predictable processes on $\mathbb{R}_+$ (resp. on $[0, T]$). $L^2_{loc}(W)$ is the subspace of $\mathcal{D}$ such that for any $t \geq 0$ a.s.

  $$\int_0^t |Z_s|^2 ds < +\infty.$$
- $\mathcal{M}_{loc}$: the set of càdlàg local martingales orthogonal to W and $\widetilde{\pi}$. If $M \in \mathcal{M}_{loc}$ then

  $$[M, W^i]_t = 0, 1 \leq i \leq k \qquad [M, \widetilde{\pi}(A, .)]_t = 0$$

  for all $A \in \mathcal{B}(\mathcal{U})$. In other words, $\mathbb{E}(\Delta M * \pi | \widetilde{\mathcal{P}}) = 0$, where the product $*$ denotes the integral process (see II.1.5 in [171]). Roughly speaking, the jumps of M and π are independent.
- $\mathcal{M}$ is the subspace of $\mathcal{M}_{loc}$ of martingales.

We refer to [171] (see also [34]) for details on random measures and stochastic integrals. Let us simple recall the next result on the decomposition of a local martingale.

Lemma 1.1 (Lemma III.4.24 in [171]) *Every local martingale has a decomposition*

$$\int_0^\cdot Z_s dW_s + \int_0^\cdot \int_{\mathcal{E}} \psi_s(e) \widetilde{\pi}(de, ds) + M$$

where $M \in \mathcal{M}_{loc}$*,* $Z \in L^2_{loc}(W)$*,* $\psi \in G_{loc}(\mu)$*.*

Let $\psi \in G_{loc}(\mu)$. Let us recall known results on the (local) martingale N given by

$$N_t = \int_0^t \int_{\mathcal{E}} \psi_s(e) \widetilde{\pi}(de, ds), \quad t \geq 0. \tag{1.1}$$

It follows that the compensator is given by

$$[N]_t = \int_0^t \int_{\mathcal{E}} |\psi_s(e)|^2 \pi(de, ds)$$

and its predictable projection by:

$$\langle N \rangle_t = \int_0^t \int_{\mathcal{E}} |\psi_s(e)|^2 \mu(de) ds.$$

For $t \geq 0$ let

$$N^*_t = \sup_{r \in [0,t]} \left| \int_0^r \int_{\mathcal{E}} \psi_s(e) \widetilde{\pi}(de, ds) \right|.$$

One fundamental inequality is the **Burkholder-Davis-Gundy inequality**: For all $p \in [1, \infty)$ there exist two universal constants c_p and C_p (not depending on N) such that for any N defined by (1.1) and for any $t \geq 0$

$$c_p \mathbb{E} \left([N]_t^{p/2} \right) \leq \mathbb{E} \left[(N^*_t)^p \right] \leq C_p \mathbb{E} \left([N]_t^{p/2} \right). \tag{1.2}$$

This inequality is proved for example in [170, Proposition 3.66] or [219, Theorem 2.1]. For $p \geq 2$, using [219, Theorem 4.1], the predictable projection can be controlled by its compensator:

$$\mathbb{E} \left(\langle N \rangle_t^{p/2} \right) \leq \hat{c}_p \mathbb{E} \left([N]_t^{p/2} \right). \tag{1.3}$$

In particular (1.2) means that martingale N is well-defined (see Chapter II in [170]) provided we can control $[N]$ in $\mathbb{L}^{p/2}(\Omega)$.

1.1.1 Suitable norms on $G_{loc}(\mu)$

For some $p \geq 1$, let us first define $\mathbb{L}^p_\mu = \mathbb{L}^p(\mathcal{E}, \mu; \mathbb{R}^d)$, the set of measurable functions $\psi : \mathcal{E} \to \mathbb{R}^d$ such that

$$\|\psi\|^p_{\mathbb{L}^p_\mu} = \int_{\mathcal{E}} |\psi(e)|^p \mu(de) < +\infty.$$

If ν is the measure defined on $\mathcal{E} \times [0,T]$ by $\nu = \mu \otimes \mathrm{Leb}$, $\mathbb{L}^p_\nu = \mathbb{L}^p(\mathcal{E} \times [0,T], \nu; \mathbb{R}^d)$ as the set of measurable functions $\psi : \mathcal{E} \times [0,T] \to \mathbb{R}^d$ such that

$$\|\psi\|^p_{\mathbb{L}^p_\mu} = \int_0^T \int_{\mathcal{E}} |\psi(e,t)|^p \nu(de \otimes dt) = \int_0^T \int_{\mathcal{E}} |\psi(e,t)|^p \mu(de) dt < +\infty.$$

On $\psi \in G_{loc}(\mu)$ we consider several norms. Again if $\nu = \mu \otimes \mathrm{Leb}$, then

$$\begin{aligned} \|\psi\|_{\mathbb{L}^p(\mathbb{L}^2_\nu)+\mathbb{L}^p(\mathbb{L}^p_\nu)} &= \inf_{\psi^1+\psi^2=\psi} \left\{ \left(\mathbb{E}\left[\left(\int_0^T \int_{\mathcal{E}} |\psi^1_s(e)|^2 \mu(de) ds \right)^{p/2} \right] \right)^{1/p} \right. \\ &\qquad \left. + \left(\mathbb{E}\left[\int_0^T \int_{\mathcal{E}} |\psi^2_s(e)|^p \mu(de) ds \right] \right)^{1/p} \right\} \end{aligned}$$

and

$$\begin{aligned} \|\psi\|_{\mathbb{L}^p(\mathbb{L}^2_\nu)\cap\mathbb{L}^p(\mathbb{L}^p_\nu)} &= \max \left\{ \left(\mathbb{E}\left[\left(\int_0^T \int_{\mathcal{E}} |\psi_s(e)|^2 \mu(de) ds \right)^{p/2} \right] \right)^{1/p}, \right. \\ &\qquad \left. \left(\mathbb{E}\left[\int_0^T \int_{\mathcal{E}} |\psi_s(e)|^p \mu(de) ds \right] \right)^{1/p} \right\}. \end{aligned}$$

Note that we can define the equivalent norms for a function ϕ defined on $\mathcal{E}$ (resp. $\mathcal{E} \times [0,T]$) w.r.t. the measure μ (resp. ν), namely $\|\phi\|_{\mathbb{L}^p_\mu + \mathbb{L}^2_\mu}$ and $\|\phi\|_{\mathbb{L}^p_\mu \cap \mathbb{L}^2_\mu}$ (resp. $\|\phi\|_{\mathbb{L}^p_\nu + \mathbb{L}^2_\nu}$ and $\|\phi\|_{\mathbb{L}^p_\nu \cap \mathbb{L}^2_\nu}$). With these norms we can define the Banach spaces $\mathbb{L}^p_\mu + \mathbb{L}^2_\mu$ and $\mathbb{L}^p_\mu \cap \mathbb{L}^2_\mu$ (resp. $\mathbb{L}^p_\nu + \mathbb{L}^2_\nu$ and $\mathbb{L}^p_\nu \cap \mathbb{L}^2_\nu$) (for the definition of the sum of two Banach spaces, see for example [202]).

These spaces appear naturally in the **Bichteler-Jacod inequality**[2]: there exist two universal constants κ_p and K_p such that for any $\psi \in G_{loc}(\mu)$, if N is defined by (1.1), then:

- For $p \in (1,2)$,

$$\kappa_p \left[\mathbb{E}\left([N]_T^{p/2} \right) \right]^{1/p} \leq \|\psi\|_{\mathbb{L}^p(\mathbb{L}^2_\nu)+\mathbb{L}^p(\mathbb{L}^p_\nu)} \leq K_p \left[\mathbb{E}\left([N]_T^{p/2} \right) \right]^{1/p}. \tag{1.4}$$

- For $p \in [2,+\infty)$,

$$\kappa_p \left[\mathbb{E}\left([N]_T^{p/2} \right) \right]^{1/p} \leq \|\psi\|_{\mathbb{L}^p(\mathbb{L}^2_\nu)\cap\mathbb{L}^p(\mathbb{L}^p_\nu)} \leq K_p \left[\mathbb{E}\left([N]_T^{p/2} \right) \right]^{1/p}. \tag{1.5}$$

The proof can be found for example in [242, Theorem 1] and the following comments pages 297 and 298.

[2]See the discussion in [242] about the name of this estimate.

1.1.2 Spaces of processes

Let us introduce the following spaces for $p \geq 1$.

- $\mathbb{L}^p(\Omega \times [0,T])$ is the space of all processes X such that

$$\mathbb{E}\int_0^T |X_t|^p dt < +\infty.$$

- $\mathbb{D}^p(0,T)$ is the space of all adapted càdlàg processes X such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty.$$

  For simplicity, $X_* = \sup_{t\in[0,T]} |X_t|$.

- $\mathbb{H}^p(0,T)$ is the subspace of all processes $X \in \mathcal{D}(0,T)$ such that

$$\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right] < +\infty.$$

- $\mathbb{M}^p(0,T)$ is the subspace of $\mathcal{M}$ of all martingales such that

$$\mathbb{E}\left[([M]_T)^{p/2}\right] < +\infty.$$

- $\mathbb{L}^p_\pi(0,T) = \mathbb{L}^p_\pi(\Omega \times (0,T) \times \mathcal{E})$: the set of processes $X \in G_{loc}(\mu)$ such that

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathcal{E}} |X_s(e)|^2 \pi(de,ds)\right)^{p/2}\right] < +\infty.$$

- $\mathcal{T}$: the set of all finite stopping times and $\mathcal{T}_T$ the set of all stopping times with values in $[0,T]$.

- $\mathcal{S}^p(0,T) = \mathbb{D}^p(0,T) \times \mathbb{H}^p(0,T) \times \mathbb{L}^p_\pi(0,T) \times \mathbb{M}^p(0,T)$.

We recall that a càdlàg adapted process Y is said to be of class (D) if the collection $\{Y_\tau,\ \tau \in \mathcal{T}\}$ is uniformly integrable. For a process Y of class (D) we set

$$\|Y\|_1 = \sup\{\mathbb{E}|Y_\tau|,\ \tau \in \mathcal{T}\}.$$

1.2 Conditions of the data of a BSDE

In the rest of the report, p is a fixed number such that $p > 1$. For this given constant $p > 1$, we use the space

$$\mathfrak{B}^2_\mu = \begin{cases} \mathbb{L}^2_\mu & \text{if } p \geq 2, \\ \mathbb{L}^1_\mu + \mathbb{L}^2_\mu & \text{if } p < 2. \end{cases} \tag{1.6}$$

For a BSDE there are two data. The generator f depends on $\omega \in \Omega$, $s \in [0,T]$, $y \in \mathbb{R}^d$, $z \in \mathbb{R}^{d\times k}$ and on $u \in \mathfrak{B}^2_\mu$. It is assumed that $f : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}^{d\times k}\times\mathfrak{B}^2_\mu \to \mathbb{R}^d$ is a random function, measurable with respect to $Prog \times \mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\mathbb{R}^{d\times k}) \times \mathcal{B}(\mathfrak{B}^2_\mu)$. For simplicity the process $f(\cdot, 0, 0, \mathbf{0})$ is denoted f^0:

$$f^0_t = f(t, 0, 0, \mathbf{0}), \tag{1.7}$$

where $\mathbf{0}$ denotes the null application from $\mathcal{E}$ to $\mathbb{R}$.

Terminal condition ξ is a $\mathcal{F}_T$-measurable random variable (with values in $\mathbb{R}^d$). And the next integrability condition is supposed to be true:

(A1) $\xi \in \mathbb{L}^p(\Omega)$ and $f^0 = f(\cdot, 0, 0, \mathbf{0})$ belongs to $\mathbb{L}^p(\Omega \times [0,T])$:

$$\mathbb{E}\left(|\xi|^p + \int_0^T |f^0_t|^p dt\right) < +\infty.$$

Several assumptions on the generator f will be used hereafter.

(A2) For every $t \in [0,T]$, $z \in \mathbb{R}^{d\times k}$ and every $u \in \mathfrak{B}^2_\mu$, the mapping $y \in \mathbb{R}^d \mapsto f(t, y, z, u)$ is continuous. Moreover there exists a constant χ such that

$$\langle f(t, y, z, u) - f(t, y', z, u), y - y'\rangle \leq \chi|y - y'|^2.$$

(A3) For every $r > 0$ the mapping $(\omega, t) \mapsto \sup_{|y|\leq r} |f(t, y, 0, \mathbf{0}) - f(t, 0, 0, \mathbf{0})|$ belongs to $\mathbb{L}^1(\Omega \times [0,T], \mathbb{P} \otimes \mathrm{Leb})$.

(A4) f is Lipschitz continuous w.r.t. z: there exists a constant $K_{f,z}$ such that for any t and y, for any $z, z' \in \mathbb{R}^{d\times k}$ and u in $\mathfrak{B}^2_\mu$

$$|f(t, y, z, u) - f(t, y, z', u)| \leq K_{f,z}|z - z'|.$$

(A5) f is Lipschitz continuous w.r.t. u: there exists a constant $K_{f,u}$ such that for any t and y, for any $z \in \mathbb{R}^{d\times k}$ and u, u' in $\mathfrak{B}^2_\mu$

$$|f(t, y, z, u) - f(t, y, z, u')| \leq K_{f,u}\|u - u'\|_{\mathfrak{B}^2_\mu}.$$

Assumption $(\mathbf{A_{ex}})$. *If the four previous conditions* **(A2)** *to* **(A5)** *are satisfied, we say that Condition* $(\mathbf{A_{ex}})$ *on the generator f holds.*

We define the constant K_f as

$$K_f^2 = K_{f,z}^2 + K_{f,u}^2.$$

The hypothesis **(A2)** is called *monotonicity condition*. Sometimes we work under restrictive assumptions.

(A2') f is Lipschitz continuous w.r.t. y: there exists a constant $K_{f,y}$ such that for any t and $y, y' \in \mathbb{R}^d$, for any $z \in \mathbb{R}^{d\times k}$ and u in $\mathfrak{B}^2_\mu$

$$|f(t, y, z, u) - f(t, y', z, u)| \leq K_{f,y}|y - y'|.$$

(A2') implies **(A2)** and **(A3)**. Sometimes the following growth assumption w.r.t. y is supposed.

(A3') There exists $q > 1$ and a jointly Borel measurable function $\Psi : [0,T] \times \Omega \to \mathbb{R}$ such that for any (t, ω, y)

$$|f(t,\omega,y,0,\mathbf{0}) - f_t^0| \leq \Psi(t,\omega)(1+|y|^q), \quad \text{and} \quad \mathbb{E}\int_0^T \Psi(t,\omega)dt < +\infty.$$

(A3') implies **(A3)** and is called *polynomial growth.*

If the assumption $(\mathbf{A_{ex}})$ is sufficient to obtain existence and uniqueness, it is in general not sufficient to obtain a comparison principle for solutions of BSDE. We reinforce Assumption **(A5)** and we assume that

(A5') For each $(y,z,u,v) \in \mathbb{R} \times \mathbb{R}^k \times (\mathfrak{B}^2_\mu)^2$, there exists a predictable process $\kappa = \kappa^{y,z,u,v} : \Omega \times [0,T] \times \mathcal{E} \to \mathbb{R}$ such that:

$$f(t,y,z,u) - f(t,y,z,v) \leq \int_{\mathcal{E}} (u(e) - v(e))\kappa_t^{y,z,u,v}(e)\mu(de)$$

with $\mathbb{P} \otimes \mathrm{Leb} \otimes \mu$-a.e. for any (y,z,u,v),

- $-1 \leq \kappa_t^{y,z,u,v}(e)$
- $|\kappa_t^{y,z,u,v}(e)| \leq \vartheta(e)$, where ϑ belongs to

$$\begin{cases} \mathbb{L}^2_\mu & \text{if } p \geq 2, \\ \mathbb{L}^\infty_\mu \cap \mathbb{L}^2_\mu & \text{if } p < 2. \end{cases}$$

Assumption $(\mathbf{A_{comp}})$ *is satisfied if* **(A2)**–**(A3)**–**(A4)** *and* **(A5')** *hold.*

The function ϑ belongs to the dual space of $\mathfrak{B}^2_\mu$ (see [202], Chapter 3, Theorem 3.1) such that the next result holds.

Lemma 1.2 *Assumption* $(\mathbf{A_{comp}})$ *implies Condition* $(\mathbf{A_{ex}})$*, that is f is Lipschitz continuous w.r.t. u with $K_{f,u} = \|\vartheta\|$.*

Proof. Indeed we take u and v in $\mathfrak{B}^2_\mu$. Thus since ϑ belongs to the dual space, then we obtain:

$$|f(t,y,z,u) - f(t,y,z,v)| \leq \|\vartheta\| \|u - v\|_{\mathfrak{B}^2_\mu}.$$

Hence f is Lipschitz continuous w.r.t. u. □

Let us emphasize that we only need $\kappa \geq -1$ and not $\kappa \geq C_\star > -1$ for some constant $C_\star$ (see for example [302] and the discussion in Section 2.1.1).

In dimension $d = 1$, the next *linearization procedure* is used several times through this report:

$$\begin{aligned} f(t,y,z,u) &= [f(t,y,z,u) - f(t,0,z,u)] + [f(t,0,z,u) - f(t,0,0,u)] \\ &\quad + [f(t,0,0,u) - f(t,0,0,\mathbf{0})] + f_t^0 \\ &= \upsilon_t y + \zeta_t z^\top + \pi_t(u) + f_t^0. \end{aligned} \tag{1.8}$$

Process υ is also progressively measurable:

$$\upsilon_t = \begin{cases} y^{-1}(f(t,y,z,u) - f(t,0,z,u)) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Under **(A2)**, this process is bounded from above: $\upsilon_t \leq \chi$. Under **(A2')**, it is bounded: $|\upsilon_t| \leq K_{f,y}$. Note that if y is replaced Y_{t-}, υ_t is still progressively measurable, but when Y_t is used instead of y, υ is only optional.

Process ζ is progressively measurable with values in $\mathbb{R}^k$, and defined as follows: for $1 \leq i \leq k$:

$$\zeta_t^i = \begin{cases} (z^i)^{-1}(f(t,0,z^{(i-1)},u) - f(t,0,z^{(i)},u)) & \text{if } z^i \neq 0 \\ 0 & \text{if } \ z^i = 0, \end{cases}$$

where $z^{(0)} = z$, $z^{(i)}$ is the k-dimensional vector such that its i first components are equal to 0 and the $k-i$ last components are equal to those of z. Under Condition **(A4)**, process ζ is bounded (by $K_{f,z}$).

And under **(A5)**, we have $|\pi_t(u)| \leq K_{f,u} \|u\|_{\mathfrak{B}^2_\mu}$, whereas with **(A5')**,

$$\int_{\mathcal{E}} u(e)\kappa_t^{0,0,0,u}(e)\mu(de) \leq \pi_t(u) \leq \int_{\mathcal{E}} u(e)\kappa_t^{0,0,u,0}(e)\mu(de).$$

1.3 SDE and Malliavin's calculus

In this report, X denotes the solution of the following SDE:

$$dX_s = b(s,X_s)ds + \sigma(s,X_s)dW_s, \quad 0 \leq s \leq T, \tag{1.9}$$

or, in the coordinate form,

$$dX_s^i = b_i(s,X_s)ds + \sum_{j=1}^k \sigma_{i,j}(s,X_s)dW_s^j.$$

Coefficients $b : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times k}$ are progressively measurable and satisfy the following standard conditions (uniformly w.r.t. ω):

(B1) There exists a constant $C \geq 0$ such that

$$|b(t,x)| + |\sigma(t,x)| \leq C(1+|x|)$$

for any $t \in [0,T]$ and $x \in \mathbb{R}^d$.

(B2) b and σ are Lipschitz continuous in x, that is there exists a constant C such that

$$|\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq C|x-x'|$$

for any $t \in [0,T]$ and $x, x' \in \mathbb{R}^d$.

These properties of σ and b imply in particular that process $X^x := (X_s^x)_{0 \leq s \leq T}$, solution of (1.9) with initial condition $x \in \mathbb{R}^d$, exists and is unique. Its infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma.\sigma^*)_{ij}(t,x)\partial^2_{ij} + \sum_{i=1}^{d} b_i(t,x)\partial_i. \tag{1.10}$$

Moreover the following estimates hold true (see e.g. [183], [297] or [300] for more details).

Proposition 1.1 *Process X^x satisfies:*

1. *For any $\theta \geq 2$, there exists a constant C such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq T} |X_s^x|^\theta\right] \leq C(1+|x|^\theta). \tag{1.11}$$

2. *There exists a constant C such that for any $t \in [0,T]$ and $x, x' \in \mathbb{R}^k$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq T} \left|X_s^x - X_s^{x'}\right|^2\right] \leq C(1+|x|^2)(|x-x'|^2). \tag{1.12}$$

If the initial time of the solution of (1.9) is now $t \in [0,T]$, the solution is denoted $X^{t,x}$ and can be extended to the whole interval: $X_s^{t,x} = x$ if $s \leq t$. Then (1.12) can be written:

$$\mathbb{E}\left[\sup_{0\leq s\leq T} \left|X_s^{t,x} - X_s^{t',x'}\right|^2\right] \leq C(1+|x|^2)(|x-x'|^2 + |t-t'|). \tag{1.13}$$

When the context is clear, to lighten the notations, $X = X^{t,x}$.

The Malliavin calculus is a useful tool in our work (see Sections 3.3.2 or 8.6.1). In what follows we borrow some notations from Nualart [258]. Recall that W is a k-dimensional Brownian motion. Let f be an element of $C_{\mathrm{p}}^\infty(\mathbb{R}^{kn})$ (the set of all infinitely many times continuously differentiable functions such that these functions and all their partial derivatives have at most polynomial growth at infinity) with

$$f(x) = f(x_1^1, \ldots, x_1^k; \ldots; x_n^1, \ldots, x_n^k).$$

We define a smooth random variable F by:

$$F = f(W(t_1), \ldots, W(t_n))$$

for $0 \leq t_1 < t_2 < \ldots < t_n \leq T$. Then the Malliavin derivative $D_t F$ is given by

$$D_t^j(F) = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i^j}(W(t_1), \ldots, W(t_n))\mathbf{1}_{[0,t_i]}(t)$$

(see Definition 1.2.1 in [258]). $D_t(F)$ is the k-dimensional vector $D_t(F) = (D_t^j(F),\ j = 1, \dots, k)$. Moreover, this derivative $D_t(F)$ is a random variable with values in the Hilbert space $L^2([0,T];\mathbb{R}^k)$. Space $\mathbb{D}_M^{1,p}$, $p \geq 1$, is the closure of the class of smooth random variables with respect to the norm

$$\|F\|_{1,p} = \left[\mathbb{E}(|F|^p) + \mathbb{E}\left(\|DF\|^p_{L^2([0,T];\mathbb{R}^k)}\right)\right]^{1/p}.$$

The subscript M is here to emphasize the Malliavin derivative. These spaces should be not confused with space $\mathbb{D}^p$ defined in Section 1.1.2. For $p = 2$, $\mathbb{D}_M^{1,2}$ is a Hilbert space. Then by induction we can define $\mathbb{D}_M^{N,p}$ the space of N-times differentiable random variables where the N derivatives are in $\mathbb{L}^p(\Omega)$. Finally

$$\mathbb{D}_M^{N,\infty} = \bigcap_{p\geq 1} \mathbb{D}_M^{N,p}, \quad \mathbb{D}_M^{\infty} = \bigcap_{N\in\mathbb{N}} \mathbb{D}_M^{N,\infty}.$$

The next result can be found in [258], Theorems 2.2.1 and 2.2.2.

Proposition 1.2 *Under conditions* **(B1)**–**(B2)**, *the coordinate X_s^i belongs to $\mathbb{D}_M^{1,\infty}$ for any $s \in [t,T]$ and $i = 1, \dots, d$. Moreover for any $j = 1, \dots, k$ and any $p \geq 1$*

$$\sup_{t\leq r\leq T} \mathbb{E}\left(\sup_{r\leq s\leq T} |D_r^j X_s^i|^p\right) < +\infty. \tag{1.14}$$

The derivative $D_r^j X_s^i$ satisfies the following linear equation:

$$D_r^j X_s^i \;=\; \sigma_{i,j}(\xi_r) + \sum_{1\leq \ell,m\leq k} \int_r^t \widetilde{\sigma}^{\ell}_{i,m}(s) D_r^j(X_s^m) dW_s^\ell + \sum_{m=1}^d \int_r^t \widetilde{b}_{i,m}(s) D_r^j(X_s^m) ds$$

for $r \leq s$ a.e. and $D_r^j X_s = 0$ for $r > s$ a.e., where for $1 \leq i \leq d$ and $1 \leq j, \ell \leq k$, $\widetilde{b}_{i,j}(s)$ and $\widetilde{\sigma}^l_{i,j}(s)$ are uniformly bounded and adapted processes such that if b and σ are differential with continuous derivatives, then:

$$\widetilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \qquad \widetilde{\sigma}^\ell_{i,j}(s) = (\partial_{x_j}\sigma_{i,\ell})(\xi_s).$$

Moreover if b and σ are of class C^1 in x with Lipschitz continuous derivatives, process X belongs to $\mathbb{D}_M^{2,\infty}$ and second derivatives $D_r^i D_s^j X_t^k$ satisfy also a linear stochastic differential equation with bounded coefficients.

Note that by induction if coefficients σ and β are infinitely differentiable functions in x with bounded derivatives of all orders greater than or equal to one, then X_t^i belongs to $\mathbb{D}_M^\infty$ for all $t \in [0,T]$ and $i = 1, \dots, d$.

Sometimes we add a jump component to the forward SDE : for any $0 \leq s \leq T$ and any $x \in \mathbb{R}^d$

$$X_s^x = x + \int_0^s b(r, X_r^x) dr + \int_0^s \sigma(r, X_r^x) dW_r + \int_0^s \int_{\mathcal{E}} \beta(r, X_{r_-}^x, e)\widetilde{\pi}(de, dr). \tag{1.15}$$

Parameters b and σ satisfy conditions **(B1)** and **(B2)**. $\beta : \Omega \times [0,T] \times \mathbb{R}^d \times \mathcal{E} \to \mathbb{R}^d$ is supposed to be measurable w.r.t. all variables and satisfy (uniformly in ω):

(B3) β is Lipschitz continuous w.r.t. x uniformly in e, i.e. there exists a constant $K_{\beta,x}$ such that for all $e \in \mathcal{E}$, for any x and y in $\mathbb{R}^d$:

$$|\beta(t,x,e) - \beta(t,y,e)| \leq K_{\beta,x}|x-y|(1 \wedge |e|).$$

(B4) There exists a constant C such that

$$|\beta(t,x,e)| \leq C(1 \wedge |e|).$$

Under these assumptions, for any $x \in [0,T] \times \mathbb{R}^d$, the forward SDE (1.15) has a unique strong solution $X^x = \{X^x_s,\ 0 \leq s \leq T\}$. Moreover for all $x \in \mathbb{R}^d$ and $p \geq 2$

$$\mathbb{E}\left[\sup_{0 \leq s \leq T} |X^x_s - x|^p\right] \leq C(1 + |x|^p). \tag{1.16}$$

These results can be found in [297], chapter V, Theorems 7 and 67.

1.4 Under a family of probability measures

In this part, we introduce the setting used for second order BSDEs (see Sections 2.4, 3.2.2 and 5.2). This framework is defined in [309, 295]. Since it is quite stodgy, the reader can skip this part if he or she is not interested by second order BSDEs.

Canonical space

Let $d \in \mathbb{N}^*$, we denote by $\Omega := C\left([0,T], \mathbb{R}^d\right)$ the canonical space of all $\mathbb{R}^d$–valued continuous paths ω on $[0,T]$ such that $\omega_0 = 0$, equipped with the canonical process X, *i.e.* $X_t(\omega) := \omega_t$, for all $\omega \in \Omega$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the canonical filtration generated by X, and by $\mathbb{F}_+ = (\mathcal{F}^+_t)_{0 \leq t \leq T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}^+_t := \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}^+_T := \mathcal{F}_T$. We equip Ω with the uniform convergence norm $\|\omega\|_\infty := \sup_{0 \leq t \leq T} \|\omega_t\|$, so that the Borel $\sigma-$field of Ω coincides with $\mathcal{F}_T$. Let $\mathbb{P}_0$ denote the Wiener measure on Ω under which X is a Brownian motion.

Let $\mathbb{M}_1$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Notice that $\mathbb{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathfrak{B}$ its Borel $\sigma-$field. Then for any $\mathbb{P} \in \mathbb{M}_1$, denote by $\mathcal{F}^{\mathbb{P}}_t$ the completed $\sigma-$field of $\mathcal{F}_t$ under $\mathbb{P}$. Denote also the completed filtration by $\mathbb{F}^{\mathbb{P}} = \left(\mathcal{F}^{\mathbb{P}}_t\right)_{t \in [0,T]}$ and $\mathbb{F}^{\mathbb{P}}_+$ the right limit of $\mathbb{F}^{\mathbb{P}}$, so that $\mathbb{F}^{\mathbb{P}}_+$ satisfies the usual conditions. Moreover, for $\mathcal{P} \subset \mathbb{M}_1$, we introduce the universally completed filtration $\mathbb{F}^U := \left(\mathcal{F}^U_t\right)_{0 \leq t \leq T}$, $\mathbb{F}^{\mathcal{P}} := \left(\mathcal{F}^{\mathcal{P}}_t\right)_{0 \leq t \leq T}$, and $\mathbb{F}^{\mathcal{P}+} := \left(\mathcal{F}^{\mathcal{P}+}_t\right)_{0 \leq t \leq T}$, defined as follows

$$\mathcal{F}^U_t := \bigcap_{\mathbb{P} \in \mathbb{M}_1} \mathcal{F}^{\mathbb{P}}_t,\ \mathcal{F}^{\mathcal{P}}_t := \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{F}^{\mathbb{P}}_t,\ t \in [0,T],\ \mathcal{F}^{\mathcal{P}+}_t := \mathcal{F}^{\mathcal{P}}_{t+},\ t \in [0,T), \text{ and } \mathcal{F}^{\mathcal{P}+}_T := \mathcal{F}^{\mathcal{P}}_T.$$

We also introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \Omega'$, where Ω' is identical to Ω. By abuse of notation, we denote by (X, B) its canonical process, *i.e.* $X_t(\bar{\omega}) := \omega_t$, $B_t(\bar{\omega}) :=$

ω'_t for all $\bar{\omega} := (\omega, \omega') \in \overline{\Omega}$, by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0 \le t \le T}$ the canonical filtration generated by (X, B), and by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0 \le t \le T}$ the filtration generated by X. Similarly, we denote the corresponding right-continuous filtrations by $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration by $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$, given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.

Semi-martingale measures

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ is a semi-martingale measure if X is a semi-martingale under $\mathbb{P}$. Then on the canonical space Ω, there is some $\mathbb{F}-$progressively measurable non-decreasing process (see e.g. Karandikar [182]), denoted by $\langle X \rangle = (\langle X \rangle_t)_{0 \le t \le T}$, which coincides with the quadratic variation of X under each semi-martingale measure $\mathbb{P}$. Denote further

$$\widehat{a}_t := \limsup_{\varepsilon \searrow 0} \frac{\langle X \rangle_t - \langle X \rangle_{t-\varepsilon}}{\varepsilon}.$$

For every $t \in [0, T]$, let $\mathcal{P}_t^W$ denote the collection of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that

- $(X_s)_{s \in [t,T]}$ is a $(\mathbb{P}, \mathbb{F})-$semi-martingale admitting the canonical decomposition (see e.g. [171, Theorem I.4.18])

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + X_s^{c,\mathbb{P}}, \; s \in [t, T], \; \mathbb{P} - a.s.,$$

  where $b^{\mathbb{P}}$ is a $\mathbb{F}^{\mathbb{P}}-$predictable $\mathbb{R}^d-$valued process, and $X^{c,\mathbb{P}}$ is the continuous local martingale part of X under $\mathbb{P}$.

- $\big(\langle X \rangle_s\big)_{s \in [t,T]}$ is absolutely continuous in s with respect to the Lebesgue measure, and $\widehat{a}$ takes values in $\mathbb{S}_d^{\geq 0}$, $\mathbb{P} - a.s.$

Given a random variable or process λ defined on Ω, we can naturally define its extension on $\overline{\Omega}$ (which, abusing notations slightly, we still denote by λ) by

$$\lambda(\bar{\omega}) := \lambda(\omega), \quad \forall \bar{\omega} = (\omega, \omega') \in \overline{\Omega}. \tag{1.17}$$

In particular, the process $\widehat{a}$ can be extended on $\overline{\Omega}$. Given a probability measure $\mathbb{P} \in \mathcal{P}_t^W$, we define a probability measure $\overline{\mathbb{P}}$ on the enlarged canonical space $\overline{\Omega}$ by $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$, so that X in $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}}, \overline{\mathbb{F}})$ is a semi-martingale with the same triplet of characteristics as X in $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{F})$, B is a $\overline{\mathbb{F}}-$Brownian motion, and X is independent of B. Then for every $\mathbb{P} \in \mathcal{P}_t^W$, there is some $\mathbb{R}^d-$valued, $\overline{\mathbb{F}}-$Brownian motion $W^{\mathbb{P}} = (W_r^{\mathbb{P}})_{t \le r \le s}$ such that (see e.g. Theorem 4.5.2 of [313])

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + \int_t^s \widehat{a}_r^{\frac{1}{2}} dW_r^{\mathbb{P}}, \quad s \in [t, T], \quad \overline{\mathbb{P}} - a.s., \tag{1.18}$$

where we extend the definition of $b^{\mathbb{P}}$ and $\widehat{a}$ on $\overline{\Omega}$ as in (1.17), and where we recall that $\widehat{a}^{\frac{1}{2}}$ has been defined in the notations above.

Notice that when $\widehat{a}_r$ is non-degenerate $\mathbb{P}-a.s.$, for all $r \in [t,T]$, then we can construct the Brownian motion $W^{\mathbb{P}}$ on Ω by

$$W_t^{\mathbb{P}} := \int_0^t \widehat{a}_s^{-\frac{1}{2}} dX_s^{c,\mathbb{P}},\ t \in [0,T],\ \mathbb{P}-a.s.,$$

and do not need to consider the above enlarged space equipped with an independent Brownian motion to construct $W^{\mathbb{P}}$.

Remark 1.2 (On the choice of $\widehat{a}^{\frac{1}{2}}$) *The measurable map $a \longmapsto a^{\frac{1}{2}}$ is fixed throughout the paper. A first choice is to take $a^{\frac{1}{2}}$ as the unique non-negative symmetric square root of a (see e.g. Lemma 5.2.1 of [313]). One can also use the Cholesky decomposition to obtain $a^{\frac{1}{2}}$ as a lower triangular matrix. Finally the reader can read [295], Remark 2.2, where the sets $\mathcal{P}(t,\omega)$ are given by the collections of probability measures induced by a family of controlled diffusion processes. In this case one can take $\widehat{a}^{\frac{1}{2}}$ in the following way:*

$$a = \begin{pmatrix} \sigma\sigma^T & \sigma \\ \sigma^T & I_n \end{pmatrix} \quad \text{and} \quad a^{\frac{1}{2}} = \begin{pmatrix} \sigma & 0 \\ I_n & 0 \end{pmatrix}, \quad \text{for some } \sigma \in \mathcal{M}_{m,n}. \tag{1.19}$$

Conditioning and concatenation of probability measures

We also recall that for every probability measure $\mathbb{P}$ on Ω and $\mathbb{F}-$stopping time τ taking value in $[0,T]$, there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}_\omega^\tau)_{\omega\in\Omega}$ (see e.g. Stroock and Varadhan [313]), satisfying:

(i) For every $\omega \in \Omega$, $\mathbb{P}_\omega^\tau$ is a probability measure on $(\Omega, \mathcal{F}_T)$.

(ii) For every $E \in \mathcal{F}_T$, the mapping $\omega \longmapsto \mathbb{P}_\omega^\tau(E)$ is $\mathcal{F}_\tau-$measurable.

(iii) The family $(\mathbb{P}_\omega^\tau)_{\omega\in\Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, *i.e.*, for every integrable $\mathcal{F}_T-$measurable random variable ξ we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}_\omega^\tau}\big[\xi\big]$, for $\mathbb{P}-a.e.\ \omega \in \Omega$.

(iv) For every $\omega \in \Omega$, $\mathbb{P}_\omega^\tau(\Omega_\tau^\omega) = 1$, where $\Omega_\tau^\omega := \big\{\overline{\omega} \in \Omega :\ \overline{\omega}(s) = \omega(s),\ 0 \le s \le \tau(\omega)\big\}$.

Furthermore, given some $\mathbb{P}$ and a family $(\mathbb{Q}_\omega)_{\omega\in\Omega}$ such that $\omega \longmapsto \mathbb{Q}_\omega$ is $\mathcal{F}_\tau-$measurable and $\mathbb{Q}_\omega(\Omega_\tau^\omega) = 1$ for all $\omega \in \Omega$, one can then define a concatenated probability measure $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot$ by

$$\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot\big[A\big] := \int_\Omega \mathbb{Q}_\omega\big[A\big]\ \mathbb{P}(d\omega),\ \forall A \in \mathcal{F}_T.$$

Hypotheses on $\mathcal{P}(t,\omega)$

We are given a family $\mathcal{P} = (\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ of sets of probability measures on $(\Omega, \mathcal{F}_T)$, where $\mathcal{P}(t,\omega) \subset \mathcal{P}_t^W$ for all $(t,\omega) \in [0,T]\times\Omega$. Denote also $\mathcal{P}_t := \cup_{\omega\in\Omega}\mathcal{P}(t,\omega)$. We make the following assumption on the family $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$.

Assumption 1 (i) *For every* $(t,\omega) \in [0,T] \times \Omega$*, one has* $\mathcal{P}(t,\omega) = \mathcal{P}(t,\omega_{\cdot\wedge t})$ *and* $\mathbb{P}(\Omega_t^\omega) = 1$ *whenever* $\mathbb{P} \in \mathcal{P}(t,\omega)$*. The graph* $[[\mathcal{P}]]$ *of* $\mathcal{P}$*, defined by* $[[\mathcal{P}]] := \{(t,\omega,\mathbb{P}) : \mathbb{P} \in \mathcal{P}(t,\omega)\}$*, is upper semi-analytic in* $[0,T] \times \Omega \times \mathbb{M}_1$*.*

(ii) $\mathcal{P}$ *is stable under conditioning, i.e. for every* $(t,\omega) \in [0,T] \times \Omega$ *and every* $\mathbb{P} \in \mathcal{P}(t,\omega)$ *together with an* $\mathbb{F}-$*stopping time* τ *taking values in* $[t,T]$*, there is a family of r.c.p.d.* $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ *such that* $\mathbb{P}_{\mathsf{w}}$ *belongs to* $\mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$*, for* $\mathbb{P}-a.e.$ $\mathsf{w} \in \Omega$*.*

(iii) $\mathcal{P}$ *is stable under concatenation, i.e. for every* $(t,\omega) \in [0,T] \times \Omega$ *and* $\mathbb{P} \in \mathcal{P}(t,\omega)$ *together with a* $\mathbb{F}-$*stopping time* τ *taking values in* $[t,T]$*, let* $(\mathbb{Q}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ *be a family of probability measures such that* $\mathbb{Q}_{\mathsf{w}} \in \mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$ *for all* $\mathsf{w} \in \Omega$ *and* $\mathsf{w} \longmapsto \mathbb{Q}_{\mathsf{w}}$ *is* $\mathcal{F}_\tau-$*measurable, then the concatenated probability measure* $\mathbb{P} \otimes_\tau \mathbb{Q}_\cdot \in \mathcal{P}(t,\omega)$*.*

We notice that for $t = 0$, we have $\mathcal{P}_0 := \mathcal{P}(0,\omega)$ for any $\omega \in \Omega$.

Spaces and norms

We now give the spaces and norms which will be needed in the rest of the paper. Fix some $t \in [0,T]$ and some $\omega \in \Omega$. In what follows, $\mathbb{G} := (\mathcal{G}_s)_{t\leq s\leq T}$ will denote an arbitrary filtration on $(\Omega, \mathcal{F}_T)$, and $\mathbb{P}$ an arbitrary element in $\mathcal{P}(t,\omega)$. Denote also by $\mathbb{G}^{\mathbb{P}}$ the $\mathbb{P}-$augmented filtration associated to $\mathbb{G}$.

For $p \geq 1$, $\mathbb{L}^p_{t,\omega}(\mathbb{G})$ (resp. $\mathbb{L}^p_{t,\omega}(\mathbb{G},\mathbb{P})$) denotes the space of all $\mathcal{G}_T-$measurable scalar random variable ξ with

$$\|\xi\|^p_{\mathbb{L}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty, \ \left(\text{resp. } \|\xi\|^p_{\mathbb{L}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty\right).$$

$\mathbb{H}^p_{t,\omega}(\mathbb{G})$ (resp. $\mathbb{H}^p_{t,\omega}(\mathbb{G},\mathbb{P})$) denotes the space of all $\mathbb{G}-$predictable $\mathbb{R}^d-$valued processes Z, which are defined $\widehat{a}_s ds - a.e.$ on $[t,T]$, with

$$\|Z\|^p_{\mathbb{H}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{\frac{1}{2}} Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < +\infty,$$
$$\left(\text{resp. } \|Z\|^p_{\mathbb{H}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{\frac{1}{2}} Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < +\infty\right).$$

$\mathbb{M}^p_{t,\omega}(\mathbb{G},\mathbb{P})$ denotes the space of all $(\mathbb{G},\mathbb{P})-$optional martingales M with $\mathbb{P}-a.s.$ càdlàg paths on $[t,T]$, with $M_t = 0$, $\mathbb{P}-a.s.$, and

$$\|M\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[[M]_T^{\frac{p}{2}}\right] < +\infty.$$

Furthermore, we say that a family $(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{M}^p_{t,\omega}((\mathbb{G}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if, for any $\mathbb{P} \in \mathbb{P}(t,\omega)$, $M^{\mathbb{P}} \in \mathbb{M}^p_{t,\omega}(\mathbb{G}^{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|M^{\mathbb{P}}\right\|_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{I}^p_{t,\omega}(\mathbb{G},\mathbb{P})$ denotes the space of all $\mathbb{G}-$predictable processes K with $\mathbb{P}-a.s.$ càdlàg and non-decreasing paths on $[t,T]$, with $K_t=0$, $\mathbb{P}-a.s.$, and

$$\|K\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K^p_T\right] < +\infty.$$

We will say that a family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{I}^p_{t,\omega}((\mathbb{G}^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if, for any $\mathbb{P}\in\mathcal{P}(t,\omega)$, $K^{\mathbb{P}}\in\mathbb{I}^p_{t,\omega}(\mathbb{G}_{\mathbb{P}},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{D}^p_{t,\omega}(\mathbb{G})$ (resp. $\mathbb{D}^p_{t,\omega}(\mathbb{G},\mathbb{P})$) denotes the space of all $\mathbb{G}-$progressively measurable $\mathbb{R}-$valued processes Y with $\mathcal{P}(t,\omega)-q.s.$ (resp. $\mathbb{P}-a.s.$) càdlàg paths on $[t,T]$, with

$$\|Y\|^p_{\mathbb{D}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq s\leq T}|Y_s|^p\right] < +\infty,\ \left(\text{resp. } \|Y\|^p_{\mathbb{D}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq s\leq T}|Y_s|^p\right] < +\infty\right).$$

For each $\xi\in\mathbb{L}^1_{t,\omega}(\mathbb{G})$ and $s\in[t,T]$ denote

$$\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{G}}_s[\xi] := \underset{\mathbb{P}'\in\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{G})}{\operatorname{ess\,sup}^{\mathbb{P}}}\ \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{G}_s] \text{ where } \mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{G}) := \left\{\mathbb{P}'\in\mathcal{P}(t,\omega),\ \mathbb{P}'=\mathbb{P} \text{ on } \mathcal{G}_s\right\}.$$

Then we define for each $p\geq\kappa\geq 1$,

$$\mathbb{L}^{p,\kappa}_{t,\omega}(\mathbb{G}) := \left\{\xi\in\mathbb{L}^p_{t,\omega}(\mathbb{G}),\ \|\xi\|_{\mathbb{L}^{p,\kappa}_{t,\omega}} < +\infty\right\},$$

where

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\underset{t\leq s\leq T}{\operatorname{ess\,sup}^{\mathbb{P}}}\left(\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{F}^+}_s[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Similarly, given a probability measure $\mathbb{P}$ and a filtration $\overline{\mathbb{G}}$ on the enlarged canonical space $\overline{\Omega}$, we denote the corresponding spaces by $\mathbb{D}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, $\mathbb{H}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, $\mathbb{M}^p_{t,\omega}(\overline{\mathbb{G}},\overline{\mathbb{P}})$, ... Furthermore, when $t=0$, there is no longer any dependence on ω, since $\omega_0=0$, so that we simplify the notations by suppressing the $\omega-$dependence and write $\mathbb{H}^p_0(\mathbb{G})$, $\mathbb{H}^p_0(\mathbb{G},\mathbb{P})$, ... Similar notations are used on the enlarged canonical space.

When there is no ambiguity (only one probability measure $\mathbb{P}$), the Brownian motion will be denoted by W and for simplicity in the notations of integrability spaces, we remove the reference to the filtration $\mathbb{F}$, the probability measure and ω: $\mathbb{D}^p_{0,\omega}(\mathbb{F},\mathbb{P})=\mathbb{D}^p$ and with the same convention $\mathbb{H}^p$, $\mathbb{M}^p$ and $\mathbb{I}^p$. Moreover for $\alpha\in\mathbb{R}$, for $(Z,M,K)\in\mathbb{H}^p\times\mathbb{M}^p\times\mathbb{I}^p$, we define

$$\begin{aligned}
\|Z\|^p_{\mathbb{H}^{p,\alpha}} &= \mathbb{E}\left[\left(\int_0^T e^{\alpha s}\|Z_s\|^2 ds\right)^{p/2}\right],\\
\|M\|^p_{\mathbb{M}^{p,\alpha}} &= \mathbb{E}\left[\left(\int_0^T e^{\alpha s} d[M]_s\right)^{p/2}\right]\\
\|K\|^p_{\mathbb{I}^{p,\alpha}} &= \mathbb{E}\left[\left(\int_0^T e^{\alpha s/2} dK_s\right)^{p}\right].
\end{aligned}$$

Part I

Singularity for monotone equations

Chapter 2

Monotone backward stochastic differential equations

The notion of linear backward stochastic differential equations (BSDE for short) was introduced by Bismut [38]. This equation describes the dynamics of the adjoint process of an optimal stochastic control problem (see among others [125, 317, 329]). Pardoux & Peng [271] extend the result for the non linear case in the Brownian setting. Let us immediately precise the notations. A solution of this equation, associated with a *terminal value* ξ and a *generator or driver* $f(t,\omega,y,z)$, is a couple of stochastic processes $(Y_t, Z_t)_{t\leq T}$ such that

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{2.1}$$

a.s. for all $t \leq T$, where W is a Brownian motion and processes $(Y_t, Z_t)_{t\leq T}$ are adapted to the natural filtration of W. In their seminal work [271], Pardoux and Peng proved existence and uniqueness of a solution under suitable assumptions, mainly square integrability of ξ and of process $(f(t,\omega,0,0))_{t\leq T}$, on the one hand, and, the Lipschitz property w.r.t. (y,z) of generator f, on the other hand. Since this first result, BSDEs have proved to be a powerful tool for formulating and solving a lot of mathematical problems arising for example in finance (see e.g. [29, 120, 301]), stochastic control and differential games (see e.g. [156, 157]), or partial differential equations (see e.g. [270, 273]). Thereby a lot of papers are devoted to weaken the conditions imposed for ξ and f in [271].

2.1 $\mathbb{L}^p$-solution under monotone conditions ([X, XV])

A huge part of the literature focuses on weakening the Lipschitz property of the coefficient f w.r.t. the y-variable. For example, Briand and Carmona [45] and Pardoux [270] consider the case of a monotone generator w.r.t. y with different growth conditions. There have been relatively few papers which deal with the problem of the existence and the uniqueness of solutions in the case where the coefficients are not square integrable. El Karoui et al. [120] and Briand et al. [48] have proved the existence and uniqueness of a solution for the standard BSDE (2.1) in the case where the data only belong to L^p for some $p \geq 1$.

Another strand of research in the theory of BSDEs concerns the underlying filtration. In [271] filtration $\mathbb{F} = \mathbb{F}^W$ is generated by the Brownian motion W. Since the work of Tang and Li [315], a lot of papers (see e.g. [24, 34, 256, 269, 302] or the books of Situ [307] or recently of Delong [91]) treat the case where the filtration is generated by the Brownian motion W and a Poisson random measure π independent of W. In most of these papers, the generator f is supposed to be Lipschitz in y, even if the monotonic case is mentioned (see [302]) and all coefficients are square integrable. The paper [326] studies the L^p case, $p > 1$, and gives existence and uniqueness result in the case where the generator is monotone but with at most linear growth w.r.t. y. [224] gives existence und uniqueness results for a fully coupled forward backward SDE under some monotonicity condition and L^p coefficients, $p \geq 2$. Note that this monotonicity condition involves the coefficients of the forward diffusion and is not the same as **(A2)**. An extension to BSDEs driven by a continuous local martingale X and an integer-valued random measure π has been studied in [325]. In this paper the author supposes that the filtration satisfies the representation property with respect to X and π (no additional orthogonal martingale term M) and that the driver is Lipschitz continuous and square integrable.

For more general filtrations, the representation property of a local martingale is no more true (see Section III.4 in [171]) and an additional (orthogonal) martingale term has to be introduced in the definition of a solution. This approach was developed in the seminal work of El Karoui and Huang [116] and by Carbone et al. [55] for càdlàg martingales. The filtration $\mathbb{F}$ is supposed to be complete, right continuous (sometimes with a quasi-left continuity condition). For a given square integrable martingale X ($\langle X \rangle$ denotes the predictable projection of the quadratic variation), the BSDE (2.1) becomes

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) d\langle X \rangle_s - \int_t^T Z_s dX_s - M_T + M_t. \tag{2.2}$$

The solution is now the triple (Y, Z, M) where M is a square integrable martingale orthogonal to X. Øksendal and Zhang [263] analyse BSDE of the form (2.2) where f does not depend on z, and apply to insider finance (see also Ceci et al. [65]). [226] also obtains results for a particular class of BSDE (2.2) on an arbitrary filtered probability space. In these papers, existence and uniqueness of the solution of (2.2) is proved for a Lipschitz continuous function f and under square integrability condition (in [263] the monotone case is treated but f does not depend on z). In [120], the authors consider the L^p-solution. The Hilbertian structure of $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is used in [80] (see also [195]). If $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ is a separable Hilbert space, then an orthogonal basis of martingales can be introduced instead of X and there is no orthogonal additional term M in (2.2). Z becomes a sequence of predictable processes. The special case of a Lévy noise is treated before by Nualart and Schoutens [260]: the orthogonal basis of martingales is explicitly given by the Teugels martingales. Klimsiak has developed the results concerning BSDEs in this general framework in two directions. First for reflected BSDE ([193, 195]), and secondly for parabolic equations ([196, 194]) with measure data. Finally let us mention the paper by Papapantolean et al. [267] (and the references in their introduction) where a more general setting is considered. The BSDE (2.2) is written under this setting as follows:

$$
\begin{aligned}
Y_t &= \xi + \int_t^T f(s, Y_s, Z_s, U_s) dC_s \\
&- \int_t^T Z_s dX_s^\circ - \int_t^T \int_{\mathcal{E}} U_s(e) \widetilde{\pi}^\natural(de, ds) - \int_t^T dM_s. \qquad (2.3)
\end{aligned}
$$

Roughly speaking, X° is a square-integrable martingale, $\widetilde{\pi}^\natural$ is integer-valued random measure, such that each component of $\langle X^\circ \rangle$ is absolutely continuous w.r.t. C and the disintegration property given C holds for the compensator $\nu^\natural$ of $\widetilde{\pi}^\natural$. The process C is only non-decreasing and càdlàg. The generator f is Lipschitz w.r.t. y, z and u and the data are square integrable.

2.1.1 Contributions

In [X, XV] we generalize many results from the works [24, 34, 256, 269, 302, 315] dealing with a filtration generated by the Brownian motion and the Poisson random measure since we allow for a more general filtration. Namely our filtration $\mathbb{F} = \{\mathcal{F}, 0 \leq t \leq T\}$ is supposed only to be complete and right continuous.

Moreover we provide existence and uniqueness of solutions in L^p-spaces, $p > 1$. In the case where the generator depends on the stochastic integrand w.r.t. a Poisson random measure, the case when $p < 2$ has to be handled carefully and can not be treated as in [48]. Indeed in this case Burkholder-Davis-Gundy inequality with $p/2 < 1$ does not apply and the $L^{p/2}$-norm of the predictable projection cannot be controlled by the $L^{p/2}$-norm of the quadratic variation (see [219]).

Compared to [55] or [325], our assumptions are in some sense more restrictive as we assume that the continuous part of the given martingale X of BSDE (2.2) is a Brownian motion W and the random measure associated to the jumps of X is a Poisson random measure π (see Equation (2.4)). However we weaken the assumptions on the driver and on the terminal condition: the generator is only supposed to be monotone and the terminal condition is allowed to be only L^p-integrable.

Note that our results have been already used in [41] and in [108].

Existence and uniqueness of the solution

We consider the following BSDE:

$$
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds - \int_t^T \int_{\mathcal{E}} U_s(e) \widetilde{\pi}(de, ds) - \int_t^T Z_s dW_s - \int_t^T dM_s. \qquad (2.4)
$$

The unknowns are (Y, Z, U, M) such that

- Y is càdlàg with values in $\mathbb{R}^d$;
- $Z \in L^2_{loc}(W)$, with values in $\mathbb{R}^{d \times k}$;
- $U \in G_{loc}(\mu)$ with values in $\mathbb{R}^d$;

- $M \in \mathcal{M}_{loc}$ with values in $\mathbb{R}^d$.

Our main result is the following.

Theorem 2.1 *Under the integrability condition* **(A1)** *for* ξ *and* f^0 *and Assumptions* $(\mathbf{A_{ex}})$ *on* f, *there exists a unique solution* (Y, Z, U, M) *in* $\mathcal{S}^p(0,T)$ *to BSDE* (2.4). *Moreover the solution satisfies the estimate:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^p + \left(\int_0^T |Z_t|^2 dt\right)^{p/2} + \left(\int_0^T \int_{\mathcal{E}} |U_s(e)|^2 \pi(de,ds)\right)^{p/2} + ([M]_T)^{p/2}\right]$$
$$\leq C_{p,\mathcal{X},K_f,T}\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(r,0,0,\mathbf{0})|^p dr\right)\right]. \tag{2.5}$$

Some comments on the result. The difference between $p \geq 2$ and $p < 2$ is crucial to define and to solve properly the BSDE. To illustrate our purpose, let us consider a stable Lévy process $X = (X_t,\ 0 \leq t \leq T)$. The Lévy measure is $\mu(de) = \frac{1}{|e|^{1+\alpha}} de$ where $e \in \mathcal{E} = \mathbb{R} \setminus \{0\}$ and $0 < \alpha < 2$. Then by the Lévy-Khintchine decomposition:

$$\begin{aligned} X_t &= \int_0^t \int_{\mathcal{E}} e \mathbb{1}_{|e|<1} \widetilde{\pi}(de,ds) + \int_0^t \int_{\mathcal{E}} e \mathbb{1}_{|e|\geq 1} \pi(de,ds) \\ &= \int_0^t \int_{\mathcal{E}} e \widetilde{\pi}(de,ds) + t \int_{\mathcal{E}} e \mathbb{1}_{|e|\geq 1} \mu(de) = \int_0^t \int_{\mathcal{E}} e \widetilde{\pi}(de,ds). \end{aligned}$$

Now $X_T \in \mathbb{L}^p(\Omega)$ if and only if $p < \alpha < 2$. We take $\xi = X_T$, $Y_t = X_t$, $U_t(e) = e$ and

$$Y_t = \xi - \int_t^T \int_{\mathcal{E}} e \widetilde{\pi}(de,ds).$$

For any $t \in [0,T]$, $p < \alpha$, Y_t is in $\mathbb{L}^p(\Omega)$ and $U_t \notin \mathbb{L}^2_\mu$. Nevertheless for any $\delta > 0$, $\phi^1_t = U_t \mathbb{1}_{|U_t|\leq \delta}$ belongs to $\mathbb{L}^2_\mu$ and $\phi^2_t = U_t \mathbb{1}_{|U_t|\geq \delta}$ to $\mathbb{L}^p_\mu$. Thus U_t is in $\mathbb{L}^p_\mu + \mathbb{L}^2_\mu$. And it is easy to check that U_t also belongs to $\mathbb{L}^1_\mu + \mathbb{L}^2_\mu$.

- Assume that $p \geq 2$. A priori estimate (2.5) together with Inequality (1.5) imply that U is in $\mathbb{L}^p(\mathbb{L}^2_\nu)$ and $\mathbb{L}^p(\mathbb{L}^p_\nu)$: for some constant C depending only on p, α, K and T

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T \int_{\mathcal{E}} |U_s(e)|^2 \mu(de) ds\right)^{p/2}\right] &\leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f^0_r|^p dr\right)\right] \\ \mathbb{E}\left[\left(\int_0^T \int_{\mathcal{E}} |U_s(e)|^p \mu(de) ds\right)\right] &\leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f^0_r|^p dr\right)\right]. \end{aligned}$$

  In particular $\mathbb{P} \otimes \mathrm{Leb}$ a.e. on $\Omega \times [0,T]$, $U_s \in \mathbb{L}^2_\mu$. Hence the generator of our BSDE will be defined on $\mathbb{L}^2_\mu$.

- But if $p < 2$, from (1.4), the solution satisfies:

$$\|U\|^p_{\mathbb{L}^p(\mathbb{L}^2_\nu)+\mathbb{L}^p(\mathbb{L}^p_\nu)} \leq C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f^0_r|^p dr\right)\right].$$

By Jensen's inequality, $\mathbb{P} \otimes \text{Leb}$-a.s. on $\Omega \times [0,T]$, U_t is in $\mathbb{L}^p_\mu + \mathbb{L}^2_\mu$. In [XV], we show that U_t is then also in $\mathbb{L}^1_\mu + \mathbb{L}^2_\mu$. Thereby for $p < 2$, our generator will be defined on $\mathbb{L}^1_\mu + \mathbb{L}^2_\mu$.

This justifies why f is defined on $\mathfrak{B}^2_\mu$, given by (1.6).

Let us describe the *outline of the proof*.

1. For $p = 2$, the arguments are rather standard. We use a truncation procedure to obtain bounded data. For bounded data:

   (a) If f is Lispchitz continuous w.r.t. y, the solution is obtained by a fixed point argument in $\mathcal{S}^2$.

   (b) Then we prove that the result holds for monotone generator with a stronger growth condition: $|f(t,y,0,\mathbf{0}) - f(t,0,0,\mathbf{0})| \leq \vartheta(|y|)$ where $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ is a deterministic continuous increasing function.

   (c) Finally we extend this for the general growth condition **(A3)**.

   Existence and uniqueness for L^2-data is then deduced using the stability result (see below) in $\mathcal{S}^2$. Roughly speaking we construct bounded data ξ^n and f^n, such that ξ^n and f^n converge to ξ and f and a sequence (Y^n, Z^n, U^n, M^n) of solutions associated to ξ^n and f^n, which converges in $\mathcal{S}^2$ to the solution.

2. For $p \geq 2$, since $x \mapsto |x|^p$ is a smooth function, we modify the last previous step to prove that the constructed $\mathcal{S}^2$-solution is in fact also in $\mathcal{S}^p$.

3. For $p < 2$, the arguments are more involved. First of all, as in [48] or [195], we explain in our setting how to deal with $x \mapsto |x|^p$ which is not smooth. Then we want to derive the a priori estimate (2.5). If generator f does not depend on the integrand in the Poisson stochastic integral ψ, then the result can be obtained by direct modifications of classical arguments. Nevertheless in general we have to control the term

   $$pK \int_0^t e^{\beta s} |Y_{s-}|^{p-1} \|U_s\|_{\mathbb{L}^1_\mu + \mathbb{L}^2_\mu} ds$$

   coming from generator f, with the quantity

   $$\int_0^t e^{\beta s} \int_{\mathcal{E}} \left[|Y_{s-} + U_s(e)|^p - |Y_{s-}|^p - p|Y_{s-}|^{p-1} \check{Y}_{s-} U_s(e) \right] \pi(de, ds)$$

   coming from the Itô formula. This point requires some *new arguments* developed in [XV] (see Section 4 of this paper and the technical results therein).

Note that as a byproduct of our results we have the stability result:

Proposition 2.1 (Stability) *Let (ξ, f) and (ξ', f') be two sets of data each satisfying the above assumptions* $(\mathbf{A_{ex}})$. *Let (Y, Z, U, M) (resp. (Y', Z', U', M')) denote a $\mathcal{S}^p$-solution of BSDE* (2.4) *with data (ξ, f) (resp. (ξ', f')). Define*

$$(\mathfrak{d}Y, \mathfrak{d}Z, \mathfrak{d}U, \mathfrak{d}M, \mathfrak{d}\xi, \mathfrak{d}f) = (Y - Y', Z - Z', U - U', M - M', \xi - \xi', f - f').$$

Then there exists a constant C depending on p, χ, K_f and T, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\mathfrak{d}Y_t|^p+\left(\int_0^T|\mathfrak{d}Z_s|^2ds\right)^{p/2}+\left(\int_0^T\int_{\mathcal{E}}|\mathfrak{d}U_s(e)|^2\pi(de,ds)\right)^{p/2}+([\mathfrak{d}M]_T)^{p/2}\right]$$
$$\leq C\mathbb{E}\left(|\mathfrak{d}\xi|^p+\int_0^T|\mathfrak{d}f(t,Y'_t,Z'_t,U'_t)|^pdt\right).$$

Comparison principle

We provide a comparison principle ([X, Proposition 4]) for BSDE of type (2.4). We assume that $d=1$ and aim at comparing two solutions Y^1 and Y^2 of BSDE (2.4) with coefficients (ξ^1,f^1) and (ξ^2,f^2). As in the papers [24, 302, 307, 299], we have to restrict the dependence of f w.r.t. ψ. Some monotonicity w.r.t. ψ is necessary, namely **(A5')**. We generalize the arguments of [299] to the situation where the filtration is not only generated by Brownian and Poisson noises. Then we prove:

Proposition 2.2 (Comparison, [X], Proposition 4) *We consider a generator f_1 satisfying ($\mathbf{A_{ex}}$) and we ask f_2 to verify ($\mathbf{A_{comp}}$). Let ξ^1 and ξ^2 be two terminal conditions for BSDE (2.4) driven respectively by f_1 and f_2. Denote by (Y^1,Z^1,U^1,M^1) and (Y^2,Z^2,U^2,M^2) the respective solutions in some space $\mathcal{S}^p(0,T)$ with $p>1$. If $\xi^1\leq\xi^2$ and $f_1(t,Y^1_t,Z^1_t,U^1_t)\leq f_2(t,Y^1_t,Z^1_t,U^1_t)$, then a.s. for any $t\in[0,T]$, $Y^1_t\leq Y^2_t$.*

Again we emphasize that the process κ of the condition **(A5')** only satisfies $\kappa\geq-1$. Thereby a strict comparison principle does not hold in general, that is $\xi_1<\xi^2$ does not imply that $Y^1_t<Y^2_t$ for all $t\in[0,T]$. Indeed we cannot use Girsanov's transform as in [302]. Moreover it is well known that the monotonicity condition **(A2)** is not sufficient to obtain strict comparison (see [276, Proposition 5.34] and the comments just after).

Random terminal time

Finally we extend the previous results when $T=\tau$ is a stopping time for filtration $\mathbb{F}$. τ is not necessarily bounded ([X, Section 6]). We want to solve the following BSDE: $\mathbb{P}$-a.s., for all $0\leq t\leq T$,

$$\begin{aligned}Y_{t\wedge\tau} &= Y_{T\wedge\tau}+\int_{t\wedge\tau}^{T\wedge\tau}f(s,Y_s,Z_s,U_s)ds-\int_{t\wedge\tau}^{T\wedge\tau}Z_sdW_s\\ &- \int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{E}}U_s(e)\widetilde{\pi}(de,ds)-\int_{t\wedge\tau}^{T\wedge\tau}dM_s\end{aligned}\tag{2.6}$$

with the condition that $\mathbb{P}$-a.s. on the set $\{t\geq\tau\}$, $Y_t=\xi$ and $Z_t=U_t=M_t=0$. On the generator, Assumptions ($\mathbf{A_{ex}}$) still hold with a monotonicity constant χ and a Lipschitz constant $K_{f,z}$ w.r.t. z and $K_{f,\psi}=\|\vartheta\|$ w.r.t. ψ (see Lemma 1.2). We denote by K the constant

$$K^2=\frac{1}{2}K_f^2=\frac{1}{2}(L^2_{f,z}+L^2_{f,\psi}).$$

However the growth condition **(A3)** is replaced by:

(A3") For any $r > 0$ and $n \in \mathbb{N}$

$$\sup_{|y|\leq r}(|f(t,y,0,\mathbf{0}) - f(t,0,0,\mathbf{0})|) \in L^1(\Omega \times (0,n)).$$

Condition **(A1)** is replaced by

(A1.1") For some $p > 1$

$$\mathbb{E}\left[e^{p\rho\tau}|\xi|^p + \int_0^\tau e^{p\rho t}|f(t,0,0,\mathbf{0})|^p dt\right] < +\infty.$$

Constant ρ satisfies

$$\rho > \nu = \nu(p) := \begin{cases} \chi + K^2 & \text{if } p \geq 2, \\ \chi + \dfrac{K^2}{p-1} + \dfrac{K_{f,u}^2}{\varepsilon(p, K_{f,u})} & \text{if } p < 2. \end{cases} \tag{2.7}$$

Constant $0 < \varepsilon(p, K_{f,u}) < p - 1$ only depends on $K_{f,u}$ and p (see [XV, Section 4]). The additional term in ν disappears if the generator does not depend on the jump part u (that is, if $K_{f,u} = 0$). Even if we cannot compute $\varepsilon(p, K_{f,u})$ explicitly, we know that

$$0 < \varepsilon(p, K_{f,u}) \leq (p-1)\left(2(\alpha(p, K_{f,u}) + 1)^2 - 1\right)^{-\frac{2-p}{2}},$$

and $\alpha(p, K_{f,u})$ has to be chosen such that for any $x \geq \alpha(p, K_{f,u})$,

$$\frac{1}{2^{p/2}}x^p - 2^{p/2} - 1 - p(2K_{f,u} + 1)x \geq 0.$$

The right-hand side is an increasing function w.r.t. $p \in (1,2)$ and decreasing w.r.t. $K_{f,u} \geq 0$. Hence when p is close to one and $K_{f,u}$ is large, ε is be very small and thus ρ becomes large. From the appendix in [XV], one choice for $\varepsilon(p, K_{f,u})$ is

$$\varepsilon(p, K_{f,u}) = \frac{p-1}{\left(C_u^{\frac{1}{p-1}} + 1\right)^{2-p}}, \qquad C_u = (4(2K_{f,u} + 2) + 1). \tag{2.8}$$

We suppose that

(A1.2") ξ is $\mathcal{F}_\tau$-measurable and

$$\mathbb{E}\left[\int_0^\tau e^{p\rho t}|f(t, e^{-\nu t}\xi_t, e^{-\nu t}\eta_t, e^{-\nu t}\gamma_t)|^p dt\right] < +\infty,$$

where $\xi_t = \mathbb{E}(e^{\nu\tau}\xi|\mathcal{F}_t)$ and (η, γ, N) are given by the martingale representation:

$$e^{\nu\tau}\xi = \mathbb{E}(e^{\nu\tau}\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s(e)\widetilde{\pi}(de, ds) + N_\tau$$

with

$$\mathbb{E}\left[\left(\int_0^\infty |\eta_s|^2 ds + \int_0^\infty \int_{\mathcal{E}} |\gamma_s(e)|^2 \pi(de, ds) + [N]_\tau\right)^{p/2}\right] < +\infty.$$

Condition **(A1.2")** was introduced in [48]. Indeed with **(A1.1")**, we know that $e^{\rho\tau}\xi$ belongs to $\mathbb{L}^p(\Omega)$, not but ξ itself. However $e^{\nu\tau}\xi$ satisfies the required integrability condition:

$$\mathbb{E}\left[e^{p\nu\tau}|\xi|^p\right] = \mathbb{E}\left[e^{p\rho\tau}|\xi|^p e^{p(\nu-\rho)\tau}\right] \leq \mathbb{E}\left[e^{p\rho\tau}|\xi|^p\right] < +\infty.$$

Hence in the proof of the next theorem, $\mathbb{E}(e^{\nu\tau}\xi|\mathcal{F}_n)$ can be used as an L^p-terminal condition at time n, but not $\mathbb{E}(\xi|\mathcal{F}_n)$ in general. This is why the assumption **(A1.2")** is required. Note that if $\rho \geq 0$, then **(A1.1")** is sufficient.

Theorem 2.2 *Under the above conditions* **(A1.1")**, **(A1.2")**, **(A2)**, **(A3")**, **(A4)** *and* **(A5)** , *BSDE* (2.6) *has a unique solution satisfying for any* $0 \leq t \leq T$

$$\mathbb{E}\left[e^{p\rho(t\wedge\tau)}|Y_{t\wedge\tau}|^p + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^p ds + \int_0^{T\wedge\tau} e^{p\rho s}|Y_s|^{p-2}|Z_s|^2 \mathbb{1}_{Y_s\neq 0} ds\right]$$
$$+\mathbb{E}\left[\int_0^{T\wedge\tau} e^{p\rho s}|Y_{s-}|^{p-2}\mathbb{1}_{Y_{s-}\neq 0} d[M]^c_s\right]$$
$$+\mathbb{E}\left[\int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{E}} e^{p\rho s}\left(|Y_{s-}|^2 \vee |Y_{s-}+U_s(e)|^2\right)^{p/2-1}\mathbb{1}_{|Y_{s-}|\vee|Y_{s-}+U_s(e)|\neq 0}|U_s(e)|^2\pi(de,ds)\right]$$
$$+\mathbb{E}\left[\sum_{0<s\leq T\wedge\tau} e^{p\rho s}|\Delta M_s|^2\left(|Y_{s-}|^2 \vee |Y_{s-}+\Delta M_s|^2\right)^{p/2-1}\mathbb{1}_{|Y_{s-}|\vee|Y_{s-}+\Delta M_s|\neq 0}\right] < +\infty.$$

Moreover

$$\mathbb{E}\left[\left(\int_0^\tau e^{2\rho s}|Z_s|^2 ds\right)^{p/2} + \left(\int_0^\tau e^{2\rho s}\int_{\mathcal{E}}|U_s(e)|^2\pi(de,ds)\right)^{p/2} + \left(\int_0^\tau e^{2\rho s}d[M]_s\right)^{p/2}\right]$$
$$\leq C\mathbb{E}\left[e^{p\rho\tau}|\xi|^p + \int_0^\tau e^{p\rho s}|f(s,0,0,0)|^p ds\right].$$

The constant C depends only on p, K and χ.

In general **(A1.2")** is not easy to check. Nonetheless if ξ is bounded, we can take $\nu = 0$ in **(A1.2")** and assume that:

$$\mathbb{E}\left[\int_0^\tau e^{p\rho t}|f(t,\xi_t,\eta_t,\gamma_t)|^p dt\right] < +\infty,$$

where $\xi_t = \mathbb{E}(\xi|\mathcal{F}_t)$ and

$$\xi = \mathbb{E}(\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty\int_{\mathcal{E}}\gamma_s(e)\widetilde{\pi}(de,ds) + N_\tau.$$

2.1.2 Open problems

The generalization of our results for BSDE of the form (2.2) or (2.3) requires some sophisticated integrability conditions to take account of the predictable projection $\langle X\rangle$ of the quadratic variation of X. Therefore it is left for future research.

Another open question concerns L^1-condition for ξ or for f^0. This problem is solved in [48, 46], when the terminal time is deterministic. The extension to a general filtration for a BSDE with jumps has also to be done. If the terminal time is random, let us simply remark that constant ρ explodes. These questions are also left for further research.

2.2 Reflected BSDE ([VII, XVII])

The one barrier reflected BSDEs have been introduced by El Karoui et al. [117]. For those BSDEs, one of the components of the solution is forced to stay above a given *barrier or obstacle* process $(L_t)_{t\leq T}$. Therefore a solution is a triple of processes $(Y_t, Z_t, K_t)_{t\leq T}$ which satisfies:

$$
\begin{aligned}
(2.9) \qquad & Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\
& Y_t \geq L_t, \quad 0 \leq t \leq T \text{ and } \int_0^T (Y_s - L_s)dK_s = 0, \ \mathbb{P} - \text{a.s.}.
\end{aligned}
$$

Here process K is non-decreasing and its role is to push upward Y in order to keep it above obstacle L. Under square integrability of the data and Lipschitz property of generator f, the authors of [117] show the existence and uniqueness of the solution. These types of equations are connected with a wide range of applications especially the pricing of American options in markets, constrained or not, mixed control, partial differential variational inequalities, real options (see e.g. [115, 117, 120] and the references therein). Another example of applications of our results is in connection with mixed control with only p-integrable coefficients (see e.g. [57]). Actually solving this latter problem turns into solving appropriate reflected BSDEs.

There have been a lot of works which deal with the issue of existence/uniqueness results under weaker assumptions than the ones of El Karoui et al [117]. However before our work, for their own reasons, authors mainly focus on weakening the Lipschitz property of the coefficient (see for example [220] for monotone assumption on f) or the regularity of barrier L and not on square integrability of the data ξ and $(f(t, \omega, 0, 0))_{t\leq T}$. When we begun to work on this problem, there have been relatively few papers which deal with the problem of existence/uniqueness of the solution for BSDEs in the case when the coefficients are not square integrable. Nevertheless El Karoui et al. [120] or Briand et al. [48] have proved the existence and uniqueness of a solution for the standard BSDE (2.1) in the case when the data belong only to L^p for some $p \in]1, 2[$. Therefore the main objective of our paper [VII] was to complete those works and to study the reflected BSDE (2.9) in the case when the terminal condition ξ and generator f are only p-integrable with $p \in]1, 2[$. In [XVII], we also extend the existence and uniqueness results for RBSDE with monotone generator and in a general filtration.

2.2.1 Reflected BSDE with L^p-data

In [VII] we show that if ξ, $\sup_{t\leq T}(L_t^+)$ and $\int_0^T |f(t, 0, 0)|dt$ belong to L^p for some $p \in]1, 2[$, then BSDE (2.9) with one reflecting barrier associated with (f, ξ, L) has a unique solution. More precisely here $\mathbb{F}$ is the natural filtration of the Brownian motion W. We suppose that **(A1)**, **(A2')** and **(A4)** hold, that is:

- The $\mathcal{F}_T$-measurable random variable ξ is in $\mathbb{L}^p(\Omega)$;
- Generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is measurable with respect to $Prog \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$ and such that:

(i) Process $\{f_t^0,\ 0 \le t \le T\}$ satisfies $\mathbb{E}\left(\int_0^T |f(t,0,0)|^p dt\right) < +\infty$.

(ii) There exists a constant K_f such that:

$$\mathbb{P} - \text{a.s.},\ |f(t,y,z) - f(t,y',z')| \le K_f(|y-y'| + |z-z'|),\ \forall t,y,y',z,z'.$$

Barrier $L = \{L_t\}_{t\in[0,T]}$ is a continuous progressively measurable $\mathbb{R}$-valued process such that $L_T \le \xi$ and $L^+ := L \vee 0 \in \mathbb{D}^p(0,T)$.

Theorem 2.3 *The reflected BSDE* (2.9) *associated with* (f,ξ,L) *has a unique* L^p*-solution, that is*

1. $\{(Y_t,Z_t),\ 0 \le t \le T\}$ *belongs to* $\mathbb{D}^p \times \mathbb{H}^p$*;*
2. $K = \{K_t,\ 0 \le t \le T\}$ *is an adapted continuous non-decreasing process s.t.* $K_0 = 0$ *and* $K_T \in L^p(\Omega)$*;*
3. $Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s,\ 0 \le t \le T$ *a.s.;*
4. $Y_t \ge L_t,\ 0 \le t \le T$*;*
5. $\int_0^T (Y_s - L_s)dK_s = 0$*,* $\mathbb{P}$*-a.s..*

We prove existence and uniqueness of the solution using penalization and Snell envelope methods. Moreover we also obtain the *a priori estimate*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^p + \left(\int_0^T |Z_s|^2 ds\right)^{p/2} + (K_T)^p\right] \\
&\quad \le C_{p,K_f}\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(s,0,0)|ds\right)^p + \left(\sup_{t\in[0,T]} (L_s^+)^p\right)\right]
\end{aligned}$$

and the *stability result*. Assume that (f,ξ,L) and (f',ξ',L') are two triplets satisfying the above conditions. Suppose that (Y,Z,K) is a solution of the RBSDE (f,ξ,L) and (Y',Z',K') is a solution of the RBSDE (f',ξ',L'). Let us set:

$$\begin{array}{lll}
\mathfrak{d}f = f - f', & \mathfrak{d}\xi = \xi - \xi' & \mathfrak{d}L = L - L' \\
\mathfrak{d}Y = Y - Y', & \mathfrak{d}Z = Z - Z' & \mathfrak{d}K = K - K'
\end{array}$$

and assume that $\mathfrak{d}L \in \mathbb{D}^p(0,T)$. Then there exists a constant C such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]} |\mathfrak{d}Y_t|^p + \frac{p(p-1)}{2}\int_0^T |\mathfrak{d}Y_s|^{p-2}\mathbf{1}_{\mathfrak{d}Y_s\neq 0}|\mathfrak{d}Z_s|^2 ds\right] \\
&\quad \le C\mathbb{E}\left[|\mathfrak{d}\xi|^p + \left(\int_0^T |\mathfrak{d}f(s,Y_s,Z_s)|ds\right)^p\right] + C(\Psi_T)^{1/p}\left[\mathbb{E}\sup_{t\in[0,T]} |\mathfrak{d}L_t|^p\right]^{\frac{p-1}{p}},
\end{aligned}$$

with

$$\Psi_T = \mathbb{E}\left[|\xi|^p + \left(\int_0^T |f(u,0,0)|du\right)^p + \left(\sup_{t\in[0,T]} (L_t^+)^p\right)\right.$$
$$\left. +|\xi'|^p + \left(\int_0^T |f'(u,0,0)|du\right)^p + \left(\sup_{t\in[0,T]} ((L'_t)^+)^p\right)\right].$$

Finally the *comparison principle* holds: if a.s. $\xi' \geq \xi$, $f' \geq f$ and $L' \geq L$, then a.s. $Y'_t \geq Y_t$ for every $t \in [0,T]$.

2.2.2 Some extensions of our paper

Let us mention some extensions of our paper. We already cite the work [220] where $p = 2$ but f is monotone w.r.t. y. Let us mention the extension for two barriers in [114]. In [192, 193, 195], Klimsiak assumes that the filtration is only right-continuous, complete and quasi left-continuous[1], $p \geq 1$, f is monotone w.r.t. y and barrier L is only progressively measurable (no regularity condition w.r.t. t). He constructs the unique solution of the reflected BSDE:

$$Y_t = \xi + \int_t^T f(s,Y_s)ds + K_T - K_t - \int_t^T dM_s, \qquad Y_t \geq L_t.$$

The case of two barriers and when f depends on z (for $p = 2$) is also developed in [195]. The minimality condition on K is: for every càdlag̃ process $\widehat{L}$ such that $L_t \leq \widehat{L}_t \leq Y_t$ for a.e. $t \in [0,T]$, $\int_0^T (Y_{t-} - \widehat{L}_{t-})dK_t = 0$. The main drawback of this condition is that Y does not satisfy:

$$Y_t = \operatorname*{esssup}_{\tau} \mathbb{E}\left[\int_t^\tau f(s,Y_s)ds + L_\tau \mathbb{1}_{\tau<T} + \xi \mathbb{1}_{\tau=T} \middle| \mathcal{F}_t\right]$$

where τ is in the set of stopping times with values in $[0,T]$. For a general barrier, in [149, 150, 197] the authors find a solution satisfying the previous relation when

- the filtration is generated by a Brownian motion W and $p \geq 1$ in [197],
- the filtration also supports a Poisson random measure π and $p = 2$ in [149, 150].

The minimality condition on K is then more delicate.

The case where the filtration is only right-continuous and complete and $p > 1$ is considered in [41] (in this paper the barrier L is càdlàg):

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s - \int_t^T dM_s + K_T - K_t, \qquad Y_t \geq L_t.$$

Generator f is Lipschitz continuous in y and z. The main difficulty in this setting is to control the jumps of the martingale part M and of the process K. Because of

[1] For any sequence τ_n of predictable stopping times such that $\tau_n \nearrow \tau$, $\mathcal{F}_\tau = \bigvee_{n\geq 1} \mathcal{F}_{\tau_n}$.

the quasi left-continuity assumption of the filtration, the martingale M cannot jump at predictable times, and thus the bracket $[M,K]$ is identically equal to 0. This is no longer true for general filtrations and it turns out to be difficult to control the term $[M,K]$ (see the introduction of [41]).

2.2.3 Monotone RBSDE and general filtration ([XVII])

As already mentioned, monotone RBSDE have been already studied in [220] or in [193]. Nevertheless in these papers, the authors assume that the filtration is generated by the Brownian motion W. If we want to study second-order BSDEs following the program established in [295], we need to generalize these results to a more general filtration. RBSDE (2.9) becomes

$$Y_t = \xi + \int_t^T f(u, Y_u, Z_u) du - \int_t^T Z_u dW_u - \int_t^T dM_u + \int_t^T dK_u \tag{2.10}$$

with $Y_t \geq L_t$ and $\int_0^T (Y_{t-} - L_{t-}) dK_t = 0$, $\mathbb{P}$-a.s. (*Skorokhod condition*). L is a càdlàg process such that $L^+ = L \vee 0$ belongs to $\mathbb{D}^p(0,T)$.

In [XVII, Appendix A.3], we study RBSDE (2.10) under the monotone condition **(A2)** for f, but with the polynomial growth assumption **(A3')**:

$$|f(t,y,0) - f(t,0,0)| \leq \Psi_t (1 + |y|^q).$$

The next result is [XVII, Proposition A.2].

Theorem 2.4 *Assume that* **(A2)**, **(A3')** *and* **(A4)** *hold where constant $q > 1$ is fixed and Ψ is in some $\mathbb{L}^\varrho((0,T) \times \Omega)$ for some $\varrho > 1$. If*

$$\mathbb{E}\left[|\xi|^{\bar{p}q} + \int_0^T |f(t,0,0)|^{\bar{p}q} + \sup_{t \in [0,T]} |L_t^+|^{\bar{p}q} \right] < +\infty$$

with $\bar{p} > \frac{\varrho}{\varrho - 1}$, then there exists a unique solution (Y,Z,M,K) such that $(Y,Z,M) \in \mathbb{D}^{\bar{p}q}(0,T) \times \mathbb{H}^p(0,T) \times \mathbb{M}^p(0,T)$ and $\mathbb{E}(|K_T|^p) < +\infty$ for $1 < p \leq \frac{\bar{p}\varrho}{\bar{p}+\varrho} \leq \bar{p}$.

Obviously the growth condition **(A3')** and this integrability assumption on ξ, $f(t,0,0)$ and L^+ are not optimal (compared to L^p-solution for non reflected BSDE); these points are left for further research.

2.3 Backward doubly stochastic differential equations ([XIII])

Backward Doubly Stochastic Differential Equations (BDSDEs for short) have been introduced by Pardoux and Peng [272] to provide a non-linear Feynman-Kac formula for classical solutions of SPDE. The main idea is to introduce in the standard BSDE a second nonlinear term driven by an external noise representing the random perturbation of the nonlinear SPDE. Hence BSDE (2.1) becomes:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\, dr + \int_t^T g(r, Y_r, Z_r)\, \overleftarrow{dB_r} - \int_t^T Z_r dW_r,\ 0 \leq t \leq T, \tag{2.11}$$

where B and W are two independent Brownian motions, and $\overleftarrow{dB_r}$ is the backward Itô integral. Pardoux and Peng [272] have proven existence and uniqueness for solutions of BDSDE (2.11) if f and g are supposed to be Lipschitz continuous functions and with square integrability condition for the terminal condition ξ and for coefficients $f(t, 0, 0)$ and $g(t, 0, 0)$. Moreover under smoothness assumptions of the coefficients, Pardoux and Peng prove existence and uniqueness of a classical solution for the related SPDE.

Motivating by singular SPDE, our first goal in [XIII] was to prove the existence and uniqueness of the solution of a BDSDE with monotone generator f. The existence of a solution relies on the solvability of the BSDE:

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr - \int_t^T Z_r dW_r,\ 0 \leq t \leq T.$$

See among others the proofs in [270, 276] or in [X]. To obtain a solution for this BSDE, the main trick is to truncate the coefficients with suitable truncation functions in order to have a bounded solution Y. This step cannot be done for a general BDSDE. Indeed take for example ($\xi = f = 0$ and $g = 1$):

$$Y_t = \int_t^T \overleftarrow{dB_r} - \int_t^T Z_r dW_r = B_T - B_t,\ 0 \leq t \leq T,$$

with $Z = 0$. Thereby Y is not bounded and in order to prove existence of a solution for (2.11), one can not directly follow the scheme of [270].

To realize this project we restrict the class of functions f: they should satisfy a polynomial growth condition (as in [45]). Until now we do not know how to extend this to general growth condition as in [48, 270, 276] or [X].

2.3.1 Setting and notations

Let us now precise our notations. W and B are independent Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^k$ and $\mathbb{R}^m$. Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where for any process η, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s;\ s \leq r \leq t\} \vee \mathcal{N}$, $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. As in [272] we define the following filtration $\mathbb{G} = (\mathcal{G}_t,\ t \in [0, T])$ by:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_{0,T}^B.$$

ξ is a $\mathcal{F}_T^W$-measurable and $\mathbb{R}^d$-valued random variable.

We modify a little bit the spaces of processes introduced in section 1.1.2. We define by $\mathbb{H}^p_{\mathbb{G}}(0, T)$ the set of (classes of $d\mathbb{P} \times dt$ a.e. equal) N-dimensional jointly measurable random processes $(X_t,\ t \geq 0)$ which satisfy:

1. $\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right] < +\infty$

2. X_t is $\mathcal{G}_t$-measurable for a.e. $t \in [0,T]$.

We denote similarly by $\mathbb{D}^p_{\mathbb{G}}(0,T)$ the set of continuous N-dimensional random processes which satisfy:

1. $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty$

2. X_t is $\mathcal{G}_t$-measurable for any $t \in [0,T]$.

Definition 2.1

- *$\mathcal{S}^p_{\mathbb{G}}(0,T)$ denotes the product space $\mathbb{D}^p_{\mathbb{G}}(0,T) \times \mathbb{H}^p_{\mathbb{G}}(0,T)$.*
- *$(Y,Z) \in \mathcal{S}^p(0,T)$ if $(Y,Z) \in \mathcal{S}^p_{\mathbb{G}}(0,T)$ and Y_t and Z_t are $\mathcal{F}_t$-measurable.*

Let us point out that $\mathcal{S}^p_{\mathbb{G}}(0,T)$ is the same space as in Section 1.1.2, expect that the filtration is $\mathbb{G}$. Recall that family $(\mathcal{F}_t,\ t \in [0,T])$ is not a filtration. This is the reason why we distinguish $\mathcal{S}^p_{\mathbb{G}}(0,T)$ (standard integrability space for BSDEs) and $\mathcal{S}^p(0,T)$, where there is the additional measurability constraint.

Now we precise our assumptions on f and g. Functions f and g are defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times k}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$ such that f^0 and g^0 are progressively measurable. Generator f satisfies Conditions **(A2)** (monotone w.r.t. y) and **(A4)** (Lipschitz continuous w.r.t. z). But we reinforce Assumption **(A3')** (and **(A3)**):

(A3*) There exist $C_f \geq 0$ and $q > 0$ such that

$$|f(t,y,z) - f(t,0,z)| \leq C_f(1 + |y|^{q+1}).$$

Concerning function g, we suppose:

(Ag1) There exist a constant $K_g \geq 0$ and $0 < \varepsilon < 1$ such that for any (t,y,y',z,z') a.s.

$$|g(t,y,z) - g(t,y',z')|^2 \leq K_g|y-y'|^2 + \varepsilon|z-z'|^2.$$

Remember that from [272] if f also satisfies **(A2')**: there exists $K_{f,y}$ such that for any (t,y,y',z) a.s.

$$|f(t,y,z) - f(t,y',z)| \leq K_{f,y}|y-y'|$$

and if $\xi \in \mathbb{L}^2(\Omega)$ f^0 and g^0 are in $\mathbb{L}^2(\Omega \times [0,T])$, then there exists a unique solution $(Y,Z) \in \mathcal{S}^2(0,T)$ to the BDSDE (2.11). Note that **(A2')** implies that

$$|f(t,y,z) - f(t,0,z)| \leq K_{f,y}|y|,$$

thus the growth assumption **(A3*)** on f is satisfied with $q = 0$.

2.3.2 Results for monotone BDSDE

In [XIII] we prove the following result.

Theorem 2.5 *Under assumptions* **(A2)**, **(A3*)** *and* **(Ag1)**, *if the data are square integrable:*

$$\mathbb{E}\left[|\xi|^2 + \int_0^T \left(|f(t,0,0)|^2 + |g(t,0,0)|^2\right) dt\right] < +\infty, \tag{2.12}$$

BDSDE (2.11) *has a unique solution* $(Y,Z) \in \mathcal{S}^2(0,T)$. *Moreover if for some* $p \geq 1$

$$\mathbb{E}\left[|\xi|^{2p} + \left(\int_0^T \left(|f(t,0,0)|^2 + |g(t,0,0)|^2\right) dt\right)^p\right] < +\infty, \tag{2.13}$$

then $(Y,Z) \in \mathcal{S}^{2p}(0,T)$.

Using the paper of Aman [9], this result can be extended to the L^p case: for $p \in (1,2)$, if

$$\mathbb{E}\left[|\xi|^p + \int_0^T \left(|f(t,0,0)|^p + |g(t,0,0)|^p\right) dt\right] < +\infty,$$

there exists a unique solution in $\mathcal{S}^p(0,T)$.

We also give (for completeness) a comparison result on the solution of BDSDE (2.11).

Proposition 2.3 *Assume that BDSDE* (2.11) *with data* (f^1, g, ξ^1) *and* (f^2, g, ξ^2) *have solutions* (Y^1, Z^1) *and* (Y^2, Z^2) *in* $\mathcal{S}^p(0,T)$, *respectively. Coefficient* g *satisfies* **(Ag1)**. *If* $\xi^1 \leq \xi^2$, *a.s., and* f^1 *satisfies Assumptions* **(A2)** *and* **(A3*)**, *for all* $t \in [0,T]$, $f^1(t, Y_t^2, Z_t^2) \leq f^2(t, Y_t^2, Z_t^2)$, *a.s. (resp.* f^2 *satisfies* **(A2)** *and* **(A3*)**, *for all* $t \in [0,T]$, $f^1(t, Y_t^1, Z_t^1) \leq f^2(t, Y_t^1, Z_t^1)$, *a.s.), then we have* $Y_t^1 \leq Y_t^2$, *a.s., for all* $t \in [0,T]$.

Let us explain the ideas of the proof of Theorem 2.5. To obtain Theorem 2.1, we assume first that f is locally Lipschitz continuous and we use a truncation argument; which leads to some bounded solution using existence result for Lipschitz continuous driver. Then we pass to the limit. And if f is only monotone in y, the existence is obtained using a weak convergence result. This boundedness step is crucial. For BDSDE, the solution can not be bounded. Instead, we follow the ideas of [45], where the convergence in $\mathbb{L}^2$ is employed, provided that the growth of f is polynomial.

The key step is to prove existence and uniqueness for the special case

$$Y_t = \xi + \int_t^T f(r, Y_r)\, dr + \int_t^T g_r \overleftarrow{dB_r} - \int_t^T Z_r dW_r, \ 0 \leq t \leq T. \tag{2.14}$$

Here f only depends on y (not on z) and g does not depend neither on y nor on z. The general case can be deduced by standard fixed point argument ([XIII, Lemma 2 and Section 2.2]). We transform BDSDE (2.14):

$$Y_t + \int_0^t g_r \overleftarrow{dB_r} = \xi + \int_0^T g_r \overleftarrow{dB_r} + \int_t^T f(r, Y_r)\, dr - \int_t^T Z_r dW_r, \ 0 \leq t \leq T.$$

Let us define:

$$U_t = Y_t + \int_0^t g_r \overleftarrow{dB_r}, \qquad \zeta = \xi + \int_0^T g_r \overleftarrow{dB_r},$$

and

$$\phi(t,y) = f\left(t, y - \int_0^t g_r \overleftarrow{dB_r}\right).$$

Then (U,Z) satisfies:

$$U_t = \zeta + \int_t^T \phi(r, U_r)\, dr - \int_t^T Z_r dW_r, \ 0 \leq t \leq T. \tag{2.15}$$

The terminal condition ζ is $\mathcal{G}_T$-measurable and generator ϕ satisfies **(A2)** and from **(A3*)**, there exists $q > 0$ such that

$$|\phi(t,y)| \leq h(t) + C_\phi(1 + |y|^{1+q}). \tag{2.16}$$

where $C_\phi = C_f 2^q$ and

$$h(t) = |f(t,0)| + 2^q \left|\int_0^t g_r \overleftarrow{dB_r}\right|^{q+1}.$$

On solution (U,Z) we impose the two measurability constraints:

1. Process (U,Z) is adapted to filtration $\mathbb{G} = (\mathcal{G}_t, \ t \geq 0)$.
2. The random variable $U_t - \int_0^t g_r \overleftarrow{dB_r}$ is $\mathcal{F}_t$-measurable for any $0 \leq t \leq T$.

Let us assume the boundedness hypothesis on ξ, g and $f(t,0)$: there exists a constant $\gamma > 0$ such that a.s. for any $t \geq 0$,

$$|\xi| + |f(t,0)| + |g_t| \leq \gamma.$$

Hence for any $p > 1$

$$\mathbb{E}\left[|\zeta|^p + \left(\int_0^T |h(t)|^p dt\right)\right] < +\infty.$$

Thus there exists a unique solution $(U,Z) \in \mathcal{S}^2_{\mathbb{G}}(0,T)$ to BSDE (2.15) such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |U_t|^2 + \left(\int_0^T |Z_r|^2 dr\right)\right] < +\infty.$$

Theorem 3.6 in [45] also gives that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |U_t|^{2(1+q)} + \left(\int_0^T |Z_r|^2 dr\right)^{1+q}\right] < +\infty.$$

But we cannot directly derive from this result that $U_t - \int_0^t g_r \overleftarrow{dB_r}$ is $\mathcal{F}_t$-measurable for any $0 \leq t \leq T$. Therefore we follow the proof of [45, Proposition 3.5] to prove the existence and uniqueness of the solution $(U,Z) \in \mathcal{S}^{2(1+q)}_{\mathbb{G}}(0,T)$ to BSDE (2.15), with the desired measurability conditions (see [XIII, Proposition 2]). Roughly speaking, we construct an approximating sequence (U^n, V^n) converging to (U, Z) such that the sequence of processes

$$Y_t^n = U_t^n - \int_0^t g_r \overleftarrow{dB_r}$$

is $\mathcal{F}_t \vee \sigma(h(s)\mathbf{1}_{h(s)\geq n},\ 0 \leq s \leq T) = \mathcal{F}_t \vee \mathcal{H}^n$-measurable. Passing through the limit, $U_t - \int_0^t g_r \overleftarrow{dB_r}$ is therefore $\mathcal{F}_t \vee \mathcal{H}^\infty$-measurable. In [XIII, Lemma 1], it is proved that the σ-algebra $\mathcal{H}^\infty$ is trivial.

It is natural to deal with a weaker growth condition on f (as for BSDEs). Suppose that there exists a non decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, y)| \leq |f(t, 0)| + \psi(|y|).$$

Using the same transformation, we have to control:

$$|\phi(t, y)| = |f(t, y + \int_0^t g_r \overleftarrow{dB_r})| \leq |f(t, 0)| + \psi(|y + \int_0^t g_r \overleftarrow{dB_r}|).$$

If it is possible to find two functions ψ_1 and ψ_2 such that $\psi(y + z) \leq \psi_1(y) + \psi_2(z)$ and if $\psi_2(|\int_0^t g_r \overleftarrow{dB_r}|)$ belongs to $\mathbb{L}^2(\Omega)$ for any bounded process g_t, it may be possible to obtain a solution with the desired properties to the BDSDE (2.11).

2.4 Second order BSDE ([XVII])

The notion of second order BSDE (2BSDE in short) has been introduced in the paper [309] (together with [308, 310]). Then it has been extended to reflected 2BSDE in [243, 244], to second order BDSDE in [249], to the jump case in [185, 186].

Our goal is to extend the results in [295] to the case where the generator f is only monotone w.r.t. y. Compared with [294], we do not assume that f is of linear growth w.r.t. y. The probabilistic setting is the same as [295]. But to overcome this difficulty induced by monotonicity, we will impose some stronger integrability conditions[2] Although the sketch is almost the same as in [295], several technical issues have to be taken into account in our setting. Moreover the monotonicity of the driver forces us to change the minimality condition on the non-decreasing process $K^{\mathbb{P}}$. The classical assumption

$$\operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \middle| \mathcal{F}_t^+ \right] = 0, \quad 0 \leq t \leq T,\ \mathbb{P} - \text{a.s.},\ \forall \mathbb{P} \in \mathcal{P}$$

is replaced by

$$\operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \exp\left(\int_t^s \lambda_u^{\mathbb{P}'} du \right) dK_s^{\mathbb{P}'} \middle| \mathcal{F}_t^+ \right] = 0, \quad 0 \leq t \leq T,\ \mathbb{P} - \text{a.s.},\ \forall \mathbb{P} \in \mathcal{P}$$

[2]Sufficient to solve the related control problem (Section 5.2). Weaker integrability assumptions are left for future research.

where $\lambda_s^{\mathbb{P}'}$ is the increment of the generator evaluated at the solution Y of the 2BSDE and at the solution $y^{\mathbb{P}'}$ of the classical BSDE under $\mathbb{P}'$. In the Lipschitz setting **(A2')**, $\lambda^{\mathbb{P}'}$ is bounded and thus can be removed, whereas under the monotone assumption, it is only bounded from above; we discuss this point later. Note that a similar condition has been introduced for second order RBSDE in [246].

2.4.1 Our assumptions

We shall consider a $\mathcal{F}_T$−measurable random variable $\xi : \Omega \longrightarrow \mathbb{R}$ and a generator function

$$f : (t, \omega, y, z, a, b) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\geq 0} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

Define for simplicity

$$\widehat{f}_s^{\mathbb{P}}(y, z) := f(s, X_{\cdot \wedge s}, y, z, \widehat{a}_s, b_s^{\mathbb{P}}) \text{ and } \widehat{f}_s^{\mathbb{P},0} := f(s, X_{\cdot \wedge s}, 0, 0, \widehat{a}_s, b_s^{\mathbb{P}}).$$

The generator function f is jointly Borel measurable and Conditions **(A2)**, **(A3')** and **(A4)** become:

(A2$\star$) For any (t, ω, z, a, b), the map $y \mapsto f(t, \omega, y, z, a, b)$ is continuous and satisfies the monotonicity assumption w.r.t. y: there exists a constant $\chi \in \mathbb{R}$ such that for every $(t, \omega, y, y', z, a, b)$

$$(f(t, \omega, y, z, a, b) - f(t, \omega, y', z, a, b))(y - y') \leq \chi(y - y')^2.$$

(A3$\star$) The polynomial growth assumption w.r.t. y holds: there exists $q > 1$ and a jointly Borel measurable function $\Psi : [0, T] \times \Omega \times \mathbb{S}_d^{\geq 0} \to \mathbb{R}$ such that for any (t, ω, a, b, y)

$$|f(t, \omega, y, 0, a, b) - f_t^0| \leq \Psi(t, \omega, a)(1 + |y|^q).$$

(A4$\star$) f is Lipschitz continuous w.r.t. z uniformly w.r.t. all other parameters, that is there exists a non-negative constant $K_{f,z}$ such that for every $(t, \omega, y, z, z', a, b)$,

$$|f(t, \omega, y, z, a, b) - f(t, \omega, y, z', a, b)| \leq K_{f,z}\|z - z'\|.$$

f_t^0 is the notation for $f(t, \omega, 0, 0, a, b)$. As for the generator, we denote

$$\widehat{\Psi}_s := \Psi(s, X_{\cdot \wedge s}, \widehat{a}_s).$$

Finally for ξ, f^0 and Ψ, we modify the integrability condition **(A1)** and we impose:

(A1.1$\star$) For some fixed constants $\varrho > 1$ and $\bar{p} > \varrho/(\varrho - 1)$, for every $(t, \omega) \in [0, T] \times \Omega$,

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \left[|\xi|^{\bar{p}q} + \int_t^T \left| \widehat{f}_s^{\mathbb{P},0} \right|^{\bar{p}q} ds + \int_t^T \left| \widehat{\Psi}_s \right|^{\varrho} ds \right] < +\infty.$$

(A1.2$\star$) There is some $\kappa \in (1, \bar{p}q]$ such that $\xi \in \mathbb{L}_0^{\bar{p}q,\kappa}$ and

$$\phi_f^{\bar{p}q,\kappa} = \sup_{\mathbb{P} \in \mathcal{P}_0} \mathbb{E}^{\mathbb{P}} \left[\operatorname*{ess\,sup}_{0 \leq s \leq T}{}^{\mathbb{P}} \left(\operatorname*{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_0(s, \mathbb{P}, \mathbb{F}_+)}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'} \left[\int_0^T |\widehat{f}_t^{\mathbb{P}',0}|^{\kappa} dt \middle| \mathcal{F}_s^+ \right] \right)^{\frac{\bar{p}q}{\kappa}} \right] < +\infty.$$

In this section, p denotes any number larger than 1; q denotes the exponent in Condition **(A3')**; $\bar{p}$ and ϱ are used in Assumptions **(A1.1⋆)** and **(A1.2⋆)** and satisfy $\bar{p} > \varrho/(\varrho-1)$ ($\bar{p}$ is greater than the Hölder conjugate of ϱ). Finally we sometimes assume that p verifies

$$1 < p \leq \frac{\varrho \bar{p}}{\varrho + \bar{p}} < \bar{p}. \tag{2.17}$$

Under this condition, $\widehat{p} = \dfrac{p\bar{p}}{(\bar{p}-p)} \leq \varrho$.

Remark 2.1 *As for BSDEs, we can suppose w.l.o.g. that $\chi = 0$ in* **(A2⋆)**.

Let us explain why we assume Condition **(A3⋆)** *together with the integrability condition* **(A1.1⋆)** *on $\widehat{\Psi}$, and not some weaker growth condition. Indeed to prove the existence of a solution for a 2BSDE we use that the solution (y,z,m) of the standard BSDE with data $\widehat{f}^{\mathbb{P}}$ and ξ is obtained by approximation with a sequence of solutions (y^n, z^n, m^n) of Lipschitz BSDEs. Moreover the fact that Ψ does not depend on b is used for regularization of the paths in order to control the downcrossings. Finally notice that this setting is sufficient to solve the related optimal control problem (see Section 5.2). Existence under weaker conditions is left for further research.*

Compared to the integrability assumption **(A1)**, **(A1.1⋆)** *looks to be too strong. As in the previous remark this hypothesis is related to the method we use to obtain existence of the solution of the 2BSDE; in particular in the Lipschitz approximation procedure and in the proof of existence of the solution of the reflected BSDE. Weaker integrability condition is also left for further research.*

2.4.2 Definition and existence and uniqueness results

We consider the 2BSDE

$$Y_t = \xi + \int_t^T \widehat{f}_u^{\mathbb{P}}(Y_u, \widehat{a}_u^{\frac{1}{2}} Z_u)du - \left(\int_t^T Z_u dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dM_u^{\mathbb{P}} + \int_t^T dK_u^{\mathbb{P}}. \tag{2.18}$$

In this equation $\left(\int_t^T Z_u dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}}$ denotes the stochastic integral of Z w.r.t. $X^{c,\mathbb{P}}$ under $\mathbb{P}$, $M^{\mathbb{P}}$ is a martingale orthogonal to $X^{c,\mathbb{P}}$ and $K^{\mathbb{P}}$ is a non-decreasing process.

Definition 2.2 *$(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ is a solution if (2.18) is satisfied $\mathcal{P}-q.s.$ and if family $(K^{\mathbb{P}},\ \mathbb{P} \in \mathcal{P})$ satisfies the minimality condition:*

(2.19)

$$\operatorname*{essinf}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)} \mathbb{E}^{\mathbb{P}'}\left[\int_t^T \exp\left(\int_t^s \lambda_u^{\mathbb{P}'} du\right) dK_s^{\mathbb{P}'} \middle| \mathcal{F}_t^+\right] = 0, \quad 0 \leq t \leq T,\ \mathbb{P}-a.s.,\ \forall \mathbb{P} \in \mathcal{P}$$

where

$$\lambda_s^{\mathbb{P}'} = \frac{\widehat{f}_s^{\mathbb{P}'}(Y_s, z_s^{\mathbb{P}'}) - \widehat{f}_s^{\mathbb{P}'}(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})}{Y_s - y_s^{\mathbb{P}'}} \mathbb{1}_{Y_s \neq y_s^{\mathbb{P}'}} \leq L_1.$$

$\mathcal{P}$-q.s. means quasi-surely, that is $\mathbb{P}-a.s.$ for any $\mathbb{P} \in \mathcal{P}$. In the above definition, $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}})$ is the solution under the probability measure $\mathbb{P}$ of the following BSDE

$$y_t = \xi + \int_t^T f(u, X_{\cdot\wedge u}, y_u, \widehat{a}_u^{\frac{1}{2}} z_u, \widehat{a}_u, b_u^{\mathbb{P}})du - \left(\int_t^T z_u dX_u^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dm_u, \ \mathbb{P}-a.s. \tag{2.20}$$

where again m is an additional martingale, orthogonal to $X^{c,\mathbb{P}}$. Moreover for $t \leq s$ and an $\mathcal{F}_s^+$-measurable random variable ζ , $y_t^{\mathbb{P}}(s,\zeta)$ is the solution of (2.20) with terminal time s and terminal condition ζ.

Let us begin with the uniqueness result, which corresponds to [295, Theorem 4.2].

Proposition 2.4 *Under Conditions* **(A1.1⋆)**, **(A1.2⋆)**, **(A2⋆)**, **(A3⋆)** *and* **(A4⋆)**, *let* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *be a solution of* (2.18) *and for any* $\mathbb{P} \in \mathcal{P}$, *let* $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}})$ *be the solution of BSDE* (2.20) *in* $\mathbb{D}_0^{\bar{p}q}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{H}_0^{\bar{p}q}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^{\bar{p}q}(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$. *Then for any* $0 \leq t_1 \leq t_2 \leq T$

$$Y_{t_1} = \underset{\mathbb{P}' \in \mathcal{P}(t_1, \mathbb{P}, \mathbb{F}_+)}{\text{ess sup}}{}^{\mathbb{P}} \, y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}). \tag{2.21}$$

Thus uniqueness holds in $\mathbb{D}_0^{\bar{p}q}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^{p}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^{p}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^{p}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *for any* $1 < p$ *satisfying Condition* (2.17).

The comparison principle ([295, Theorem 4.3]) , the *a priori* estimate ([295, Theorem 4.4]) and the stability result ([295, Theorem 4.5]) for 2BSDE remain unchanged here. Indeed it is a direct consequence of the comparison principle and the stability result on BSDEs and the formula (2.21). The other arguments follow exactly the proofs in [295].

Now we come to the existence result (equivalent to [295, Theorems 4.1 and 4.4]).

Proposition 2.5 *Under Conditions* **(A1.1⋆)**, **(A1.2⋆)**, **(A2⋆)**, **(A3⋆)** *and* **(A4⋆)**, *there exists a solution* $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ *to 2BSDE* (2.18) *in the space* $\mathbb{D}_0^{\bar{p}q}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^{p}(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^{p}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^{p}((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ *for any* $p > 1$ *satisfying* (2.17). *More precisely there exists a constant* C *depending on* $\bar{p}$, q T, χ, $K_{f,z}$ *such that*

$$\|Y\|_{\mathbb{D}_0^{\bar{p}q}}^{\bar{p}q} + \|Z\|_{\mathbb{H}_0^p}^p + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left(K_T^{\mathbb{P}}\right)^p + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left(\left[M^{\mathbb{P}}\right]_T\right)^{p/2} \leq C\left(\|\xi\|_{\mathbb{L}_0^{\bar{p}q}}^{\bar{p}q} + \phi_f^{\bar{p}q,\kappa}\right). \tag{2.22}$$

Discussion and comparison with [294]

When f is Lipschitz continuous w.r.t. y, process λ is bounded also from below. Thus our minimality condition is equivalent to the classical one:

$$\underset{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\text{essinf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'}\left[K_T^{\mathbb{P}'} - K_t^{\mathbb{P}'} \middle| \mathcal{F}_t^+\right] = 0, \quad 0 \leq t \leq T, \ \mathbb{P}-a.s., \ \forall \mathbb{P} \in \mathcal{P}_0. \tag{2.23}$$

In general we only have that the classical condition (2.23) implies (2.19).

If there is only one probablity measure $\mathbb{P}$ in $\mathcal{P}_0$, the minimality condition (2.19) imposed on $K^{\mathbb{P}}$ should imply that $K^{\mathbb{P}}$ is equivalent to zero. In the Lipschitz setting this is a direct consequence of (2.23). In our setting it is still true but the arguments are not direct. From the proof of Proposition 2.4, (2.19) implies uniqueness of the solution. But if $\mathcal{P}_0$ is the singleton, solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, 0)$ of the classical BSDE (2.20) becomes a solution of 2BSDE (2.18). By uniqueness, $K^{\mathbb{P}} \equiv 0$.

The monotone case was already studied in [294]. Generator f satisfies Conditions **(A2⋆)** and **(A4⋆)**, is uniformly continuous in y, uniformly in (t, ω, z, a) and has the linear growth property:

$$|f(t, \omega, y, 0, a)| \leq |f(t, \omega, 0, 0, a)| + C(1 + |y|).$$

Then under some integrability condition for ξ and $\widehat{f}_s^{\mathbb{P},0}$, from [294, Theorem 2.2], there exists a unique solution of 2BSDE (2.18) such that $K^{\mathbb{P}}$ satisfies the minimality condition (2.23).

Therefore if the generator f satisfies the assumptions of [294] and Conditions **(A2⋆)** and **(A4⋆)**, then the solution obtained by [294] with minimality condition (2.23) is also the solution given by Propositions 2.4 and 2.5 with minimality criterion (2.19). Let us emphasize that the ways to obtain the solution are completely different. Indeed in [294] the generator is approximated by a sequence of Lipschitz generators f_n. For any fixed n using [309], there exists a unique solution $(Y^n, Z^n, M^{n,\mathbb{P}}, K^{n,\mathbb{P}})$ to 2BSDE (2.18) with generator f_n and process $K^{n,\mathbb{P}}$ verifies (2.23). Then the core of the paper [294] consists to show that sequence $(Y^n, Z^n, M^{n,\mathbb{P}}, K^{n,\mathbb{P}})$ converges to $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ and that (2.23) is preserved through the limit. The uniform continuity and the linear growth conditions of f w.r.t. y are crucial there.

2.5 Summary

This chapter extends some existence and uniqueness results when the generator is monotone for

- BSDEs with jumps and L^p-data;
- BDSDEs, RBSDEs and 2BSDEs without jumps and a polynomial growth assumption on the generator.

These results are the foundation to the next chapter where a terminal singularity is added.

Chapter 3

Terminal singularity

This chapter is devoted to the study of backward stochastic differential equations with *singular* terminal condition. The basic idea comes from the ODE theory. If the generator f is a deterministic function of y and if the terminal condition is deterministic, BSDE (2.4) is in fact an ODE:

$$y(t) = \xi + \int_t^T f(y(s))ds \Longleftrightarrow y'(t) = \dot{y}(t) = -f(y(t)), \quad y(T) = \xi.$$

Assume for example that $\xi = x > 0$ and $f(y) = -y|y|$. The explicit solution of this ODE is:

$$y(t) = \frac{1}{T - t + \frac{1}{x}}.$$

Letting x go to $+\infty$, we obtain

$$y^\infty(t) = \frac{1}{T-t}.$$

Let us emphasize that this function is bounded and solves the ODE on any interval $[0, T-\varepsilon]$. The singularity only appears at time T, since $\lim_{t\to T} y^\infty(t) = +\infty$. Direct computations show that the same property holds if f satisfies for some $c \in \mathbb{R}$:

$$\int_c^{+\infty} -\frac{1}{f(y)} dy < +\infty.$$

The natural question in the starting papers [I] and [II] was: is it possible to extend this result for BSDE such that the terminal condition is singular ? Singular means in the sense of

Definition 3.1 (Singular terminal condition) *The terminal condition ξ is called singular if the positive part ξ^+ of ξ does not belong to any $\mathbb{L}^p(\Omega)$. In particular if*

$$\mathbb{P}(\xi = +\infty) > 0. \tag{3.1}$$

A second motivation was the relation between BSDE and PDE, which is a extension for semi-linear PDE of the Feynman-Kac formula. If $X^{t,x}$ is the solution of SDE (1.9) starting from x at time t, if $\xi = g(X_T^{t,x})$, then $Y_t^{t,x} = u(t,x)$ is a deterministic function and a viscosity solution of the semi-linear PDE

$$\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^*(t,x)\nabla u(t,x)) = 0, \tag{3.2}$$

$\mathcal{L}$ being the infinitesimal generator of $X^{t,x}$. The exact result can be found in [276, Section 5.4]. Among all semi-linear PDEs, a particular form has been widely studied: for some $q > 1$

$$\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^{q-1} = 0. \tag{3.3}$$

Baras & Pierre [21], Marcus & Veron [241] (and many other papers) have given existence and uniqueness results for this PDE, which is close to the Lane-Emden equation in astrophysics. In [241] it is shown that every positive solution of (3.3) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S}_\infty, \mu)$ where $\mathcal{S}_\infty$ is a closed subset of $\mathbb{R}^d$ and μ a non negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}_\infty$. The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^d, A \text{ Borel}, \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S}_\infty \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}_\infty$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view, Dynkin & Kuznetsov [106] and Le Gall [216] have proved similary results for PDE (3.3) in the case $1 < q \leq 2$ using the theory of superprocesses. Now if we want to represent the solution u of (3.3) using a forward backward SDE, we have to deal with the generator $f(y) = -y|y|^{q-1}$ and a *singular* terminal condition ξ, where $\{\xi = +\infty\}$ corresponds to the set $\mathcal{S}_\infty$.

We have already mention that BSDEs have proved to be a powerful tool to solve stochastic optimal control problems (see e.g. the survey article [118] or the book [288]). Let us consider the problem of minimizing the cost functional [1]

$$J(X) = \mathbb{E}\left[\int_0^T (\eta_s|\alpha_s|^p + \gamma_s|X_s|^p)\, ds + \xi|X_T|^p\right] \tag{3.4}$$

over all progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_0^s \alpha_r dr.$$

Here $p > 1$ and the processes η and γ are non negative progressively measurable. Again the $\mathcal{F}_T$-measurable random variable ξ takes the value ∞ with positive probability. This singularity imposes the terminal state constraint on the set of strategies α. Indeed, any strategy X that does not satisfy this terminal constraint creates infinite costs. In particular, such a strategy cannot be optimal if there exists some strategy that creates

[1] We define $0 \cdot \infty := 0$.

finite costs (which will always be the case under the assumptions that we impose). The analysis of optimal control problems with state constraints on the terminal value is motivated by models of optimal portfolio liquidation under stochastic price impact. In [10], the filtration is generated by the Brownian motion W and $\xi = \infty$ a.s., that is there is an a.s. constraint $X_T = 0$. The authors characterize optimal strategies and the value function of this control problem with the BSDE

$$dY_t = (p-1)\frac{Y_t^q}{\eta_t^{q-1}}dt - \gamma_t dt \tag{3.5}$$

with $\liminf_{t\to T} Y_t = +\infty$. Here $q > 1$ is the Hölder conjugate of p. Note that if we modify ξ, the liquidation constraint is relaxed in the following way. Instead of enforcing the condition $X_T = 0$ a.s., that is the position has to be closed imperatively, the model is flexible enough to allow for a specification of a set of market scenarios $\mathcal{S}_\infty \subset \mathcal{F}_T$ where liquidation is mandatory: $X_T \mathbf{1}_{\mathcal{S}_\infty} = 0$. On the complement $\mathcal{R}$ a penalization depending on the remaining position size can be implemented.

These previous points motivate the study of BSDE (2.4), but with a singularity at time T in the sense of Definition 3.1. Our main contribution are the papers [I], [II] and [XI]. We prove the existence of a minimal (super-) solution $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$. This supersolution is constructed via approximation from below. For each $L > 0$ we consider a truncated version of (2.4) with terminal condition $\xi \wedge L$. We impose that driver f satisfies a monotonicity assumption in the y-variable and is Lipschitz continuous with respect to (z, ψ). Then existence, uniqueness and comparison results for a solution (Y^L, Z^L, U^L, M^L) to the truncated BSDE can be deduced from Section 2.1. We obtain the minimal supersolution $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ with singular terminal condition by passing to the limit $L \to \infty$. The crucial task is to establish suitable a priori estimates for Y^L guaranteeing that when passing to the limit the solution $Y^{\min}$ does *not* explode before time τ. To this end, the generator f cannot be Lipschitz continuous w.r.t. y. Hence we impose that f is monotone and decreases sufficiently fast in the y-variable. The detailed results are in Section 3.1.

The next part 3.2 is devoted to BDSDE and 2BSDE with singularity at time T. For doubly stochastic equations, the study of PDE (3.2) with a random noise (SPDE) was our motivation. If SPDE have been widely developed, the paper [XIII] was the first attempt to add a terminal singularity for these equations. For second order BSDEs, we were motivated by Knightian uncertainty for some control problem (see Section 5.2). Roughly speaking we need to extend the existence and uniqueness result for solution of a 2BSDE of [309] and [310] to monotone generators.

From the theoretical point of view, there are two main unclear points concerning this minimal solution. The first one concerns the behavior of $Y^{\min}$ at time T (see Section 3.3). Indeed in general we only know that a.s.

$$\liminf_{t\to T} Y_t^{\min} \geq \xi = Y_T^{\min}.$$

Hence there are two questions:

- Does the limit exist?
- Is there an equality? This question is called "*continuity problem*"[2].

We already studied these questions in [I, II] for a particular setting. Paper [XII] extends the results. Roughly speaking the limit exists under structural (but rather general) conditions on the generator f. And the equality holds under the half-Markovian setting, that is if $\xi = g(X_T)$ where $g : \mathbb{R}^d \to [0,+\infty]$. The first non-Markovian cases were considered in [XVI], where we used a stopping time and a PDE with singularity on the boundary in the first paper; and in [XIX] with the Itô functional calculus to provide other non-Markovian examples where the equality holds.

The second problem concerns uniqueness. In [148], a uniqueness result is proved using the link with an optimal control problem. In [XXI] we obtain uniqueness under weaker conditions. However in the general case, it remains an open question.

In Part 4, we make the link with PDE [I, II], integro-partial differential equations (IPDE) [XIV] and SPDE [XIII] with terminal singularity. In other words we enlarge the kind of PDE for which there is a blow-up at time T. Let us emphasize that for SPDE we use backward doubly stochastic differential equations (BDSDE) with a monotone driver f and a singular terminal data ξ.

3.1 BSDE with terminal singularity ([I, II, XI])

In this part, we consider BSDE (2.4)

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T \int_{\mathcal{E}} U_s(e)\widetilde{\pi}(de, ds) - \int_t^T Z_s dW_s - \int_t^T dM_s,$$

where the driver f satisfies **(A2)**–**(A3)**–**(A4)** and **(A5')**. Hence the conclusions of Theorem 2.1 and Proposition 2.2 hold if the integrability condition **(A1)** is verified.

We are interesting in the case where ξ is such that (3.1) holds:

$$\mathbb{P}(\xi = +\infty) > 0,$$

and ξ^-, the negative part of ξ, belongs to $\mathbb{L}^\ell(\Omega)$ for some $\ell > 1$. But more generally, our results hold for any singular ξ (Definition 3.1) such that $\xi^- \in \mathbb{L}^\ell(\Omega)$.

To deal with a singular terminal condition, we need to modify the definition of a solution.

Definition 3.2 (Supersolution for singular terminal condition) *A triple of processes* (Y, Z, U, M) *is a supersolution to BSDE* (2.4) *with singular terminal condition* $Y_T = \xi$ *if it satisfies:*

[2]Even if it concerns only the lim inf

1. *There exists some $\ell > 1$ such that for all $\varepsilon > 0$ and all $t \geq 0$*

$$\mathbb{E}\left[\sup_{s\in[0,T-\varepsilon]} |Y_s|^\ell + \left(\int_0^{T-\varepsilon} |Z_s|^2 ds\right)^{\ell/2} \right.$$
$$\left. + \left(\int_0^{T-\varepsilon}\int_{\mathcal{E}} |U_s(e)|^2 \pi(de,ds)\right)^{\ell/2} + [M]_{T-\varepsilon}^{\ell/2}\right] < +\infty.$$

2. *Y is bounded from below by a process $\widetilde{Y} \in \mathbb{D}^\ell(0,T)$.*

3. *For all $0 \leq t \leq s < T$:*

$$Y_t = Y_s + \int_t^s f(r, Y_r, Z_r, U_r)dr - \int_t^s Z_r dW_r - \int_t^s \int_{\mathcal{E}} U_r(e)\widetilde{\pi}(de,dr) - \int_t^s dM_r.$$

4. *A.s.*

$$\liminf_{t\to T} Y_t \geq \xi = Y_T. \tag{3.6}$$

We say that $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ is the minimal supersolution of BSDE (2.4) if for any other supersolution (Y', Z', U', M'), we have a.s.: $Y_t^{\min} \leq Y_t'$ for any $t > 0$.

Let us precise our setting. Again, f_s^0 denotes $f(s, 0, 0, \mathbf{0})$ and we assume that

(C1) *There exists a constant $\ell > 1$ such that*

$$\mathbb{E}\left[(\xi^-)^\ell + \int_0^T \left((f_s^0)^-\right)^\ell ds\right] < +\infty.$$

This condition is used to control the negative part of the solution. Moreover

(C2) There exists a positive process η and some constant $q > 1$ such that for any $y \geq 0$

$$f(t, y, z, u) - f(t, 0, z, u) \leq -\frac{1}{\eta_t} y|y|^{q-1}.$$

(C3) There exists some $\ell > 1$ such that

$$\mathbb{E}\int_0^T \left[((p-1)\eta_s)^{p-1} + (T-s)^p \left(f_s^0\right)^+\right]^\ell ds < +\infty$$

where p is the Hölder conjugate of q.

(C4) There exists $k > \max(2, \ell/(\ell-1))$ such that

$$\int_{\mathcal{E}} |\vartheta(e)|^k \mu(de) < +\infty.$$

Recall that ϑ appears in **(A5')**.

We suppose that the generator $(t,y) \mapsto -y|y|^{q-1}/\eta_t$ satisfies conditions **(A2)** and **(A3)**, which means that η satisfies the next integrability condition:

$$\mathbb{E}\int_0^T \frac{1}{\eta_t} dt < +\infty. \tag{3.7}$$

Note that this bound on η is necessary to have an optimal control in (3.4) (see Example 1.1 in [10]).

Let us stress that generator f can be also "singular" at time T provided Assumption **(C3)** holds. Indeed

$$\mathbb{E}\int_0^T |f_s^0| ds = +\infty$$

can occur with our framework. Note that BSDEs with singular generator were already studied in [174] and [175], but the setting is completely different (we come back on this topic in Section 3.1.1). Several examples are detailled in [XII], among other:

$$f(t,y) = -(T-t)^{\varsigma} y|y|^{q-1} + \frac{1}{(T-t)^{\varpi}}$$

where ς and ϖ are two real numbers. Since $1/\eta_t = (T-t)^{\varsigma}$ is in $L^1(0,T)$, ς must be greater than -1. Condition **(C3)** imposes that

$$\int_0^T \left[(T-t)^{-\ell\varsigma/(q-1)} + (T-t)^{\ell(p-\varpi)} \right] dt < +\infty.$$

This implies the following bounds:

$$-1 < \varsigma < q-1, \qquad \varpi < 1 + \frac{1}{q-1} + 1/\ell,$$

with $1 \leq \ell$ and $\ell < (q-1)/\varsigma$ if $\varsigma > 0$. The singularity ot time T of the generator has to be not too important (upper bound on ϖ) and the coefficient $1/\eta$ before $y|y|^{q-1}$ can degenerate at time T, but not too quickly (upper bound on ς).

We prove an existence property of the minimal supersolution.

Theorem 3.1 *Under Conditions* **(A2)**–**(A5')** *and* **(C1)**–**(C4)** *and if the filtration* $\mathbb{F}$ *is left-continuous at time* T*, there exists a minimal supersolution* $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ *in the sense of Definition 3.2.*

Let us briefly explain how we obtain this solution by a penalization scheme. For any $L \geq 0$ we consider the BSDE

$$dY_t^L = -f^L(t, Y_t^L, Z_t^L, U_t^L)dt + Z_t^L dW_t + \int_{\mathcal{E}} U_t^L(e)\widetilde{\pi}(de,dt) + dM_t^L \tag{3.8}$$

with bounded terminal condition $Y_T^L = \xi \wedge L$ and where

$$f^L(t,y,z,u) = (f(t,y,z,u) - f_t^0) + f_t^0 \wedge L. \tag{3.9}$$

Under our setting, there exists for every $L > 0$ a unique solution $(Y^L, Z^L, U^L, M^L) \in \mathcal{S}^\ell(0,T)$ to (3.8). Moreover there exists a process $\widetilde{Y}$ in $\mathbb{D}^\ell(0,T)$, independent of L, such that a.s. for any $t \in [0,T]$, $\widetilde{Y}_t \leq Y^L_t$. If $(f^0_t)^- = \xi^- = 0$, then $\widetilde{Y}_t = 0$, Y^L_t is non-negative and $(Y^L, Z^L, U^L, M^L) \in \mathcal{S}^\infty(0,T)$.

The key point is the existence of a upper bound for family Y^L which is independent of L.

Proposition 3.1 *For every $t \in [0,T]$ the random variable Y^L_t is bounded from above by $L(1+T)$ and for $t \in [0,T)$ the following estimate holds:*

$$(3.10) \qquad Y^L_t \leq \frac{K_{\ell,\vartheta,K_{f,z}}}{(T-t)^p} \left[\mathbb{E} \left(\int_t^T \left(((p-1)\eta_s)^{p-1} + (T-s)^p (f^0_s)^+ \right)^\ell ds \middle| \mathcal{F}_t \right) \right]^{1/\ell}$$

where $K_{\ell,\vartheta,K_{f,z}}$ is a constant depending on constants $K_{f,z}$ in **(A4)** *and ϑ in* **(A5')**.

Constant $K_{\ell,\vartheta,K_{f,z}}$ is a non-decreasing function of $K_{f,z}$ and ϑ and a non-increasing function of ℓ. Constants $K_{\ell,\vartheta,K_{f,z}}$ and $\ell > 1$ come from the growth condition on f w.r.t. z and ψ. If we assume that $f(t,0,z,u)$ is uniformly bounded from above by κ, in (3.10) we can take $\ell = 1$ and $K_{\ell,\vartheta,K_{f,z}} = 1$ and we add $\frac{\kappa}{2+1/q}(T-t)$.

Remark 3.1 *This a priori estimate is optimal. Indeed Section 5.1 provides some example where $Y^{\min}$ is equal to the upper bound in* (3.10).

The comparison principle for BSDEs (Proposition 2.2) implies that if $L_1 \leq L_2$, then a.s.

$$\forall t \in [0,T], \quad Y^{L_1}_t \leq Y^{L_2}_t.$$

Thus we define $Y^{\min}$ as the increasing limit of Y^L:

$$\forall t \in [0,T], \quad Y^{\min}_t = \lim_{L \to +\infty} Y^L_t.$$

Proposition 3.1 and Condition **(C3)** imply that a.s. $Y^L_t \leq Y^{\min}_t < +\infty$ on $[0,T)$. Then by Itô's formula we show that for any fixed $\varepsilon > 0$, (Y^L, Z^L, U^L, M^L) is a Cauchy sequence in $\mathcal{S}^\ell(0, T-\varepsilon)$ and we deduce that limit $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ verify the first three properties of Definition 3.2.

To obtain the fourth property, let us remark that in general

$$\lim_{t \to T} Y^L_t = \xi - \Delta M^L_T.$$

In other words if T is a thin time for the filtration $\mathbb{F}$, the orthogonal martingale may have a jump at time T. We need to impose an extra condition on the filtration $\mathbb{F}$ to ensure that a martingale cannot have a jump at time T. An usual and enough condition is: the filtration $\mathbb{F}$ is quasi left-continuous. For example if $\mathbb{F}$ is generated by the Brownian motion and the Poisson random measure, this hypothesis is true. A sufficient and less strong condition is: the filtration $\mathbb{F}$ is left-continuous at time T (see the proof of [181, Proposition 25.19]). The reader could find examples on non quasi left-continuous filtrations in [3, Remark 1.9] (see also the references therein, in particular [173]). Note that in [20], the authors assume that ξ is $\mathcal{F}_{T-}$-measurable to avoid this problem.

The minimality of this solution is proved in [XI, Proposition 4]. This fact implies that limit $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ does not depend on the particular choice of our truncation procedure.

3.1.1 Uniqueness and asymptotic behaviour

Concerning this minimal solution, several questions are still open. The first (and perhaps the most important) is the uniqueness. This property is known to be an issue for the related PDE (see [241]). Our method only provides a minimal supersolution. The asymptotic method developed by Graewe et al. [148] for viscosity solution of the related PDE doesn't improve this result. However in the papers [148, 163], uniqueness is proved for a particular kind of generators.

If $\xi = +\infty$ a.s. in the Brownian framework

Recently [XXI] brings together these ideas (BSDE technics and asymptotic method) and extends some results. We assume that filtration $\mathbb{F}$ is generated by the Brownian motion, that T is deterministic, and that generator f has the following form:

$$f(\omega, t, y) = \frac{1}{\eta_t(\omega)} g(y) + \lambda_t(\omega) \tag{3.11}$$

where:

1. *Processes η and λ are bounded: there exist three constants $0 < \eta_\star < \|\eta\|$ and $\|\lambda\| \geq 0$ such that a.s. for any t*
$$\eta_\star \leq \eta_t \leq \|\eta\|, \qquad 0 \leq \lambda_t \leq \|\lambda\|.$$
2. *Function g is continuous and non-increasing, with $g(0) = 0$ and with continuous derivative.*
3. *For any $x > 0$, the function*
$$G(x) := \int_x^\infty \frac{1}{-g(t)}\, dt$$
is well-defined on $(0, \infty)$.

In [148, 163], the authors consider the particular case $g(y) = -y|y|^{q-1}$, $q > 1$. Using the same arguments as in the proof of Theorem 3.1, we show that there exists a minimal solution $(Y^{\min}, Z^{\min})$ to the BSDE

$$dY_t = -f(t, Y_t)dt + Z_t dW_t \tag{3.12}$$

with singular terminal condition $\xi = +\infty$ a.s. ([XXI, Proposition 1]).

In [148, 163], that is for $g(y) = -y|y|^{q-1}$, the authors prove that uniqueness holds. The proof is based of two ingredients. First they show that any solution satisfies the a priori estimate (3.10). Secondly they prove that the first component Y of the solution is the value function of the related control problem (5.8), which proves uniqueness. Our arguments are different and don't rely on the control problem. Thus we can deal with more general functions g. Essentially, g should be concave, together with technical conditions on G. The functions $y \mapsto \exp(-ay) - 1$ or $y \mapsto \exp(-ay^2)$ verify the required conditions. If we also assume some exponential integrability à la Novikov on the Malliavin derivative of $1/\eta$, then uniqueness holds. The arguments are based uniquely on BSDEs technics and on the asymptotic development of Y near the terminal time T.

Let us define the two functions:

$$\phi(x) := G^{-1}(x) > 0, \qquad \psi(x) := -\phi'(x) > 0. \tag{3.13}$$

Function ϕ being decreasing and C^2 on $(0, \infty)$ solves $\phi' = f \circ \phi$. Let us denote

$$A_t = \mathbb{E}\left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t \right]. \tag{3.14}$$

Note that A satisfies the BSDE:

$$-dA_t = \frac{1}{\eta_t} dt + Z_t^A dW_t \tag{3.15}$$

The main statement of [XXI] is:

Theorem 3.2 *The minimal solution $(Y^{\min}, Z^{\min})$ of the BSDE with generator f and terminal value $+\infty$ is given by:*

$$Y_t^{\min} = \phi(A_t) + \psi(A_t) H_t,$$

where (H, Z^H) is the minimal non-negative solution of the BSDE with terminal condition 0, but with a singular generator F: a.s.

$$\int_0^T F(t, h, z) dt = +\infty.$$

Such BSDEs with singular generator are studied in [175]. Let us clarify the meaning of solution for these BSDEs:

- H is non-negative and essentially bounded: for any $0 \leq t < T$, $0 \leq \sup_{s \in [0,t]} H_s < +\infty$ a.s. and
$$\mathbb{E} \int_0^T |F(s, H_s, Z_s^H)| ds < +\infty.$$
- The process Z^H belongs to $\mathbb{H}^1(0, T) \cap \mathbb{H}^p(0, T - \theta)$ for any $\theta > 0$ and $p > 1$.
- For any $0 \leq t \leq T$
$$H_t = \int_t^T F(s, H_s, Z_s^H) ds - \int_t^T Z_s^H dW_s.$$

  In particular a.s.
$$\lim_{t \to T} H_t = 0 = H_T.$$

- For any other solution $(\widehat{H}, \widehat{Z})$ of the same BSDE, a.s. for any $t \in [0,T]$, $\widehat{H}_t \geq H_t$.

Now if $(\widehat{Y}, \widehat{Z})$ is a solution of the BSDE (3.12), then we define $\widehat{H} = (\widehat{Y} - \phi(A))/(-\phi'(A))$. We can prove that this process $\widehat{H}$ and the related $\widehat{Z}^H$ solve the BSDE with generator F and terminal condition 0, that is it satisfies the previous properties. This one-to-one correspondence between the solutions of BSDEs with singular terminal condition resp. with a singular generator allows us to describe precisely the behaviour at time T and as a by product uniqueness.

Corollary 3.1 *If g is concave*[3]*, then H, and thus $Y^{\min}$, are unique.*

In other words the respective BSDEs have a unique solution.

In the power case $g(y) = -y|y|^{q-1}$, we even prove that H is obtained by a Picard iteration procedure in the space

$$\mathcal{H}^\delta := \{H \in L^\infty(\Omega; C([T-\delta, T]; \mathbb{R})) : \|H\|_{\mathcal{H}^\delta} < +\infty\}$$

endowed with the weighted norm

$$\|H\|_{\mathcal{H}^\delta} = \left\| \sup_{t \in [T-\delta, T)} (T-t)^{-2} |H_t| \right\|_\infty .$$

Let us mention that we have no uniqueness result for a general generator.

Back to the general case.

Another question concerns the behavior of $Y^{\min}$ at time T on the singular set. In other words, is it possible to determine the rate of explosion of $Y^{\min}$? To obtain such rate, we need a lower bound on $Y^{\min}$. We consider again BSDE (2.4), but the generator is given by (3.11). Under the previous setting of [XXI], we prove that a.s.

$$Y_t^{\min} \geq \phi(A_t), \quad \forall t \in [0,T]. \tag{3.16}$$

This estimate together with Theorem 3.2 leads to an explosion rate of order $\phi((T-t)/\eta_\star)$.

In the power case, we can provide a better estimate. Assume that

$$f(\omega, t, y) = -\frac{1}{\eta_t(\omega)} y|y|^{q-1} + \lambda_t(\omega)$$

such that λ and ξ are non-negative and **(C3)** holds. Following the ideas of [I, Proposition 11] and [10, Proposition 3.1], the minimal supersolution $Y^{\min}$ verifies a.s.

$$Y_t^{\min} \geq \mathbb{E}\left[\left(\frac{1}{(q-1)\int_t^T \frac{1}{\eta_s} ds + \xi^{1-q}} \right)^{p-1} \middle| \mathcal{F}_t \right], \quad \forall t \in [0,T].$$

[3]Again we skip here the technical conditions. The interested reader can found them in [XXI].

Hence if the process η is a constant (or is bounded from below by some constant), we get:

$$Y_t^{\min} \geq \mathbb{E}\left[\left(\frac{\xi^{q-1}\mathbf{1}_{\xi<+\infty}}{(q-1)\xi^{q-1}\frac{1}{\eta}(T-t)+1} \right)^{p-1} \middle| \mathcal{F}_t \right] + \mathbb{E}\left[\left(\frac{1}{(q-1)\frac{1}{\eta}(T-t)} \right)^{p-1} \mathbf{1}_{\xi=+\infty} \middle| \mathcal{F}_t \right].$$

Thereby we deduce that

Lemma 3.1 *If η is bounded from below by $\eta_\star$, on the set $\{\xi = +\infty\}$,*

$$\liminf_{t\to T} (T-t)^{p-1} Y_t^{\min} \geq \left(\frac{\eta_\star}{q-1} \right)^{p-1} = ((p-1)\eta_\star)^{p-1}.$$

Since f does not depend neither on z nor on ψ, the upper bound (3.10) can be written:

$$Y_t^{\min} \leq \frac{1}{(T-t)^p} \mathbb{E}\left[\int_t^T \left(((p-1)\eta_s)^{p-1} + (T-s)^p(\lambda_s) \right) ds \middle| \mathcal{F}_t \right].$$

Again for constant processes η and λ (or bounded from above), we have:

$$\limsup_{t\to T} (T-t)^{p-1} Y_t^{\min} \leq ((p-1)\eta)^{p-1}.$$

In other words if η is a constant, we have an exact rate of explosion: on the set $\{\xi = +\infty\}$,

$$\lim_{t\to T} (T-t)^{p-1} Y_t^{\min} = ((p-1)\eta)^{p-1}.$$

Recall that the terminal condition ξ is related to the minimal supersolution $Y^{\min}$ through the relation: a.s.

$$\liminf_{t\to T} Y_t^{\min} \geq \xi.$$

The upper bound (3.10) or the preceding lower bound don't lead to any conclusion concerning the existence of a limit, nor the equality between this limit and the condition ξ. These problems are studied in Section 3.3.

3.2 BDSDEs and 2BSDEs with singularity ([XIII] and [XVII])

In [XIII] and [XVII], we study the existence of a minimal solution for BDSDEs and 2BSDEs with singular terminal condition. Roughly speaking, we obtain the same result but under some restrictions, since the existence of L^p-solution is not proved under a very general setting and since we need some useful a priori estimate.

3.2.1 For BDSDEs

In this part, we extend the results concerning singular BSDEs to BDSDEs of the form (2.11). Let us consider generators f and g such that **(A2)**, **(A3$\star$)**,**(A4)** and **(Ag1)** hold. From now on we assume that the terminal condition ξ is singular (Definition 3.1). It could satisfy (3.1):

$$\mathbb{P}(\xi = +\infty) > 0.$$

And we suppose that **(C1)** for ξ and **(C2)** for f are verified. However to use our existence theorem 2.5 for monotone BDSDE, we reinforce **(C2)**:

$$f(t,y,z) - f(t,0,z) \leq -\frac{1}{\eta} y|y|^{q-1},$$

for some positive constant η.

By the comparison principle (Proposition 2.3), we know that for any L, if $(Y^L, Z^L) \in \mathcal{S}^\ell(0,T)$ denotes the solution of the BDSDE (2.11) with terminal condition $\xi \wedge L \in \mathbb{L}^\ell(\Omega)$, then for $L \leq L'$:

$$\Xi_t^0 \leq Y_t^L \leq Y_t^{L'} \leq \Xi_t^{L'}. \tag{3.17}$$

Here Ξ^0 is the first component of the unique solution (Ξ^0, Θ^0) in $\mathcal{S}^\ell(0,T)$ of (2.11) with terminal condition $-\xi^-$. And Ξ^L is the first component of the unique solution (Ξ^L, Θ^L) in $\mathcal{S}^\ell(0,T)$ of the BDSDE:

$$\begin{aligned} Y_t &= L - \frac{1}{\eta}\int_t^T Y_r|Y_r|^{q-1}dr + \int_t^T [f(r,0,Z_r) - f(r,0,0)]\,dr \\ &+ \int_t^T ((f_r^0)^+ \wedge L)dr + \int_t^T g\,(r,Y_r,Z_r)\,\overleftarrow{dB_r} - \int_t^T Z_r dW_r,\ 0 \leq t \leq T. \end{aligned} \tag{3.18}$$

In fact (Ξ^L, Θ^L) belongs to any $\mathcal{S}^r(0,T)$, $r > 1$. In order to have explicit and useful bound on Y^L, we add some restrictions on f and g.

- Assume that f^0 is a deterministic function and that $g(t,y,0) = 0$ for any (t,y) a.s. In this case for $L \geq 1$, $\Theta^L = 0$ and Ξ^L solves an ODE, which can be controlled by an adapted version of (3.10):

$$\Xi_t^L \leq \frac{\eta}{(T-t)^{p-1}} + \frac{1}{(T-t)^p}\int_t^T \left((T-s)^p (f_s^0)^+\right) ds.$$

- If g is linear and does not depend on z:

$$g(t,y,z) = g_0(t) + g_1(t)y,$$

then arguing as in the proof of [10, Proposition 3.1] we have for any $y \geq 0$ and $a \geq 0$

$$(p-1)y^q - pa^{q-1}y + a^q \geq 0,$$

thus with $a = (\eta/(q-1))^{p-1}/(T-t)^{p/q}$

$$-\frac{1}{\eta}y^q + (f_t^0)^+ \wedge L \leq -p\frac{1}{(T-t)}y + \left(\frac{\eta}{(q-1)(T-t)}\right)^p + (f_t^0)^+ \wedge L.$$

We define the vector $\zeta^L(s) = [f(s,0,\Theta_s^L) - f(s,0,0)]/\Theta_s^L$ (see (1.8) for the precise definition) and

$$\begin{aligned}
\Gamma_{t,r} &= \exp\left[\int_t^r \frac{-p}{T-s}ds + \int_t^r g_1(s)\overleftarrow{dB_s} + \int_t^r \zeta^L(s)dW_s - \frac{1}{2}\int_t^r ((\zeta^L(s))^2 - g_1(s)^2)ds\right] \\
&= \frac{(T-r)^p}{(T-t)^p}\exp\left[\int_t^r g_1(s)\overleftarrow{dB_s} + \int_t^r \zeta^L(s)dW_s - \frac{1}{2}\int_t^r ((\zeta^L(s))^2 - g_1(s)^2)ds\right] \\
&= \frac{(T-r)^p}{(T-t)^p}\gamma_{t,r}^L.
\end{aligned}$$

Let us apply the formula for linear BDSDE (see [18, Proposition 2.1]):

$$\begin{aligned}
\Xi_t^L &\leq \mathbb{E}\left[\frac{\varepsilon^p}{(T-t)^p}\gamma_{t,T-\varepsilon}^L \Xi_{T-\varepsilon}^L \middle| \mathcal{G}_t\right] \\
&+ \mathbb{E}\left[\int_t^{T-\varepsilon} \gamma_{t,r}^L \frac{(T-r)^p}{(T-t)^p}\left(\left(\frac{\eta}{(q-1)(T-r)}\right)^p + (f_r^0)^+ \wedge L + g_1(r)g_0(r)\right)dr \middle| \mathcal{G}_t\right] \\
&+ \mathbb{E}\left[\int_t^{T-\varepsilon} \gamma_{t,r}^L \frac{(T-r)^p}{(T-t)^p} g_1(r)\overleftarrow{dB_r}\middle| \mathcal{G}_t\right].
\end{aligned}$$

Let ε go to zero with the dominated convergence theorem:

$$\begin{aligned}
\Xi_t^L &\leq \frac{1}{(T-t)^p}\mathbb{E}\left[\int_t^T \gamma_{t,r}^L \left(\left(\frac{\eta}{q-1}\right)^p + (T-r)^p\left((f_r^0)^+ + g_1(r)g_0(r)\right)\right)dr \middle| \mathcal{G}_t\right] \\
&+ \frac{1}{(T-t)^p}\mathbb{E}\left[\int_t^T \gamma_{t,r}^L (T-r)^p g_1(r)\overleftarrow{dB_r}\middle| \mathcal{G}_t\right].
\end{aligned}$$

Since **(Ag1)** and **(A4)** hold, g_1 and ζ^L are bounded processes (resp. by K_g and by $K_{f,z}$). The martingale $\gamma_{t,\cdot}^L$ has finite ϱ–moments for any $\varrho \geq 1$. Thus the right-hand side is well-defined and finite provided that $(T-\cdot)^p((f_\cdot^0)^+ + g_0(\cdot))$ belongs to some $\mathbb{L}^\ell((0,T)\times\Omega)$.

To summarize we can find several sufficient non trivial conditions on f and g, such that sequence Y^L has an upper bound independent of L and finite on any interval $[0, T-\varepsilon]$, $\varepsilon > 0$. Then as with BSDEs with singular terminal conditions we prove

Theorem 3.3 *Under* **(A2)**, **(A3$\star$)**, **(A4)**, **(Ag1)**, **(C1)** *and* **(C2)** *with a positive constant η, if f^0 and g satisfy*

- *either f^0 is deterministic and $g(t,y,0) = 0$;*
- *or $g(t,y,z) = g_0(t) + g_1(t)y$ with a bounded process g_1, and if $(T-\cdot)^p((f_\cdot^0)^+ + g_0(\cdot))$ belongs to some $\mathbb{L}^\ell((0,T)\times\Omega)$*

there exists a process $(Y^{\min}, Z^{\min})$ *satisfying:*

1. *For all* $t \in [0,T[$,

$$\mathbb{E}\left[\sup_{0\le s\le t} |Y_s^{\min}|^\ell + \left(\int_0^t |Z_r^{\min}|^2 dr\right)^{\frac{\ell}{2}}\right] < +\infty.$$

2. *There exists a process* $\bar{Y} \in \mathbb{D}^\ell(0,T)$ *such that a.s.* $\bar{Y}_t \le Y_t^{\min}$.

3. *For all* $0 \le s \le t < T$*:*

$$Y_s^{\min} = Y_t^{\min} + \int_s^t f(r, Y_r^{\min}, Z_r^{\min})dr + \int_s^t g(r, Y_r^{\min}, Z_r^{\min})\overleftarrow{dB_r} - \int_s^t Z_r^{\min} dW_r.$$

4. $\mathbb{P}$*-a.s.* $\liminf_{t\to T} Y_t^{\min} \ge \xi$.

Moreover this solution is minimal: if $(\widetilde{Y}, \widetilde{Z})$ *is another supersolution bounded from below by some process in* $\mathbb{D}^\ell(0,T)$*, then a.s. for any* t*,* $\widetilde{Y}_t \ge Y_t^{\min}$.

Let us emphasize that the conditions imposed on f^0 and g are sufficient to obtain an a priori estimate. But they are surely not necessary.

3.2.2 For second order BSDEs

In [XVII], we study 2BSDEs with singular terminal condition, in order to solve a robust control problem. Thus our generator is very specific but we remark that the extension to more general driver is possible. Let us describe here this extension.

We work under the setting described in Sections 1.4 and 2.4. We consider a $\mathcal{F}_T$-Borel measurable random variable ξ such that for any $\mathbb{P} \in \mathcal{P}_0$, ξ is a.s. non-negative[4]. We denote by $\mathcal{S}_\infty$ the singular set $\{\xi = +\infty\}$. We define a Borel measurable function

$$\eta : \quad (t,\omega,a) \in [0,T] \times \Omega \times \mathbb{S}_d^{\ge 0} \longrightarrow \mathbb{R}_+^*.$$

Note that here η does not depend on the drift of X. We define for simplicity

$$\widehat{\eta}_s := \eta(s, X_{\cdot\wedge s}, \widehat{a}_s).$$

Finally we assume that there exists $\varrho > 1$ such that for any $(t,\omega) \in [0,T] \times \Omega$

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \int_t^T \left(\frac{1}{\widehat{\eta}_s}\right)^\varrho ds < \infty. \tag{3.19}$$

We shall also consider a generator function

$$f : (t,\omega,y,z,a,b) \in [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_d^{\ge 0} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

[4]This condition can be relaxed provided that the negative part of ξ satisfies **(C1)** uniformly w.r.t. $\mathbb{P}$. For simplicity we only consider the non-negative case.

Define for simplicity

$$\widehat{f}_s^{\mathbb{P}}(y,z) := f(s, X_{\cdot\wedge s}, y, z, \widehat{a}_s, b_s^{\mathbb{P}}) \text{ and } \widehat{f}_s^{\mathbb{P},0} := f(s, X_{\cdot\wedge s}, 0, 0, \widehat{a}_s, b_s^{\mathbb{P}}).$$

Generator f satisfies all assumptions of Part (2.4.1), together with the growth condition **(C2)**: there exists a constant $q > 1$ such that for any $y \geq 0$

$$\widehat{f}^{\mathbb{P}}(t,y,z) \leq -\frac{1}{\widehat{\eta}_t} y|y|^{q-1} + \widehat{f}^{\mathbb{P}}(t,0,z). \tag{3.20}$$

For simplicity as for the terminal condition ξ, we suppose that $\widehat{f}^{\mathbb{P},0}$ is non-negative for any $\mathbb{P} \in \mathcal{P}_0$. Since Condition **(A2$\star$)** should hold also for the generator $-\frac{1}{\eta(t,\omega,a)} y|y|^{q-1}$, this is the reason why η does not depend on the drift of X (and why (3.19) holds). Also see Remark 2.1. Now Assumption **(C3)** takes the following form: there exists $\ell > 1$ and $\kappa \in (1,\ell)$ such that for any (t,ω)

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left[(p-1)\widehat{\eta}_s^{p-1} + (T-s)^p \ \widehat{f}_s^{0,\mathbb{P}} \right]^{\ell} ds \right] < \infty, \tag{3.21}$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}} \left[\operatorname*{ess\,sup}_{0\leq t\leq T}^{\mathbb{P}} \left(\mathbb{E}^{\mathbb{P}} \left[\int_0^T \left[(p-1)\widehat{\eta}_s^{p-1} + (T-s)^p \ \widehat{f}_s^{0,\mathbb{P}} \right]^{\kappa} ds \middle| \mathcal{F}_t^+ \right] \right)^{\frac{\ell}{\kappa}} \right] < \infty. \tag{3.22}$$

From the results established in Section 2.4, we deduce that there exists a unique solution $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ to the second order BSDE: for any $0 \leq t \leq T$ and any $\mathbb{P}$

$$\begin{aligned} Y_t^L &= (\xi\wedge L) + \int_t^T \left[\widehat{f}_s^{\mathbb{P}}(Y_s^L, Z_s^L) - \widehat{f}_u^{0,\mathbb{P}} \right] du + \int_t^T (\widehat{f}_u^{0,\mathbb{P}} \wedge L) du \\ &\quad - \left(\int_t^T Z_s^L dX_s^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_t^T dM_s^{L,\mathbb{P}} + (K_T^{L,\mathbb{P}} - K_t^{L,\mathbb{P}}), \quad \mathbb{P}-a.s., \end{aligned} \tag{3.23}$$

such that:

- For any $p > 1$, Y^L belongs to $\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$.
- For any $1 < p < \varrho$, $(Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ is in $\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$.
- $K^{L,\mathbb{P}}$ is a $\mathbb{P}-a.s.$ non-decreasing process satisfying the minimality condition (2.19).

Moreover we have the representation formula

$$Y_t^L = \operatorname*{esssup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}^{\mathbb{P}} y_t^{L,\mathbb{P}'} \tag{3.24}$$

where $(y^{L,\mathbb{P}}, z^{L,\mathbb{P}}, m^{L,\mathbb{P}})$ is the solution under $\mathbb{P}$ of the BSDE

$$dy_t^{L,\mathbb{P}} = \left[\widehat{f}_t^{\mathbb{P}}(y_t^{L,\mathbb{P}}, z_t^{L,\mathbb{P}}) - \widehat{f}_t^{0,\mathbb{P}} \right] dt - (\widehat{f}_t^{0,\mathbb{P}} \wedge L) dt + z_t^{L,\mathbb{P}} dX_t^{c,\mathbb{P}} + dm_t^{L,\mathbb{P}}.$$

Note that by comparison principle for standard BSDEs, these solutions $y^{L,\mathbb{P}}$ satisfy the inequality: $\mathbb{P}$-a.s.

$$0 \le y_t^{L,\mathbb{P}} \le L(T+1), \quad \forall t \in [0,T].$$

Thus $\mathcal{P}_0$-q.s.

$$0 \le Y_t^L \le L(T+1) \quad \forall t \in [0,T].$$

Moreover for $L \le L'$ and any $\mathbb{P} \in \mathcal{P}_0$, we have $\mathbb{P} - a.s.$ for any $t \in [0,T]$

$$y_t^{L,\mathbb{P}} \le y_t^{L',\mathbb{P}} \le Y_t^{L'}.$$

Hence $\mathcal{P}_0$-q.s., $Y_t^L \le Y_t^{L'}$ for $t \in [0,T]$ (also see the comparison result [295, Theorem 4.3]).

In order to pass to the limit on L and to get a finite limit, as for BSDEs or BDSDEs, we need some a priori estimate of Y^L. The estimate (3.10) gives for any $\mathbb{P} \in \mathcal{P}_0$

$$0 \le y_t^{L,\mathbb{P}} \le \frac{K_{\ell,K_{f,z}}}{(T-t)^p} \left[\mathbb{E}^{\mathbb{P}} \left(\int_t^T \left(\widehat{\eta}_s + (T-s)^p (\widehat{f}_s^{0,\mathbb{P}})^+ \right)^\ell ds \middle| \mathcal{F}_t \right) \right]^{1/\ell} = \frac{K_{\ell,K_z}}{(T-t)^p} \, (u_t^{\mathbb{P}})^{1/\ell}.$$

The process $(u^{\mathbb{P}}, v^{\mathbb{P}}, n^{\mathbb{P}})$ is the solution of the BSDE

$$u_t^{\mathbb{P}} = \int_t^T \left(\widehat{\eta}_s + (T-s)^p \widehat{f}_s^{0,\mathbb{P}} \right) ds - \left(\int_t^T v_s^{\mathbb{P}} dX_s^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_t^T dn_s^{\mathbb{P}}.$$

Then using (3.21), (3.22) and [295, Theorem 4.1], there exists a unique solution $(U, V, \mathcal{N}^{\mathbb{P}}, \mathcal{K}^{\mathbb{P}})$ to the 2BSDE:

$$U_t = \int_t^T \left(\widehat{\eta}_s + (T-s)^p \widehat{f}_s^{0,\mathbb{P}} \right) ds - \left(\int_t^T V_s dX_s^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_t^T d\mathcal{N}_s^{\mathbb{P}} + (\mathcal{K}_T^{\mathbb{P}} - \mathcal{K}_t^{\mathbb{P}}),$$

such that $U \in \mathbb{D}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0})$ and $(V, \mathcal{N}^{\mathbb{P}}, \mathcal{K}^{\mathbb{P}})$ is in $\mathbb{H}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$. Moreover for any $\mathbb{P} \in \mathcal{P}_0$ and any $t \in [0,T]$, we have the representation formula:

$$\underset{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} \; u_t^{\mathbb{P}'} = U_t, \quad \mathbb{P} - a.s.$$

Thus there exists $U \in \mathbb{D}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0})$ such that for any $0 \le t \le T$, $\mathcal{P}_0$-q.s.

$$0 \le Y_t^L \le \frac{K_{\ell,K_{f,z}}}{(T-t)^p} (U_t)^{1/\ell}. \tag{3.25}$$

Let us emphasize that the right-hand side does not depend on L and is finite on $[0,T)$. From this a priori estimate, we deduce that for any $\varepsilon > 0$, sequence $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ converges, when L goes to $+\infty$, to $(Y^{\min}, Z^{\min}, (M^{\min})^{\mathbb{P}}, (K^{\min})^{\mathbb{P}})$ in the space $\mathbb{D}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ on $[0, T-\varepsilon]$, which means that all processes are restricted on this time interval. Moreover $(Y^{\min}, Z^{\min}, (M^{\min})^{\mathbb{P}}, (K^{\min})^{\mathbb{P}})$ satisfies the dynamics: for any $\mathbb{P} \in \mathcal{P}_0$, and any $0 \le s \le t < T$:

$$\begin{aligned} Y_s^{\min} = Y_t^{\min} &+ \int_s^t \widehat{f}_u^{\mathbb{P}}(Y_u^{\min}, \widehat{a}_u^{\frac{1}{2}} Z_u^{\min}) du - \left(\int_s^t Z_u^{\min} dX_u^{c,\mathbb{P}} \right)^{\mathbb{P}} \\ &- \int_s^t d(M^{\min})_u^{\mathbb{P}} + (K^{\min})_t^{\mathbb{P}} - (K^{\min})_s^{\mathbb{P}}. \end{aligned} \tag{3.26}$$

Finally $Y^{\min}$ satisfies the representation property: for any $t < T$ and any $\mathbb{P} \in \mathcal{P}_0$,

$$Y_t^{\min} = \underset{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{esssup}^{\mathbb{P}}} y_t^{\mathbb{P}'}, \quad \mathbb{P} - a.s.$$

where $y^{\mathbb{P}}$ is the minimal super-solution of

$$dy_t^{\mathbb{P}} = -\widehat{f}_t^{\mathbb{P}}(y_t^{\mathbb{P}}, z_t^{\mathbb{P}})dt + z_t^{\mathbb{P}} dX_t^{c,\mathbb{P}} + dm_t^{\mathbb{P}}.$$

Minimality condition (2.19) on $(K^{\min})^{\mathbb{P}}$ becomes: for any $\varepsilon > 0$

$$\underset{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{\operatorname{essinf}^{\mathbb{P}}} \mathbb{E}^{\mathbb{P}'} \left[\int_t^{T-\varepsilon} \exp\left(\int_t^s \lambda_u^{\mathbb{P}'} du \right) d(K^{\min})_s^{\mathbb{P}'} \middle| \mathcal{F}_t^+ \right] = 0, \quad 0 \leq t \leq T - \varepsilon, \ \mathbb{P} - a.s.,$$

where

$$\lambda_u^{\mathbb{P}'} = \frac{\widehat{f}_s^{\mathbb{P}'}(Y_s^{\min}, z_s^{\mathbb{P}'}) - \widehat{f}_s^{\mathbb{P}'}(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'})}{Y_s^{\min} - y_s^{\mathbb{P}'}} \mathbb{1}_{Y_s^{\min} \neq y_s^{\mathbb{P}'}}.$$

Recall that for singular BSDEs, we need the left-continuity of the filtration at time T to avoid the thin time case. Here we require that

- **Left-continuity condition:** for any probability measure $\mathbb{P} \in \mathcal{P}_t$, filtration $\mathbb{F}_+^{\mathbb{P}}$ is left continuous at time T.

This condition implies that for any $\mathbb{P} \in \mathcal{P}_0$

$$\liminf_{s \to T} y_s^{\mathbb{P}} \geq \xi, \quad \mathbb{P} - a.s.$$

Hence from the representation formula, the same inequality holds for $Y^{\min}$. In [XVII], we give an example of probability measure family $\mathcal{P}$ satisfying this hypothesis.

Finally arguing as for the case of BSDEs, we show that:

Theorem 3.4 *Process* $(Y^{\min}, Z^{\min}, (M^{\min})^{\mathbb{P}}, (K^{\min})^{\mathbb{P}})$ *is the minimal non-negative supersolution of the 2BSDE* (3.26) *with singular terminal condition* ξ.

3.3 Continuity problem

In Sections 3.1 and 3.2, the existence of a minimal super-solution of BSDE (2.4) (or BDSDEs or 2BSDEs) with singular terminal condition is established under some conditions. The main requirement is that f decreases w.r.t. y sufficiently fast when y is large and that the filtration is left-continuous at time T. The final condition on $Y^{\min}$ is (3.6):

$$\liminf_{t \to T} Y^{\min}(t) \geq \xi.$$

In the classical setting ($\xi \in \mathbb{L}^p(\Omega)$), $Y^{\min}$ has a limit as t increases to T since the process is solution of BSDE (2.4) and thus is càdlàg. Moreover this limit is equal to ξ a.s. if filtration $\mathbb{F}$ is left-continuous at time T. For the related control problem (see Chapter 5), this weak behaviour (3.6) at time T of the minimal process $Y^{\min}$ is sufficient to obtain the optimal control and the value function. Nevertheless two natural questions arise here:

1. Does the limit of $Y^{\min}$ at time T exist ?

2. Can the inequality (3.6) be an equality if the filtration is left-continuous at time T ? We call this question *continuity problem.*

We call continuity problem the equality: a.s.

$$\liminf_{t\to T} Y_t^{\min} = \xi. \tag{3.27}$$

Here we assume again that the underlying filtration $\mathbb{F}$ is left-continuous at time T to avoid the case where T could be a thin time (see the framework of Theorem 3.1).

Despite the very theoretical aspect of these questions, there are several applications. Minimal supersolutions of BSDE (2.4) with singular terminal conditions can be used to represent the value function of a corresponding stochastic optimal control problem with constraints, see Chapter 5 (or [10, X] and [XVI, Section 4]) for the precise formulation of the optimal control problem and a detailed discussion. In this connection between the BSDE and its corresponding stochastic optimal control problem, changing the terminal condition of the BSDE corresponds to changing the terminal payoff and the constraints of the problem. A natural question: when these change, do the value function and the optimal control of the control problem change? The continuity results we prove establishing that a minimal supersolution is a solution in the sense of:

$$\lim_{t\to T} Y^{\min}(t) = \xi \tag{3.28}$$

provides an answer as follows. Suppose $Y^{(1)}$ and $Y^{(2)}$ are minimal supersolutions of BSDE (2.4) for two distinct terminal conditions $\xi^{(1)}$, $\xi^{(2)}$. Suppose that $Y^{(i)}$ are solutions to the BSDE with these terminal conditions in the sense of (3.28), i.e., that $Y^{(i)}$ are both continuous at time T. This and $\xi^{(1)} \neq \xi^{(2)}$ imply that $Y^{(1)}$ and $Y^{(2)}$ are distinct processes. To rephrase this in terms of the control interpretation: changing the constraint and terminal value of the control problem from $\xi^{(1)}$ to $\xi^{(2)}$ leads to distinct value functions (and hence optimal controls) for the control problem. And in other words, we think that the continuity problem is a first step (and a necessary condition) for uniqueness of the solution.

We explain a further implication of the continuity results to optimal control through the following example. Let X denote the state process of the corresponding optimal control problem. As explained in [10, X] the terminal condition $\xi^{(1)} = \infty$ corresponds to the constraint $X_T = 0$. Let us relax this constraint to requiring $X_T = 0$ only when $\{\tau > T\}$ where τ is a stopping time of the filtration. The corresponding terminal condition $\xi = \infty \cdot \mathbf{1}_{\{\tau > T\}}$ belongs to the class we treat in Section 3.3.4. Two questions:

1. Does this relaxation lead to a lower value function? This question is a special case of the question discussed in the previous paragraph, i.e., whether the same BSDE with distinct terminal conditions have distinct solutions, and we know that continuity of the solution implies that the solutions will be distinct.

2. A more delicate question: is the optimal control tight, i.e., is it the case that, under the optimal control $X_T = 0$ if and only if $\{\tau < T\}$? The continuity of the minimal supersolution implies that the answer to this question is also affirmative. In finance applications a non-tight optimal control can be interpreted as a strictly super-hedging trading strategy. Continuity results overrule such strategies.

As a last point in connection with optimal control and optimal liquidation we note that the continuity of the minimal supersolution at terminal time appears in [20], as a condition for the solution of an optimal targeting problem.

To summarize the obtained results on this topic, we distinguish the two questions:

1. The existence of a limit at time T is proved under a structural condition on generator f ([XII, Theorem 3.1]). Roughly speaking $Y^{\min}$ is a non-linear continuous transform of a non-negative supermartingale.

2. The equality in (3.6) is obtained under some sufficient conditions on the terminal data ξ and on the growth of f w.r.t. y.

In this section, we call Y the solution of a BSDE, even if we should precise the first component Y of the solution. Indeed we are essentially interested by the behavior of Y at time T.

Before going into details, we mention

Lemma 3.2 *Let $Y^{\min}$ be the minimal supersolution of the BSDE with singular terminal condition ξ. Suppose that continuity condition holds for $Y^{\min}$:*

$$\lim_{t\to T} Y_t^{\min} = \xi.$$

Then we have a.s. on $\{\xi < \infty\}$

$$\sup_{t\in[0,T]} \frac{\mathbb{E}[\xi = \infty|\mathcal{F}_t]}{(T-t)^{p-1}} < \infty.$$

This result is related to the notion of fractional smoothness developed by [323, 160], by [136] for applications in finance and by [135, 137] in the context of BSDEs. Indeed let A be an element of $\mathcal{F}_T$ and assume that continuity holds both for $\xi = +\infty\mathbf{1}_A$ and for $\xi = +\infty\mathbf{1}_{A^c}$. From the preceding lemma, we get the following inequality: it holds $\mathbb{P}$-a.s.

$$\forall t \in [0,T], \quad |\mathbf{1}_A - \mathbb{E}(\mathbf{1}_A|\mathcal{F}_t)| \leq C(T-t)^{p-1}.$$

Following [135, Definition 1], this means that $\mathbf{1}_A$ belongs to $B^{2(p-1)}_{q,\infty}(W)$ for any $1 < q < \infty$.

Singularity of the generator

Note that generator f of the BSDE (2.4) can be *singular* in the sense that Condition **(C3)** implies

$$\mathbb{E}\int_0^T (T-s)^{\ell p}((f^0(s))^+)^{\ell} ds < +\infty.$$

Thus $(f^0)^+ \in \mathbb{L}^1((0,T-\varepsilon)\times\Omega)$ for any $\varepsilon > 0$, but we could have $(f^0_T)^+ = +\infty$ and/or $(f^0)^+ \notin \mathbb{L}^1((0,T)\times\Omega)$. For example if $f^0_t = (T-t)^{-\varpi}$ with $1 \leq \ell$ and $\varpi < 1+1/q+1/\ell$. Hence for $\varpi \geq 1$, then $f^0 \notin \mathbb{L}^1((0,T)\times\Omega)$. The next result shows that Equality (3.27)

$$\liminf_{t\to T} Y^{\min}_t = \xi$$

may be false.

Lemma 3.3 *Assume that the generator is given by: $f(t,y,z,u) = f(t,y) = -y|y|^{q-1} + f^0_t$ with f^0 non-negative, deterministic and not in $\mathbb{L}^1(0,T)$. Then a.s. $\lim_{t\to T} Y^{\min}_t = +\infty$.*

The proof can be found in [XII]. Hence Equality (3.27) cannot be true whatever the terminal condition ξ is. Hence in the rest of this section, we assume that

$$(f^0)^+ \in \mathbb{L}^1((0,T)\times\Omega). \tag{3.29}$$

Hence with Condition **(C1)**, f^0 belongs to $\mathbb{L}^1((0,T)\times\Omega)$.

Sharper estimate on $Z^{\min}$ and $U^{\min}$

The understanding of the behavior of $Y^{\min}$ at time T cannot be separated from the study of the martingale part $(Z^{\min}, U^{\min}, M^{\min})$. However since $M^{\min}$ does not appear in the generator, we only focus on $(Z^{\min}, U^{\min})$. In the construction of the minimal solution, sequences Z^L and U^L converge in a suitable integrability space on $[0,T-\varepsilon]$ for any $\varepsilon > 0$. Here we want to obtain an estimate on limits $Z^{\min}$ and $U^{\min}$ on the whole time interval $[0,T]$.

To get a better estimate on $(Z^{\min}, U^{\min})$, we reinforce the condition **(C3)**:

(C3*) There exist $\varpi < 1$ and $\ell > 1$ such that

$$\mathbb{E}\int_0^T (T-s)^{-1+\varpi}\left[((p-1)\eta_s)^{p-1} + (T-s)^p(f^0_s)^+\right]^{\ell} ds < +\infty.$$

Some remarks concerning this assumption:

- Since $1/\eta$ belongs to $\mathbb{L}^1((0,T)\times\Omega)$, necessarily

$$\varpi > -\frac{\ell}{q-1}.$$

- If **(C3)** holds from some $\ell' > 1$, then by Hölder's inequality for any $1 < \ell < \ell'$ and $\frac{\ell}{\ell'} \leq \varpi < 1$, **(C3*)** holds.

- In particular for bounded coefficients, **(C3*)** is verified for any $0 < \varpi < 1$.

The key result on $(Z^{\min}, U^{\min})$ is the following (see [XII, Proposition 3.3]). Recall that (Y^L, Z^L, U^L, M^L) is the solution of the BSDE (3.8).

Proposition 3.2 *Under Assumption* **(C3*)***, there exists a constant C independent of L such that process (Z^L, U^L) satisfies:*

$$\mathbb{E}\left[\int_0^T (T-s)^\rho \left(|Z_s^L|^2 + \|U_s^L\|^2_{\mathbb{L}^2_\pi}\right) ds\right]^{\ell/2} \leq C.$$

Constant ρ is given by:

$$\rho = \frac{2}{q-1} + 2\left(1 - \frac{1}{\ell}\right) + \frac{2\varpi}{\ell}. \tag{3.30}$$

Under our general framework, constant C is not explicitly given, but it depends on q, ℓ, ϖ, the Lipschitz constants $K_{f,z}$ and $K_{f,\psi}$ of the generator f, and η and $(f^0)^+$ through the following integral:

$$\mathbb{E}\int_0^T (T-u)^{-2+\eta} \left(\int_u^T \left[((p-1)\eta_s)^{p-1} + (T-s)^p (f_s^0)^+\right]^\ell ds\right) du < +\infty.$$

See [XIV, Lemma 3.6] for the finiteness of this quantity. However if $f(y) = -y|y|^{q-1}$, we can take $\ell = 1$ and $\varpi = 0$, in other words $\rho = 2/(q-1)$. Constant C is explicitly given by: $C = 16\left(\frac{1}{q-1}\right)^{\frac{2}{q-1}}$ (see [I, Proposition 10]). Moreover we extend this result for BDSDE in [XIII, Proposition 4].

3.3.1 Existence of a limit at time T [XII]

As mentioned before, the existence of a limit essentially depends on generator f. We assume that all conditions **(A2)**–**(A5')** and **(C1)**–**(C4)** hold. But we do not suppose that the filtration is left-continuous at time T. We prove that the left limit of $Y^{\min}$ at time T exists provided we know the precise behavior of the generator w.r.t. y. In other words we show that $Y^{\min}$ is càdlàg on $[0, T]$. In some sense our generator has to be more specific to control the behaviour of the supersolution at time T.

The main hypothesis is the following: the generator satisfies

$$b_t g(y) \leq f(t, y, z, u) - f(t, 0, z, u), \quad \forall y \geq 0, \ \forall (t, z, u), \tag{3.31}$$

where

- b is positive and $b \in \mathbb{L}^1((0,T) \times \Omega)$;
- g is a negative, decreasing and of class C^1 function and concave on $\mathbb{R}_+$ with $g(0) < 0$ and $g'(0) < 0$.

Since Condition **(C2)** should hold, from (3.31) we deduce that $b_t g(y) \leq -a_t y|y|^{q-1}$ ($a = 1/\eta$) for any $t \in [0,T]$ and y. Thus w.l.o.g. $g(y) \leq -y|y|^{q-1}$ and $b_t \geq (-1/g(1))a_t = Ca_t$ for some positive constant C. We can always add to g a linear function like $-y-1$ such that $g(0) < 0$ and $g'(0) < 0$.

We decompose f as follows:

$$f(t,y,z,u) = \phi(t,y,z,u) + \pi(t,z,u) + f_t^0$$

where $f_t^0 = f(t,0,0,\mathbf{0})$ and

$$\begin{aligned} \phi(t,y,z,u) &= f(t,y,z,u) - f(t,0,z,u) \\ \pi(t,z,u) &= f(t,0,z,u) - f(t,0,0,\mathbf{0}). \end{aligned}$$

Our main result is the following:

Theorem 3.5 *Assumptions* **(A2)–(A5')**, **(C1)–(C4)** *and* (3.31) *hold. Moreover one of the next three cases holds:*

- **Case 1.** *f does not depend on u or $\pi(t,0,u) \geq 0$;*
- **Case 2.** *$\vartheta \in \mathbb{L}^1_\lambda(\mathcal{E})$ and there exists a constant $\kappa_* > -1$ such that $\kappa_s^{0,0,u,0}(e) \geq \kappa_*$ a.e. for any (s,u,e);*
- **Case 3.** *λ is a finite measure on $\mathcal{E}$.*

Then the minimal supersolution $Y^{\min}$ has a left limit at time T.

Let us add some comments.

1. This result shows that the process $Y^{\min}$ is càdlàg on $[0,T]$ when filtration $\mathbb{F}$ is complete and right-continuous. No additional assumption (left-continuity) on the filtration is needed here.
2. If Inequality (3.6) holds, then a.s. $\lim_{t\to T} Y_t^{\min} \geq \xi$.
3. The second condition on κ in Case 2 is quite classical. Indeed a stronger version is used to prove the comparison principle for BSDE with jumps in [24] or in [302].

In [XI], the control problem leads to a generator f (see BSDE (5.9) and (5.10)) which satisfies due to the Lipschitz continuity of Θ w.r.t. y, for $y \geq 0$:

$$\begin{aligned} f(t,y,z,u) - f(t,0,z,u) &= -\frac{y|y|^{q-1}}{(q-1)\alpha_t^{q-1}} - \Theta(t,y,u) + \Theta(t,0,u) \\ &\geq -\frac{y|y|^{q-1}}{(q-1)\alpha_t^{q-1}} - L|y| \\ &\geq -\left(\frac{1}{(q-1)\alpha_t^{q-1}} \vee L\right)\left(y^{1+q} + y\right) \geq b_t g(y). \end{aligned}$$

if

$$b_t = \frac{1}{(q-1)\alpha_t^{q-1}} \vee L, \qquad g(y) = -y^q - y - 1.$$

Let us just give the trick of the proof of the previous theorem. If b_t is deterministic, consider the ordinary differential equation $y' = -f(t,y) = -b_t g(y)$. To solve it, we can separate the variables and with $G' = 1/g$, we write formally:

$$G(y(T)) - G(y(t)) = -\int_t^T \frac{y'(s)}{g(y(s))} ds = -\int_t^T b_s ds$$

which gives:

$$y(t) = \Theta^{-1} \left(G(y(T)) + \int_t^T b_s ds \right).$$

We follow the same idea: we apply the Itô formula with function G to process $Y_t^{\min}$. Then we cancel the martingale part with the conditional expectation and we have to control the terms of finite variations. The positive parts give a non negative supermartingale, which has always a limit at time T. The negative parts have to be more carefully studied to prove that they have a limit at time T. This is the reason why we impose these extra conditions on f, κ or λ. Let us emphasize that the same trick was used in [I] or in [XIII] on singular BDSDEs, for a simpler generator (see therein for more details).

The key step is given in the next lemma.

Lemma 3.4 *Assume that the conditions of Theorem 3.5 are satisfied. Then the process $Y^{\min}$ can be written as follows:*

$$Y_t^{\min} = G^{-1}\left(\mathbb{E}^{\mathcal{F}_t}\left[G(\xi)\right] + \psi_t^- - \psi_t^+\right)$$

where ψ^+ and ψ^- are two non-negative càdlàg supermartingales with a.s. $\lim_{t\to T} \psi_t^- = 0$.

The details of the proof are in [XII]. Theorem 3.5 can be immediately proved. ψ^+ being a non-negative càdlàg supermartingale, we can deduce the existence of the following limit:

$$\psi_{T-}^+ := \lim_{t\nearrow T} \psi_t^+$$

And so $Y_{T-}^{\min}$ exists and is equal to:

$$Y_{T-}^{\min} := \lim_{t\nearrow T} Y_t^{\min} = G^{-1}\left(G(\xi) - \psi_{T-}^+\right).$$

Note that we have no idea how to prove that $\psi_{T-}^+ = 0$ (even if filtration $\mathbb{F}$ is left-continuous at time T). This property would directly give that

$$Y_{T-}^{\min} := \lim_{t\nearrow T} Y_t^{\min} = \xi = Y_T^{\min}.$$

In [XIII, section 3.2], we show that this idea can easily be adapted to BDSDEs with singular terminal condition.

3.3.2 Half-Markovian framework $\xi = \Phi(X_T)$

We assume now that ξ is a deterministic function Φ of the terminal value of a diffusion process X: $\xi = \Phi(X_T)$. We call this case half-Markovian since we do not impose some similar condition on f (as in the classical Markovian setting for BSDEs).

To obtain the desired result (3.27), some quantities of the form:

$$\int_0^T |\phi(s, X_s) Z_s^L| ds$$

have to be controlled. We use Hölder's and Young's inequalities to obtain:

$$\begin{aligned} \int_0^T |\phi(s, X_s) Z_s^L| ds &\leq \left[\int_0^T (T-s)^\rho |Z_s^L|^2 ds \right]^{1/2} \left[\int_0^T \frac{|\phi(s, X_s)|^2}{(T-s)^\rho} ds \right]^{1/2} \\ &\leq \frac{1}{\ell} \left[\int_0^T (T-s)^\rho |Z_s^L|^2 ds \right]^{\frac{\ell}{2}} + \frac{\ell - 1}{\ell} \left[\int_0^T \frac{|\phi(s, X_s)|^2}{(T-s)^\rho} ds \right]^{\frac{\ell-1}{2\ell}}. \end{aligned}$$

For bounded function ϕ, the last integral is finite if and only if $\rho < 1$. Hence in this section, we assume that

$$\rho = \frac{2}{q-1} + 2\left(1 - \frac{1}{\ell}\right) + \frac{2\varpi}{\ell} < 1 \tag{3.32}$$

Condition $\rho < 1$ is a balance between the non linearity q and the singularity of the generator f. Let us remark that:

- Under **(C3)** with constant ℓ', then **(C3*)** holds for any $1 < \ell < \ell'$ and $\frac{\ell}{\ell'} < \varpi < 1$. And (3.32) holds if
$$\frac{1}{q-1} + \frac{1}{\ell'} < \frac{1}{2},$$
that is $\ell' > 2$, $q - 1 > 2$ and $(q-3)(\ell' - 2) > 4$.
- In particular if η and $(f^0)^+$ are bounded ($\ell' = +\infty$), then we can take ϖ close to zero and ℓ close to 1: $\rho < 1 \Leftrightarrow q - 1 > 2$. This bound on q is also supposed in [I], in [XIII] and in [XIX].

Now we define the function Φ on $\mathbb{R}^d$ with values in $[0, +\infty] = \mathbb{R}_+ \cup \{+\infty\}$ and with

$$\mathcal{S}_\infty = \{x \in \mathbb{R}^d \quad s.t. \quad \Phi(x) = \infty\}$$

the set of singularity points for the terminal condition induced by Φ. $\mathcal{S}_\infty$ is supposed to be closed. We also denoted by $\partial \mathcal{S}_\infty$ the boundary of $\mathcal{S}_\infty$.

Our terminal condition ξ satisfies:

$$\xi = \Phi(X_T). \tag{3.33}$$

and for all closed set $\mathcal{K} \subset \mathbb{R}^d \setminus \mathcal{S}_\infty$

$$\Phi(X_T) \mathbf{1}_{\mathcal{K}}(X_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}). \tag{3.34}$$

Process X is the solution of the SDE (1.15) with jumps:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s + \int_0^t \int_{\mathcal{E}} \beta(s, X_{s_-}, e) \tilde{\mu}(de, ds).$$

The coefficients b, σ and β satisfy **(B1)** to **(B4)**.

Continuity problem for large value of q

In the proof of the continuity of $Y^{\min}$, there is a term due to the covariance between the jumps of the SDE (1.15) and the jumps of the BSDE (2.4). To control this additional part, we make a link between the singularity set $\mathcal{S}_\infty$ and the jumps of the forward process X. More precisely we assume

(D1) The boundary $\partial \mathcal{S}_\infty$ is compact and of class C^2.

(D2) For any $x \in \mathcal{S}_\infty$, any $s \in [0,T]$ and λ-a.s.

$$x + \beta(s,x,e) \in \mathcal{S}_\infty.$$

Furthermore there exists a constant $\nu > 0$ such that if $x \in \partial\mathcal{S}_\infty$, then for any $s \in [0,T]$, $d(x+\beta(s,x,e), \partial\mathcal{S}_\infty) \geq \nu$, λ-a.s.

These assumptions mean in particular that if $X_{s^-} \in \mathcal{S}_\infty$, then $X_s \in \mathcal{S}_\infty$ a.s. Moreover if X_{s^-} belongs to the boundary of $\mathcal{S}_\infty$, and if there is a jump at time s, then X_s is in the interior of $\mathcal{S}_\infty$. Let us now state [XII, Theorem 3.5]:

Theorem 3.6 *Under Conditions* (3.29), **(C3*)** *with* (3.32), (3.33), (3.34) *and* **(D1)**-**(D2)**, *the minimal supersolution $Y^{\min}$ satisfies a.s.*

$$\liminf_{t\to T} Y_t^{\min} = \xi.$$

For the details of the proof, we refer to [I] (when there is only the Brownian motion) and [XII]. We simply give the start and the sketch of the proof. We consider (Y^L, Z^L, U^L, M^L) the solution of the BSDE (3.8) with terminal condition $\xi \wedge L$ and generator f^L. Let ϕ be a non negative function in $C^2_b(\mathbb{R})$, the set of bounded smooth functions of class C^2, with bounded derivatives, and such that the support of ϕ is included in $\mathcal{R} = \mathcal{S}_\infty^c$. We apply Itô's formula to process $Y^L\phi(X)$ between 0 and t and then take the expectation:

$$\begin{aligned}(3.35)\ \mathbb{E}[Y_t^L\phi(X_t)] &= \mathbb{E}[Y_0^L\phi(X_0)] - \mathbb{E}\left[\int_0^t \phi(X_{s-})f^L(s,Y_s^L,Z_s^L,U_s^L)ds\right] \\ &+\mathbb{E}\left[\int_0^t Y_{s-}^L\mathcal{L}\phi(s,X_s)ds\right] + \mathbb{E}\left[\int_0^t Y_{s-}^L\mathcal{I}(s,X_{s-},\phi)ds\right] \\ &+\mathbb{E}\left[\int_0^t \nabla\phi(X_s)\sigma(s,X_s)Z_s^L ds\right] + \mathbb{E}\left[\int_0^t\int_{\mathcal{E}}(\phi(X_s)-\phi(X_{s-}))U_s^L(e)\mu(de)ds\right].\end{aligned}$$

Operators $\mathcal{L}$ and $\mathcal{I}$ are defined on $C^2(\mathbb{R})$ by:

$$\mathcal{L}\phi(t,x) = \nabla\phi(x)b(t,x) + \frac{1}{2}\mathrm{Trace}(D^2\phi(x)(\sigma\sigma^*)(t,x))$$

and

$$\mathcal{I}(t,x,\phi) = \int_{\mathcal{E}}[\phi(x+\beta(t,x,e)) - \phi(x) - (\nabla\phi)(x)\beta(t,x,e)]\mu(de). \tag{3.36}$$

The linearization argument (w.r.t. z and u, see Equation (1.8)) for BSDE implies that (3.35) is equal to:

$$
\begin{aligned}
(3.37) \qquad \mathbb{E}[Y_t^L \phi(X_t)] &= \mathbb{E}[Y_0^L \phi(X_0)] + \mathbb{E}\left[\int_0^t \phi(X_{s-})(f_s^0 \wedge L) ds\right] \\
&-\mathbb{E}\left[\int_0^t \phi(X_{s-})(f(s, Y_s^L, 0, 0) - f_s^0) ds\right] \\
&+\mathbb{E}\left[\int_0^t Y_{s-}^n \mathcal{L}\phi(s, X_s) ds\right] + \mathbb{E}\left[\int_0^t Y_{s-}^L \mathcal{I}(s, X_{s-}, \phi) ds\right] \\
&+\mathbb{E}\left[\int_0^t \left(\nabla\phi(X_s)\sigma(s, X_s) - \phi(X_s)\mathfrak{z}_s^L\right) Z_s^L ds\right] \\
&+\mathbb{E}\left[\int_0^t \left[\int_{\mathcal{E}} [(\phi(X_s) - \phi(X_{s-}))] U_s^L(e)\lambda(de) - \phi(X_{s-})\mathfrak{u}_s^L\right] ds\right].
\end{aligned}
$$

where $\mathfrak{z}^L$ is a bounded process (by $K_{f,z}$, **(A4)**) and $\mathfrak{u}_s^L = f(s, Y_s^L, Z_s^L, U_s^L) - f(s, Y_s^L, Z_s^L, 0)$ can be controlled uniformly (by $K_{f,u}$, **(A5)**).

First we prove that we can pass to the limit on L in (3.37) and that the limits have suitable integrability conditions on $[0, T] \times \Omega$. Our assumptions **(C3*)** and (3.32) are used to control the last two terms, whereas (3.33) and (3.34) give a uniform (w.r.t. L) upper bound on the first two terms, and (3.29) on the third one. The expectation with $\mathcal{L}$ can be estimated exactly as in [I] by some technics developed in [241]; the function ϕ has to be chosen a little bit more carefully. Finally our hypotheses **(D1)** and **(D2)** are used to give a bound on the term with the non local operator $\mathcal{I}$ ([XII, Lemma 3.8]).

Secondly we rewrite (3.37) between t and T and we pass to the limit when t goes to T. The conclusion comes from Fatou's lemma.

For small value of q with the Malliavin calculus

If q is too small (that is when (3.32) fails), we loose our control on Z^L and U^L. However in the Brownian setting, where X is the solution of (1.9), we have the following representation of Z: Z_s is the Malliavin derivative D_s of Y_s. More precisely we consider the BSDE:

$$Y_t = h(X_T) + \int_t^T f(r, Y_r, Z_r) dr - \int_t^T Z_r dW_r,$$

where $h : \mathbb{R}^d \to \mathbb{R}$ is a bounded and Lipschitz function. The existence and uniqueness of the solution (Y, Z) comes from the conditions **(A2)**, **(A3)** and **(A4)**, and Y is a bounded process if f^0 is bounded. In [I] or [XIII] we consider deterministic generators $f(y) = -y|y|^{q-1}$. The extension of our setting requires additional conditions on f, namely:

- **(A6)** For any (t, z), the map $y \mapsto f(t, y, z)$ is locally Lipschitz continuous.
- **(A7)** For each (y, z), $f(\cdot, y, z)$ is in $\mathbb{L}_{1,2}^a(\mathbb{R})$ with Malliavin derivative denoted by $D_\theta f(t, y, z)$ and $\int_0^T \|D_\theta f(t, Y_t, Z_t)\|_{\mathbb{H}^2(0,T)}^2 d\theta < +\infty$.

(A8) For any $t \in [0,T]$, $R \geq 0$ and any (y, y', z, z') s.t. $|y| + |y'| \leq R$,

$$|D_\theta f(t, y', z') - D_\theta f(t, y, z)| \leq K_\theta(t, \omega)(|y - y'| + |z - z'|)$$

where for a.e. θ, $\{K_\theta(t, \cdot),\ 0 \leq t \leq T\}$ is an $\mathbb{R}_+$-valued adapted process satisfying $\int_0^T \|K_\theta\|^4_{\mathbb{H}^4(0,T)} d\theta < +\infty$.

$\mathbb{L}^a_{1,2}(\mathbb{R})$ denotes the set of $\mathbb{R}$ -valued progressively measurable processes $\{u(t, \omega),\ 0 \leq t \leq T;\ \omega \in \Omega\}$ such that:

- For a.e. $t \in [0,T]$, $u(t, \cdot)$ belongs to $\mathbb{D}_{1,2}$.
- $(t, \omega) \mapsto Du(t, \omega) \in (\mathbb{L}^2(0,T))^k$ admits a progressively measurable version.
- $\|u\|^a_{1,2} = \int_0^T |u(t)|^2 dt + \int_0^T \int_0^T |D_\theta u(t)|^2 d\theta dt < +\infty$.

(A8) is a locally Lipschitz continuous property of the Malliavin derivative. If $f(y) = -y|y|^{q-1}$, these properties are trivially true.

Lemma 3.5 *If* **(A5)** *to* **(A8)** *hold together with*

$$\mathbb{E} \int_0^T |f^0_t|^4 dt < +\infty, \tag{3.38}$$

for all $1 \leq i \leq k$, $\{D^i_s Y_s, 0 \leq s \leq T\}$ is a version of Z^i.

$Z^i = \{(Z^i_s), 0 \leq s \leq T\}$ denotes the i-th component of Z. This result comes from the [120, Proposition 5.3]. Here, $D^i_s Y_s$ has the following sense:

$$D^i_s Y_s = \lim_{\substack{r \to s \\ r < s}} D^i_r Y_s.$$

Indeed from Proposition 1.2, we know that X_T belongs to $\mathbb{D}^{1,\infty}$, and since h is Lipschitz, with the [258, Proposition 1.2.3], $\xi = h(X_T) \in \mathbb{D}^{1,2}$. Moreover, since h is bounded, Y is also bounded. From **(A6)**, **(A7)** and **(A8)**, the conclusion of [120, Proposition 5.3] holds.

Additionally to **(B1)** and **(B2)**, let us assume the next conditions on the coefficients of the diffusion process X:

(B5) σ and b are bounded: there exists a constant K s.t.

$$\forall (t, x) \in [0,T] \times \mathbb{R}^d, \quad |b(t,x)| + \|\sigma(t,x)\| \leq K;$$

(B6) The second derivatives of $\sigma\sigma^*$ belongs to $\mathbb{L}^\infty$:

$$\frac{\partial^2 \sigma\sigma^*}{\partial x_i \partial x_j} \in L^\infty([0,T] \times \mathbb{R}^d).$$

(B7) $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$\forall y \in \mathbb{R}^d,\ \sigma\sigma^*(t,x) y.y \geq \lambda |y|^2.$$

The next result is proved in [I, Proposition 16].

Lemma 3.6 *Under assumptions* **(B5)** *to* **(B7)***, for each function ϕ in the class $C^2\left(\mathbb{R}^d\right)$ with a compact support, there exists a real Borel function ψ defined on $]0,T] \times \mathbb{R}^d$ s.t. for all $t > 0$, $\mathbb{E}\left(|Y_t \psi(t,X_t)|\right) < +\infty$ and*

$$\mathbb{E}\left[Z_t . \nabla \phi(X_t) \sigma(t,X_t)\right] = -\ \mathbb{E}\left[Y_t \psi(t,X_t)\right].$$

The function ψ is given by the following formula:

$$(3.39) \qquad \psi(t,x) = \sum_{i=1}^k (\nabla \phi \sigma)_i(x) \frac{div\ (p\sigma^i)(t,x)}{p(t,x)} + Trace\left(D^2\phi(x)\sigma\sigma^*(t,x)\right) + \sum_{i=1}^k \nabla \phi(x).[(\nabla \sigma^i)\sigma^i](t,x);$$

where σ^i is the i-th column of the matrix σ and p is the density of the process X.

Now we are able to obtain continuity of $Y^{\min}$ at time T. We assume again that our terminal condition ξ satisfies (3.33) together with (3.34). But we also suppose that Φ is continuous from $\mathbb{R}^d$ to $\overline{\mathbb{R}_+}$ and:

$$(3.40) \qquad \forall M \geq 0,\ \Phi \text{ is a Lipschitz function on the set } \mathcal{O}_M = \{|\Phi| \leq M\}.$$

This hypothesis implies that $\Phi \wedge L$ is a Lipschitz function on $\mathbb{R}^d$. Indeed if we define

$$K_L = \sup\left\{\frac{|\Phi(x)-\Phi(y)|}{|x-y|};\ \Phi(x) \vee \Phi(y) \leq L\right\},$$

then assumption (3.40) implies that $\Phi \wedge L$ has a Lipschitz norm smaller than K_{L+1}. Therefore solution (Y^L, Z^L) of BSDE (3.8) with terminal condition $\xi \wedge L$ and generator f^L satisfies the Lemma 3.5: $Z^L = D.Y^L$. Coming back to (3.37), we have

$$\begin{aligned}\mathbb{E}[Y_t^L \phi(X_t)] &= \mathbb{E}[Y_0^L \phi(X_0)] + \mathbb{E}\left[\int_0^t \phi(X_{s-})(f_s^0 \wedge L)ds\right] \\ &- \mathbb{E}\left[\int_0^t \phi(X_{s-})(f(s,Y_s^L,0) - f_s^0)ds\right] + \mathbb{E}\left[\int_0^t Y_{s-}^L \mathcal{L}\phi(s,X_s)ds\right] \\ &+ \mathbb{E}\left[\int_0^t \left(\nabla\phi(X_s)\sigma(s,X_s) - \phi(X_s)\mathfrak{z}_s^L\right) Z_s^L ds\right].\end{aligned}$$

where $\mathfrak{z}^L$ is a bounded process (by $K_{f,z}$, **(A4)**). Here we remark that we should add some restriction on the generator. Indeed the integration by parts works only if $\mathfrak{z}^L$ is a smooth and bounded function of X, that is if for any (t,y,z)

$$f(t,y,z) - f(t,y,0) = \mathfrak{z}(X_t)z.$$

Thus we can apply Lemma 3.6 and (3.37) becomes

$$\begin{aligned}
\mathbb{E}[Y_t^L \phi(X_t)] &= \mathbb{E}[Y_0^L \phi(X_0)] + \mathbb{E}\left[\int_0^t \phi(X_s)(f_s^0 \wedge L) ds\right] \\
&- \mathbb{E}\left[\int_0^t \phi(X_s)(f(s, Y_s^L, 0) - f_s^0) ds\right] + \mathbb{E}\left[\int_0^t Y_s^L \mathcal{L}\phi(s, X_s) ds\right] \\
&+ \mathbb{E}\left[\int_0^t \Psi(s, X_s) Y_s^L ds\right].
\end{aligned}$$

The rest of the proof can be deduced by the same arguments of the case q large.

Proposition 3.3 *For $q > 1$ and if $\mathbb{F}$ is generated by W, under conditions* **(A6)** *to* **(A8)** *and* **(B5)** *to* **(B7)**, *the minimal supersolution $(Y^{\min}, Z^{\min})$ satisfies: a.s.*

$$\liminf_{t \to T} Y_t^{\min} = \xi.$$

Extension for BDSDE

For the BDSDE with singular terminal condition (Theorem 3.3), the same questions appear: does the left limit of $Y^{\min}$ at time T exist ? And is the limit equal to ξ ? In [XIII], we study the case where $f(t, y, z) = -y|y|^{q-1}$ and we prove that a.s.

$$\lim_{t \to T} Y_t^{\min} = \xi.$$

We extend the technics of the previous sections to this case: for $q > 3$ with a suitable control on Z and for $q \leq 3$ with Malliavin's calculus.

The generalization to different generators f, with the setting of Theorem 3.3 on g, has not been checked in details. Nevertheless the existence of the limit can be obtained as for Theorem 3.5 using the assumption (3.31) on f, that is:

$$b_t \widehat{f}(y) \leq f(t, y, z) - f(t, 0, z), \quad \forall y \geq 0, \ \forall (t, z),$$

for some negative, decreasing and of class C^1 function $\widehat{f}$ which is concave on $\mathbb{R}_+$ with $\widehat{f}(0) < 0$ and $\widehat{f}'(0) < 0$. The transformation used for classical BSDE should be suitable also here. The continuity problem (3.27), $\liminf_{t \to T} Y_t^{\min} = \xi$, should be obtained without a main trouble and with minor modifications.

We guess that we could also apply the same trick for 2BSDEs, where Equation (3.37) would be computed under any $\mathbb{P} \in \mathcal{P}_0$. But we didn't verify the details.

3.3.3 Beyond the Markovian setting: smooth functional [XIX]

In the previous cases, ξ is supposed to be given by $\Phi(X_T)$. Is it possible to remove this condition ? For small values of q, we didn't extend the prior result. However if q is sufficiently large, we succeeded to enlarge the setting on ξ. In this section, we assume that **(C3*)** and (3.32) hold.

Indeed our proof for large q is essentially based on the Itô formula applied to $Y\phi(X)$ for well-chosen functions ϕ. As long as Itô's formula remains valid, the same scheme can be used. In [XIX], we study the case where ξ is a smooth functional of the paths of X, namely:

$$\xi = \Phi(F(T, X_T, A_T)),$$

for some measurable function $\Phi : \mathbb{R} \to [0, +\infty]$ and $F \in \mathbb{C}_b^{1,2}$. Let us precise the notations for this subsection. If ϕ is a function from $[0, T]$ to $\mathbb{R}^d$, $\phi(t)$ is the value of ϕ at time t, whereas ϕ_t is the stopped path of ϕ. Thus it implies that

$$\|\phi_t - \psi_t\|_\infty = \sup\{|\phi(u) - \psi(u)|,\ 0 \le u \le t\}.$$

Now X is the solution of the SDE

$$X(t) = \zeta(t) + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW(s) \tag{3.41}$$

The coefficients $b(\cdot, \cdot, \phi) : \Omega \times [0, T] \to \mathbb{R}^d$ and $\sigma(\cdot, \cdot, \phi) : \Omega \times [0, T] \to \mathbb{R}^{d\times k}$ are defined for every continuous function ϕ and satisfy the standard conditions:

- b, σ are Lipschitz continuous w.r.t. ϕ uniformly in t and ω, i.e. there exists a constant $K_{b,\sigma}$ such that for any $(\omega, t) \in \Omega \times [0, T]$, for any ϕ and ψ in $C([0, T]; \mathbb{R}^d)$: a.s.
$$|b(t, \phi) - b(t, \psi)| + |\sigma(t, \phi) - \sigma(t, \psi)| \le K_{b,\sigma}\|\phi_t - \psi_t\|_\infty.$$
- b and σ growth at most linearly:
$$|b(t, 0)| + |\sigma(t, 0)| \le C_{b,\sigma}.$$

These conditions are extension of **(B2)** and **(B5)**. Here ζ is a progressively measurable continuous stochastic process such that $\zeta \in \mathbb{D}^{\ell^*}(0, T)$, ℓ^* being the Hölder conjugate of ℓ of Condition **(C3)**.

Let us emphasize that compared to the solution of (1.9), X is not a Markovian process since the drift and the volatility matrix may depend on the whole trajectory of X. Under the above assumptions, the forward SDE (3.41) has a unique strong continuous solution X (see [276, Theorem 3.17]), which is a semimartingale with

$$[X](t) = \int_0^t \sigma(s, X_s)\sigma^*(s, X_s)ds = \int_0^t A(s)ds.$$

Space $\mathbb{C}_b^{1,2}$ is defined in [81, 82] and denotes the set of left-continuous functionals F such that

- F admits a horizontal derivative $\mathcal{D}F(t, \omega)$ for all $(t, \omega) \in \Lambda_T$, and the map $\mathcal{D}F(t, \cdot) : (D([0, T], \mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$ is continuous for each $t \in [0, T[$;
- F is two times vertically differentiable with $\nabla_\omega^j F \in \mathbb{C}_l^{0,0}$;
- $\mathcal{D}F$, $\nabla_\omega F$ and $\nabla_\omega^2 F$ belong to $\mathbb{B}(\Lambda_T)$ (boundedness-preserving functionals).

Let us recall the change of variable formulas [82, Theorem 4.1]. Let $F \in \mathbb{C}_b^{1,2}$ such that F is a non-anticipative functional with *predictable dependence* with respect to v: $F(t,x,v) = F(t,x_t,v_{t-})$. Then for $t \in [0,T]$

$$\begin{aligned}(3.42) \quad F(t,X_t,A_t) &= F(0,X_0,A_0) + \int_0^t \mathcal{D}F(u,X_u,A_u)du \\ &+ \int_0^t \nabla_\omega F(u,X_u,A_u)dX(u) + \frac{1}{2}\int_0^t \text{Trace}\left(\nabla^2_\omega F(u,X_u,A_u)d[X](u)\right).\end{aligned}$$

Hence even if ξ is no more Markovian, the Itô formula can be still applied.

As before $\mathcal{R} = \{\Phi < +\infty\}$ is supposed to be an open subset of $\mathbb{R}$. We suppose that $\mathbb{P}(\xi = \infty) > 0$ and that for any compact set $\mathcal{K} \subset \mathcal{R}$, $\mathbb{E}(\xi \mathbf{1}_{\mathcal{K}}(F(T,X_T,A_T))) < +\infty$. We also require some integrability conditions on $\mathcal{X} = F(\cdot,X,A)$, continuous semimartingale for any $F \in \mathbb{C}_b^{1,2}$. Let us recall that the classical norm on semimartingales is defined in [90, Section VII.3 (98.1)-(98.2)] or [297, Section V.2]. Nevertheless this norm is not sufficient in our case and we follow the ideas of [81, Section 7.5]. For $p \geq 1$, $\mathcal{A}^p(\mathbb{F})$ is defined as the set of continuous $\mathbb{F}$-predictable absolutely continuous processes $H = H(0) + \int_0^\cdot h(t)dt$ with finite variation such that

$$\|H\|^p_{\mathcal{A}^p} = \mathbb{E}\left(|H(0)|^p + \int_0^T |h(t)|^p dt\right) < +\infty.$$

We consider the direct sum

$$\mathcal{SM}^p = \mathbb{M}^p(0,T) \oplus \mathcal{A}^p(\mathbb{F}).$$

Any process $S \in \mathcal{SM}^p$ is an $\mathbb{F}$-adapted special semimartingale with a unique decomposition $S = M + H$, where $M \in \mathbb{M}^p(0,T)$ with $M(0) = 0$ and $H \in \mathcal{A}^p(\mathbb{F})$ with $H(0) = 0$. Let us remark that by Jensen's inequality, the norm defined on $\mathcal{SM}^p$ is stronger than the norm of semimartingales defined in [90]. Moreover if $S \in \mathcal{SM}^p$, then $S \in \mathbb{D}^p(0,T)$ by the Burkhölder-Davis-Gundy inequality. The interested reader can find in [81, Chapter 7] how the vertical and horizontal derivatives can be defined on this space $\mathcal{SM}^p$. Our assumptions are:

- $F(\cdot,X,A)$ is in $\mathcal{SM}^p$ for $p = \frac{q}{q-1}\ell^*$, where q comes from **(C2)** and ℓ^* is the Hölder conjugate of the constant $\ell > 1$ of Condition **(C3)**.
- $\nabla_\omega F$ is in $\mathbb{D}^{\ell^*}(0,T)$.

Here we only consider the continuous case, that is we suppose that the truncated BSDE (3.8) has the form

$$Y^L(t) = \xi \wedge L + \int_t^T f^L(s,Y^L(s),Z^L(s))ds - \int_t^T Z^L(s)dW(s) - M^L(T) + M^L(t)$$

and all martingales have continuous paths. The extension to càdlàg semimartingales is certainly possible but the Itô formula (3.42) becomes more cumbersome.

Note that **(C3*)** and (3.32) hold. Hence if $q > 3$, our conditions are true if

$$p > 2 + \frac{6}{q-3}.$$

In particular, if q is close to 2, p is large.

Theorem 3.7 *Under the previous hypotheses, the minimal supersolution $Y^{\min}$ satisfies a.s.*

$$\liminf_{t\to T} Y_t^{\min} = \xi.$$

Let us finish this part with some examples. First we can recover the Markovian case if for some smooth function $h \in C^{1,2}([0,T]\times\mathbb{R}^d)$

$$F(t, X_t, A_t) = h(t, X(t)),$$

and if X satisfies the SDE (1.9). Then $\mathcal{D}F(s, X_s, A_s) = \partial_t h(t, X(t))$, $\nabla_\omega F(s, X_s, A_s) = \nabla_x h(t, X(t))$ and $\nabla_\omega^2 F(s, X_s, A_s) = D_x^2 h(t, X(t))$, where D_x^2 is the Hessian matrix w.r.t. x. If we assume that h and its derivatives are of linear growth w.r.t. x, uniformly in time and ω, then all integrability conditions are satisfied.

As a second example, we consider the case where X is the solution of (3.41) and

$$F(t, X_t, A_t) = \int_0^t h(s, X(s))A(s)ds$$

where h is a continuous function on $[0,T]\times\mathbb{R}^d$. Then $\mathcal{D}F(s, X_s, A_s) = h(s, X(s))A(s)$, $\nabla_\omega F(s, X_s, A_s) = 0$. Our Itô formula can be simplified:

$$\begin{aligned} Y_t^L \phi(F(t, X_t, A_t)) &= Y_0^L \phi(F(0, X_0, A_0)) + \int_0^t \phi(F(s, X_s, A_s))\left[Z^L(s)dW(s) + dM^L(s)\right] \\ &- \int_0^t f^L(s, Y_s^L, Z_s^L)\phi(F(s, X_s, A_s))ds + \int_0^t Y_s^L \phi'(F_s(X_s, A_s))h(s, X(s))A(s)ds. \end{aligned}$$

The vertical derivatives are vanishing.

Other examples are given by [82, Examples 4 and 5], namely

$$F(t, x_t, v_t) = x(t)^2 - \int_0^t v(u)du, \qquad F(t, x_t, v_t) = \exp\left(x(t) - \frac{1}{2}\int_0^t v(u)du\right).$$

Conditions on b and σ can be easily found such that the desired integrability conditions hold, especially if X is given by (1.9).

Let us finish with the weak Euler-Maruyama scheme as in [83]. We still consider the SDE (3.41) with $b = 0$ and the non-anticipative functional X^n given by the recursion

$$X^n(t_{j+1}) = X^n(t_j) + \sigma(t_j, X_{t_j}^n)(W(t_{j+1}) - W(t_j)).$$

For a Lipschitz functional $g : D([0,T], \mathbb{R}^d) \to \mathbb{R}$, consider the "weak Euler approximation"

$$F_n(t) = \mathbb{E}\left[g(X_T^n)|\mathcal{F}_t^W\right]$$

of the conditional expectation $\mathbb{E}\left[g(X_T)|\mathcal{F}_t^W\right]$, where $\mathbb{F}^W$ is the filtration generated by the Brownian motion W. This weak approximation is computed by initializing the scheme on $[0,t]$ with ω (a path of the Brownian motion) and then iterating the scheme with the increments of the Wiener process between t and T. Then $F_n \in \mathbb{C}^{1,\infty}_{\text{loc}}$ (see [83, Theorem 3.1]). Under our setting and thanks to [83, Theorem 4.1], the first integrability condition on $F(\cdot,X,A)$ holds. The second one does not hold on the whole interval $[0,T]$. Nevertheless this functional is locally regular ([83, Definition 7]) and on our neighbourhood of T, one can easily get this hypotehsis provided that g is bounded for example.

3.3.4 Other non-Markovian cases [XVI, XXIV]

In [XVI], we study also the continuity problem but for a non smooth functional and using a different scheme. We consider only the Brownian setting, the dimension $d=1$ and the generator $f(y)=-y|y|^{q-1}$. In [XXIV], the general BSDE (2.4) is considered.

We consider the following class of terminal conditions:

$$\xi_1 = \infty \cdot \mathbf{1}_{\{\tau_1 \leq T\}}$$

where τ_1 is any stopping time with a bounded density in a neighborhood of T and

$$\xi_2 = \infty \cdot \mathbf{1}_{A_T}$$

where A_t, $t\in[0,T]$ is a decreasing sequence of events adapted to the filtration $\mathbb{F}$ that is continuous in probability at T (equivalently, $A_T=\{\tau_2>T\}$ where τ_2 is any stopping time such that $\mathbb{P}(\tau_2=T)=0$). We prove that the minimal non-negative supersolutions of the BSDE are in fact solutions, i.e., they attain almost surely their terminal values:

$$\lim_{t\to T} Y_t^{\min} = \xi.$$

We note that the first exit time from a time varying domain of a d-dimensional diffusion process driven by the Brownian motion with strongly elliptic covariance matrix does have a continuous density. Therefore such exit times can be used as τ_1 and τ_2 to define the terminal conditions ξ_1 and ξ_2.

Let us go into details. We assume that Conditions **(A2)** to **(A5')** and **(C2)** to **(C4)** hold. To avoid extra technical problems, we suppose that ξ and f^0 are non-negative, such that **(C1)** is verified. And f^0 satisfies

$$\mathbb{E}\int_0^T \left(f_s^0\right)^\ell ds < +\infty.$$

where $\ell>1$ is the constant in assumption **(C3)**. The next result is [XXIV, Theorem 1] and generalizes the continuity result [XVI, Theorem 2.1]

Theorem 3.8 *Under the prior conditions, if the distribution of the stopping time τ is given by a bounded density in a neighborhood of T, $\ell>2$ and $q>2+\dfrac{2}{\ell-2}$, then the minimal supersolution with terminal condition ξ_1 satisfies*

$$\lim_{t\to T} Y_t^{\min} = \xi_1 \tag{3.43}$$

almost surely.

If the filtration $\mathbb{F}$ is assumed to be generated by W and π alone then we can describe the solution $Y^{\min}$ as follows. Let Y^∞ be the minimal supersolution of BSDE (2.4) with terminal condition $Y_T = \infty$ (if $f(y) = -y|y|^{q-1}$, then $Y^\infty_t = ((q-1)(T-t))^{-\frac{1}{q-1}}$). Define

$$\xi_1^{(\tau)} \doteq \mathbf{1}_{\{\tau<T\}} Y^\infty_\tau.$$

Let $Y^{1,\tau}$ be the solution of BSDE (2.4) in the time interval $[\![0, \tau \wedge T]\!]$ with terminal condition $\xi_1^{(\tau)}$ (again we can apply Theorem 2.2). Following the idea of [XVI, Theorem 2.1], let us define

$$\widehat{Y}_t \doteq \begin{cases} Y^{1,\tau}_t, & t \leq \tau \wedge T \\ Y^\infty_t, & \tau < t \leq T \end{cases}$$

where we assume that τ is an $\mathbb{F}^W$ stopping time, that is it just depends on the paths of W, and is predictable (exit times of a diffusion process X solution of (1.9) are a particular case). The jump times of $Y^{1,\tau}_t$ and of Y^∞ coincide with the jump times of the Poisson random measure or of the orthogonal martingale component. A consequence of the Meyer theorem (see [297, Chapter 3, Theorem 4]) implies that the jump times of π are totally inaccessible, hence a.s. cannot be equal to τ. However we cannot exclude that the orthogonal martingale may have a jump at time τ if filtration $\mathbb{F}$ is general. The second issue is the definition of the martingale part (Z, U). For the first two components, we can easily paste them together

$$\widehat{Z}_t \doteq \begin{cases} Z^{1,\tau}_t, & t \leq \tau \wedge T \\ Z^\infty_t, & \tau < t \leq T \end{cases}, \qquad \widehat{U}_t(e) \doteq \begin{cases} U^{1,\tau}_t(e), & t \leq \tau \wedge T \\ U^\infty_t(e), & \tau < t \leq T \end{cases}.$$

Since τ is predictable, these two processes are also predictable and the stochastic integrals

$$\int_0^\cdot \widehat{Z}_t dW_t, \qquad \int_0^\cdot \int_{\mathcal{E}} \widehat{U}_t(e) \widetilde{\pi}(de, dt)$$

are well-defined and are local martingales on $[0, T)$. Nonetheless if we define $\widehat{M}$ similarly, we cannot ensure that this process is still a local martingale. For the parts with Z and U, the local martingale property is due to the representation as a stochastic integral. Based on these observations we provide the following result on the pasting method under the assumption that the filtration is generated by W and π alone; the approach in the proof of this proposition is the generalization of the approach used in [XVI].

Proposition 3.4 *Assume that filtration $\mathbb{F}$ is generated by W and π. Then $\widehat{Y}_t$ solves BSDE (2.4) on $[0, T]$ with terminal condition $\widehat{Y}_T = \xi_1$ and satisfies the continuity property at time T. Moreover $\widehat{Y} = Y^{\min}$.*

To illustrate this result, let us consider the Markovian framework of [XVI]. Here $f(y) = -y|y|^{q-1}$:

$$Y_s = Y_t - \int_s^t Y_r |Y_r|^{q-1} dr + \int_s^t Z_r dW_r, \quad 0 < s < t < T. \tag{3.44}$$

The random time τ is the exit time:

$$\tau \doteq \inf\{t \in [0,\infty) : W_t \in \{0, L\}\}, \quad W_0 = x, \ x \in (0, L).$$

Then Y^∞ is deterministic and equal to

$$Y_t^\infty = y_t := ((q-1)(T-t))^{1-p}, \ Z_t^\infty = 0, \quad t \in (\tau, T].$$

Hence $(Y^{1,\tau}, Z^{1,\tau})$ is described with the solution of the parabolic equation:

$$\partial_t V + \frac{1}{2}\partial_{xx} V - V^q = 0; \tag{3.45}$$

with the following boundary conditions to accompany the PDE:

$$V(0,t) = V(L,t) = y_t, t \in [0,T], \quad V(x,T) = 0, 0 < x < L. \tag{3.46}$$

This result is contained in [XVI] and is equivalent to the previous proposition.

Proposition 3.5 *If $q > 2$ then there is a function u which is C^∞ in the x variables and C^1 in the t variable and continuous on $\bar{D} \setminus \{(L,T),(0,T)\}$ satisfying PDE* (3.45) *with the boundary condition* (3.46) *such that*

1. $$Y_t = \begin{cases} u(W_t, t) & , t < \tau \wedge T, \\ y_t & , \tau \leq t \leq T, \end{cases} \qquad Z_t = \begin{cases} u_x(W_t, t) & , t < \tau \wedge T, \\ 0 & , \tau \leq t \leq T. \end{cases} \tag{3.47}$$

   solve BSDE (3.44) *with terminal condition $\xi = \xi_1 = \infty \cdot \mathbf{1}_{\tau \leq T}$; in particular, Y is continuous on* $[0,T]$,

2. *The minimal supersolution $(Y^{\min}, Z^{\min})$ is equal to (Y, Z); in particular the continuity problem holds.*

The main part of the proof is devoted to the construction of the smooth function u, obtained by approximation and regularization as the limit of $u_{n,m}$. The density of the stopping time τ is also a key ingredient.

Let us give several numerical examples and simulation of our results. The left side of Figure 3.1 shows the graph of $u_{m,n}$ with $L = 3$ and $T = 1$, $m = 100$ and $n = 50$ computed using a finite difference approximation of the PDE with $\Delta x = 0.1$ and $\Delta t = 0.01$. The right side of the same figure shows the graph of $u_{m,n}$ over the line $x = L/2 = 1.5$ for $m = 100$ and $n = 10$ and $n = 150$ as well as the graph of y_t; note $u_{100,10}(1.5,t) < u_{100,1000}(1.5,t) < y_t$ in the figure, as expected. Figure 3.2 shows two randomly sampled sample paths of the Brownian motion W with $W_0 = L/2 = 3/2$ and the corresponding path for Y, computed using (3.47) where we use a numerical approximation of $u_{m,n}$ with $m = 100$ and $n = 1000$ to approximate u.

In [XVI, XXIV], we also study the case

$$\xi = \xi_2 = \infty \cdot \mathbf{1}_{A_T},$$

where A_t is a decreasing sequence of events adapted to our filtration: for any $s \leq t$, $A_t \subset A_s$ and $A_t \in \mathcal{F}_t$. If τ_0 is a stopping time, the set $A_t = \{\tau_0 > t\}$ provides an example. We also assume that:

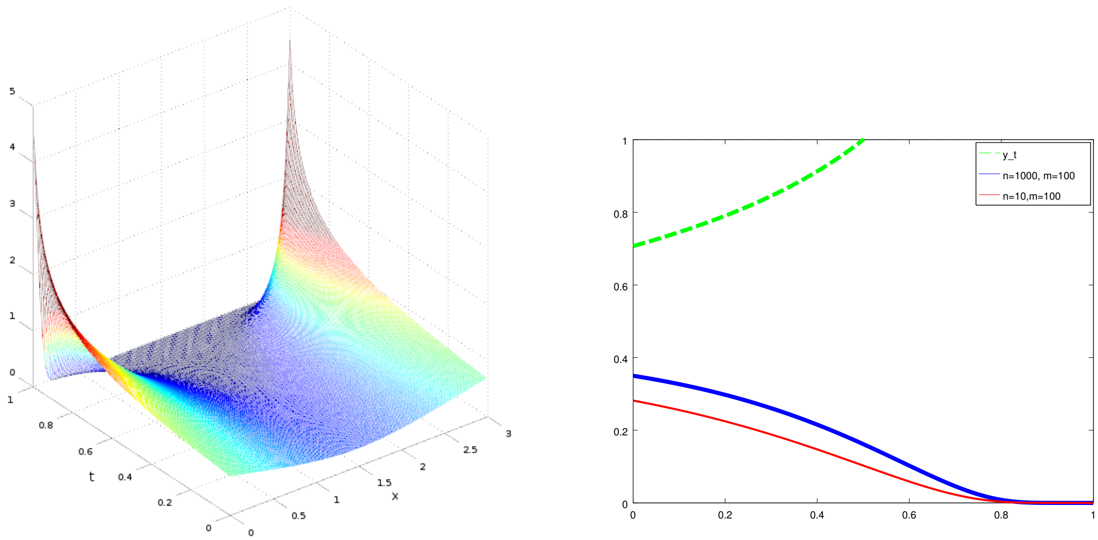


Figure 3.1: On the left, the graph of $u_{m,n}$ with $m = 100$ and $n = 50$; on the right, the graph of $u_{m,n}$ over $x = 1.5$ for $m = 100$, $n = 10$ (thin) and $n = 1000$ (thick), and y_t (dashed line). In all computations $L = 3$ and $T = 1$

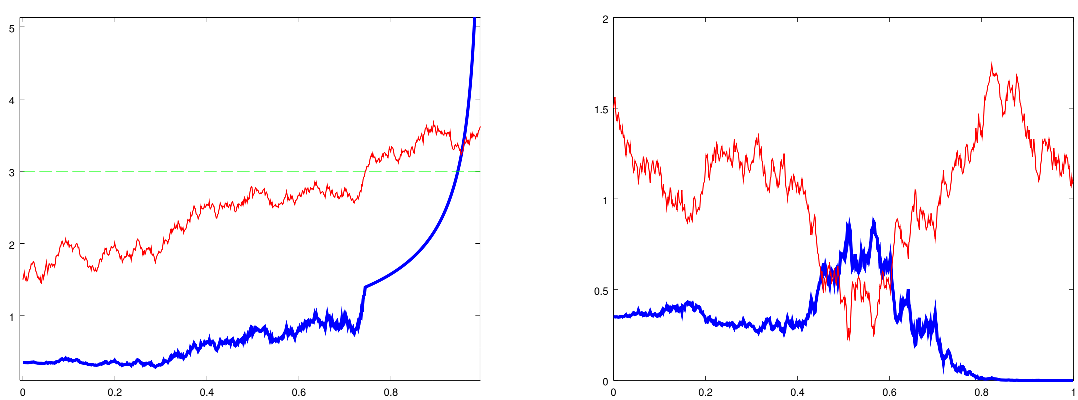


Figure 3.2: Numerically computed trajectories of W (thin light path) and Y (thick dark) (left with explosion, right without); Y is computed using (3.47) with $u_{m,n}$ approximating u with $m = 100$ and $n = 1000$; $L = 3$ and $T = 1$

- The sequence is left continuous at time T in probability:

$$\mathbb{P}\left(\bigcap_{t<T} A_t \setminus A_T\right) = 0.$$

If A_t is defined as $A_t = \{\tau_0 > t\}$ through a stopping time τ_0, the first assumption is equivalent to: $\mathbb{P}(\tau_0 = T) = 0$. In particular if τ_0 has a density this condition is satisfied. Therefore, as in the previous section, if τ_0 is the jump time of an $\mathbb{F}$-adapted compound Poisson process, then it generates a sequence A_t satisfying the first condition. The same comment applies to the exit times of some diffusion processes X.

On filtration $\mathbb{F}$, we suppose that

- There exists an increasing sequence $(t_n,\ n \in \mathbb{N})$, $t_n < T$ for all n, $\lim_{n\to+\infty} t_n = T$, and the filtration $\mathbb{F}$ is left continuous at time t_n for any n. Recall that we already assume left continuity of $\mathbb{F}$ at time T.

If filtration $\mathbb{F}$ is quasi left-continuous, then this condition holds for any sequence t_n. In particular our hypothesis is valid if $\mathbb{F}$ is generated by W and π. The notion of jumps for a filtration has been studied in [173] (see also [298, Section 2]). Let us note that we are not able to construct a counter example, that is a filtration such that this hypothesis does not hold.

Theorem 3.9 *Under the previous conditions, the minimal supersolution with terminal condition ξ_2 satisfies*

$$\lim_{t\to T} Y_t^{\min} = \xi_2 \tag{3.48}$$

almost surely.

Let us emphasize that no particular condition is required on q here. In the particular case of BSDE (3.44), we consider PDE (3.45) with the boundary condition

$$V(0,t) = V(L,t) = 0, t \in [0,T], \quad V(x,T) = \infty, 0 < x < L. \tag{3.49}$$

Then we prove:

Proposition 3.6 *There exists a function $\bar{v}$ which is C^∞ in the x variable and C^1 in the t variable and continuous on $\bar{D} \setminus \{(L,T),(0,T)\}$ and which solves the PDE* (3.45) *with the boundary condition* (3.49) *such that*

1. *The processes*

$$Y_t = \begin{cases} \bar{v}(W_t,t) & ,t < \tau \wedge T, \\ 0 & ,\tau \le t \le T, \end{cases} \qquad Z_t = \begin{cases} \bar{v}_x(W_t,t) & ,t < \tau \wedge T, \\ 0 & ,\tau \le t \le T. \end{cases} \tag{3.50}$$

solve BSDE (3.44) *with terminal condition $\xi = \infty \cdot \mathbf{1}_{B(0,L)}$, and in particular, Y is continuous on $[0,T]$,*

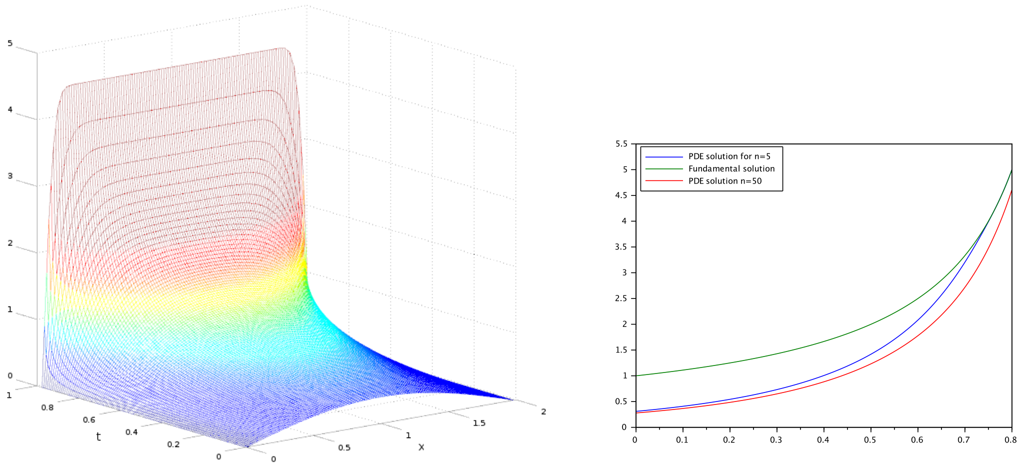


Figure 3.3: On the left, graph of $\bar{u}_{50}$; on the rights graphs of $\bar{u}_5(1,t)$, $\bar{u}_{50}(1,t)$ and y_t, $t \in [0,1]$; $T = 1$ and $L = 2$

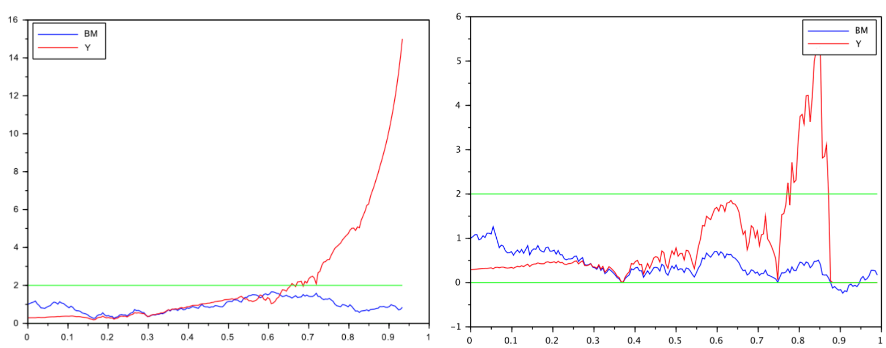


Figure 3.4: Two trajectories of W and Y (left with explosion, right without).

2. Again $(Y^{\min}, Z^{\min})$ *is equal to* (Y, Z) *and the continuity problem holds.*

We illustrate the computations above with several numerical examples in Figures 3.3 and 3.4. The left side of Figure 3.3 shows the graph of $\bar{u}_{50}$, computed numerically using finite differences; the right side of the same figure shows the graphs of $\bar{u}_5(1,t)$ and $\bar{u}_{50}(1,t)$ and y_t. Figure 3.4 shows two sets of sample paths of W and Y with $W_0 = L/2 = 1$ and where Y is approximated by $\bar{u}_{50}(W_t, t)$ for $t < \tau$; in all computations $L = 2$ and $T = 1$.

3.4 BSDE with singular terminal condition and terminal stopping time

We also study the case of a random terminal time, that is the BSDE (2.6)

$$\begin{aligned} Y_{t\wedge\tau} &= Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s \\ &- \int_{t\wedge\tau}^{T\wedge\tau}\int_{\mathcal{E}} U_s(e)\widetilde{\pi}(de,ds) - \int_{t\wedge\tau}^{T\wedge\tau} dM_s \end{aligned}$$

with the condition that $\mathbb{P}$-a.s. on the set $\{t \geq \tau\}$, $Y_t = \xi$ and $Z_t = U_t = M_t = 0$. The assumptions **(A2)**–**(A3")**–**(A4)** and **(A5')**, together with the integrability conditions **(A1.1")** and **(A1.2")**, hold such that the existence and uniqueness results stated in Theorem 2.2 are true.

We first need to adapt the definition of a solution when there is a singularity at time τ. Very recently in [E], we develop the following notions.

Definition 3.3 (Supersolution for singular terminal condition) *We say that a process* (Y,Z,U,M) *is a supersolution to BSDE* (2.6) *with singular terminal condition* $Y_\tau = \xi$ *if it satisfies:*

1. *There exists some* $\ell > 1$ *and an increasing sequence of stopping times* τ_n *converging to* τ *such that for all* $n > 0$ *and all* $t \geq 0$

$$\mathbb{E}\left[\sup_{s\in[0,t]} |Y_{s\wedge\tau_n}|^\ell + \left(\int_0^{t\wedge\tau_n} |Z_s|^2 ds\right)^{\ell/2} \right. \\ \left. + \left(\int_0^{t\wedge\tau_n}\int_{\mathcal{E}} |U_s(e)|^2 \pi(de,ds)\right)^{\ell/2} + [M]^{\ell/2}_{t\wedge\tau_n}\right] < +\infty;$$

2. Y *is bounded from below by a process* $\widetilde{Y} \in \mathbb{D}^\ell(0,T\wedge\tau)$ *for any* T.*;*

3. *for all* $0 \leq t \leq T$ *and* $n > 0$*:*

$$\begin{aligned} Y_{t\wedge\tau_n} = Y_{T\wedge\tau_n} &+ \int_{t\wedge\tau_n}^{T\wedge\tau_n} f(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\tau_n}^{T\wedge\tau_n} Z_s dW_s \\ &- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_{\mathcal{E}} U_s(e)\widetilde{\pi}(de,ds) - \int_{t\wedge\tau_n}^{T\wedge\tau_n} dM_s. \end{aligned} \tag{3.51}$$

4. *On the set* $\{t \geq \tau\}$*:* $Y_t = \xi, Z = U = M = 0$ *a.s. and*

$$\liminf_{t\to+\infty} Y_{t\wedge\tau} \geq \xi, \quad a.s. \tag{3.52}$$

We say that (Y,Z,U,M) *is a minimal supersolution to the BSDE* (2.4) *or* (2.6) *if for any other supersolution* (Y',Z',U',M') *we have* $Y_t \leq Y'_t$ *a.s. for any* $t > 0$.

We next introduce a concept that we think provides a general and natural framework for the study of BSDE (2.6) with singular terminal conditions when the terminal time is a stopping time:

Definition 3.4 *A stopping time τ will be called solvable with respect to BSDE* (2.6) *if the filtration $\mathbb{F}$ is left-continuous at time τ and if BSDE* (2.6) *has a supersolution on the time interval $[\![0,\tau]\!]$ with terminal condition $Y_\tau = \infty$ that is defined as the limit of the solution of the same BSDE with terminal condition equal to the constant k, as k tends to ∞.*

From [XI] and Section 3.1, we know that every deterministic time τ is solvable. [XI, Example 1] show that any stopping time that has a strictly positive density around 0 is non-solvable. And we will show below that first exit times of classical diffusion processes from smooth bounded domains are also solvable.

The case of random terminal time is considered in [II] where the generator f is equal to $f(y) = -y|y|^{q-1}$ for some $q > 1$ and the filtration is generated by a Brownian motion. The paper [XI] is an extension of this first paper. Compared to the deterministic case, there are two main issues:

- The truncation procedure has to be done carefully since the integrability conditions **(A1.1")** and **(A1.2")** mix ξ and τ and are not true for any bounded value ξ.
- The derivation of the a priori estimate is more involved than in the deterministic case (see (3.10)).

In other words, solvability of a random terminal time is much more complex (no surprise!).

We already saw in Section 2.1.1 that some assumptions have to be modified if the terminal time is random. Here we suppose that **(A2)**, **(A3")**, **(A4)** and **(A5')** hold. and we denote by $K_{f,\psi} = \|\vartheta\|$.

Bounded terminal value

The first step consists to obtain the existence of a solution (Y^L, Z^L, U^L, M^L) to the BSDE (3.8) with terminal condition $Y_\tau = \xi \wedge L$. More precisely for any $0 \leq t \leq T$

$$
\begin{aligned}
Y^L_{t\wedge\tau} &= Y^L_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f^L(s, Y^L_s, Z^L_s, U^L_s) ds \\
&\qquad - \int_{t\wedge\tau}^{T\wedge\tau} Z^L_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \int_{\mathcal{E}} U^L_s(e) \widetilde{\pi}(de, ds) - \int_{t\wedge\tau}^{T\wedge\tau} dM^L_s,
\end{aligned}
$$

and $Y^L_t = \xi \wedge L$ on the set $\{t \geq \tau\}$. Since we also want an a priori estimate independent of L, we will assume that **(C2)** holds and:

(C5) The data ξ^- and f^0 and the process η are bounded.

(C6) There exists $\delta > \delta^*$ such that $\mathbb{E}\left[e^{\delta\tau}\right] < +\infty$. The constant δ^* depends on χ, $K_{f,z}$ and $K_{f,u}$.

(C7) There exists $m > m^*$ such that for any j

$$\mathbb{E} \int_0^\tau |U_t(j)|^m dt < +\infty.$$

Here $U_t(j) = \sup_{|y| \leq j} |f(t, y, 0, 0) - f_t^0|$. The value of m^* depends on χ, $K_{f,z}$ and $K_{f,u}$ and also on δ and δ^*.

Note that **(C5)** implies that **(C1)**, **(C3)** and **(C4)** hold immediately. We can take f instead of f^L in (3.8). Furthermore Hypotheses **(C2)** and **(C7)** imply that

$$\mathbb{E} \int_0^\tau \frac{1}{\eta_s^m} ds < +\infty. \tag{3.53}$$

Let us roughly explain the conditions **(C6)** and **(C7)**. They are sufficient to get the integrability assumptions **(A1.1")** and **(A1.2")** with $\xi \wedge L$ for any constant L. Let us consider the simple case where $\xi = +\infty$ and $f^0 \equiv 0$ a.s.. Then **(A1.1")** becomes: for some $r > 1$ and $\rho > \nu$ (due to (2.7)):

$$\mathbb{E}\left[e^{r\rho\tau}\right] < +\infty.$$

In other words if in **(A1.1")** ξ and τ are related, here we need to assume some integrability on τ, whatever ξ is. For **(A1.2")**, we need:

$$\mathbb{E}\left[\int_0^\tau e^{r\rho t} |U_t(L)|^r dt\right] < +\infty.$$

By Hölder's inequality, we obtain

$$\mathbb{E}\left[\int_0^\tau e^{r\rho t} |U_t(L)|^r dt\right] \leq \left[\mathbb{E} \int_0^\tau e^{r\rho h t} dt\right]^{\frac{1}{h}} \left[\mathbb{E} \int_0^\tau |U_t(L)|^{r\bar{h}} dt\right]^{\frac{1}{\bar{h}}}$$

If there exists m sufficiently large (related to r and ρ) such that

$$\mathbb{E} \int_0^\tau |U_t(L)|^m dt < +\infty,$$

then the conclusion follows. If we summarize, we need to find δ and m such that there exists $r > 1$, $\rho > \nu(r)$ (Hypothesis (2.7)) such that $r\rho < \delta$, and $h > 1$ such that $r\bar{h} < m$ which is equivalent to $\dfrac{r\delta}{\delta - r\rho} < m$. The computational difficulty comes from the expression of $\nu(r)$ in (2.7). However we can find δ^* and m^* such that the next lemma, namely [X, Proposition 5], holds.

Lemma 3.7 *Under Conditions* **(A2)**, **(A3")**, **(A4)** *and* **(A5')**, *together with* **(C2)**, **(C5)**, **(C6)**, **(C7)**, *BSDE* (3.8) *with terminal condition* $Y_\tau = \xi \wedge L$ *has a unique solution* (Y^L, Z^L, U^L, M^L) *in* $\mathcal{S}^r(0,\tau)$. *The value of* r *depends on the coefficients* χ, $K_{f,z}$ *and* $K_{f,\psi}$ *(but not on* L*).*

The values of δ^* and m^* are postponed and discussed in 9.1.

A priori estimate and convergence

We can use the comparison principle to obtain the existence of a limit process Y such that a.s. $\lim_{L\to+\infty} Y^L_t = Y_t$. In the deterministic case, the key point is the a priori estimate (3.10). For a general random time τ, we show (see [XI, Example 1]) that the limit process Y may be infinite before time τ: for example if τ is the first jump time of a Poisson process or if $\mathbb{E}(1/\tau) = +\infty$, then $\liminf_{L\to+\infty} Y^L_0 = +\infty$.

In order to ensure the finiteness of $Y_{t\wedge\tau}$ on the set $\{t < \tau\}$, we prove a *Keller-Osserman type inequality* ([XI, Proposition 6]). In the proof we compare the solution (Y^L, Z^L, U^L, M^L) with the solution $(\mathcal{Y}, \mathcal{Z}, 0, \mathcal{M})$ of a BSDE with generator:

$$g(t, y, z) = -\frac{y}{\eta_t}|y|^{q-1} + f(t, 0, z, \mathbf{0}).$$

Note that g satisfies **(A2)** since $y \mapsto g(t, y, z)$ is non-increasing and the conditions $(\mathbf{H_{comp}})$ hold for g with the same constant $L_{g,z} = L_{f,z}$, but with $\chi = 0$ and $L_{g,\psi} = 0$. If for generator f, constant χ in **(A2)** is negative, then we can modify g in a neighborhood of zero (add a linear trend), such that we keep this negative constant χ. In other words from (3.53), **(C7)** holds also for g. And we also use **(C6)** with the same constants δ^* and m^*. Thereby **the constants δ^* and m^* only depend on χ,, $L_{f,z}$ and $L_{f,\psi}$.**

Let us assume that process Ξ in $\mathbb{R}^d$ is the unique solution to SDE (1.9)

$$d\Xi_t = b(\Xi_t)dt + \sigma(\Xi_t)dW_t$$

with some initial value $\Xi_0 \in \mathbb{R}^d$. Functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ satisfy **(B1)** and **(B2)**. Let D be an open bounded subset of $\mathbb{R}^d$, whose boundary is at least of class C^2 (see for example [140], Section 6.2, for the definition of a regular boundary). We introduce the signed distance function dist : $\mathbb{R}^d \to \mathbb{R}$ of D, which is defined by $\mathrm{dist}(x) = \inf_{y\notin D} \|x-y\|$ if $x \in D$ and $\mathrm{dist}(x) = -\inf_{y\in D} \|x-y\|$ if $x \notin D$. Let us denote by R the diameter of D:

$$R = \sup\{\|x - y\|,\ (x, y) \in D^2\},$$

by $\|\sigma\|$ the spectral norm of σ

$$\|\sigma\| = \sup_{x\in\mathbb{R}^d} \sup_{v\in\mathbb{R}^d,\ |v|\leq 1} v.(\sigma(x)\sigma^*(x))v,$$

and by $\|b\|$ the sup norm of b:

$$\|b\| = \sup_{x\in\mathbb{R}^d} |b(x)|.$$

Remark that the regularity of b and σ imply that they are bounded functions on the compact set $\overline{D}$. Since we will only consider Ξ on the random time interval $[\![0, \tau]\!]$, that is when Ξ belongs to D, using a truncation argument outside $\overline{D}$, we can assume w.l.o.g. that: $\|b\| + \|\sigma\| < +\infty$.

From now on, Ξ_0 is fixed and supposed to be in D. We define stopping time τ as the first exit time of D, i.e.

$$\tau = \tau_D = \inf\{t \geq 0, \quad \Xi_t \notin D\}. \tag{3.54}$$

The condition **(C6)** imposes some implicit hypotheses between the generator f, the set D and the coefficients of the SDE (1.9). Define j_d to be equal to $\pi^2/4$ if $d = 1$ and to be equal to the first positive zero of the Bessel function of first kind $J_{d/2-1}$ if $d \geq 2$ (for $d = 2$, $j_2 \approx 2.4048$). The next lemma is [XI, Lemma 2].

Lemma 3.8

1. *Assume that there exists $b_\star > 0$ and $v \in \mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$ it holds that $b(x).v \geq b_\star > 0$. If $\delta^* < \frac{b_\star^2}{\|\sigma\|}$, then Condition* **(C6)** *holds for all $\delta \in (\delta^*, \frac{b_\star^2}{\|\sigma\|})$.*

2. *Assume that $b = 0$ (there is no drift) and $\sigma\sigma^*$ is uniformly elliptic, that is there exists a constant $\alpha > 0$ such that $(\sigma\sigma^*)(x) \geq \alpha \mathrm{Id}_d$ for all $x \in \mathbb{R}^d$. If $\delta^* < \frac{2\alpha}{R^2}(j_d)^2$, then Condition* **(C6)** *holds for all $\rho \in (\delta^*, \frac{2\alpha}{R^2}(j_d)^2)$.*

Remark that the bound $\frac{b_\star^2}{\|\sigma\|}$ respectively $\frac{2\alpha}{R^2}(j_d)^2$ gives a minimal value for the parameter m^* in **(C7)**.

The next result is a *Keller-Osserman type inequality* (c.f. (3.55) and see [187, 264]). Using analytical properties of the diffusion near boundary ∂D, allows us to bound at each time t the value of process Y_t^L against the distance of diffusion Ξ to boundary ∂D, denoted $\mathrm{dist}(\Xi_t)$.

Proposition 3.7 *If τ is the exit time given by* (3.54)*, under the assumptions of Lemma 3.7, the solution processes Y^L are bounded uniformly in L: there exists a process $\bar{Y} \in \mathbb{S}^r(0,\tau)$ and a constant C such that:*

$$\bar{Y}_{t\wedge\tau} \leq Y_{t\wedge\tau}^L \leq \frac{C}{\mathrm{dist}(\Xi_{t\wedge\tau})^{2(p-1)}}. \tag{3.55}$$

Following Definition 3.3 we set

$$\tau_n = \inf\{t \geq 0, \mathrm{dist}(\Xi_t) \leq 1/n\}. \tag{3.56}$$

Hence on the random interval $[\![0,\tau_n]\!]$, Y^L is bounded uniformly w.r.t. L by $Cn^{2(p-1)}$. From this proposition and passing to the limit, we prove the main result of this part:

Theorem 3.10 *If τ is the exit time given by* (3.54)*, there exists a minimal supersolution $(Y^{\min}, Z^{\min}, U^{\min}, M^{\min})$ to the BSDE* (2.6) *with singular terminal condition $Y_\tau^{\min} = \xi$.*

Behavior at time τ

Evoke that we already have the a priori estimate (3.55) on solution $Y^{\min}$:

$$Y_{t\wedge\tau}^{\min} \leq \frac{C}{\mathrm{dist}(\Xi_{t\wedge\tau})^{2(p-1)}}.$$

If driver f is equal to $-y|y|^{q-1}$, in [II, Proposition 5] we derive a lower bound on Y: if

$$\Upsilon_t = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{(q-1)(\tau - \tau\wedge t) + \frac{1}{\xi^{q-1}}}\right)^{p-1}\right],$$

then a.s. for all $t \geq 0$, $\Upsilon_t \leq Y_t$. Furthermore if $\rho(x)$ denotes the distance of $x \in D$ to the boundary ∂D and if τ is the exit time from $\overline{D}$ of the diffusion X, then we show in [II, Lemma 3] that under the conditions **(B1)**, **(B2)** and **(B7)**, there exist two positive constants C_1 and C_2 which depend on D, q, σ and b such that for all $x \in D$,

$$C_1 \leq \rho(x)^{2(p-1)} \mathbb{E}_x \left[\left(\frac{1}{\tau} \right)^{p-1} \right] \leq C_2.$$

Remark that if the diffusion matrix σ is degenerate, the result on the lower bound may be false. Suppose that $\sigma \equiv 0$ and b is bounded by k. Then

$$\rho(x)^{2(p-1)} \mathbb{E}_x \left(\frac{1}{\tau^{p-1}} \right) \leq k \rho(x)^{p-1}$$

and the limit, as $\rho(x)$ goes to zero, is zero. Under the same setting, we deduce that on $\{\xi = +\infty\}$ the explosion rate of Y is in the order of $\rho^{-2(p-1)}(X_{t\wedge\tau})$: there exists a positive constant $\widetilde{C}$ depending on D, q, the bound on b and σ in **(B1)** and on the constant λ in **(B7)**, such that

$$\liminf_{t \to +\infty} \rho^{2(p-1)}(X_{t\wedge\tau}) Y_{t\wedge\tau} \geq \widetilde{C} \quad \text{a.s. on } \{\xi = +\infty\}.$$

Remark that we cannot prove that constants C in (3.55) and $\widetilde{C}$ are equal. However it proves that the rate of explosion is of order $1/\rho^{2(p-1)}(X)$.

Continuity problem when the terminal time is random

Uniqueness and a similar asymptotic behavior as in [XXI] are still open questions. And as for a deterministic terminal time, concerning the terminal condition (3.52)

$$\liminf_{t \to +\infty} Y^{\min}_{t\wedge\tau} \geq \xi, \quad \text{a.s.}$$

existence of the left-limit and equality are natural questions. They have been investigated in [II] for $f(y) = -y|y|^{q-1}$ in the Brownian framework. In the working paper [E], we try to obtain similar results as in Section 3.3.

3.5 Summary

If ξ is singular (in the sense of Definition 3.1), there exists a minimal supersolution to BSDE (2.4) with deterministic final time T (Theorem 3.1) or to BSDE (2.6) with a random time τ given by (3.54) (Theorem 3.10). The key conditions are **(C2)** and the left-continuity of filtration $\mathbb{F}$ at time T or τ and the main ingredient is the a priori estimate (3.10) or (3.55).

For a deterministic final time T, the condition

$$\lim_{t \to T} Y^{\min}_t = \xi$$

holds under structural conditions on generator f (Theorem 3.5, existence of the limit) and for a large class of terminal value ξ:

- Markovian setting $\xi = \Phi(X_T)$ (Theorem 3.6 for large values of q and Proposition 3.3 in the Brownian setting for q small).
- Smooth functional of X (Theorem 3.7).
- Some specific cases (Theorems 3.8 and 3.9).

The general case is still an open question. The same questions where T is replaced by τ, are investigating.

Uniqueness and asymptotic behavior are essentially developed for a deterministic time T and in the Brownian setting (Theorem 3.2 and Corollary 3.1).

For BDSDE and 2BSDE, existence of a minimal supersolution is proved in Theorems 3.3 and 3.4.

Chapter 4

Related (I)PDE and SPDE ([XIII, XIV])

We already mentioned that one particular interest for the study of BSDE is the application to partial differential equations (PDEs). Indeed as proved by Pardoux & Peng in [273], BSDEs can be seen as generalization of the Feynman-Kac formula for non linear PDEs. Roughly speaking, if we can solve a system of two SDEs with one forward in time and one backward in time, then the solution is a deterministic function and is a (weak) solution of the related PDE. This is a method of characteristics to solve the parabolic PDE. The converse assertion can be proved provided we can apply Itô's formula, that is if the solution of the PDE is regular enough. Since then a large literature has been developped on this topic (see in particular the books [91], [118], [276], [331] and the references therein).

In [24], Barles, Buckdahn & Pardoux show that we can add in the system of forward backward SDE a Poisson random measure and if we can find a solution to this system, again the solution is a weak solution of a IPDE:

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u,(\nabla u)\sigma,\mathcal{B}(t,x,u)) = 0 \tag{4.1}$$

with terminal condition $u(T,.) = h$. Here $\mathcal{L}$ is a local second-order differential operator corresponding to the infinitesimal generator of the continuous part of the forward SDE and $\mathcal{I}$ and $\mathcal{B}$ are two integro-differential operators. $\mathcal{I}$ is the discontinuous part of the infinitesimal generator of the forward SDE, and $\mathcal{B}$ is related to the generator of the BSDE. In [24], weak solution means viscosity solution. Since this paper, several authors have weaken the assumptions of [24]. The book [91] (Chapter 4) gives a nice review of these results (and several references on this topic).

Among all semi-linear PDEs, the particular form (3.3) has been widely studied:

$$\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^{q-1} = 0.$$

Baras & Pierre [21], Marcus & Veron [241] (and many other papers) have given existence and uniqueness results for this PDE. In [241] it is shown that every positive solution of (3.3) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S}_\infty, \mu)$ where $\mathcal{S}_\infty$ is a closed subset of $\mathbb{R}^d$ and μ a non-negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}_\infty$. The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^d, A \text{ Borel}, \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S}_\infty \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

Set $\mathcal{S}_\infty$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view Dynkin & Kuznetsov [106] and Le Gall [216] have proved similary results for PDE (3.3) in the case $1 < q \leq 2$ using the theory of superprocesses. Now if we want to represent the solution u of (3.3) using a FBSDE, we have to deal with a *singular* terminal condition ξ in the BSDE, which means that $\mathbb{P}(\xi = +\infty) > 0$. This singular case and the link between the solution of the BSDE with singular terminal condition and the viscosity solution of the PDE (3.3) have been studied first in [I]. Recently it was used to solve a stochastic control problem for portfolio liquidation (see [10] or [147]). In [X] we enlarge the known results on this subject for more general generator f.

In the first section of this chapter, our goal is to present the results of [I, XIV] on IPDE (4.1) when the terminal condition $u(T, .) = h$ is *singular* in the sense that h takes values in $\mathbb{R}_+ \cup \{+\infty\}$ and the set

$$\mathcal{S}_\infty = \{x \in \mathbb{R}^d, \quad h(x) = +\infty\}$$

is a non empty closed subset of $\mathbb{R}^d$. Again in the non singular case, if the terminal function h is of linear growth, the relation between the FBSDE and the IPDE is obtained in [24]. Moreover several papers have studied the existence and the uniqueness of the solution of such IPDE (see among others [8], [27], [35] or [159]). The novelty is that we gather the papers [24], [X], [I] and [XII] and we obtain non trivial conditions (for example between $\mathcal{I}$ and $\mathcal{S}_\infty$) for the existence and minimality of the viscosity solution of (4.1) with singularity at time T.

In the second section, we present similar results obtained in [XIII] concerning stochastic PDEs (SPDEs) with singularity:

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T \left[\mathcal{L}u(s,x) + f(s,x,u(s,x),((\nabla u)\sigma)(s,x))\right] ds \\ &+ \int_t^T g(s,x,u(s,x),((\nabla u)\sigma)(s,x)) \overleftarrow{dB_s}. \end{aligned}$$

Here there is no integro-differential operator, but a random noise given by the backward Itô integral $\overleftarrow{dB}$.

In Section 4.3, BSDE with terminal stopping time (see Part 3.4) and elliptic PDEs with singularity on the boundary are studied, following the papers [107, 215, 239, 240].

4.1 BSDE and (Integro) Partial Differential Equation

Let us emphasize that in this section, filtration $\mathbb{F}$ is generated by the Brownian motion W and the Poisson random measure π, that is there is no additional martingale term M in the BSDEs.

Since the paper of [273] and the extension to the jump case in [24], it is well known that a system of a SDE and a BSDE provides a probabilistic representation of the solution of a PDE (if there is no jump) or a IPDE (when there is a jump part). More precisely, we consider the forward SDE (1.15) starting at time t from the point $x \in \mathbb{R}^d$: for $t \leq s \leq T$

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r + \int_t^s \int_{\mathcal{E}} \beta(r, X_{r-}^{t,x}, e)\widetilde{\pi}(de, dr). \tag{4.2}$$

Coefficients b, σ and β satisfy Assumptions **(B1)** to **(B4)**. We consider BSDE (2.4): for $t \leq s \leq T$

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr \\ &- \int_s^T \int_{\mathcal{E}} U_r^{t,x}(e)\widetilde{\pi}(de, dr) - \int_s^T Z_r^{t,x} dW_r. \end{aligned} \tag{4.3}$$

System (4.2) and (4.3) is called a forward backward SDE (FBSDE in short). Generator f of BSDE (4.3) is a deterministic function from $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k \times \mathcal{B}^2_\mu$ to $\mathbb{R}$ and has the special structure for ψ: there exists a function γ from $\mathbb{R}^d \times \mathcal{E}$ to $\mathbb{R}$ such that

$$f(t,x,y,z,u) = f\left(t,x,y,z, \int_{\mathcal{E}} u(e)\gamma(x,e)\mu(de)\right). \tag{4.4}$$

For simplicity we denote with the same function f the right and the left hand side and for notational convenience $f_r^0 = f_r^{0,t,x} = f(r, X_r^{t,x}, 0, 0, \mathbf{0})$.

Setting on f

Structure (4.4) holds and in the rest f is the function on the right-hand side of this structural hypothesis. We still assume that **(A2)** and **(A4)** hold: there exists $\chi \in \mathbb{R}$ such that for any $t \in [0,T]$, $x \in \mathbb{R}^d$, $z \in \mathbb{R}^k$ and $u \in \mathbb{R}$

$$(f(t,x,y,z,u) - f(t,x,y',z,u))(y-y') \leq \chi(y-y')^2;$$

and there exists $K_{f,z} \geq 0$ such that for any (t,x,y,u), z and z':

$$|f(t,x,y,z,u) - f(t,x,y,z',u)| \leq K_{f,z}|z-z'|.$$

Condition **(A3)** is replaced by the stronger one: f is locally Lipschitz continuous w.r.t. y: for all $R > 0$, there exists L_R such that for any y and y' and any (t,x,z,u)

$$|y| \leq R, |y'| \leq R \Longrightarrow |f(t,x,y,z,u) - f(t,x,y',z,u)| \leq L_R|y-y'|.$$

Hypothesis **(A5)** becomes here: the function $u \in \mathbb{R} \mapsto f(t,x,y,z,u)$ is Lipschitz continuous and non decreasing for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k$:

$$\forall u \leq u', \quad 0 \leq f(t,x,y,z,u') - f(t,x,y,z,u) \leq L(u'-u).$$

And finally there exists a function $\vartheta \in \mathbb{L}^2_\mu$ such that for all $(x,e) \in \mathbb{R}^d \times \mathcal{E}$

$$0 \leq \gamma(x,e) \leq \vartheta(e).$$

Now to deal to the singularity at time T, **(C2)** still holds: there exists a constant $q > 1$ and a positive measurable function $\eta : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$f(r, X_r^{t,x}, y, z, u) \leq -\frac{1}{\eta(r, X_r^{t,x})} y|y|^{q-1} + f(r, X_r^{t,x}, 0, z, u).$$

To avoid extra difficulty to manage the negative part of the solution, we suppose that the process $f^{0,t,x}$ is non negative for any $(t,x) \in [0,T] \times \mathbb{R}^d$ and $\xi \geq 0$ a.s. Thus **(C1)** holds. To derive the a priori estimate, the following assumptions hold:

- The function
$$(t,x) \mapsto \eta(t,x)^{\frac{1}{q-1}} + f(t,x,0,0,\mathbf{0})$$
belongs to $\Pi_{pg}(0,T)$, the space of functions $\phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^k$ of polynomial growth, i.e. for some non-negative constants δ and C
$$\forall (t,x) \in [0,T] \times \mathbb{R}^d, \quad |\phi(t,x)| \leq C(1+|x|^\delta).$$

- There exists $\ell > 1$ such that the function ϑ belongs to $\mathbb{L}^{\ell^*}_\mu$ with $\ell^* = \ell/(\ell-1)$.

Note that **(C3)** and **(C4)** are consequences of these hypotheses. Sometimes to lighten the notations, $\eta(r, X_r^{t,x})$ is denoted η_r or $\eta_r^{t,x}$ if we do not need to precise the variables t and x.

Since we want to work on the link with IPDE, in order to use the work [24], we need extra assumptions on the regularity of f w.r.t. t and x.

- The function $t \mapsto f(t,x,y,z,u)$ is continuous on $[0,T]$.

- For all $R > 0$, $t \in [0,T]$, $|x| \leq R$, $|x'| \leq R$, $|y| \leq R$, $z \in \mathbb{R}^k$, $u \in \mathbb{R}$,
$$|f(t,x,y,z,u) - f(t,x',y,z,u)| \leq \varpi_R(|x-x'|(1+|z|)),$$
where $\varpi_R(s) \to 0$ when $s \searrow 0$.

- There exists $C_\gamma > 0$ such that for all $(x,x') \in (\mathbb{R}^d)^2$, $e \in \mathcal{E}$,
$$|\gamma(x,e) - \gamma(x',e)| \leq C_\gamma |x-x'|(1 \wedge |e|^2).$$

In the rest of this part, we assume that f satisfies all preceding conditions. Note that under this setting, the condition **(C3*)** holds for any $\varpi > 0$.

Viscosity solution

Our IPDE contains three operators:

- $\mathcal{L}$ is the local second-order differential operator, due to the continuous part of the forward SDE:

$$\mathcal{L}(t,x,\phi) = \frac{1}{2}\sum_{i,j=1}^{d}((\sigma\sigma^*)(t,x))_{i,j}\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^{d}(b(t,x))_i \frac{\partial \phi}{\partial x_i}. \tag{4.5}$$

- $\mathcal{I}$ is an integro-differential operator and comes from the jump part of the forward SDE:

$$\mathcal{I}(t,x,\phi) = \int_{\mathcal{E}}[\phi(t,x+\beta(t,x,e)) - \phi(t,x) - (\nabla\phi)(t,x)\beta(t,x,e)]\mu(de).$$

- $\mathcal{B}$ is also an integral operator coming from the generator of the BSDE:

$$\mathcal{B}(t,x,\phi) = \int_{\mathcal{E}}[\phi(t,x+\beta(t,x,e)) - \phi(t,x)]\gamma(x,e)\mu(de).$$

The equation is now (4.1):

$$\frac{\partial}{\partial t}u(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u,(\nabla u)^\top\sigma, \mathcal{B}(t,x,u)) = 0$$

with terminal condition $u(T,.) = h$.

For a locally bounded function v in $[0,T]\times\mathbb{R}^d$, we define its upper (resp. lower) semicontinuous envelope v^* (resp. v_*) by:

$$v^*(t,x) = \limsup_{(s,y)\to(t,x)} v(s,y) \quad (\text{resp. } v_*(t,x) = \liminf_{(s,y)\to(t,x)} v(s,y)).$$

We introduce the notion of viscosity solution as in [8] (see also Definition 3.1 in [24] or Definitions 1 and 2 in [27]). Since we do not assume the continuity of the involved function u, we adapt the definition of discontinuous viscosity solution (see Definition 4.1 and 5.1 in [159]).

Definition 4.1 *A locally bounded function v is*

1. ***a viscosity subsolution*** *of* (4.1) *if it is upper semicontinuous (usc) on $[0,T)\times\mathbb{R}^d$ and if for any $\phi \in C^2([0,T]\times\mathbb{R}^d)$ wherever $(t,x)\in[0,T)\times\mathbb{R}^d$ is a global maximum point of $v-\phi$,*

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}(t,x,\phi) - f(t,x,v,(\nabla\phi)\sigma,\mathcal{B}(t,x,\phi)) \leq 0.$$

2. ***a viscosity supersolution*** *of* (4.1) *if it is lower semicontinuous (lsc) on $[0,T)\times\mathbb{R}^d$ and if for any $\phi \in C^2([0,T]\times\mathbb{R}^d)$ wherever $(t,x)\in[0,T)\times\mathbb{R}^d$ is a global minimum point of $v-\phi$,*

$$-\frac{\partial}{\partial t}\phi(t,x) - \mathcal{L}\phi(t,x) - \mathcal{I}(t,x,\phi) - f(t,x,v,(\nabla\phi)\sigma,\mathcal{B}(t,x,\phi)) \geq 0.$$

3. ***a viscosity solution*** *of* (4.1) *if its upper envelope v^* is a subsolution and if its lower envelope v_* is a supersolution of* (4.1).

4.1.1 IPDE with singular terminal condition

If our terminal condition ξ satisfies (3.33) and (3.34):

$$\xi = h(X_T), \qquad h(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in \mathbb{L}^1(\Omega, \mathcal{F}_T, \mathbb{P})$$

for every closed set $\mathcal{K} \subset \mathbb{R}^d \setminus \mathcal{S}_\infty$, and if h is a continuous function from $\mathbb{R}^d$ to $[0, +\infty]$, then the solution[1] $(Y^{L,t,x}, Z^{L,t,x}, U^{L,t,x})$ of the BSDE (3.8) provides a function u^L defined by

$$u^L(t,x) := Y_t^{L,t,x}.$$

Moreover from [24, Theorems 3.4 and 3.5], we directly have the next result.

Lemma 4.1 *Function $u^L(t,x) := Y_t^{L,t,x}$, $(t,x) \in [0,T] \times \mathbb{R}^d$, is the unique bounded (by $L(T+1)$) continuous viscosity solution of* (4.1) *with generator f^L and with terminal condition $u^L(T,.) = h(\cdot) \wedge L$.*

Now the minimal solution $Y^{t,x}$ of the singular BSDE (2.4) is obtained as the increasing limit of $Y^{L,t,x}$: for any $t \leq s \leq T$

$$\lim_{L \to +\infty} Y_s^{L,t,x} = Y_s^{t,x}.$$

And it is well known that viscosity solutions are stable by monotone limit. That is the reason why we use this notion of weak solutions.

We define the function u by:

$$u(t,x) = Y_t^{t,x}.$$

Therefore sequence $u^L(t,x)$ converges to $u(t,x)$. Since η and f^0 only depend on $X^{t,x}$, the a priori estimate (3.10) becomes: there exists two constants $K > 0$ and $\delta > 0$ such that for all $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$0 \leq u^L(t,x) \leq u(t,x) \leq \frac{K}{(T-t)^{\frac{1}{q-1}}}(1 + |x|^\delta). \tag{4.6}$$

Since u^L is a continuous function, the function u is lower semi-continuous on $[0,T] \times \mathbb{R}^d$ and satisfies for all $x_0 \in \mathbb{R}^d$:

$$\liminf_{(t,x) \to (T,x_0)} u(t,x) \geq h(x_0). \tag{4.7}$$

The next theorems were proved in [I] in the Brownian setting (no jump) and for the generator $f(y) = -y|y|^{q-1}$. In [XIV] we generalize them. The first result is:

Theorem 4.1 *Under our framework, $u(t,x) = Y_t^{t,x}$ is a viscosity solution of the IPDE* (4.1) *on $[0,T[\times\mathbb{R}^d$. Moreover u is the minimal viscosity solution among all non-negative solutions satisfying* (4.7).

[1] Again there is no martingale part M due to the restriction imposed on the filtration.

Note that we do not prove the continuity of u because of the lack of uniform convergence of the approximating sequence u^L. But we are also not able to show that u is discontinuous (see below more results on regularity of u).

As for the singular BSDE (2.4), the main difficulty is to show that

$$\limsup_{(t,x)\to(T,x_0)} u(t,x) \leq h(x_0) = u(T,x_0).$$

On the set $\mathcal{S}_\infty = \{h = +\infty\}$, we already have (4.7). Hence we concentrate ourselves on $\mathcal{R} = \{h < +\infty\}$. We overcome this problem in two steps:

- We prove that u^* is locally bounded on a neighbourhood of T on the open set $\mathcal{R}$.
- We deduce that u^* is a subsolution with relaxed terminal condition and we apply this to demonstrate that $u^*(T,x) \leq h(x)$ if $x \in \mathcal{R}$.

To obtain the local boundedness of u^*, we add a link between the singularity set $\mathcal{S}_\infty$ and the jumps of the forward process X. More precisely we assume that **(D1)** and **(D2)** hold. As for the continuity problem, hypothesis (3.32) is supposed to be true.

Theorem 4.2 *Under all these conditions, we have*

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = h(x_0).$$

4.1.2 Regularity of the viscosity solution

Function u is the minimal non-negative viscosity solution of the IPDE (4.1). From (4.6) we know that u is finite on $[0,T[\times\mathbb{R}^d$, and for $\varepsilon > 0$ u is bounded on $[0,T-\varepsilon]\times\mathbb{R}^d$ by $K(1+|x|^\delta)\varepsilon^{-1/(q-1)}$. We cannot expect regularity on $[0,T]\times\mathbb{R}^d$, but only on $[0,T-\varepsilon]\times\mathbb{R}^d$ for any $\varepsilon > 0$. In order to obtain a smoother solution u, some assumptions are imposed on the coefficients. We distinguish three different conditions.

- *Sobolev regularity.* The viscosity solution is a weak solution in the Sobolev sense if the coefficients on the forward SDE (4.2) are smooth and if the *linkage operator* $x \mapsto x + \beta(x,e)$ is a C^2-diffeomorphism. These extra assumptions are used to control the stochastic flow generated by $X^{t,x}$ (see [249, Proposition 2]).
- *Hölder regularity.* There have been several papers [25, 26, 52, 67, 68, 306] (among many others) dealing with C^α estimates and regularity of the solution of the IPDE (4.1). We need some non degeneration assumption on the operators $\mathcal{L}$ or $\mathcal{I}$. Roughly speaking, the viscosity solution is locally Hölder continuous if h is Hölder continuous function on the set $\mathcal{O}_M = \{|h| \leq M\}$ for all $M \geq 0$.

  The classical assumption is the uniform ellipticity of $\mathcal{L}$ **(B7)**: there exists $\lambda > 0$ s.t. for all $x \in \mathbb{R}^d$
  $$\forall y \in \mathbb{R}^d,\ \sigma\sigma^*(x)y\cdot y \geq \lambda|y|^2.$$
  Nevertheless when the local second order differential operator $\mathcal{L}$ becomes degenerated, the non-local operator $\mathcal{I}$ can take over from $\mathcal{L}$. In [XIV] we give some assumptions on $\mathcal{I}$ such that the strict ellipticity is involved by the non local terms and such that the assumptions denoted by *(J1)* to *(J5)* in [25] hold.

- *Strong regularity.* Under the uniform ellipticity condition of $\mathcal{L}$, u can be a classical solution under different settings.
  - If the measure μ is finite we can transform the IPDE (4.1) into some PDE without non local operator (arguments developed in [237] or [287]) and then use regularity arguments for such PDE.
  - In the setting of [131], i.e. for some $\gamma < 2$

$$\int_{\mathcal{E}} (1 \wedge |e|^\gamma)\mu(de) < +\infty \tag{4.8}$$

    and the linkage operator satisfies

$$\det(\mathrm{Id}_d + \nabla_x \beta(x, e)) \geq c_1 > 0, \tag{4.9}$$

    the existence of a Green function with suitable properties will ensure a regularizing effect of the operator $\mathcal{L} + \mathcal{I}$.

Of course, none of these settings gives necessary conditions and other sufficient assumptions could be exhibited.

4.2 Stochastic PDE with singularity at time T

One of the goal of [XIII] was to study non linear SPDE with explosion at time T. In some sense we wanted to obtain similar results[2] as [241], but for PDE with noise.

Pardoux and Peng [272] have proven existence and uniqueness for solutions of BDSDE (2.11) if f and g are supposed to be Lipschitz continuous functions and with square integrability condition on the terminal condition ξ and on the coefficients $f(t, 0, 0)$ and $g(t, 0, 0)$. Moreover under smoothness assumptions of the coefficients, Pardoux and Peng prove existence and uniqueness of a classical solution for the SPDE: for $(t, x) \in [0, T] \times \mathbb{R}^d$

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T [\mathcal{L}u(s,x) + f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))]\,ds \\ &+ \int_t^T g(s,x,u(s,x),(\nabla u \sigma)(s,x))\overleftarrow{dB_r}. \end{aligned} \tag{4.10}$$

$\mathcal{L}$ is again the second-order differential operator defined by (4.5)

$$\mathcal{L}\phi = b\nabla\phi + \frac{1}{2}\mathrm{Trace}(\sigma\sigma^* D^2).$$

and they also establish connection with solutions of BDSDE (2.11). Several generalizations to investigate the weak solution of SPDE (4.10) have been developed following different approaches:

- The technics of stochastic flow (Bally and Matoussi [18], Matoussi et al. [248, 250], Kunita [210], El Karoui and Mrad [119]);

[2] Let us stay humble, the notion of trace developed in [241] is much more intricate.

- The approach based on Dirichlet forms and their associated Markov processes (Denis and Stoica [94], Bally et al. [19], Stoica [312]);
- Stochastic viscosity solution for SPDEs (Buckdahn and Ma [50, 51], Lions and Souganidis [227, 228]).

Above approaches have allowed the study of numerical schemes for the Sobolev solution of semilinear SPDEs via Monte-Carlo methods (time discretization and regression schemes: [13, 14, 247]). For some general references on SPDE, see among others [85, 208, 296, 322].

One goal of [XIII] is to extend the results of [18] and of [I] for the SPDE with singular terminal condition h: for any $0 \leq t \leq T$

$$
\begin{aligned}
u(t,x) &= h(x) + \int_t^T \left(\mathcal{L}u(s,x) - u(s,x)|u(s,x)|^{q-1}\right) ds \\
&\quad + \int_t^T g(s,x,u(s,x),\sigma(s,x)\nabla u(s,x))\overleftarrow{dB_s}
\end{aligned}
\tag{4.11}
$$

where we will assume that $\mathcal{S}_\infty = \{h = +\infty\}$ is a closed non empty set. Roughly speaking we want to show that there is a (minimal) solution u in the sense that

- u belongs to some Sobolev space and is a weak solution of the SPDE on any interval $[0, T-\delta]$, $\delta > 0$,
- u satisfies the terminal condition: $u(t,x)$ goes to $h(x)$ also in a weak sense as t goes to T.

Of course we use all results contained in Section 3.2.1 on BDSDE. Under monotonicity assumption on f, we also prove that the SPDE (4.10) has a unique weak solution (as in [18]), given by the solution of the BDSDE (2.11), if h is in $\mathbb{L}^{2q}(\mathbb{R}^d, \rho^{-1}dx)$ or if $\xi = h(X_T^{t,x})$ belongs to $\mathbb{L}^{2q}(\Omega)$. Then we extend the existence of a solution when $\mathcal{S}_\infty$ is non empty.

Let us summarize the setting (similar to [18]) and our results. In order to define the space of solutions, we choose a continuous positive weight function $\rho : \mathbb{R}^d \to \mathbb{R}$. We require only that the derivatives of ρ are in $C_b^1(\mathbb{R}^d;\mathbb{R})$ on the set $\{|x| > R\}$ for some R. For example ρ can be $(1+|x|)^\kappa$, $\kappa \in \mathbb{R}$. We assume that functions $f : [0,T]\times\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times d} \to \mathbb{R}^N$ and $g : [0,T]\times\mathbb{R}^d\times\mathbb{R}^N\times\mathbb{R}^{N\times d} \to \mathbb{R}^{N\times m}$ are measurable in (t,x,y,z) and w.r.t. (y,z). Recall that k (resp. m) is the dimension of the Brownian motion W (resp. B) (in the forward (resp. backward) Itô integral) in BDSDE (2.11). d is the dimension of the solution $X^{t,x}$ of the forward SDE (4.2) (without the jump part β):

$$
X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \text{ for } s \in [t,T],
\tag{4.12}
$$

whereas N is the dimension of $Y^{t,x}$: for $t \leq s \leq T$

$$
\begin{aligned}
Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr \\
&\quad + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\overleftarrow{dB_r} - \int_s^T Z_r^{t,x} dW_r.
\end{aligned}
\tag{4.13}
$$

Coefficients f and g satisfy Assumptions **(A2)**, **(A3$\star$)**, **(A4)** and **(Ag1)**. The only difference with [18] is that we do not assume that f is Lipschitz continuous w.r.t. y. We also assume that

$$\int_{\mathbb{R}^d}\left[|h(x)|^{2q}+\int_0^T\left(|f(t,x,0,0)|^{2q}+|g(t,x,0,0)|^{2q}\right)dt\right]\rho^{-1}(x)dx<+\infty. \tag{4.14}$$

We define space $\mathcal{H}(0,T)$ as in [18].

Definition 4.2 *$\mathcal{H}(0,T)$ is the set of the random fields $\{u(t,x);\ 0\leq t\leq T,\ x\in\mathbb{R}^d\}$ such that $u(t,x)$ is $\mathcal{F}^B_{t,T}$-measurable for each (t,x), u and $\sigma^*\nabla u$ belong to $\mathbb{L}^2((0,T)\times\Omega\times\mathbb{R}^d;\ ds\otimes d\mathbb{P}\otimes\rho^{-1}(x)dx)$. On $\mathcal{H}(0,T)$ we consider the following norm*

$$\|u\|_2^2=\mathbb{E}\int_{\mathbb{R}^d}\int_0^T\left(|u(s,x)|^2+|(\sigma^*\nabla u)(s,x)|^2\right)\rho^{-1}(x)dsdx.$$

Let us also evoke the definition of a weak solution.

Definition 4.3 *u is a weak solution of SPDE (4.10) if the following conditions are satisfied.*

1. *For some $\delta>0$*

$$\sup_{s\leq T}\mathbb{E}\left[||u(s,.)||^{1+\delta}_{\mathbb{L}^2_{\rho^{-1}}(\mathbb{R}^d)}\right]<\infty. \tag{4.15}$$

2. *For every test-function $\phi\in C^\infty(\mathbb{R}^d)$, $dt\otimes d\mathbb{P}$ a.e.*

$$\lim_{s\uparrow t}\int_{\mathbb{R}^d}u(s,x)\phi(x)dx=\int_{\mathbb{R}^d}u(t,x)\phi(x)dx. \tag{4.16}$$

3. *Finally u satisfies for every function $\Psi\in C_c^{1,\infty}([0,T]\times\mathbb{R}^d;\mathbb{R})$*

$$\begin{aligned}\int_t^T\int_{\mathbb{R}^d}u(s,x)\partial_s\Psi(s,x)dxds&+\int_{\mathbb{R}^d}u(t,x)\Psi(t,x)dx-\int_{\mathbb{R}^d}h(x)\Psi(T,x)dx\\&-\frac{1}{2}\int_t^T\int_{\mathbb{R}^d}(\sigma^*\nabla u)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds\\&-\int_t^T\int_{\mathbb{R}^d}u(s,x)div\ \left(\left(b-\widetilde{A}\right)\Psi\right)(s,x)dxds\\=\int_t^T\int_{\mathbb{R}^d}&\Psi(s,x)f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))dxds\\&+\int_t^T\int_{\mathbb{R}^d}\Psi(s,x)g(s,x,u(s,x),(\sigma^*\nabla u)(s,x))dx\overleftarrow{dB_s}.\end{aligned} \tag{4.17}$$

Here

$$\widetilde{A}_i=\frac{1}{2}\sum_{j=1}^d\frac{\partial(\sigma\sigma^*)_{j,i}}{\partial x_j}.$$

Concerning SPDE with monotone coefficient f, we get the following result.

Proposition 4.1 *Under this setting and if Condition* (4.14) *holds, then the random field defined by $u(t,x) = Y_t^{t,x}$ is in $\mathcal{H}(0,T)$ with*

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^T |u(s,x)|^{2q}\rho^{-1}(x)dsdx. \tag{4.18}$$

Moreover u is the unique weak solution of the SPDE (4.10).

Note that Condition (4.14) is important to ensure that (4.18) holds and therefore the quantity $F_s = f(s,x,u(s,x),(\sigma^*\nabla u)(s,x))$ is in $\mathcal{H}'_{0,\rho}$, the dual space of $\mathbb{L}^2(\mathbb{R}^d,\rho(x)dx)$ (see [18]), which is crucial to prove the existence of a weak solution. To prove the proposition, we use the existence theorem 2.5 and we sketch the proof of [18, Theorem 3.1] step by step.

Now suppose that the conditions of Theorem 3.3 hold. Condition (4.14) now becomes:

$$\int_{\mathbb{R}^d}\left[\int_0^T \left(|f(t,x,0,0)|^{2q} + |g(t,x,0,0)|^{2q}\right)dt\right]\rho^{-1}(x)dx < +\infty. \tag{4.19}$$

For any $L \in \mathbb{N}^*$, let $(Y^{L,t,x}, Z^{L,t,x})$ be the solution of the BDSDE (4.13) with terminal condition $h(X_T^{t,x}) \wedge L$. From (3.17) if $L \leq L'$

$$\Xi_s^{0,t,x} \leq Y_s^{L,t,x} \leq Y_s^{L',t,x} \leq \Xi_s^{L',t,x}.$$

If we assume that (3.40) holds for h, then $h \wedge L$ is a Lipschitz and bounded function on $\mathbb{R}^d$. Hence $h \wedge L$ belongs to $\mathbb{L}^2(\mathbb{R}^d,\rho^{-1}(x)dx)$ provided the function $\rho^{-1} \in \mathbb{L}^1(\mathbb{R}^d,dx)$. From the preceding proposition 4.1, we have

Proposition 4.2 *There exists a unique weak solution $u^L \in \mathcal{H}(0,T)$ of SPDE* (4.11) *with terminal function $h \wedge L$. Moreover $u^L(t,x) = Y_t^{L,t,x}$ and*

$$Y_s^{L,t,x} = u^L(s,X_s^{t,x}), \quad Z_s^{L,t,x} = (\sigma^*\nabla u^L)(s,X_s^{t,x}).$$

Remember that we have defined a process $(Y^{t,x}, Z^{t,x})$ solution of the backward doubly stochastic differential equation (4.13) with singular terminal condition h (see Theorem 3.3). Process Y is obtained as the increasing limit of processes Y^L:

$$Y_s^{t,x} = \lim_{L\to+\infty} Y_s^{L,t,x} \quad \text{a.s..}$$

Therefore we can define the following random field u as follows: for all $(t,x) \in [0,T]\times\mathbb{R}^d$

$$u(t,x) = Y_t^{t,x} = \lim_{L\to+\infty} Y_t^{L,t,x} = \lim_{L\to+\infty} u^L(t,x).$$

The result obtained is a direct extension of [XIII]:

Theorem 4.3 *Under the conditions of Theorem 3.3 and with Hypothesis* (4.19)*, the random field u defined by $u(t,x) = Y_t^{t,x}$ belongs to $\mathcal{H}(0,T-\delta)$ for any $\delta > 0$ and is a weak solution of SPDE* (4.11) *on $[0,T-\delta]\times\mathbb{R}^d$. At time T, u satisfies a.s. $\liminf_{t\to T} u(t,x) \geq h(x)$.*

Moreover under the same assumptions of Theorem 3.6 or Proposition 3.6 (see also the subsection on the continuity problem for BDSDE), for any function $\phi \in C_c^\infty(\mathbb{R}^d)$ with support included in $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}_\infty$, then

$$\lim_{t\to T}\mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\phi(x)dx\right) = \int_{\mathbb{R}^d} h(x)\phi(x)dx.$$

Finally u is the minimal non-negative solution of (4.11)*.*

The almost sure continuity of u at time T is still an open question. In [I, XIV], this property is proved using viscosity solution arguments (relaxation of the boundary condition). Here we cannot do the same trick. This point will be investigated in further publications.

In [XIII], we only study the case $f(y) = -y|y|^{q-1}$ and $g(t,y,0)=0$. Then the proof is based on the a priori estimate

$$0 \leq Y_s^{L,t,x} \leq \left(\frac{1}{(q-1)(T-s)+\frac{1}{L^{q-1}}}\right)^{\frac{1}{q-1}} \leq \left(\frac{1}{(q-1)(T-s)}\right)^{\frac{1}{q-1}}.$$

In particular for any (t,x)

$$0 \leq u^L(t,x) \leq \left(\frac{1}{(q-1)(T-t)}\right)^{\frac{1}{q-1}}$$

and hence u satisfies the same estimate. Thus u is bounded on $[0,t]\times\mathbb{R}^d$ and in $\mathbb{L}^2_\rho(\mathbb{R}^d)$. By dominated convergence theorem, for any $\delta > 0$, u satisfies (4.15) and (4.16) for any $0 \leq s \leq t \leq T-\delta$. Moreover we have $Z_s^{L,t,x} = (\sigma^*\nabla u^L)(s, X_s^{t,x})$ and from the proof on Theorem 3.3, the sequence of processes $(Z_s^{L,t,x},\ s\geq t)$ converges in $\mathbb{L}^2((0,T-\delta)\times\Omega)$ for any $\delta > 0$ to $Z^{t,x}$. Hence the sequence u^L converges in $\mathcal{H}(0,t)$ to u. As for Proposition 3.2 we have the a priori estimate:

$$\mathbb{E}\int_0^T (T-s)^{\frac{2}{q-1}}|Z_s^{L,t,x}|^2 ds \leq \frac{8+K_gT}{1-\varepsilon}\left(\frac{1}{q-1}\right)^{\frac{2}{q-1}}$$

(as usual $Z_s^{L,t,x} = 0$ if $s < t$). Therefore using [18, Proposition 5.1] we deduce

$$\mathbb{E}\int_{\mathbb{R}^d}\int_0^T (T-s)^{\frac{2}{q-1}}|(\sigma^*\nabla u^L)(s,x)|^2\rho^{-1}(x)dxds \leq C$$

where the constant C does not depend on L. With the Fatou lemma we have the same inequality for u. Now for every function $\Psi \in C_c^{1,\infty}([0,T]\times\mathbb{R}^d)$, u^L satisfies (4.17), therefore for every $0 \le r \le t < T$, u^L satisfies also:

$$
\begin{aligned}
(4.20) \quad & \int_r^t \int_{\mathbb{R}^d} u^L(s,x)\partial_s\Psi(s,x)dxds + \int_{\mathbb{R}^d} u^L(r,x)\Psi(r,x)dx - \int_{\mathbb{R}^d} u^L(t,x)\Psi(t,x)dx \\
& \quad -\frac{1}{2}\int_r^t \int_{\mathbb{R}^d} (\sigma^*\nabla u^L)(s,x)(\sigma^*\nabla\Psi)(s,x)dxds \\
& \quad -\int_r^t \int_{\mathbb{R}^d} u^L(s,x)\mathrm{div}\ \left(\left(b-\widetilde{A}\right)\Psi\right)(s,x)dxds \\
& = -\int_r^t \int_{\mathbb{R}^d} \Psi(s,x)u^L(s,x)|u^L(s,x)|^{q-1}dxds \\
& \quad +\int_r^t \int_{\mathbb{R}^d} \Psi(s,x)g(s,x,u^L(s,x),\sigma^*\nabla u^L(s,x))dx\overleftarrow{dB_s}.
\end{aligned}
$$

But using monotone convergence theorem or the convergence of u^L to u in $\mathcal{H}(0,t)$, we can pass to the limit as L goes to $+\infty$ in (4.20) and we obtain that u is a weak solution of (4.11) on $[0, T-\delta] \times \mathbb{R}^d$ for any $\delta > 0$.

To extend the conclusion to more general generators, from (3.17), it is sufficient to have an upper bound on $\Xi^{L,t,x}$ uniformly in (L,x). In (4.20), the term

$$
-\int_r^t \int_{\mathbb{R}^d} \Psi(s,x)u^L(s,x)|u^L(s,x)|^{q-1}dxds
$$

should be replaced by

$$
\int_r^t \int_{\mathbb{R}^d} \Psi(s,x)f(s,x,u^L(s,x),,\sigma^*\nabla u^L(s,x))dxds.
$$

Again we do not claim that it can be done for all generators f and g. But we easily can obtain sufficient conditions for other generators such that the conclusion of the previous theorem holds.

The only trouble concerns the behavior of u near T. By monotonicity we obtain easily that a.s.

$$
(4.21) \qquad \liminf_{t\uparrow T} \int_{\mathbb{R}^d} u(t,x)\psi(x)dx \ge \int_{\mathbb{R}^d} h(x)\psi(x)dx.
$$

To get the converse inequality, evoke that we have proved for the BDSDE with suitable integrability condition on all terms:

$$
\begin{aligned}
\mathbb{E}(h(X_T^{t,x})\theta(X_T^{t,x})) &= \mathbb{E}(u(t,x)\theta(x)) - \mathbb{E}\int_t^T \theta(X_r^{t,x})f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr \\
&+ \mathbb{E}\int_t^T Y_r^{t,x}\mathcal{L}\theta(X_r^{t,x})dr + \mathbb{E}\int_t^T Z_r^{t,x}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})dr
\end{aligned}
$$

for any smooth functions θ such that its compact support is strictly included in $\mathcal{R} = \{h < +\infty\}$. Integrate this w.r.t. dx and letting t go to T, together with the dominated convergence theorem and Fatou's lemma, yield to:

$$
\mathbb{E}\left(\liminf_{t\to T}\int_{\mathbb{R}^d} u(t,x)\theta(x)dx\right) \le \lim_{t\to T}\mathbb{E}\left(\int_{\mathbb{R}^d} u(t,x)\theta(x)dx\right) = \int_{\mathbb{R}^d} h(x)\theta(x)dx
$$

for any function $\theta \in C_c^2(\mathbb{R}^d)$ with $\mathrm{supp}(\theta) \cap \mathcal{S}_\infty = \emptyset$. With Inequality (4.21), we obtain that for any $\psi \in C_c^\infty(\mathbb{R}^d)$ a.s.

$$\liminf_{t \to T} \int_{\mathbb{R}^d} u(t,x)\psi(x)dx = \int_{\mathbb{R}^d} h(x)\psi(x)dx.$$

Remark 4.1 *If g does not depend of Z (or on ∇u), and if $g \in C_b^{0,2,3}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R}^d)$, then from [50], u^L is a stochastically bounded viscosity solution of the SPDE* (4.11) *on $[0,T] \times \mathbb{R}^d$ and u is also a stochastically bounded viscosity solution of the SPDE* (4.11) *on $[0, T-\delta] \times \mathbb{R}^d$ for any $\delta > 0$.*

Minimality is deduced from the minimality of the solution of the BDSDE.

4.3 Elliptic PDE with singularity on the boundary

Considering the system of SDEs (4.2) and (4.3), but with the random terminal time τ corresponding to the exit time of the diffusion process X, leads to an elliptic PDE with Dirichlet condition (see among many others [87, 276]).

In [II] we restrict ourselves to the continuous case. Let D be an open bounded subset of $\mathbb{R}^d$, whose boundary is at least of class C^2 (see [213] for the definition of a regular boundary). For all $x \in \mathbb{R}^d$, let X^x denote the solution of the SDE:

$$X_t^x = x + \int_0^t b(X_r^x)dr + \int_0^t \sigma(X_r^x)dW_r, \text{ for } t \geq 0. \tag{4.22}$$

The functions b and σ are defined on $\mathbb{R}^d$, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times k}$, and are measurable such that:

- **(B2)** holds for σ:

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \ \|\sigma(x) - \sigma(y)\| \leq K|x-y|;$$

- The boundedness condition **(B5)** is verified:

$$\forall x \in \mathbb{R}^d, \ |b(x)| + \|\sigma(x)\| \leq K;$$

- Uniform ellipticity **(B7)**: there exists a constant $\lambda > 0$ such that

$$\forall x \in \mathbb{R}^d, \ \sigma\sigma^*(x) \geq \lambda \mathrm{Id}.$$

Under these assumptions, from a result of Veretennikov [320] and [321], the equation (4.22) has a unique strong solution X^x. For each $x \in \overline{D}$, we define the stopping time

$$\tau = \tau_x = \inf\left\{t \geq 0, \ X_t^x \notin \overline{D}\right\}. \tag{4.23}$$

Our stopping time satisfies the following two properties. Since D is bounded and from our setting

$$\text{every point } x \in \partial D \text{ is regular.} \tag{4.24}$$

In particular, if $x \in \partial D$, $\tau_x = 0$ a.s. (see [31, Corollary 3.2]). Assumption (4.24) is important to define a singular solution. Moreover we have the following result (see [290, Theorem 2.1] and [270, Remark 5.6]): for all $x \in \overline{D}$, $\tau_x < +\infty$ a.s. and there exists $\beta > 0$ such that

$$\sup_{x \in \overline{D}} \mathbb{E}\left(e^{\beta \tau_x}\right) < \infty. \tag{4.25}$$

This property is used to construct solutions of the BSDE for bounded terminal conditions ξ.

From the papers [87, 270, 281], we know that the BSDE:

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(Y_r, Z_r)dr - \int_{t\wedge\tau}^{\tau} Z_r dW_r, \tag{4.26}$$

with terminal time equal to $\tau = \tau_x$ and final data equal to $\xi = h(X^x_{\tau_x})$ is associated with the following elliptic PDE with Dirichlet condition h:

$$\begin{cases} -\mathcal{L}u - f(u, \sigma^\top \nabla u) = 0 & \text{on } D, \\ u = h & \text{on } \partial D; \end{cases} \tag{4.27}$$

where $\mathcal{L}$ is defined by (4.5): for all $\phi \in C^2_0(\mathbb{R}^d)$,

$$\forall x \in \mathbb{R}^d, \quad \mathcal{L}\phi(x) = \frac{1}{2}\text{Trace}\left(\sigma\sigma^*(x)D^2\phi(x)\right) + b(x)\nabla\phi(x).$$

If (Y^x, Z^x) denotes the solution of BSDE (4.26) with terminal data $h(X^x_{\tau_x})$, the connection is given by the formula

$$u(x) = Y^x_0.$$

The particular case $f(y, z) = -y|y|^{q-1}$ has been extensively studied. Le Gall [215] succeeded in describing all solutions of the equation $\Delta u = u^2$ in the unit disk D in $\mathbb{R}^2$ by a purely probabilistic method. He established a 1-1 correspondence between all solutions and all pairs $(\mathcal{S}_\infty, \nu)$, where $\mathcal{S}_\infty$ is a closed subset of ∂D and ν is a Radon measure on $\partial D \setminus \mathcal{S}_\infty$. $\mathcal{S}_\infty$ is the set of singular points of ∂D where the solution explodes badly: roughly speaking, near points of $\mathcal{S}_\infty$, the solution behaves like the inverse of the squared distance to the boundary. Measure ν can be interpreted as the "boundary value" of u on $\partial D \setminus \mathcal{S}_\infty$. The solution corresponding to $(\mathcal{S}_\infty, \nu)$ is expressed in terms of the Brownian snake (a path-valued Markov process). In [215] the results announced in [217] are proved in detail and are extended to a general smooth domain in $\mathbb{R}^2$.

Pair $(\mathcal{S}_\infty, \nu)$ is called the boundary trace for positive solution of the PDE (4.27). The definition of boundary trace in general was provided by Marcus and Véron [240] who showed by analytic methods that every positive solution of (4.27) posseses a unique trace. The trace can be described by a (possibly unbounded) positive regular Borel measure $\tilde{\nu}$ on ∂D. The correspondence between $(\mathcal{S}_\infty, \nu)$ and $\tilde{\nu}$ is given by

$$\tilde{\nu}(A) = \begin{cases} \nu(A) & \text{if } A \subseteq (\partial D \setminus \mathcal{S}_\infty) \\ \infty & \text{if } A \cap \mathcal{S}_\infty \neq \emptyset \end{cases}$$

for every Borel subset A of ∂D.

The corresponding boundary value problem is presented in [240] in the subcritical case $0 < q-1 < 2/(d-1)$ and in [239] in the supercritical case $q-1 \geq 2/(d-1)$. In the subcritical case, for every pair $(\mathcal{S}_\infty, \nu)$, the problem has a unique solution. Remark that in [215, 217], $q = 2$ and $d = 2$, that is, the subcritical case is studied: $q - 1 = 1 < 2/(2-1) = 2/(d-1)$. In the supercritical case, Marcus and Véron derive necessary and sufficient conditions for the existence of a maximal solution. Similar conditions were obtained by Dynkin and Kuznetsov [107] for $q \leq 2$. Their method relies on probabilistic techniques and is not extendable to $q > 2$, because the main tool is the q-superdiffusion which is not defined for $q > 2$. In our case there is no restriction on $q > 1$.

Here we provide another probabilistic representation of the solution of PDE (4.27) in terms of the solution of related BSDE (4.26). In general, a solution of the PDE has a "blow-up" set $\mathcal{S}_\infty$. Therefore, the final data ξ of the BSDE must be allowed to be infinite with positive probability and the set $\{\xi = +\infty\}$ corresponds to $\mathcal{S}_\infty$. Hence, we use Section 3.4, where a solution of (4.26) is defined when ξ is infinite with positive probability.

We already define viscosity solution for a parabolic PDE in Definition 4.1. Let us adapt the definition of a viscosity solution in the elliptic case (which can be found in [22, 23] or [270], [84] for v continuous). If v is a function defined on $\overline{D}$, we denote by v^* (respectively v_*) the upper- (respectively lower-) semicontinuous envelope of v: for all $x \in \overline{D}$

$$v^*(x) = \limsup_{x' \to x,\ x' \in \overline{D}} v(x') \quad \text{and} \quad v_*(x) = \liminf_{x' \to x,\ x' \in \overline{D}} v(x').$$

Definition 4.4

- *$v : \overline{D} \to \mathbb{R}$ is called a* ***viscosity subsolution*** *of* (4.27) *if $v^* < +\infty$ on $\overline{D}$ and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \overline{D}$ is a point of local maximum of $v^* - \phi$,*

$$\begin{aligned} -\mathcal{L}\phi(x) - f(v^*(x), \sigma^\top(x)\nabla\phi(x)) &\leq 0 \quad \text{if} \quad x \in D; \\ \min\left(-\mathcal{L}\phi(x) - f(v^*(x), \sigma^\top(x)\nabla\phi(x)), v^*(x) - h(x)\right) &\leq 0 \quad \text{if} \quad x \in \partial D. \end{aligned}$$

- *$v : \overline{D} \to \mathbb{R}$ is called a* ***viscosity supersolution*** *of* (4.27) *if $v_* > -\infty$ on $\overline{D}$ and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \overline{D}$ is a point of local minimum of $v_* - \phi$,*

$$\begin{aligned} -\mathcal{L}\phi(x) - f(v_*(x), \sigma^\top(x)\nabla\phi(x)) &\geq 0 \quad \text{if} \quad x \in D; \\ \max\left(-\mathcal{L}\phi(x) - f(v_*(x), \sigma^\top(x)\nabla\phi(x)), v(x) - h(x)\right) &\geq 0 \quad \text{if} \quad x \in \partial D. \end{aligned}$$

- *$v : \overline{D} \to \mathbb{R}$ is called a* ***viscosity solution*** *of* (4.27) *if it is both a viscosity sub- and supersolution.*

The [270, theorem 5.3] states that:

Proposition 4.3 *Under our setting, if h is continuous on ∂D, if Y^x is the solution of BSDE* (4.26) *with terminal condition $h(X^x_{\tau_x})$, then $u(x) = Y^x_0$ is continuous on $\overline{D}$ and it is a viscosity solution of the elliptic PDE* (4.27) *with boundary data h.*

In particular we can apply this result for the truncated case.

For unbounded boundary condition, we cannot apply this result. Moreover the condition $v^* < +\infty$ in the definition of a viscosity solution cannot be satisfied on $\overline{D}$. Therefore we restrict ourselves to the case where f only depends on y (for example $f(y) = -y|y|^{q-1}$) and we change the definition of a solution.

Definition 4.5 (Unbounded viscosity solution) *We say that v is a viscosity solution of PDE* (4.27)

$$\left\{ \begin{array}{rl} -\mathcal{L}v - f(v) = 0 & \text{on } D; \\ v = h & \text{on } \partial D; \end{array} \right.$$

with unbounded terminal data h if v is a viscosity solution on D in the sense of the previous definition and if

$$h(x) \leq \lim_{\substack{x' \to x \\ x' \in D,\ x \in \partial D}} v_*(x') \leq \lim_{\substack{x' \to x \\ x' \in D,\ x \in \partial D}} v^*(x') \leq h(x).$$

Remark that this definition implies that $v^* < +\infty$ and $v_* > -\infty$ on D.

Our main result is the following:

Theorem 4.4 *Let us assume:*

- *Function $h : \mathbb{R}^d \to \overline{\mathbb{R}_+}$ is such that $\mathcal{S}_\infty = \{h = +\infty\} \cap \partial D$ is a closed set.*
- *On $\mathbb{R}^d \setminus \mathcal{S}_\infty$, h is locally bounded, that is, for all compact set $\mathcal{K} \subset \mathbb{R}^d \setminus \mathcal{S}_\infty$,*

$$h\mathbf{1}_{\mathcal{K}} \in \mathbb{L}^\infty(\mathbb{R}^d).$$

- *Boundary ∂D belongs to C^3.*
- *Generator f only depends on y and satisfies* **(C2)** *with a deterministic constant η.*

Minimal solution (Y^x, Z^x) of BSDE (4.26) *exists and is continuous:*

$$\lim_{t \to +\infty} Y^x_{t \wedge \tau} = \xi = h(X^x_{\tau_x}) \quad \mathbb{P} - a.s.$$

Moreover if we define

$$u(x) = Y^x_0,$$

u is the minimal non-negative viscosity solution of PDE (4.27) *with Dirichlet condition Φ.*

Here we do not suppose that a viscosity solution is continuous. But under some stronger assumptions on the operator $\mathcal{L}$, we also give some regularity properties of the solution u.

Proof. The existence of Y^x can be deduced from the results in Section 3.4. The continuity problem can be solved using a local version of the Keller-Osserman inequalilty (Proposition 3.7). □

Note that there are some differences between our work and the results of Le Gall or Dynkin and Kuznetsov. With the superprocesses (see [217, 107]), it should be assumed that $q \leq 2$. Moreover, the Dirichlet boundary condition for PDE (4.27) is not taken in the same sense in the two approaches. With the notion of the *boundary trace* (see [107], [217], [240] and [239]), there always exists a maximal positive solution; if $q < 2/(d-1)$, this solution is unique, and if $q \geq 2/(d-1)$, problem (4.27) may possess more than one positive solution. More precisely, assume that D is the unit ball in $\mathbb{R}^d$, that $q \geq 2/(d-1)$ and denote by μ_∞ the Borel measure on ∂D which assigns the value $+\infty$ to every nonempty set. Then for every $\varepsilon > 0$, there exists a positive solution of (4.27) such that $u(0) < \varepsilon$ and the trace of u is μ_∞ (see [239, Proposition 5.1]). In our case the Dirichlet condition in (4.27) is taken in the viscosity sense (Definition 4.5). The results are rather different: there exists a minimal positive viscosity solution. But we are unable to give conditions to ensure uniqueness of the solution.

Remark that the condition **(B7)** can be relaxed. For example we can work under the condition: b is continuous and satisfies the monotonicity condition: there exists $\mu \in \mathbb{R}$ such that

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d,\ \langle x-y | b(x) - b(y) \rangle \leq \mu |x-y|^2;$$

here $\langle . | . \rangle$ denotes the scalar product in $\mathbb{R}^d$. Under these assumptions, equation (4.22) has a unique strong solution X^x. For each $x \in \overline{D}$, we define the stopping time

$$\tau = \tau_x = \inf \left\{ t \geq 0,\ X_t^x \notin \overline{D} \right\}.$$

We also assume that the conditions (4.24) and (4.25) hold. However under these conditions, we are unable to control the explosion rate of Y, nor to prove that the viscosity solution u is continuous on D without the ellipticity condition. Indeed, we use this assumption in order to control the Green function $G(x, .)$ associated to the process X^x killed at τ_x. Under **(B7)**, this function $G(x, .)$ is continuous on D except at the point x, and is integrable on D. This assumption on G can replace **(B7)** (see, for example, [290] for more details on G).

4.4 Some open problems

Here we address two questions concerning PDE with singularity. In [XVI], we study the parabolic PDE (3.45):

$$\partial_t V + \frac{1}{2} \partial_{xx} V - V|V|^{q-1} = 0$$

for $(t,x) \in [0,T] \times [0,L]$ with the boundary conditions (3.46):

$$V(0,t) = V(L,t) = ((q-1)(T-t))^{1/(1-q)}, t \in [0,T], \quad V(x,T) = 0, 0 < x < L$$

or (3.49)

$$V(0,t) = V(L,t) = 0, t \in [0,T], \quad V(x,T) = \infty, 0 < x < L.$$

In Propositions 3.5 and 3.6, the existence of a smooth solution v is proved. The extension to more general domain or generators should be possible and is investigating in [E].

In [XVII] (see Section 3.2.2), existence of a minimal solution to 2BSDE (3.26) is obtained. It is known that this type of BSDEs is related to fully non-linear PDE; see among others [331, Chapter 11], [282, 283, 284] or [110, 111, 112]. It could be interesting to develop singularity for these fully non-linear PDEs.

Chapter 5

Related control problems

The basic problem of the calculus of variations consists in minimizing an integral functional over a set of functions satisfying an initial and terminal condition. Let us consider a version of the basic problem, where the Lagrangian is convex and subject to random influences supported by a Brownian motion W on a probability space $(\Omega, \mathcal{F}, P)$. More precisely, let $T \in (0, \infty)$ and $f : \Omega \times [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ be a function, convex in the last two variables, such that for all $(x, a) \in \mathbb{R}^2$, mapping $(\omega, t) \mapsto j(\omega, t, x, a)$ is progressively measurable with respect to $\mathbb{F}^W = (\mathcal{F}_t^W)_{t \in [0,\infty)}$, the augmented filtration generated by the Brownian motion W. The goal is to show the existence of a solution of the following problem:

$$\text{(5.1)} \qquad \begin{array}{l} \text{Minimize } J(X) = \mathbb{E}\left[\int_0^T j(t, X_t, \dot{X}_t) dt\right] \text{ over all absolutely continuous} \\ \text{and progr. mb. processes } X \text{ satisfying } X_0 = x_0 \in \mathbb{R} \text{ and } X_T = 0. \end{array}$$

We interpret t as time, X_t as the state and $\dot{X}_t$ as the velocity at time t.

Minimizing $J(X)$ is a classical problem with many applications e.g. in physics, economics and engineering. We refer to the scripts of Clarke [79], Evans [124] and Gelfand and Fomin [138] for explicit applications and an overview on the *deterministic* version of the basic problem. Stochastic examples of problem (5.1) have been recently analyzed in the context of closing financial asset positions in illiquid markets (see e.g. the introduction in [203] for an overview). In these applications j includes transaction costs, depending on the liquidation rate $\dot{X}$; moreover j can incorporate measures of the risk exposure, depending on the volume X_t of the remaining position.

In order to prove existence of a process X minimizing the functional $J(X)$ we study also a related control problem without the constraint $X_T = 0$, but with an additional term in the cost functional penalizing any deviation of X_T from zero. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function, convex in the second variable, such that for all $x \in \mathbb{R}$ the mapping $\omega \mapsto g(\omega, x)$ is $\mathcal{F}_T$-measurable. Under some nice analytic assumptions, the following unconstrained control problem is considered:

$$\text{(5.2)} \qquad \begin{array}{l} \text{Minimize } \widetilde{J}(X) = \mathbb{E}\left[\int_0^T j(t, X_t, \dot{X}_t) dt + g(X_T)\right] \text{ over all absolutely} \\ \text{continuous and progr. mb. processes } X \text{ satisfying } X_0 = x_0. \end{array}$$

In the following, we explain that by setting the penalty function equal to $g(x) = Lx^p$ and letting $L \to \infty$ one can reduce the constrained problem (5.1) to the unconstrained one (5.2).

By following a classic Bellman approach for solving (5.2) (at least if j and g are deterministic functions), one obtains a non-linear Hamilton-Jacobi-Bellman (HJB) equation that is difficult to solve. By the Pontryagin maximum principle an optimal solution of (5.2) can be characterized in terms of a forward-backward stochastic differential equation (FBSDE), where the forward component describes the optimal state dynamics and the backward component the dynamics of the so-called costate. The FBSDE for (5.2) takes the form

$$\begin{array}{rcl} X_t &=& x - \int_0^t j_y^*(s, X_s, Y_s) ds, \\ Y_t &=& g'(X_T) - \int_t^T Z_s dW_s + \int_t^T j_x(s, X_s, j_y^*(s, X_s, Y_s)) ds, \end{array} \tag{5.3}$$

where $j^*(t, x, \cdot)$ denotes the convex conjugate of the function $a \mapsto j(t, x, a)$, j_y^* its derivative w.r.t. y and j_x the derivative of j w.r.t. x. Notice that the FBSDE (5.3) is fully coupled, i.e. the forward dynamics depend on the backward component Y, and the backward dynamics on the forward part X. It is a longstanding challenge to find conditions guaranteeing that a fully coupled FBSDE possesses a solution. Sufficient conditions are provided e.g. in [233, 277, 236, 285, 89, 234] (and the references therein). The method of decoupling fields, developped in [128] (also see the precursor articles [235], [129] and [234]), is practically useful for determining whether a solution exists. A decoupling field describes the functional dependence of the backward part Y on the forward component X. If the coefficients of a fully coupled FBSDE satisfy a Lipschitz condition, then there exists a maximal non-vanishing interval possessing a solution triplet (X, Y, Z) and a decoupling field with nice regularity properties. The method of decoupling fields consists in analyzing the dynamics of the decoupling field's gradient in order to determine whether the FBSDE has a solution on the whole time interval $[0, T]$.

Solutions of problem (5.1) and (5.2) have been obtained under some additional structural assumptions on the function j. One focus of the literature so far is set on cost functions j that are *additive* and *homogeneous*. In [10] it is assumed that j takes the form $j(t, x, a) = \gamma_t |x|^p + \eta_t |a|^p$, where $p > 1$ and (η, γ) is a pair of non-negative progressively measurable processes. The particular form allows to decouple the FBSDE (5.3), after a variable change. As the penalty of any deviation of X_T from 0 increases to infinity, the backward part of the decoupled FBSDE converges to a solution of a BSDE with singular terminal condition. This additive–homogeneous case has been studied in the linear-quadratic case in [11] and [20]. It has been extended to

- Poisson random measure as an additional source of randomness in [148, 147] and in [XI];
- Random terminal time or general filtration or terminal condition in [XI].

These results are developed in the next subsection 5.1. But let us present already the model. For some $p > 1$, we want to minimize the functional cost

$$J(X) = \mathbb{E}\left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p + \int_{\mathcal{E}} \lambda_s(e) |\beta_s(e)|^p \mu(de)\right) ds + \xi |X_T|^p\right]$$

over all progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_0^s \alpha_r dr + \int_0^s \int_{\mathcal{E}} \beta_r(e)\pi(de, dr)$$

This model is motivated by the analysis of optimal control problems with state constraints on the terminal value and is a theoretical model for optimal portfolio liquidation under stochastic price impact. The traditional assumption that all trades can be settled without impact on market dynamics is not always appropriate when investors need to close large positions over short time periods. In recent years models of optimal portfolio liquidation have been widely developed, see, e.g. [5, 6, 126, 133, 162, 201], among many others. The book [151] is a nice survey on this topic. Variants of the position targeting problem (3.4) have been studied in [10, 11, 147, 148, 304].

Here the state process X denotes the agent's position in the financial market. She has two means to control her position. At each point in time t she can trade in the primary venue at a rate α_t which generates costs $\eta_t|\alpha_t|^p$ incurred by the stochastic price impact parameter η_t. The term $\gamma_t|X_t|^p$ can be understood as a measure of risk associated to the open position. Moreover, she can submit passive orders to a secondary venue ("dark pool"). These orders get executed at the jump times of the Poisson random measure π and generate so called slippage costs $\int_{\mathcal{Z}} \lambda_t(z)|\beta_t(z)|^p\mu(dz)$. We refer to [201] for a more detailed discussion. $J(X)$ thus represents the overall expected costs for closing an initial position x over the time period $[0, T]$ using strategy X.

This terminal constraint is described by the $\mathcal{F}_T$-measurable non-negative random variable ξ such that $\mathcal{S} = \{\xi = +\infty\}$. Thus for a binding liquidation $X_T = 0$, we take $\xi = +\infty$ a.s. For excepted scenarios, we can consider $\xi = \infty \mathbf{1}_{\mathcal{S}}$ with for example $\mathcal{S} = \{\max_{t\in[0,T]} \eta_t \leq H\}$ or $\mathcal{S} = \{\int_0^T \eta_t dt \leq H\}$ for a given threshold $H > 0$. This means that liquidation is only mandatory if the maximal price impact (or the average price impact) is small enough throughout the liquidation period. If the illiquidity of the market is too high, the trader has not obligatorily to close his position.

For more general convex costs (but in the Brownian setting), the FBSDE (5.3) can be shown to possess a solution by using the so-called continuation method, developed in [164, 285, 327]. In particular the problem (5.2) has been solved already using this method (see e.g. [59, Section 5]). The continuation method, however, does not provide the existence of a decoupling field, which is fundamental for passing to the limit as the penalty converges to infinity and for solving Problem (5.1). Indeed in [XVIII] we use decoupling fields since they provide an additional structure enabling to pass to the limit when the constant L of the penalty function $g(x) = Lx^2$ converges to infinity. Indeed, we show that the corresponding decoupling fields u^L are non-decreasing in L. We can thus identify a limit u^∞, which we further use for solving Problem (5.1). In addition, from the convergence of u^L we infer convergence of the corresponding solution processes (X^L, Y^L, Z^L) to a process triplet $(X^\infty, Y^\infty, Z^\infty)$. We show that $(X^\infty, Y^\infty, Z^\infty)$ can be characterized as the unique solution of a pair of stochastic differential equations, where an initial and terminal condition is imposed on the forward equation, but no condition on the second. One can interpret the system as an FBSDE with a free backward component. To the best of our knowledge, this type of FBSDE is new since it cannot be reduced to the case studied in [328]. Moreover this FBSDE characterizes an optimal control for problem (5.1) as (5.3) does for problem (5.2) (see Part 5.3)

Let us mention that the articles [252, 314] reformulate mass transportation problems as control problems imposing a constraint on the terminal state and hence bearing similarities with Problem (5.1). In contrast to the present article, the position process in both articles is assumed to be disturbed by some Brownian noise; in [252] with constant and in [314] with freely controllable diffusion coefficient. In addition, [252] links the unique optimal control to an FBSDE related to the FBSDE (5.3). In contrast to our approach, the authors derive the FBSDE from the existence of an optimal control, but do not use it for proving existence in the first place.

Coming back to the additive-homogeneous case, we add some uncertainty on the model. For some $p > 1$, we want to minimize the functional cost

$$J(X, \mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left[\int_0^T (\eta_s |\alpha_s|^p + \gamma_s |X_s|^p)\, ds + \xi |X_T|^p \right]$$

over all progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_0^s \alpha_u du.$$

When there is only one probability measure $\mathbb{P}$, the optimal strategies and the value function of this control problem are characterized in [10] and [XI] by the minimal supersolution $(Y^{\min}, M^{\min})$ of the BSDE (3.5) with singular terminal condition:

$$dY_t = \frac{Y_t^q}{(q-1)\eta_t^{q-1}} dt - \gamma_t dt + dM_t$$

with $\liminf_{t \to T} Y_t \geq \xi$. Here $q > 1$ is the Hölder conjugate of p and $M^{\min}$ is a martingale. By the verification theorem based on a penalization argument, it is proved that $\inf_X J(X, \mathbb{P}) = Y_0^{\min}$.

When $\mathbb{P}$ is not unique, we need to solve

$$\widehat{J}(X) = \sup_{\mathbb{P} \in \mathcal{P}} J(X, \mathbb{P}) = \sup_{\mathbb{P} \in \mathcal{P}} Y_0^{\mathbb{P}} \tag{5.4}$$

where $Y^{\mathbb{P}}$ is the minimal supersolution of (3.5) under the probability measure $\mathbb{P}$. Minimizing (5.4) corresponds for an agent to compute the worst case scenario for the liquidation of her portfolio. From the theory of second order BSDE (2BSDE in short) introduced by [309, 310] (see Sections 2.4 and 3.2.2), our problem can be solved with this useful tool. From our results on 2BSDEs (Section 2.4), we can now obtain directly the value function and an optimal control for the unconstrained problem. For the constrained problem, a known difficulty concerns the filtration. Indeed to avoid the possibility of an uncontrolled jump for the orthogonal martingale part at the terminal time T, some additional hypothesis on the filtration is needed (see Theorem 3.1). Under this technical condition on the filtration, we prove that the 2BSDE with singular terminal condition has a minimal super-solution and that we can solve (5.4) using this super-solution.

Mean field games (MFGs) are a powerful tool to analyse strategic interactions in large populations when each individual player has only a small impact on the behavior of other players. In the economics literature, mean-field-type (or anonymous) games were first considered by Jovanovic and Rosenthal [179] and later analyzed by many authors including [39, 88, 161]. In the mathematical literature MFGs were independently introduced by Huang, Malhamé and Caines [167] and Lasry and Lions [214]. MFGs have been successfully applied to various economic problems, ranging from systemic risk management [62] to principal agent problems [122, 261] and from portfolio optimization [211] to optimal exploitation of exhaustible resources [66].

In the last section of this part, corresponding to [XX], we analyze a novel class of MFGs that arise in models of optimal portfolio liquidation. For single-player portfolio liquidation models, the controlled state sequence follows a dynamics of the form

$$X_t = x - \int_0^t \alpha_s \, ds,$$

where $x > 0$ is the initial portfolio that a trader needs to unwind, and α is the trading rate. The set of admissible controls is confined to those processes α that satisfy almost surely the liquidation constraint $X_T = 0$. It is typically assumed that the unaffected benchmark price process follows a one-dimensional Brownian motion W (or some Brownian martingale) and that the trader's transaction price is given by

$$S_t = s_0 + \int_0^t \sigma_s \, dW_s - \int_0^t \kappa_s \alpha_s \, ds - \eta_t \alpha_t$$

where σ is a (sufficiently regular) stochastic volatility process. The integral term accounts for permanent price impact, i.e. the impact of past trades on current prices, while the term $\eta_t \alpha_t$ accounts for the instantaneous impact that does not affect future transactions. The expected cost functional is typically of the linear-quadratic form

$$\mathbb{E}\left[\int_0^T \Big(\kappa_s \alpha_s X_s + \eta_s \alpha_s^2 + \gamma_s X_s^2\Big) \, ds\right]$$

where κ, η and γ are one-dimensional bounded adapted and non-negative processes. The process γ describes the trader's degree of risk aversion or her belief about the volatility process; it penalizes slow liquidation. The process η describes the degree of market illiquidity; it penalizes fast liquidation. The process κ describes the impact of past trades on current transaction prices. This problem is studied in the next section 5.1. If the transactions are not directly observable, then it is natural to assume that the permanent impact is driven by the markets expectation about the traders transactions as in [30], given the publicly observable information. It leads to a mean field control problem, which is not the focus of our paper.

We consider a MFG of optimal portfolio liquidation among asymmetrically informed players. Each player observes the realization of her initial portfolio and knows the distribution of all the other initial portfolios. The transaction price for each player $i = 1, ..., N$ is given by

$$S_t^i = s_0^i + \int_0^t \sigma_s^i \, dW_s^0 - \int_0^t \frac{\kappa_s^i}{N} \sum_{j=1}^N \alpha_s^j \, ds - \eta_t^i \alpha_t^i.$$

In particular, the permanent price impact depends on the players' average trading rate. Given her initial portfolio $\mathcal{X}^i = x^i$ the optimization problem of player $i = 1, ..., N$ is to minimize the cost functional

$$J^{N,i}(\vec{\alpha}) = \mathbb{E}\left[\int_0^T \left(\frac{\kappa_t^i}{N} \sum_{j=1}^N \alpha_t^j X_t^i + \eta_t^i (\alpha_t^i)^2 + \gamma_t^i (X_t^i)^2\right) dt \big| \mathcal{X}^i = x^i\right]$$

subject to the state dynamics

$$\begin{aligned} dX_t^i &= -\alpha_t^i \, dt, \\ X_0^i &= \mathcal{X}^i, \quad X_T^i = 0. \end{aligned}$$

Our game is different from the majority of the MFG literature in at least three respects. First, as in [63, 146] the players interact through the impact of their strategies rather than states on the other players' payoff functions. Second, all players observe the common Brownian motion W^0 that drives the benchmark price process. Hence, ours is a MFG with common noise. While MFGs with common noise have been investigated before (see, e.g. [61]) the nature of both the common and the idiosyncratic noise in our model is very different from the existing literature. Third, the individual state dynamics are subject to a terminal state constraint arising from the liquidation requirement. MFGs with terminal state constraint have been considered before in the literature by means of so-called mean field (game) planning problems (MFGP) introduced by Lions in his lectures at Collège de France (2009–2010). In these problems the terminal state constraint is given by a target density of the state at the terminal time. While our problem formally belongs to the literature on MFGP, see e.g. [1, 145, 293] and the references therein, ours seems to be the first paper that considers a MFG with strict terminal state constraint.

We apply the probabilistic method to solve the MFG with terminal constraint (5.35). In a first step we show how the analysis of our MFG can be reduced to the analysis of a conditional mean-field type FBSDE. The forward component describes the optimal portfolio process; hence both its initial and terminal condition are known. The backward component describes the optimal trading rate; its terminal value is unknown. Making an affine ansatz, we show that the mean-field type FBSDE with unkown terminal condition can be replaced by a coupled FBSDE with known initial and terminal condition, yet singular driver. Proving the existence of a small time solution to this FBSDE by a fixed point argument is not hard. The challenge is to prove the existence of a global solution on the whole time interval. Under a weak interaction condition that has been used in the game theory literature before (see, e.g. [161]) we prove the existence and uniqueness of a global solution by a generalization of the method of continuation established in [165, 286] to linear-quadratic FBSDE systems with singular driver. Under the additional assumption that all players share the same cost structure, we prove that each player's best response to the mean-field equilibrium μ^* is of the form $\xi^{*,i} = \phi(\mathcal{X}^i, W^0, W^i)$ for some function ϕ and that the resulting homogeneous action profile forms an ϵ-equilibrium in the original N-player game.

The common information case where all the cost coefficients are measurable with respect to the common factor can be analysed in greater detail. When different players hold different initial portfolios, then the optimal portfolio processes are given as weighted averages of the players' initial portfolios and the differences of their own and the average initial portfolio. In this case, we show that if the average initial portfolio is positive and a player holds an above average initial portfolio, then her optimal portfolio process is always positive. If, however, a player holds a positive yet well below average initial portfolio, then it is optimal to quickly unwind the position, to then take a negative position and to buy the stock back by the end of the trading period. This is intuitive as players with negative portfolios benefit from the negative price trends generated by other players while the cost of unwinding a small portfolio is low. As such, our result suggests that traders with small portfolios act as liquidity providers in equilibrium even if their initial holds are positive.

The benchmark case of deterministic coefficients can be solved in closed form. For this case we show that when the strength of interaction κ in (5.35) is large and all players share the same initial portfolio, the players initially trade very fast in equilibrium to avoid the negative drift generated by the mean field interaction. Our model thus provides a possible explanation for large price drops in markets with many strategically interacting homogenous investors. We also show that the deterministic case is equivalent to a single player model with suitably adjusted cost terms.

The three papers closest to our model are Cardaliaguet and Lehalle [56], Carmona and Lacker [63], Huang, Jaimungal and Nourin [168]. In [63], the authors propose a specific portfolio liquidation model where each players portfolio is subject to exogenous fluctuations (customer flow) described by independent Brownian motions. As such, their model is much closer to a standard MFG than ours, but no liquidation constraint is possible in their framework. The papers [56] and [168] consider mean field models parameterized by different preferences and with major-minor players, respectively. Again, no liquidation constraint is allowed. The model introduced in [56] is extended to portfolios of correlated assets in [218] where the effect of trading flows on naive estimates of intraday volatility and correlations is analyzed.

5.1 Additive-homogeneous control problem ([XI])

The results of this section were already obtained in [10] in a more restricted setting. We consider the additive-homogeneous problem (5.1), but in a general filtration. Let us now describe exactly the stochastic control problem.

We assume that the framework of Sections 1.2 and 3.1 is given. Moreover, we suppose that **the measure μ is finite**. As in Section 3.1 we fix some $p > 1$ and denote by $q = 1/(1 - 1/p)$ its Hölder conjugate. Let τ be a $\mathbb{F}$ stopping time. For any $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, we denote by $\mathcal{A}(t, x)$ the set of progressively measurable processes $(X_s)_{s \geq 0}$ that satisfy the dynamics

$$X_s = x + \int_t^{s\vee t} \alpha_r dr + \int_t^{s\vee t} \int_{\mathcal{E}} \beta_r(e)\pi(de,dr) \tag{5.5}$$

for any $s \geq 0$ and for some $\alpha \in \mathbb{L}^1(t,\infty)$ a.s. and $\beta \in G_{loc}(\pi)$. Observe that for all $X \in \mathcal{A}(t,x)$ it holds that $X_s = x$ for all $s \leq t$. We consider the stochastic control problem to minimize the functional[1]

$$J(t,X) = \mathbb{E}\left[\int_{t\wedge\tau}^{\tau} \left(\eta_s|\alpha_s|^p + \gamma_s|X_s|^p + \int_{\mathcal{E}} \lambda_s(e)|\beta_s(e)|^p\mu(de)\right) ds + \xi|X_\tau|^p \middle| \mathcal{F}_t\right] \tag{5.6}$$

over all $X \in \mathcal{A}(t,x)$. The random variable ξ is supposed to be non-negative and may take the value ∞ with positive probability. Observe that if for $x > 0$ there exists $X \in \mathcal{A}(t,x)$ such that $J(t,X) < \infty$, then $\tau > t$ a.s. and X satisfies almost surely that

$$X_\tau \mathbb{1}_{\xi=\infty} = 0. \tag{5.7}$$

This way we impose implicitly a terminal state constraint on the set of admissible controls. As before we define the set $\mathcal{S}_\infty$ by $\mathcal{S}_\infty = \{\xi = +\infty\}$. Coefficient processes $(\eta_t)_{t\geq 0}$, $(\gamma_t)_{t\geq 0}$ and $(\lambda_t)_{t\geq 0}$ are non-negative progressively measurable càdlàg processes. The process λ is $\widetilde{\mathcal{P}}$-measurable with values in $[0,+\infty]$.

We introduce the random field v that represents for each initial condition (t,x) the minimal value of J

$$v(t,x) = \operatorname{essinf}_{X\in\mathcal{A}(t,x)} J(t,X). \tag{5.8}$$

A formal stochastic maximum principle for (5.6) leads to a FBSDE similar to (5.3). However the homogeneity of the control problem implies that the forward and the backward equations can be decoupled and that the value function v is of the form

$$v(t,x) = |x|^p Y_t,$$

where Y is the solution of a BSDE with singular terminal condition ξ of the form:

$$dY_t = (p-1)\frac{Y_t^q}{\eta_t^{q-1}}dt + \Theta(t,Y_t,U_t)dt - \gamma_t dt + \int_{\mathcal{E}} U_t(e)\widetilde{\pi}(de,dt) + dM_t, \tag{5.9}$$

where the function Θ is given by

$$\Theta(t,y,\psi) = \int_{\mathcal{E}} (y+\psi(e))\left(1 - \frac{\lambda_t(e)}{((y+\psi(e))^{q-1} + \lambda_t(e)^{q-1})^{p-1}}\right) \mathbb{1}_{y+\psi(e)\geq 0}\, \mu(de). \tag{5.10}$$

See the discussion in [10, Section 2] and in [148, Section 2.2], when $\tau = T$ is deterministic.

The scheme is the following. First we show that the BSDE (5.9) has a minimal solution using the results in Section 3.1. Then a verification argument proves that this minimal solution gives the value function and an optimal control. We distinguish two

[1] We use the convention that $0 \cdot \infty := 0$

cases. In the first case we assume that τ is deterministic and impose some integrability assumptions on the coefficient processes $(\eta_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$.

Deterministic case. Stopping time τ is a.s. equal to a deterministic constant $T > 0$. Process η is positive, process γ is non-negative, such that for some $\ell > 1$

$$\mathbb{E}\left[\int_0^T (\eta_t + (T-t)^p \gamma_t)^\ell dt\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^T \frac{1}{\eta_t^{q-1}} dt\right] < \infty.$$

$\diamond$

In the second case we assume that $\tau = \tau_D$ is given by (3.54) as the first hitting time of a diffusion. We need to impose some stronger boundedness conditions on η and γ compared to the deterministic case.

Random case. We have $\tau = \tau_D$ and there exists $\rho > \mu(\mathcal{E})$ such that $\mathbb{E}e^{\rho\tau} < \infty$. Processes η and γ are bounded from above, η is positive and satisfies the integrability conditions

$$\mathbb{E}\left[\int_0^n \frac{1}{\eta_t^{q-1}} dt\right] + \mathbb{E}\left[\int_0^\tau \frac{1}{\eta_t^{m(q-1)}} dt\right] < \infty \tag{5.11}$$

for all $n \in \mathbb{N}$ and for some m satisfying:

$$m > \frac{2\rho}{\rho - \mu(\mathcal{E}) + (\sqrt{\rho} - \sqrt{2\mu(\mathcal{E})})\mathbf{1}_{\rho > 2\mu(\mathcal{E})}}.$$

Process γ is non-negative. $\diamond$

Lemma 3.8 gives sufficient conditions on the coefficients of the forward SDE (1.9) such that $\mathbb{E}e^{\rho\tau} < \infty$ holds. From Theorems 3.1 and 3.10, there exists a minimal supersolution $(Y^{\min}, U^{\min}, M^{\min})$ to (5.9) with singular terminal condition $Y_\tau = \xi$. Set $Y_s = \xi$ for all $s \geq \tau$.

Theorem 5.1 *For all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ it holds $\mathbb{P}$-a.s. that $v(t,x) = Y_t^{\min}|x|^p$. Moreover, for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, process X satisfying the linear dynamics*

$$X_s = x - \int_t^{s\vee t} \left(\frac{Y_r^{\min}}{\eta_r}\right)^{q-1} X_r dr - \int_t^{s\vee t} X_{r-} \int_{\mathcal{E}} \zeta_r(e)\pi(de,dr),$$

with

$$\zeta_r(z) = \frac{(Y_{r-}^{\min} + U_r^{\min}(e))^{q-1}}{[(Y_{r-}^{\min} + U_r^{\min}(e))^{q-1} + \lambda_r(e)^{q-1}]}$$

belongs to $\mathcal{A}(t,x)$, satisfies the terminal state constraint (5.7) *if $t < \tau$ and is optimal in* (5.8).

The optimal process X^* is given explicitly by

$$X_s^* = x \exp\left[-\int_t^{s\vee t} \left(\frac{Y_r^{\min}}{\eta_r}\right)^{q-1} dr\right] \exp\left[\int_t^{s\vee t} \int_{\mathcal{E}} \ln\left(1 - \zeta_r(e)\right)\pi(de,dr)\right]. \tag{5.12}$$

Proof. To prove Theorem 5.1, we consider a variant of the minimization problem (5.8), where we penalize any non-zero terminal state by $(\xi \wedge L)|X_\tau|^p$ and thus omit the constraint $X_\tau \mathbb{1}_{\mathcal{S}_\infty} = 0$ on the set of admissible controls. Moreover precisely by Theorems 2.1 and 2.2, there exists a unique solution (Y^L, U^L, M^L) of the truncated BSDE

$$dY_t^L = (p-1)\frac{(Y_t^L)^q}{\eta_t^{q-1}}dt + \Theta(t, Y_t^L, U_t^L)dt - (\gamma_t \wedge L)dt + \int_{\mathcal{E}} U_t^L(e)\widetilde{\pi}(de, dt) + dM_t^L \tag{5.13}$$

with terminal condition $Y_\tau^L = \xi \wedge L$. Process $(Y^{\min}, U^{\min}, M^{\min})$ is the limit as L goes to $+\infty$ of (Y^L, U^L, M^L) and is the minimal (super-)solution of the BSDE (5.9). We show that optimal controls for this unconstrained minimization problem: for $L > 0$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$

$$\begin{aligned} v^L(t,x) &= \operatorname{essinf}_{X \in \mathcal{A}(t,x)} J^L(t,X) \\ &= \operatorname{essinf}_{X \in \mathcal{A}(t,x)} \mathbb{E}\left[\int_{t\wedge\tau}^{\tau} \left(\eta_s|\alpha_s|^p + (\gamma_s \wedge L)|X_s|^p + \int_{\mathcal{E}} \lambda_s(e)|\beta_s(e)|^p \mu(de)\right) ds \right. \\ &\qquad\qquad \left. + (\xi \wedge L)|X_\tau|^p \middle| \mathcal{F}_t \right]. \end{aligned} \tag{5.14}$$

admit a representation in terms of the solutions Y^L of a truncated version of (5.9):

Proposition 5.1 *Let (Y^L, U^L, M^L) be the solution to (5.13) with terminal condition $Y_\tau = \xi \wedge L$. Let $Y_s^L = L \wedge \xi$ for all $s \geq \tau$. Then for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ the process X^L satisfying the linear dynamics*

$$X_s^L = x - \int_t^{s\vee t} \left(\frac{Y_r^L}{\eta_r}\right)^{q-1} X_r^L dr - \int_t^{s\vee t} X_{r-}^L \int_{\mathcal{E}} \zeta_r^L(e)\pi(de, dr),$$

with

$$\zeta_r^L(e) = \frac{(Y_{r^-}^L + \psi_r(e))^{q-1}}{\left[(Y_{r^-}^L + \psi_r^L(e))^{q-1} + \lambda_r(e)^{q-1}\right]}$$

is optimal in (5.14). Moreover, we have $v^L(t,x) = Y_t^L |x|^p$.

To prove this proposition we make use of the two following auxiliary results ([XI, Lemmata 3 and 4]). Firstly we show that in the case $x \geq 0$ we can without loss of generality restrict attention to monotone strategies[2]. We introduce set $\mathcal{D}(t,x)$, the subset of $\mathcal{A}(t,x)$ containing only processes X that have non-increasing sample paths (i.e. $\alpha_t \leq 0$ and $\beta_t(z) \leq 0$), and that remain non-negative. We prove that for $x \geq 0$, every control $X \in \mathcal{A}(t,x)$ can be modified to a control $\underline{X} \in \mathcal{D}(t,x)$ such that $J^L(t,X) \geq J^L(t,\underline{X})$ (see Lemma 5.2 below). In particular, $v^L(t,x) = \operatorname{essinf}_{X\in\mathcal{D}(t,x)} J^L(t,X)$. Secondly we provide the dynamics of two auxiliary processes: $\eta|\alpha^L|^{p-1}$ and $Y^L(X^L)^p$.

We then use this result to derive an optimal control for (5.8), by a verification argument. We use the auxiliary processes to get a non-negative local martingale, thus a non-negative supermartingale θ, such that

$$0 \leq X_s^* \leq \left(\frac{\theta_s}{pY_s^{\min}}\right)^{q-1}.$$

Since $Y^{\min}$ satisfies the terminal condition $\liminf_{s\nearrow\tau} Y_s^{\min} \mathbb{1}_{\mathcal{S}_\infty} = \infty$ we have a.s. on the set $\{t < \tau\} \cap \mathcal{S}_\infty$:

$$0 \leq X_s^* \leq \left(\frac{\theta_s}{pY_s^{\min}}\right)^{q-1} \to 0$$

when s goes τ . It follows that X satisfies the terminal constraint if $t < \tau$.

[2]It is straightforward to show that $v(t,x) = v(t,-x)$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$. Therefore, we restrict attention to the case $x \geq 0$ in the sequel.

Appealing once more our result on auxiliary processes, we observe that for all $n > 0$ (τ_n is given by (3.56)):

$$\begin{aligned} Y_t^{\min}|x|^p \quad \geq \quad & \mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^{\tau_n\vee t}\left\{\eta_u|\alpha_u^*|^p+\gamma_u(X_u^*)^p+\int_{\mathcal{E}}\lambda_u(e)|\beta_u^*(e)|^p\mu(de)\right\}du+\mathbb{1}_{\{\xi<\infty\}}Y_{\tau_n\vee t}^{\min}|X_{\tau_n\vee t}^*|^p\right] \\ & +\mathbb{1}_{\{t\geq\tau\}}J(t,X^*) \end{aligned}$$

Appealing to monotone convergence theorem yields

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^{\tau_n\vee t}\left\{\eta_u|\alpha_u^*|^p+\gamma_u(X_u^*)^p+\int_{\mathcal{E}}\lambda_u(e)|\beta_u^*(e)|^p\mu(de)\right\}du\right] \\ &\qquad =\mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathcal{F}_t}\left[\int_t^{\tau}\left\{\eta_u|\alpha_u^*|^p+\gamma_u(X_u^*)^p+\int_{\mathcal{E}}\lambda_u(e)|\beta_u^*(e)|^p\mu(de)\right\}du\right] \end{aligned}$$

We want to show that $\liminf_{n\to\infty}\mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathcal{F}_t}\left[Y_{\tau_n\vee t}^{\min}|X_{\tau_n\vee t}|^p\right]\geq\mathbb{1}_{\{t<\tau\}}\mathbb{E}^{\mathcal{F}_t}\left[\xi|X_\tau|^p\right]$, where $\infty\cdot 0:=0$. By Fatou's lemma it suffices to show that $\liminf_{n\to\infty}Y_{\tau_n}^{\min}|X_{\tau_n}|^p\geq\xi|X_\tau|^p$ a.s. From the definition of the supermartingale θ and since the limit $\lim_{n\to\infty}\theta_{\tau_n}\in\mathbb{R}$ exists, it follows that also the limit $\lim_{n\to\infty}Y_{\tau_n}^{\min}|X_{\tau_n}|^{p-1}\in\mathbb{R}$ exists and that $X_\tau=\lim_{n\to\infty}|X_{\tau_n}|=0$ if $\liminf_{n\to\infty}Y_{\tau_n}^{\min}=\infty$. Let us distinguish two cases. First assume that $\liminf_{n\to\infty}Y_{\tau_n}^{\min}=\infty$. Then

$$\liminf_{n\to\infty}Y_{\tau_n}^{\min}|X_{\tau_n}|^p=(\lim_{n\to\infty}Y_{\tau_n}^{\min}|X_{\tau_n}|^{p-1})(\lim_{n\to\infty}|X_{\tau_n}|)=0=\xi|X_\tau|^p$$

(for the last equality we use that $\infty\cdot 0:=0$). Next assume that $\liminf_{n\to\infty}Y_{\tau_n}^{\min}<\infty$. Then it follows that $\liminf_{n\to\infty}Y_{\tau_n}^{\min}|X_{\tau_n}|^p\geq\xi|X_\tau|^p$. This proves the claim and altogether we obtain that $Y_t^{\min}|x|^p\geq J(t,X^*)$.

Next, note that for every $X\in\mathcal{A}(t,x)$ we have $J(t,X)\geq J^L(t,X)$. This implies $v(t,x)\geq v^L(t,x)$ for every $L>0$. By the previous proposition: $Y_t^L|x|^p=v^L(t,x)$. Minimality of $Y^{\min}$ implies

$$Y_t^{\min}|x|^p=\lim_{L\nearrow\infty}Y_t^L|x|^p=\lim_{L\nearrow\infty}v^L(t,x)\leq v(t,x).$$

Consequently we obtain

$$Y_t^{\min}|x|^p\geq J(t,X^*)\geq v(t,x)\geq Y_t^{\min}|x|^p$$

and thus optimality of X^*. □

Note that for deterministic final time T we only know (in general) that

$$\liminf_{t\to T}v(t,x)\geq|x|^p\xi.$$

However since μ is finite, from Theorem 3.5, we obtain the existence of the limit at time T. The value function v is càdlàg on $[0,T]$. But there may be is an extra cost at time T. The *continuity problem* corresponds to the question: is the left limit of the value function v at time T equal to the penalization cost ξ ? A positive answer is assumed in [20]. The section 3.3 provides several examples of penalty ξ such that we have continuity at time T.

5.2 Extension to Knightian uncertainty ([XVII])

Here we want to minimize the functional cost

$$J(\mathcal{X})=\sup_{\mathbb{P}\in\mathcal{P}}J(\mathcal{X},\mathbb{P})=\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\left[\int_0^T(\eta_s|\alpha_s|^p+\gamma_s|\mathcal{X}_s|^p)\,ds+\xi|\mathcal{X}_T|^p\right]\tag{5.15}$$

over all progressively measurable processes $\mathcal{X}$ that satisfy the dynamics

$$\mathcal{X}_s = x + \int_0^s \alpha_u du.$$

State process is denoted by $\mathcal{X}$ whereas X is the canonical process (see Part 1.4). Minimizing (5.15) corresponds for an agent to compute the worst case scenario for the liquidation of her portfolio. For a fixed $\mathbb{P}$, we know that the infimum of $J(\mathcal{X}, \mathbb{P})$ is given by the solution $Y^{\mathbb{P}}$ of a BSDE. Then up to the inversion of a supremum (over $\mathbb{P}$) and an infimum (over α), the solution of (5.15) should be given by $\sup_{\mathbb{P}} Y^{\mathbb{P}}$, that is by the solution of a 2BSDE.

The general setting is the same as in Section 3.2.2. We consider a $\mathcal{F}_T$-Borel measurable random variable ξ such that for any $\mathbb{P} \in \mathcal{P}_0$, ξ is a.s. non-negative. We denote by $\mathcal{S}_\infty$ the singular set $\{\xi = +\infty\}$. We define the two Borel measurable functions

$$\begin{aligned} \eta : \quad & (t, \omega, a) \in [0,T] \times \Omega \times \mathbb{S}_d^{\geq 0} \longrightarrow \mathbb{R}_+^*, \\ \gamma : \quad & (t, \omega, a) \in [0,T] \times \Omega \times \mathbb{S}_d^{\geq 0} \longrightarrow \mathbb{R}_+. \end{aligned}$$

Here η and γ (and thus the generator of our BSDE) do not depend on the drift of X. This condition is sufficient to obtain an optimal control independent of the probability measure $\mathbb{P}$. This hypothesis is similar to the setting in [245]. We define for simplicity

$$\widehat{\eta}_s := \eta(s, X_{\cdot \wedge s}, \widehat{a}_s) \quad \text{and} \quad \widehat{\gamma}_s := \gamma(s, X_{\cdot \wedge s}, \widehat{a}_s).$$

Finally we assume that there exists $\varrho > 1$ such that for any $(t, \omega) \in [0,T] \times \Omega$

$$\sup_{\mathbb{P} \in \mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}} \int_t^T \left(\frac{1}{\widehat{\eta}_s} \right)^{\varrho(q-1)} ds < \infty. \tag{5.16}$$

To simplify we only consider the case $x \geq 0$ and we define the following control sets:

- $\mathcal{A}(t,x)$ is the set of processes $\mathcal{X} = (\mathcal{X}_s,\ 0 \leq s \leq T)$ such that $\mathcal{X}_s = x$ if $s \leq t$ and for any $\mathbb{P} \in \mathcal{P}_t$, $\mathbb{P}-a.s.$, $\mathcal{X}$ is absolutely continuous, that is: $\mathcal{X}_s(\omega) = x + \int_t^s \alpha_u(\omega) du$ with $\int_t^T |\alpha_u(\omega)| du < +\infty$.
- For a fixed $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{A}^{\mathbb{P}}(t,x)$ is the set of processes $\mathcal{X} = (\mathcal{X}_s,\ 0 \leq s \leq T)$ such that $\mathcal{X}_s = x$ if $s \leq t$ and $\mathbb{P} - a.s.$, $\mathcal{X}$ is absolutely continuous, that is: $\mathcal{X}_s(\omega) = x + \int_t^s \alpha_u(\omega) du$ with $\int_t^T |\alpha_u(\omega)| du < +\infty$.

Set $\mathcal{A}^{\mathbb{P}}(t,x)$ depends of $\mathbb{P}$, whereas $\mathcal{A}(t,x)$ depends only on the probability set $\mathcal{P}_t$. Of course $\mathcal{A}(t,x)$ is included in $\mathcal{A}^{\mathbb{P}}(t,x)$. Next for any $L \geq 0$ we define the following unconstrainted control problems

(5.17)

$$J^L(t,x) = \underset{\mathcal{X} \in \mathcal{A}(t,x)}{\operatorname{essinf}} \ \underset{\mathbb{P} \in \mathcal{P}_t}{\operatorname{ess\,sup}} \ \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left(\widehat{\eta}_s |\alpha_s|^p + (\widehat{\gamma}_s \wedge L) |\mathcal{X}_s|^p \right) ds + (L \wedge \xi) |\mathcal{X}_T|^p \middle| \mathcal{F}_t^+ \right],$$

together with

$$I^L(t,x) = \underset{\mathbb{P} \in \mathcal{P}_t}{\operatorname{ess\,sup}} \ \underset{\mathcal{X} \in \mathcal{A}(t,x)}{\operatorname{essinf}} \ \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left(\widehat{\eta}_s |\alpha_s|^p + (\widehat{\gamma}_s \wedge L) |\mathcal{X}_s|^p \right) ds + (L \wedge \xi) |\mathcal{X}_T|^p \middle| \mathcal{F}_t^+ \right],$$

and

$$H^L(t,x) = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} \operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}^{\mathbb{P}}(t,x)} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\alpha_s|^p + (\widehat{\gamma}_s \wedge L)|\mathcal{X}_s|^p)\, ds + (L\wedge\xi)|\mathcal{X}_T|^p \bigg| \mathcal{F}_t^+\right].$$

Immediately $H^L(t,x) \leq I^L(t,x) \leq J^L(t,x)$. From the standard formulation (see Section 5.1) we have

$$H^L(t,x) = x^p \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} y_t^{L,\mathbb{P}} = x^p\, Y_t^L.$$

Indeed from Propositions 2.4 and 2.5, we deduce that there exists a unique solution $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ to the second order BSDE: for any $0 \leq t \leq T$ and any $\mathbb{P}$

$$\begin{aligned} Y_t^L &= (\xi\wedge L) - \int_t^T \frac{|Y_u^L|^{q-1}Y_u^L}{(q-1)(\widehat{\eta}_u)^{q-1}}du + \int_t^T (\widehat{\gamma}_u \wedge L)du \\ &\quad - \left(\int_t^T Z_s^L dX_s^{c,\mathbb{P}}\right)^{\mathbb{P}} - \int_t^T dM_s^{L,\mathbb{P}} + (K_T^{L,\mathbb{P}} - K_t^{L,\mathbb{P}}), \quad \mathbb{P}-a.s., \end{aligned} \tag{5.18}$$

such that:

- For any $p > 1$, Y^L belongs to $\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$.
- For any $1 < p < \varrho$, $(Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ is in $\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$.
- $K^{L,\mathbb{P}}$ is a $\mathbb{P}-a.s.$ non-decreasing process satisfying the minimality condition (2.19).

Lemma 5.1 *For any (t,x), $J^L(t,x) \leq H^L(t,x)$.*

Proof. We define

$$\beta_s^L = -\left(Y_s^L/\widehat{\eta}_s\right)^{q-1}, \qquad d\mathcal{X}_s^{*,L} = \beta_s^L \mathcal{X}_s^{*,L} ds = \alpha_s^L ds.$$

Let us apply the Itô formula under the probability $\mathbb{P}$ to $Y^L(\mathcal{X}^{*,L})^p$, we integrate the result from t to T and we take the conditional expectation w.r.t. $\mathbb{P}$:

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}\left[(\xi\wedge L)(\mathcal{X}_T^{*,L})^p + \int_t^T \left[\widehat{\eta}_s\left(\alpha_s^L\right)^p + (\widehat{\gamma}_s\wedge L)(\mathcal{X}_s^{*,L})^p\right] ds \bigg| \mathcal{F}_t^+\right] \\ &\quad = Y_t^L x^p - \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\mathcal{X}_s^{*,L})^p dK_s^{L,\mathbb{P}} \bigg| \mathcal{F}_t^+\right] \leq Y_t^L x^p \end{aligned}$$

since $K^{L,\mathbb{P}}$ is non-decreasing. Therefore

$$\operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[(\xi\wedge L)(\mathcal{X}_T^{*,L})^p + \int_t^T \left[\widehat{\eta}_s\left(\alpha_s^L\right)^p + (\widehat{\gamma}_s\wedge L)(\mathcal{X}_s^{*,L})^p\right] ds \bigg| \mathcal{F}_t^+\right] \leq Y_t^L x^p.$$

Moreover the process $\mathcal{X}^{*,L}$ is in $\mathcal{A}(t,x)$:

$$\mathcal{X}_s^{*,L} = x - \int_t^s \left(\frac{Y_u^L}{\widehat{\eta}_u}\right)^{q-1} \mathcal{X}_u^{*,L} du.$$

This implies that

$$J^L(t,x) \leq Y_t^L x^p = H^L(t,x).$$

$\square$

Therefore we deduce that $H^L(t,x) \leq I^L(t,x) \leq J^L(t,x) \leq H^L(t,x)$ and the first result:

Proposition 5.2 *The unconstrainted problem* (5.17) *satisfies*

$$\begin{aligned}
&\operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}(t,x)} \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\alpha_s|^p + \widehat{\gamma}_s|\mathcal{X}_s|^p)\,ds + (\xi\wedge L)|\mathcal{X}_T|^p \middle| \mathcal{F}_t^+\right] \\
&\quad = \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} \operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}^{\mathbb{P}}(t,x)} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\alpha_s|^p + \widehat{\gamma}_s|\mathcal{X}_s|^p)\,ds + (\xi\wedge L)|\mathcal{X}_T|^p \middle| \mathcal{F}_t^+\right]
\end{aligned}$$

and the solution of the 2BSDE (5.18)*, denoted by Y^L, gives the optimal process $\mathcal{X}^{*,L}$:*

$$d\mathcal{X}_s^{*,L} = \left[-\left(Y_s^L/\widehat{\eta}_s\right)^{q-1}\mathcal{X}_s^{*,L}\right] ds.$$

For the constrained problem under uncertainty, we denote by $\mathcal{A}_0(t,x)$ the set of admissible controls $\mathcal{X}\in\mathcal{A}(t,x)$ such that $\mathcal{X}_T\mathbb{1}_{\mathcal{S}}=0$, $\mathcal{P}_t$-q.s. ($\mathcal{P}_t$-q.s means $\mathbb{P}-a.s.\ \forall\mathbb{P}\in\mathcal{P}_t$) and $\mathcal{A}_0^{\mathbb{P}}(t,x)$ the set of admissible controls $\mathcal{X}\in\mathcal{A}^{\mathbb{P}}(t,x)$ such that $\mathcal{X}_T\mathbb{1}_{\mathcal{S}}=0$ $\mathbb{P}-a.s.$ Now consider

$$J(t,x) = \operatorname*{essinf}_{\mathcal{X}\in\mathcal{A}_0(t,x)} \operatorname*{ess\,sup}_{\mathbb{P}\in\mathcal{P}_t} \mathbb{E}^{\mathbb{P}}\left[\int_t^T (\widehat{\eta}_s|\alpha_s|^p + \widehat{\gamma}_s|\mathcal{X}_s|^p)\,ds + \xi|\mathcal{X}_T|^p \middle| \mathcal{F}_t^+\right]. \tag{5.19}$$

Again we use the convention that $0\times\infty=0$. As mentioned for the standard formulation, a left-continuity condition is imposed on the underlying filtration to have the desired terminal condition[3]. In our present setting we add the next assumption on our set of probability measures $\mathcal{P}_t^W$:

- **Left-continuity condition:** for any probability measure $\mathbb{P}\in\mathcal{P}_t^W$, the filtration $\mathbb{F}_+^{\mathbb{P}}$ is left-continuous at time T.

As in Section 3.2.2, let us now assume that there exists $\ell>1$ and $\kappa\in(1,\ell)$ such that for any (t,ω)

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\int_t^T [\widehat{\eta}_s + (T-s)^p\,\widehat{\gamma}_s]^\ell\,ds\right] < \infty, \tag{5.20}$$

and

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}_{0\leq t\leq T}{}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T [\widehat{\eta}_s + (T-s)^p\,\widehat{\gamma}_s]^\kappa\,ds\middle|\mathcal{F}_t^+\right]\right)^{\frac{\ell}{\kappa}}\right] < \infty. \tag{5.21}$$

Then there exists $U\in\mathbb{D}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0})$ such that for any $0\leq t\leq T$, $\mathcal{P}_0$-q.s.

$$0 \leq Y_t^L \leq \frac{1}{(T-t)^p}U_t, \tag{5.22}$$

[3]This technical condition can be avoided if ξ is $\mathcal{F}_{T-}$-measurable (see for example [20]).

where U is the first part of the solution of the 2BSDE:

$$U_t = \int_t^T (\widehat{\eta}_s + (T-s)^p \widehat{\gamma}_s)\, ds - \left(\int_t^T V_s dX_s^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_t^T d\mathcal{N}_s^{\mathbb{P}} + (\mathcal{K}_T^{\mathbb{P}} - \mathcal{K}_t^{\mathbb{P}}).$$

Let us emphasize that the right-hand side of (5.22) does not depend on L and is finite on $[0,T)$. Hence for any $\varepsilon > 0$, the sequence $(Y^L, Z^L, M^{L,\mathbb{P}}, K^{L,\mathbb{P}})$ converges, when L goes to $+\infty$, to $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ in the space $\mathbb{D}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{H}_0^\ell(\mathbb{F}_+^{\mathcal{P}_0}) \times \mathbb{M}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}) \times \mathbb{I}_0^\ell((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ on $[0, T-\varepsilon]$, which means that all processes are restricted on this time interval. Moreover $(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ satisfies the dynamics: for any $\mathbb{P} \in \mathcal{P}_0$, and any $0 \le s \le t < T$:

$$(5.23)\quad Y_s = Y_t - \int_s^t \frac{Y_u^q}{(q-1)(\widehat{\eta}_u)^{q-1}} du + \int_s^t \widehat{\gamma}_u du - \left(\int_s^t Z_u dX_u^{c,\mathbb{P}} \right)^{\mathbb{P}} - \int_s^t dM_u^{\mathbb{P}} + K_t^{\mathbb{P}} - K_s^{\mathbb{P}}.$$

Finally Y satisfies the representation property: for any $t < T$ and any $\mathbb{P} \in \mathcal{P}_0$,

$$Y_t = \operatorname*{esssup}_{\mathbb{P}' \in \mathcal{P}(t,\mathbb{P},\mathbb{F}_+)}{}^{\mathbb{P}}\ y_t^{\mathbb{P}'}, \quad \mathbb{P} - a.s.$$

We can now obtain an optimal solution for the control problem (5.19).

Proposition 5.3 *The constrainted problem* (5.19) *has an optimal state process $\mathcal{X}^*$ defined by*

$$\mathcal{X}_s^* = x - \int_t^s \left(\frac{Y_u}{\widehat{\eta}_u} \right)^{q-1} \mathcal{X}_u^* du,$$

Moreover the value function is given by: $J(t,x) = |x|^p Y_t$.

The proof is similar to the case without uncertainty.

5.3 Extension to convex cost ([XVIII])

Now we come back to the initial problem (5.1), together with the unconstrained control problem (5.2) and the FBSDE (5.3). Let us explain the setting. Here $T \in (0,\infty)$ is a deterministic finite time horizon. Let $W = (W_t)_{t\in[0,T]}$ be a d - dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote by $\mathbb{F} = \mathbb{F}^W = (\mathcal{F}_t)_{t\in[0,T]}$ the smallest filtration satisfying the usual conditions and containing the filtration generated by W.

Let $A \subseteq \mathbb{R}$ be a closed and connected set of possible control values satisfying $\inf A \le 0 < \sup A$. Let

$$g : \Omega \times \mathbb{R} \to \mathbb{R}$$

be measurable and

$$j : \Omega \times [0,T] \times \mathbb{R} \times A \to \mathbb{R}$$

be measurable such that for all $(x,a) \in \mathbb{R} \times A$ the mapping $(\omega,t) \mapsto j(\omega,t,x,a)$ is progressively measurable. We make the following additional assumptions on j and g:

(E1) For every fixed pair $(\omega, t) \in \Omega \times [0,T]$ the mappings $(x,a) \mapsto j(t,x,a)$ and $x \mapsto g(x)$ are convex[4], with j being strictly convex in a.

(E2) The mappings $A \ni a \mapsto j(t,x,a)$ and $\mathbb{R} \ni x \mapsto j(t,x,a)$ attain a minimum at zero (for all ω, t, x and all ω, t, a respectively). We also assume that $j(t,0,0) = 0$ for all $t \in [0,T]$. Observe that j is then non-negative.

(E3) j is coercive, i.e. there exist $p > 1$ and $b > 0$ such that

$$\forall (\omega, t, x, a) \in \Omega \times [0,T] \times \mathbb{R} \times A : j(t,x,a) \geq b|a|^p.$$

(E4) $g(\cdot)$ restricted to $[0,\infty)$ is twice continuously differentiable, $j(t,\cdot,\cdot)$ restricted to $[0,\infty) \times (A \cap [0,\infty))$ is continuously differentiable, while $j_x(t,\cdot,\cdot)$ and $j_a(t,\cdot,\cdot)$ are continuously differentiable on $[0,\infty) \times A_+$, where $A_+ := A \cap (0,\infty)$. All second derivatives are bounded on compacts in $[0,\infty) \times A_+$ and all first derivatives are bounded on compacts in $[0,\infty) \times (A \cap [0,\infty))$, uniformly in (ω, t).

(E5) The mapping $x \mapsto g(x)$ attains its minimum at zero (for all ω). We also assume that $g(0) = 0$. Observe that g is then non-negative.

Remark 5.1 *The assumptions that $j(t,0,0) = 0$ for all $t \in [0,T]$ and $g(0) = 0$ can be relaxed to the assumptions that $j(\cdot,0,0) \in \mathbb{L}^1(\Omega \times [0,T])$ and $g(0) \in \mathbb{L}^1(\Omega)$. Indeed, in this case one can consider the problems* (5.1) *and* (5.2) *with $\tilde{j}(t,x,a) = j(t,x,a) - j(t,0,0)$ and $\tilde{g}(x) = g(x) - g(0)$ instead of j and g and add $\mathbb{E}\left[\int_0^T j(t,0,0)dt + g(0)\right]$ outside the minimization problem.*

For $t \in [0,T]$ we define $\mathcal{A}(t)$ as the set of all progressively measurable $\alpha : \Omega \times [t,T] \to A$ such that a.s. $u \mapsto \alpha(\cdot, u)$ is integrable. Hence for $(t,x) \in [0,T] \times \mathbb{R}$, the process

$$X_s^{t,x,\alpha} := x - \int_t^s \alpha_u du$$

is well-defined for all $s \in [t,T]$. The dynamic version of problem (5.2) then reads

$$\text{(5.24)} \qquad \begin{array}{l} \text{Minimize } J(t,x,\alpha) := \mathbb{E}\left[\int_t^T j(s, X_s^{t,x,\alpha}, \alpha_s)ds + g(X_T^{t,x,\alpha}) \middle| \mathcal{F}_t\right] \\ \text{over all } \alpha \in \mathcal{A}(t). \end{array}$$

The value function $v \colon \Omega \times [0,T] \times \mathbb{R} \to [0,\infty]$ is the random field that satisfies for all $(t,x) \in [0,T] \times \mathbb{R}$

$$v(t,x) = \operatorname{essinf}_{\alpha \in \mathcal{A}(t)} J(t,x,\alpha).$$

The next result shows that when starting with a non-negative initial position, then it can not be optimal to choose α such that the position process is increasing or negative at some time point. This result is coherent with the absence of transaction-triggered price manipulation (see [4]).

[4]Note that here and in the sequel we follow the usual convention and omit the function argument ω.

Lemma 5.2 *Let $(t,x) \in [0,T] \times [0,\infty)$. If $\alpha \in \mathcal{A}(t)$ is optimal in* (5.24)*, then $X^\alpha_s = x - \int_t^s \alpha_r dr$, $s \in [0,T]$, is non-increasing and non-negative. Moreover, for any $\alpha \in \mathcal{A}(t)$ there exists $\beta \in \mathcal{A}(t)$ such that X^β is non-increasing and non-negative and $J(t,x,\beta) \leq J(t,x,\alpha)$.*

Observe that by symmetry, when starting in a negative position, one can restrict the analysis to non-positive controls and positions, with straightforward adjustments in the hypotheses (differentiability condition for non-positive values). In the following we consider only the positive case and always assume that any positions and controls are non-negative.

The so-called Hamiltonian of the control problem is defined by

$$\mathcal{H}(t,x,a,y) := -ay + j(t,x,a),$$

for $t \in [0,T]$ and $(x,a,y) \in \mathbb{R} \times A \times \mathbb{R}$. Notice that

$$\min_{a \in A} \mathcal{H}(t,x,a,y) = -j^*(t,x,y),$$

where $j^*(t,x,\cdot)$ is the convex conjugate of $j(t,x,\cdot)$. The properties of j^* (definition, continuity, differentiability) are detailed in [XVIII, Remarks 1.3 and 1.4].

Next we consider for $(t,x) \in [0,T] \times [0,\infty)$ the so-called adjoint forward-backward stochastic differential equation (FBSDE) for the control problem (5.24), given by

$$\begin{aligned} X^{t,x}_s &= x - \int_t^s j^*_y(r, X^{t,x}_r, Y^{t,x}_r)dr, \\ Y^{t,x}_s &= g'(X^{t,x}_T) - \int_s^T Z^{t,x}_r dW_r + \int_s^T j_x(r, X^{t,x}_r, j^*_y(r, X^{t,x}_r, Y^{t,x}_r))dr, \end{aligned} \tag{5.25}$$

for all $s \in [t,T]$. To simplify the notations, when there is no ambiguity, $(X^{t,x}, Y^{t,x}, Z^{t,x})$ will be denoted by (X,Y,Z). In this section we mean by a solution to (5.25) a triplet $(X,Y,Z) = (X^{t,x}, Y^{t,x}, Z^{t,x})$ of progressively measurable processes with values in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$ such that

1. X and Y are continuous and non-negative processes,
2. the processes X, Y and $s \mapsto j^*_y(s, X_s, Y_s)$ are bounded and, finally,
3. the two equations (5.25) are satisfied a.s. for every fixed $s \in [t,T]$.

Note that under our framework, the stochastic integral $\int_0^\cdot Z^{t,x}_r dW_r$ is a BMO martingale (see e.g. Proposition 1.1 in [28]). In particular for any $p \geq 1$ it holds that

$$\mathbb{E}\left[\left(\int_0^T |Z^{t,x}_r|^2 dr\right)^{p/2}\right] < +\infty. \tag{5.26}$$

Constructing solutions to the above FBSDE is important for the following reason.

Lemma 5.3 *Let $(t,x) \in [0,T] \times [0,\infty)$. If there exists a solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ of* (5.25)*, then process $\alpha = (\alpha_s)_{s \in [t,T]}$ satisfying $\alpha_s = j^*_y(s, X^{t,x}_s, Y^{t,x}_s)$, $s \in [t,T]$, is an optimal control for problem* (5.24) *with finite expected costs, i.e.*

$$v(t,x) = J(t,x,\alpha) < \infty.$$

It is a longstanding challenge to find conditions guaranteeing that a fully coupled FBSDE possesses a solution. The method of decoupling fields, developped in [128] is practically useful for determining whether a solution exists. A decoupling field describes the functional dependence of the backward part Y on the forward component X:

$$Y_t(\omega) = u(\omega, t, X_t(\omega)).$$

If the coefficients of a fully coupled FBSDE satisfy a Lipschitz condition, then there exists a maximal non-vanishing interval possessing a solution triplet (X, Y, Z) and a decoupling field with nice regularity properties. The method of decoupling fields consists in analyzing the dynamics of the decoupling field's gradient in order to determine whether the FBSDE has a solution on the whole time interval $[0, T]$.

The FBSDE (5.25) can be shown to possess a solution by using the so-called continuation method, developed in [165, 285, 327]. In particular the problem (5.24) has been solved already using this method (see e.g. [59, Section 5]). The continuation method, however, does not provide the existence of a decoupling field, which is fundamental in the present article for passing to the limit as the penalty converges to infinity and for solving Problem (5.1).

Concerning this unconstrained problem, we prove in [XVIII, Theorem 2.10] that

Proposition 5.4 *Under Assumptions* **(E1)** *to* **(E5)***, the maximal interval associated with FBSDE* (5.25) *satisfies* $I_{\max} = [0, T]$. *Furthermore, the unique weakly regular decoupling field* u *on* $[0, T]$ *satisfies* $u(t, x) = 0$ *for all* $x \le 0$ *and* $t \in [0, T]$. *For all* $t \in [0, T]$ *and* $x \in [0, \infty)$ *there exists a unique solution* (X, Y, Z) *of FBSDE* (5.25). *The processes* X *and* Y *are both bounded and non-negative (and* Z *is BMO).*

In the additive-homogeneous case (see Section 5.1), the decoupling field is given by:

$$u(\omega, t, x) = pY_t^{\min}(\omega)x|x|^{p-1},$$

where $Y^{\min}$ is the solution of the BSDE (5.13) (or the BSDE (5.9) in the constrained case). In other words, we have an explicit decoupling field and thus we can separate the forward and the backward SDE.

Now we consider the dynamic version of the constrained problem (5.1)

$$\begin{aligned} &\text{Minimize } \hat{J}(t, x, \alpha) = \mathbb{E}\left[\int_t^T j(s, X_s^{t,x,\alpha}, \alpha_s)dt \middle| \mathcal{F}_t\right] \\ &\text{over all } \alpha \in \mathcal{A}(t) \text{ such that } X_T^{t,x,\alpha} = 0. \end{aligned} \tag{5.27}$$

In the rest of this part, for all $(t, x) \in [0, T) \times \mathbb{R}$ let

$$\mathcal{A}^0(t, x) := \{\alpha \in \mathcal{A}(t) \,|\, X_T^{t,x,\alpha} = 0 \text{ a.s.}\}.$$

As written in the introduction we solve this problem via a penalization method. Using Proposition 5.4, tor every $(t_0, x_0) \in [0, T) \times (0, \infty)$ and every penalty function $g^L(x) = Lx^2$, $L > 0$, we have a unique solution (X^L, Y^L, Z^L) to the FBSDE (5.25) with initial condition $X_{t_0} = x_0$, as well as a unique weakly regular decoupling field u^L associated with (5.25). For every $L \in (0, \infty)$ let $v^L : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be the value function of problem (5.24) with penalty function g^L. The strategy $\alpha_t^L := j_y^*(t, X_t^L, Y_t^L)$, $t \in [t_0, T]$, minimizes the cost J^L, i.e. it holds that

$$v^L(t_0, x_0) = J^L(t_0, x_0, \alpha^L) \tag{5.28}$$

for non-negative x_0.

In order to pass to the limit on L, we assume that the next conditions are verified.

(E6) The functions $(x, y) \mapsto j_y^*(t, x, y)$ and $(x, y) \mapsto j_x(t, x, f_y^*(t, x, y))$ are Lipschitz continuous on $[0, \infty) \times [0, \infty)$, uniformly in $(\omega, t) \in \Omega \times [0, T]$.

(E7) It holds that $j_x(t, 0, 0) = 0$ for all $(\omega, t) \in \Omega \times [0, T]$.

(E8) It holds that $\sup A = \infty$ and $j_a(t, x, 0) = 0$ for all $(\omega, t, x) \in \Omega \times [0, T] \times [0, \infty)$.

(E9) The whole Hessian matrix $D^2 j(t, x, a)$ of j w.r.t. $(x, a) \in [0, \infty) \times (0, +\infty)$ is uniformly bounded independently of $(\omega, t, x, a) \in \Omega \times [0, T] \times [0, \infty) \times (0, +\infty)$.

Roughly speaking, these conditions imply that the coefficients of the FBSDE are Lipschitz continuous.

Under the above assumptions, α^L converges for $L \to \infty$ to an admissible strategy $\alpha^\infty \in \mathcal{A}^0(t_0, x_0)$, which minimizes $\hat{J}(t_0, x_0, \cdot)$, i.e. provides an optimal strategy for problem (5.27). We do so by first proving convergence of u^L to some limit u^∞ and then showing convergence of X^L to a limit X^∞. This will finally lead us to the limit $\alpha^\infty \in \mathcal{A}^0(t_0, x_0)$. Let us summarize the main result.

Theorem 5.2 *Under Assumptions* **(E1)** *to* **(E9)***, the decoupling field u^L converges to u^∞. If X^∞ is the unique solution to the ODE*

$$X_t^\infty := x_0 - \int_0^t j_y^*(s, X_s^\infty, u^\infty(s, X_s^\infty))ds, \quad t \in [0, T),$$

and if $Y^\infty = u^\infty(\cdot, X^\infty)$, then (X^L, Y^L, Z^L) converges to $(X^\infty, Y^\infty, Z^\infty)$ and for any $0 \le t \le r < T$

$$Y_t^\infty = Y_r^\infty + \int_t^r j_x(s, X_s^\infty, j_y^*(s, X_s^\infty, Y_s^\infty))ds - \int_t^r Z_s^\infty dW_s.$$

Process $(X^\infty, Y^\infty, Z^\infty)$ is the unique solution of the preceding FBSDE satisfying $X^\infty, Y^\infty \ge 0$, $Z^\infty \in \mathbb{L}^2((t_0, t) \times \Omega, \mathbb{R}^d)$ for all $t \in [t_0, T)$ and $j_y^*(\cdot, X^\infty, Y^\infty) \in \mathbb{L}^2((t_0, T) \times \Omega, \mathbb{R})$.

Finally $X_T^\infty = 0$ and if we define $\alpha_s^\infty := j_y^*(s, X_s^\infty, u^\infty(s, X_s^\infty))$, for $s \in [0, T)$, while setting $\alpha_T^\infty := 0$, then strategy α^∞ minimizes $\hat{J}(x_0, \cdot)$.

Proof. Let us give the main steps of the proof. To get the convergence of u^L, the two key lemmas are the following. The first one is equivalent to the a priori estimate (3.10) of Proposition 3.1 and the lower bound (3.16) for $Y^{\min}$.

Lemma 5.4 *There exist constants $C_1, C_2 \in (0,\infty)$, which depend on T and the norm of the second derivatives of f^* and f only and are monotonically increasing in these values, such that for all $L > 0$ we have the following estimates for u^L_x:*

$$\frac{1}{C_1\left(\frac{1}{2L} + (T-t)\right)} =: \kappa^L_t \leq u^L_x(t,x) \leq \gamma_t := C_2\left(1 + \frac{1}{T-t}\right),$$

for all $t \in [0,T)$ and a.a. $x > 0$. As a consequence

$$x \cdot \kappa^L_t \leq u^L(t,x) \leq x \cdot \gamma_t,$$

for all $t \in [0,T)$ and $x > 0$.

Again note that these bounds correspond to the estimates on the solution $Y^{\min}$ of a singular BSDE (2.4) with generator f controlled by $-y|y|$ in **(C2)**. It means that expect close to the terminal time, the Lipschitz constant of u^L is bounded w.r.t. L (and we could apply the Arzelà–Ascoli theorem). The second result is:

Lemma 5.5 *The mapping $(t,\omega,L,x) \mapsto u^L(t,\omega,x)$ is progressively measurable while being continuous and non-decreasing in L.*

Now we can define $u^\infty : \Omega \times [0,T) \times \mathbb{R}$ via $u^\infty(s,x) := \lim_{L\to\infty} u^L(s,x)$. Note that u^∞ inherits progressive measurability from u^L. Also note that for all $s < T$ the mapping $u^L(s,\cdot)$ is Lipschitz continuous w.r.t. x with Lipschitz constant γ_s, which does not depend on L. Therefore, $u^\infty(s,\cdot)$ is also Lipschitz continuous with the same Lipschitz constant. Finally, note that for all $s < T$ and all $x > 0$

$$u^\infty(s,x) \geq x \cdot \lim_{L\to\infty} \kappa^L_s = \frac{x}{C_1(T-s)}.$$

Now let $x_0 > 0$. Since u^∞ restricted to $\Omega \times [0, T-\varepsilon] \times \mathbb{R}$ is progressively measurable and uniformly Lipschitz continuous in the last component for every $\varepsilon \in (0,T)$, we can define X^∞ as the unique solution to the ODE

$$X^\infty_t := x_0 - \int_0^t j^*_y(s, X^\infty_s, u^\infty(s, X^\infty_s))ds, \quad t \in [0,T),$$

which is motivated by passing to the limit $L \to \infty$ in

$$X^L_t = x_0 - \int_0^t j^*_y(s, X^L_s, u^L(s, X^L_s))ds.$$

Note that X^∞ is defined on $[0,T)$ only. It holds that $\lim_{L\to\infty} X^L = X^\infty$ and $\lim_{L\to\infty} u^L(\cdot, X^L_\cdot) \to u^\infty(\cdot, X^\infty_\cdot)$ almost everywhere on $\Omega \times [0,T)$. Note that X^L and therefore X^∞ is non-negative. Furthermore, X^∞ is non-increasing, since j^*_y is non-negative. Therefore, $\lim_{t\to T} X^\infty_t$ exists and we can continuously extend the process X^∞ to the whole of $[0,T]$ via $X^\infty_T := \lim_{t\to T} X^\infty_t$. Remark that $X^\infty_T \geq 0$ a.s. It holds that $X^\infty_T = 0$. If we define $\alpha^\infty_s := j^*_y(s, X^\infty_s, u^\infty(s, X^\infty_s))$, for $s \in [0,T)$, while setting $\alpha^\infty_T := 0$, then $\alpha^\infty \in \mathcal{A}^0(x_0)$.

Lemma 5.6 *The strategy α^∞ minimizes $\hat{J}(x_0,\cdot)$.*

Let $Y^\infty \colon [0,T) \times \Omega \to \mathbb{R}$ satisfy for all $t \in [0,T)$ a.s. that $Y^\infty_t = u^\infty(t, X^\infty_t)$. Then sequence Z^L converges in $L^2((0,t)\times\Omega)$ for any $t < T$ to Z^∞ and process $(X^\infty, Y^\infty, Z^\infty)$ satisfies for all $0 \leq t \leq r < T$ a.s. that

$$\begin{aligned} X^\infty_t &= x_0 - \textstyle\int_0^t j^*_y(s, X^\infty_s, Y^\infty_s)ds, \quad X^\infty_T = 0 \\ Y^\infty_t &= Y^\infty_r + \textstyle\int_t^r j_x(s, X^\infty_s, j^*_y(s, X^\infty_s, Y^\infty_s))ds - \int_t^r Z^\infty_s dW_s. \end{aligned}$$

□

Remark 5.2 *From the lower estimate of u^L, and thus on u^∞, we obtain*

$$\liminf_{t\to T}\left(\frac{Y_t^\infty}{X_t^\infty}\right) = +\infty.$$

This behaviour is similar to the weak terminal condition (1.3) in [10]. Moreover from the preceding system, the process

$$Y^\infty + \int_0^\cdot j_x(s, X_s^\infty, f_y^*(s, X_s^\infty, Y_s^\infty))ds$$

is a non-negative local martingale on $[0,T)$. Hence its limit at time T exists in $[0,\infty)$ a.s. By the monotone convergence theorem, the integral has also a limit. Since both terms are non-negative, we deduce that Y^∞ has a limit at time T a.s. and Y_T^∞ is finite a.s. Note that here Y_T^∞ is not a given terminal condition, but part of the solution.

Let us give an example. Let $C \in (1,\infty)$ and let $\eta, \gamma\colon [0,T]\times\Omega \to [0,\infty)$ be progressively measurable stochastic processes such that for all $t \in [0,T]$ it holds a.s. that $\eta_t \geq \frac{1}{C}$ and $\max(\eta_t, \gamma_t) \leq C$. Assume that for all $t \in [0,T]$, $x \in \mathbb{R}$ and $a \in \mathbb{R}$ it holds that $j(t,x,a) = \eta_t \frac{|a|^3+2|a|^2}{|a|+1} + \gamma_t |x|^2$. Then all required conditions are satisfied. In particular it holds for all $t \in [0,T]$, $x \in \mathbb{R}$ and $a \in \mathbb{R}$ a.s. that $\frac{2}{C} \leq 2\eta_t \leq j_{aa}(t,x,a) \leq 4\eta_t \leq 4C$. And thus $[0,\infty)^2 \ni (x,y) \mapsto j_y^*(t,x,y)$ is uniformly Lipschitz continuous. Therefore, it follows from the preceding theorem that α^∞ is an optimal control in Problem (5.27).

Our assumptions essentially imply that j is almost quadratic. Removing or weakening these conditions is still an open question.

5.3.1 A more abstract setting

We consider the FBSDE

$$\begin{aligned} X_s^{t,x} &= x + \int_t^s \mu(r, X_r^{t,x}, Y_r^{t,x})dr, \\ Y_s^{t,x} &= \xi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r. \end{aligned} \tag{5.29}$$

We assume throughout that:

- (μ, f) are Lipschitz continuous in (x,y) with Lipschitz constant $\mathcal{L}$,
- $\|(|\mu| + |f|)(\cdot,\cdot,0,0)\|_\infty < \infty$,
- $\xi\colon \Omega\times\mathbb{R} \to \mathbb{R}$ is measurable such that $\|\xi(\cdot,0)\|_\infty < \infty$ and $\mathcal{L}_{\xi,x} < \infty$.

Moreover, we impose the following conditions.

- For all $x, \bar{x}, y, \bar{y} \in \mathbb{R}$, $t \in [0,T]$ it holds a.s. that

  $$(\mu(t,x,y) - \mu(t,x,\bar{y}))(y-\bar{y}) \leq 0, \quad (f(t,x,y) - f(t,\bar{x},y))(x-\bar{x}) \geq 0,$$

  and

  $$(\xi(x) - \xi(\bar{x}))(x-\bar{x}) \geq 0.$$

- The coefficients μ, f and ξ are either continuously differentiable or at least differentiable such that we can use the chain rule argument developed in [128, Lemma A.3.1].

Theorem 5.3 *Under the previous conditions, there exists a unique strongly regular decoupling field u on $I_{\max} = [0,T]$. In particular, for all $t \in [0,T]$ and $x \in \mathbb{R}$ there exists a unique solution (X,Y,Z) of FBSDE* (5.29).

The proof is based on the dynamics of $\Psi_t = u_x(t,X_t)$ and we show that Ψ is non-negative and bounded.

Remark 5.3 *The first condition is sufficient to get that the maximal interval is equal to $[0,T]$. This assumption is equivalent to Condition (H2.2) in [285] with $G = 1$ and $\beta_1 = \beta_2 = 0$. Notice, however, that in [285] it is required that $\beta_1 + \beta_2 > 0$.*

In the setting of the control problem, it is natural to add conditions on $(\xi,(\mu,f))$ to get the desired sign for the solutions X and Y. We assume

- For all $t \in [0,T]$, $x \leq 0$, it holds a.s. that $\xi(x) = \mu(t,x,0) = f(t,x,0) = 0$.
- For all $x,y \in \mathbb{R}$, $t \in [0,T]$ it holds a.s. that $\mu(t,x,y) \leq 0$, $f(t,x,y) \geq 0$.

Theorem 5.4 *The unique weakly regular decoupling field u on $[0,T]$ satisfies $u(t,x) = 0$ for all $x \leq 0$, $t \in [0,T]$. For all $(t,x) \in [0,T] \times [0,+\infty)$, processes X and Y from the solution of* (5.29) *are both bounded and non-negative.*

As before we obtain the positivity of X_s on $[t,T]$. Now we assume that the terminal condition ξ is of the form $\xi(x) = \xi^L(x) = 2Lx$, $x \geq 0$, for $L > 0$. Hence we obtain a solution (X^L, Y^L, Z^L, u^L) and we want to pass to the limit on L. We require the following condition, which partly strengthens (A1).

- There exists $\eta > 0$ such that for all $x,y \in (0,\infty)$, $t \in [0,T]$ it holds a.s. that $\mu_y(t,x,y) \leq -\eta < 0$.

Under this setting we get the same a priori estimate on u^L independent of L and that the mapping $(t,\omega,L,x) \mapsto u^L(t,\omega,x)$ is progressively measurable while being continuous and non-decreasing in L. Hence we can define

$$u^\infty(s,x) := \lim_{L\to\infty} u^L(s,x), \quad (s,x) \in [0,T) \times \mathbb{R}. \tag{5.30}$$

X^L converges to X^∞ and we obtain

Theorem 5.5 *Let $(t_0,x_0) \in [0,T) \times (0,\infty)$. For every $L \in (0,\infty)$ let (X^L,Y^L,Z^L) be the solution of the FBSDE* (5.29) *with initial condition $X^L_{t_0} = x_0$ and terminal condition ξ^L and let u^L be the associated decoupling field. Let u^∞ and X^∞ be the limits of u^L and X^L as $L \to \infty$. Let $Y^\infty\colon \Omega \times [0,T) \to \mathbb{R}$ satisfy for all $t \in [0,T)$ a.s. that $Y^\infty_t = u^\infty(t,X^\infty_t)$. Then the sequence Z^L converges in $\mathbb{L}^2((t_0,t)\times\Omega,\mathbb{R}^d)$ for any $t \in [t_0,T)$ to Z^∞ and the process $(X^\infty,Y^\infty,Z^\infty)$ satisfies for all $t_0 \leq t \leq r < T$ a.s. that*

$$\begin{array}{rcl} X^\infty_t &=& x_0 + \int_{t_0}^t \mu(s,X^\infty_s,Y^\infty_s)ds, \quad X^\infty_T = \lim_{s\to T} X^\infty_s = 0 \\ Y^\infty_t &=& Y^\infty_r + \int_t^r f(s,X^\infty_s,Y^\infty_s)ds - \int_t^r Z^\infty_s dW_s. \end{array} \tag{5.31}$$

Note that Remark 5.2 also holds in this case. However the uniqueness of (5.31) remains an open question.

5.4 A mean field liquidation problem ([XX])

In [XX], we consider a mean field game (MFG in short) of optimal portfolio liquidation among asymmetrically informed players. In order to introduce the game, we fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ that carries independent standard Brownian motions $W^0, W^1, ..., W^N$ and independent and identically distributed random variables $\mathcal{X}^1, ..., \mathcal{X}^N$ with law ν that are independent of the Brownian motions. The Brownian motion W^0 describes a commonly observed random factor that drives the unaffected benchmark price process; the Brownian motion W^i is private information to player i and determines that player's cost function. We may think of W^i as measuring a player's individual degree of market impact or as capturing hedging effects when computing the risk of the portfolio that the player intends to liquidate. The random variables $\mathcal{X}^1, ..., \mathcal{X}^N$ specify the respective players' initial portfolios. We assume that each player observes the realization of her initial portfolio and knows the distribution of all the other initial portfolios.

Given her initial portfolio $\mathcal{X}^i = x^i$ the optimization problem of player $i = 1, ..., N$ is to minimize the cost functional

$$J^{N,i}(\vec{\alpha}) = \mathbb{E}\left[\int_0^T \left(\frac{\kappa_t^i}{N}\sum_{j=1}^N \alpha_t^j X_t^i + \eta_t^i(\alpha_t^i)^2 + \gamma_t^i(X_t^i)^2\right) dt \big| \mathcal{X}^i = x^i\right] \tag{5.32}$$

subject to the state dynamics

$$\begin{aligned} dX_t^i &= -\alpha_t^i \, dt, \\ X_0^i &= \mathcal{X}^i, \quad X_T^i = 0. \end{aligned} \tag{5.33}$$

Here, $\vec{\alpha} = (\alpha^1, \cdots, \alpha^N)$ is the vector of strategies of all the players. We assume that the cost coefficients $(\kappa^i, \eta^i, \gamma^i)$ have the same distribution across players and are adapted to the filtration

$$\mathbb{F}^i := (\mathcal{F}_t^i, 0 \le t \le T), \quad \text{with} \quad \mathcal{F}_t^i := \sigma(\mathcal{X}^i, W_s^0, W_s^i, 0 \le s \le t). \tag{5.34}$$

The MFG

In order to specify the resulting MFG, let W^0 and W be independent Brownian motions of dimension 1 and $m-1$, respectively, and $\mathcal{X}$ be an independent random variable with law ν defined on some probability space, again denoted $(\Omega, \mathcal{G}, \mathbb{P})$. Let $\mathbb{F}^0 := (\mathcal{F}_t^0, 0 \le t \le T)$ with $\mathcal{F}_t^0 = \sigma(W_s^0, 0 \le s \le t)$ be the filtration generated by W^0 and let $\mathbb{F} := (\mathcal{F}_t, 0 \le t \le T)$ with $\mathcal{F}_t := \sigma(\mathcal{X}, W_s^0, W_s, 0 \le s \le t)$. The MFG associated with the N-player game (5.32) and (5.33) is then given by:

$$
(5.35)\qquad \begin{cases} \text{1. fix a } \mathbb{F}^0 \text{ progressively measurable process } \mu \text{ (in some suitable space);} \\ \text{2. solve the corresponding constrained stochastic control problem :} \\ \quad \inf_\alpha \mathbb{E}\left[\int_0^T (\kappa_s \mu_s X_s + \eta_s \alpha_s^2 + \gamma_s X_s^2)\, ds \middle| \mathcal{X} \right] \\ \ \text{subsect to} \\ \quad dX_t = -\alpha_t\, dt, X_0 = \mathcal{X} \ \text{ and } X_T = 0; \\ \text{3. search for the fixed point } \mu_t = \mathbb{E}[\alpha_t^{*,\mathcal{X}} | \mathcal{F}_t^0], \text{ for } a.e.\ t \in [0,T] \end{cases}
$$

where $\alpha^{*,\mathcal{X}}$ is the optimal strategy from 2 and the processes (κ, η, γ) are adapted to the filtration $\mathbb{F}$.

The set of admissible controls for the representative player's liquidation problem is given by

$$
\mathcal{A}_{\mathbb{F}}(\mathcal{X}) := \left\{ \alpha \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}),\ \int_0^T \alpha_s\, ds = \mathcal{X} \text{ a.s.} \right\}.
$$

For a given process $\mu \in \mathbb{L}^2_{\mathbb{F}^0}([0,T] \times \Omega; \mathbb{R})$, the corresponding cost and value functions are given by

$$
J(\mathcal{X}, \alpha; \mu) := \mathbb{E}\left[\int_0^T \left(\kappa_s X_s \mu_s + \eta_s \alpha_s^2 + \gamma_s X_s^2 \right) ds \middle| \mathcal{X} \right],
$$

and

$$
V(\mathcal{X}; \mu) = \inf_{\alpha \in \mathcal{A}_{\mathbb{F}}(\mathcal{X})} J(\mathcal{X}, \alpha; \mu),
$$

respectively. We denote by Y the adjoint process to the controlled state process X. The Hamiltonian is

$$
H(t, \alpha, X, Y; \mu) = -\alpha Y + \kappa_t \mu X + \eta_t \alpha^2 + \gamma_t X^2,
$$

and the stochastic maximum principle suggests that the solution to the optimization problem can be characterised in terms of the FBSDE

$$
(5.36)\qquad \begin{cases} dX_t = -\,\alpha_t\, dt, \\ -dY_t = (\kappa_t \mu_t + 2\gamma_t X_t)\, dt - Z_t\, d\widetilde{W}_t, \\ X_0 = \mathcal{X} \\ X_T = 0, \end{cases}
$$

where $\widetilde{W} = (W^0, W)$ is a m-dimensional Brownian motion. The liquidation constraint $X_T = 0$ results in a singularity of the value function at liquidation time; see [10, 148]. As a result, the terminal condition for Y cannot be determined a priori (see [XVIII] and the previous section). In particular, the first equation holds on $[0,T]$ while the second equation holds on $[0,T)$. A standard approach (see Theorem 5.1) yields the candidate optimal control

$$
(5.37)\qquad \alpha_t^* = \frac{Y_t}{2\eta_t}.
$$

Taking the equilibrium condition into account suggests that the analysis of the MFG reduces to the analysis of the following conditional mean-field type FBSDE:

$$
\tag{5.38}
\begin{cases}
dX_t = -\dfrac{Y_t}{2\eta_t}\,dt, \\
-dY_t = \left(\kappa_t \mathbb{E}\left[\dfrac{Y_t}{2\eta_t}\middle|\,\mathcal{F}^0_t\right] + 2\gamma_t X_t\right) dt - Z_t\, d\widetilde{W}_t, \\
X_0 = \mathcal{X} \\
X_T = 0.
\end{cases}
$$

We establish the existence and uniqueness of a solution to the preceding FBSDE in the following space of weighted stochastic processes.

Definition 5.1 *For $\ell \in \mathbb{R}$, we introduce the space*

$$
\mathcal{H}_\ell := \{Y \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) : \ (T-.)^{-\ell}Y. \in \mathbb{S}^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\})\},
$$

which we endowed with the norm

$$
\|Y\|_{\mathcal{H}_\ell} := \|Y\|_\ell := \left(\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{Y_t}{(T-t)^\ell}\right|^2\right]\right)^{\frac{1}{2}},
$$

and the space

$$
\mathcal{M}_\ell := \{Y \in \mathcal{P}_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\}) : \ (T-.)^{-\ell}Y. \in \mathbb{L}^\infty_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}\cup\{\infty\})\},
$$

which we endowed with the norm

$$
\|Y\|_{\mathcal{M}_\ell} := \ \text{ess sup}_{\ (t,\omega)\in[0,T]\times\Omega}\frac{|Y_t|}{(T-t)^\ell}.
$$

Let us remark that such space is similar to the space used in [XXI] and defined after Corollary 3.1.

We assume throughout that the cost coefficients are bounded and that the dependence of an individual player's cost function on the average action is weak enough. The weak interaction condition is consistent with the game theory literature on mean-field type games where some form of moderate dependence condition is usually required to prove the existence and uniqueness of Nash equilibria; see [161] and references therein. The condition is also consistent with the monotonicity condition for FBSDE systems originally proposed by [165, 285] and slightly weaker than the generalizations to mean-field type FBSDEs established in [36, 58]. Specifically, we assume that the following condition is satisfied.

(F1) The processes κ, γ, $1/\gamma$, η and $1/\eta$ belong to $\mathbb{L}^\infty_{\mathbb{F}}([0,T]\times\Omega;[0,\infty))$ and $\mathcal{X}\in\mathbb{L}^2(\Omega)$ is independent of W and W^0.

(F2) There exists a constant $\theta > 0$ such that

$$
\tag{5.39}
\frac{\|\kappa\|}{4\eta_\star} < \theta < \frac{4\gamma_\star}{\|\kappa\|}.
$$

The following quantity will be important in our subsequent analysis:

$$\lambda := \eta_\star/\|\eta\| \in (0,1]. \tag{5.40}$$

We are now ready to state our first major result.

Theorem 5.6 *Under Assumptions* **(F1)** *and* **(F2)***, there exists a unique solution*

$$(X,Y,Z) \in \mathcal{H}_\lambda \times \mathbb{L}^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R}) \times \mathbb{L}^2_{\mathbb{F}}([0,T-]\times\Omega;\mathbb{R}^m)$$

to the FBSDE (5.38)*. Moreover, the process*

$$\alpha^* = \frac{Y}{2\eta}$$

is an optimal control for the representative player and the aggregation effect given by

$$\mu^* := \mathbb{E}[\alpha^*|\mathcal{F}^0], \quad t \in [0,T)$$

is the solution to the MFG (5.35)*.*

Proof. Decoupling the FBSDE (5.38) by $Y = AX + B$ yields the following system of Riccati type equations:

$$\begin{cases} -dA_t = \left(2\gamma_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A \, d\widetilde{W}_t, \\ -dB_t = \left(\kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}(A_tX_t + B_t)\middle| \mathcal{F}_t^0\right] - \dfrac{A_tB_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t, \\ A_T = \infty \\ B_T = 0. \end{cases} \tag{5.41}$$

The existence of a unique solution $A \in \mathcal{M}_{-1}$ to the first equation is established in Theorem 3.1, together with [148, Theorems 6.1 and 6.3]. Namely, there exists a unique process (A, Z^A) such that $A \in \mathcal{M}_{-1}$, $Z^A \in \mathbb{L}^2_{\mathcal{F}}([0,T-];\mathbb{R}^m)$, the dynamics is given on any interval $[0,\tau]$, $\tau < T$ by the first equation of (5.41) and $\lim_{t\to T} A_t = +\infty = A_T$. Moreover A satisfies the a priori estimate (3.10), namely in this case

$$\frac{1}{\mathbb{E}\left[\int_t^T \frac{1}{2\eta_s}\, ds \middle| \mathcal{F}_t\right]} \le A_t \le \frac{1}{(T-t)^2}\mathbb{E}\left[\int_t^T 2\eta_s + 2(T-s)^2\gamma_s \, ds \middle| \mathcal{F}_t\right]. \tag{5.42}$$

The upper estimate is the same as (3.10) adapted to this particular setting. The lower bound is given by (3.16).

Hence we need to solve the following FBSDE:

$$\begin{cases} dX_t = -\dfrac{1}{2\eta_t}(A_tX_t + B_t)\, dt, \\ -dB_t = \left(\kappa_t \mathbb{E}\left[\dfrac{1}{2\eta_t}(A_tX_t + B_t)\middle| \mathcal{F}_t^0\right] - \dfrac{A_tB_t}{2\eta_t}\right) dt - Z_t^B \, d\widetilde{W}_t, \\ X_0 = \mathcal{X} \\ B_T = 0. \end{cases} \tag{5.43}$$

Our approach is based on an extension of the method of continuation that accounts for the singularity of the process A at the terminal time and hence for the singularity in the driver of the FBSDE. We apply to the method of continuation to the triple $(X, B, Y = AX + B)$ rather than the pair (X, B), and search for solutions

$$(X, B, Y = AX + B) \in \mathcal{H}_\lambda \times \mathcal{H}_\zeta \times \mathbb{L}^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}),$$

where λ was defined in (5.40) and ζ is any constant such that: $0 < \zeta < \lambda \wedge 1/2$. Specifically, the method of continuation will be applied to the FBSDE

$$\begin{cases} dX_t = -\dfrac{1}{2\eta_t}(A_t X_t + B_t)\,dt, \\ -dB_t = \left(\kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}(A_t X_t + B_t)\middle|\mathcal{F}^0_t\right] + f_t - \dfrac{A_t B_t}{2\eta_t}\right)dt - Z^B_t\,d\widetilde{W}_t, \\ dY_t = \left(-2\lambda_t X_t - \kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{A_t X_t + B_t}{2\eta_t}\middle|\mathcal{F}^0_t\right] - f_t\right)dt + Z^Y_t\,d\widetilde{W}_t, \\ X_0 = \mathcal{X} \\ B_T = 0, \end{cases} \tag{5.44}$$

where $\mathfrak{p} \in [0,1]$, $f \in \mathbb{L}^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$. We emphasise that the first two equations hold on $[0,T]$, while the third equation holds on $[0,T)$.

Note that we have an a priori estimate on the processes Z^B and Z^Y. Indeed assume that $f \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$ and that there exists a solution (X, B, Y, Z^B, Z^Y) to (5.44) such that

$$(X, B, Y) \in \mathcal{H}_\lambda \times \mathcal{H}_\zeta \times S^2_{\mathbb{F}}([0,T-] \times \Omega, \mathbb{R}).$$

Then

$$(Z^B, Z^Y) \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$$

and there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\int_0^T |Z^B_t|^2\,dt\right] \le C\left(\|B\|^2_\zeta + \|X\|^2_\lambda + \mathbb{E}\left[\int_0^T |f_t|^2\,dt\right]\right)$$

and such that for each $\tau < T$

$$\mathbb{E}\left[\left|\int_0^\tau |Z^Y_s|^2\,ds\right|^2\right] \le C\left(\mathbb{E}\left[\sup_{0\le t\le\tau} |Y_t|^2\right] + \|X\|^2_\lambda + \|B\|^2_\zeta + \mathbb{E}\left[\int_0^T |f_t|^2\,dt\right]\right).$$

In particular, $\int_0^\cdot Z^B_s\,d\widetilde{W}_s$ is a true martingale on $[0,T]$ and $\int_0^\cdot Z^Y_s\,d\widetilde{W}_s$ is a true martingale on $[0,\tau]$, for each $\tau < T$. Hence in the proof, we "forget" Z^B and Z^Y.

Now for $\mathfrak{p} = 0$ there exists for every given data $f \in L^2_{\mathbb{F}}([0,T]\times\Omega; \mathbb{R})$ a unique solution $(X, B, Y, Z^B, Z^Y) \in \mathcal{H}_\lambda \times \mathcal{H}_\zeta \times \mathbb{D}^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$ to (5.44). It is given by

$$\begin{cases} B_t = \mathbb{E}\left[\int_t^T f_s e^{-\int_t^s (2\eta_r)^{-1} A_r\,dr}\,ds\middle|\mathcal{F}_t\right], \quad t \in [0,T] \\ X_t = \mathcal{X} e^{-\int_0^t (2\eta_r)^{-1} A_r\,dr} - \int_0^t (2\eta_s)^{-1} B_s e^{-\int_s^t (2\eta_r)^{-1} A_r\,dr}\,ds, \quad t \in [0,T] \\ Y_t = A_t X_t + B_t, \quad t \in [0,T), \end{cases}$$

and $Z^B \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m)$ and $Z^Y \in L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$ are given by the martingale representation theorem.

Then we prove that if for some $\mathfrak{p} \in [0,1]$ the FBSDE (5.44) is for every data $f \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R})$ uniquely solvable in $\mathcal{H}_\lambda \times \mathcal{H}_\zeta \times \mathbb{D}^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$, then this holds also for $\mathfrak{p} + \mathfrak{d}$ with $\mathfrak{d} > 0$ small enough (independent of $\mathfrak{p}$ and f).

Using our a priori estimates on Z^B and Z^Y, and by induction on $\mathfrak{p}$, we obtain that there exists a unique solution $(X, B, Y, Z^B, Z^Y) \in \mathcal{H}_\lambda \times \mathcal{H}_\zeta \times D^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \times L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$

to the FBSDEs (5.38) and (5.43). Moreover, there exists a constant $C > 0$ depending on η, γ, κ, T and $\|\mathcal{X}\|_{L^2}$, such that

$$\|X\|_{\mathcal{H}_\lambda} + \|B\|_{\mathcal{H}_\zeta} + \mathbb{E}\left[\int_0^T |Y_t|^2\, dt\right] \leq C.$$

From the equation (5.37) we obtain the following candidates of the optimal portfolio process and the optimal trading strategy for the representative player:

$$\begin{aligned} X_t^* &= \mathcal{X} e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} - \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\, dr}\, ds, \\ \xi_t^* &= \mathcal{X} e^{-\int_0^t \frac{A_r}{2\eta_r}\, dr} \frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} - \frac{A_t}{2\eta_t} \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r}\, dr}\, ds. \end{aligned} \tag{5.45}$$

By construction, $X_T^* = 0$ and hence ξ^* is an admissible liquidation strategy. We need to show that it is indeed the optimal liquidation strategy and that its conditional expectation defines the desired equilibrium for our MFG, to finish the proof of the theorem. The arguments are somehow similar to [10, 148]. □

Our verification argument implies that the value function is given by

$$V(\mathcal{X}; \mu^*) = \frac{1}{2} A_0 \mathcal{X}^2 + \frac{1}{2} B_0 \mathcal{X} + \frac{1}{2} \mathbb{E}\left[\int_0^T \kappa_s X_s^* \mu_s^*\, ds \middle| \mathcal{X}\right]. \tag{5.46}$$

Particular cases

The benchmark case where all players share the same information, except for their initial value can be analyzed in greater detail. In this section we therefore assume that all randomness is generated by the common Brownian motion W^0 and the initial value $\mathcal{X}$.

(F3) The processes κ, γ, η and $1/\eta$ belong to $\mathbb{L}^\infty_{\mathbb{F}^0}([0, T] \times \Omega; [0, \infty))$.

The weak interaction condition (5.39) is not required here. Under the common information assumption the conditional mean-field FBSDE (5.38) reduces to the following FBSDE:

$$\begin{cases} dX_t = -\dfrac{Y_t}{2\eta_t}\, dt, \\ -dY_t = \left(\dfrac{\kappa_t}{2\eta_t} \mathbb{E}\left[Y_t \middle| \mathcal{F}_t^0\right] + 2\gamma_t X_t\right) dt - Z_t\, dW_t^0, \\ X_0 = \mathcal{X}, \\ X_T = 0. \end{cases} \tag{5.47}$$

If we further assume that the initial portfolio is common to all players, i.e. $\mathcal{X} = x \in \mathbb{R}$, then all processes are $\mathbb{F}^0$-adapted and the consistency condition reads $\mu = \alpha^*$. The mean-field FBSDE (5.47) simplifies to a regular FBSDE. The linear decoupling $Y = A^\kappa X$ yields to a BSDE with singular terminal condition:

$$-\, dA_t^\kappa = \left(2\gamma_t + \frac{\kappa_t A_t^\kappa}{2\eta_t} - \frac{(A_t^\kappa)^2}{2\eta_t}\right) dt - Z_t^{A^\kappa} dW_t^0, \quad A_T^\kappa = \infty. \tag{5.48}$$

This equation has a unique solution (see [XX]). And

$$X_t^* = x e^{-\int_0^t \frac{A_r^\kappa}{2\eta_r}\,dr}.$$

The processes A^κ, X^*, $Y^* = A^\kappa X^*$ and $\alpha^* = \mu^* = \frac{Y^*}{2\eta}$ have the same sign as x. $\alpha^*(= \mu^*)$ is an admissible optimal control as well as the equilibrium to MFG (5.35).

Let us now return to the problem (5.47). From the solution to the complete common information problem, we deduce that

$$\mu_t^* = \frac{1}{2\eta_t}\mathbb{E}(Y_t \big| \mathcal{F}_t^0) = \frac{\mathbb{E}[\mathcal{X}]}{2\eta_t} A_t^\kappa e^{-\int_0^t \frac{A_r^\kappa}{2\eta_r}\,dr},$$

where A^κ solves the BSDE (5.48). Making the affine ansatz that $Y = AX + B$, we obtain (see equation (5.41) below) that

$$\begin{cases} -dA_t = \left(2\gamma_t - \dfrac{A_t^2}{2\eta_t}\right) dt - Z_t^A\, dW_t^0, \quad A_T = +\infty \\ -dB_t = \left(\kappa_t \mu_t^* - \dfrac{A_t B_t}{2\eta_t}\right) dt - Z_t^B\, dW_t^0, \quad B_T = 0. \end{cases} \tag{5.49}$$

Note that A and B are $\mathbb{F}^0$-adapted and that $A = A^0$. Thereby we have an explicit solution: for $t \in [0, T]$

$$\begin{cases} B_t = \mathbb{E}\left[\displaystyle\int_t^T \kappa_s \mu_s^* e^{-\int_t^s (2\eta_r)^{-1} A_r\,dr}\, ds \middle| \mathcal{F}_t\right], \\ X_t^* = \mathcal{X} e^{-\int_0^t (2\eta_r)^{-1} A_r\,dr} - \displaystyle\int_0^t (2\eta_s)^{-1} B_s e^{-\int_s^t (2\eta_r)^{-1} A_r\,dr}\, ds, \\ Y_t^* = A_t X_t^* + B_t. \end{cases}$$

From Theorem 5.6 the system (5.49) has a unique solution from which we deduce that the optimal state process for a given initial position $\mathcal{X} = x \in \mathbb{R}$ is given by:

$$X_t^{*,x} = (x - \mathbb{E}[\mathcal{X}]) e^{-\int_0^t (2\eta_r)^{-1} A_r\,dr} + \mathbb{E}[\mathcal{X}] e^{-\int_0^t (2\eta_r)^{-1} A_r^\kappa\,dr}. \tag{5.50}$$

Thus, if different players hold different initial portfolios, then a trader's optimal position consists of a weighted sum of the competitors' average portfolio size $\mathbb{E}[\mathcal{X}]$ and the deviation of the own initial position from that average.

Note that the process A^κ is increasing in κ. In particular, $A^\kappa \geq A$. Moreover $A_0^\kappa > A_0$ if $\kappa > 0$ on some set of positive measure. Hence the dependence of the optimal portfolio process decreases if $\mathbb{E}[\mathcal{X}] > 0$. It also suggests that - contrary to the previous case - the sign of the optimal portfolio process X^* may change on the interval $[0, T]$. In fact, if $\mathbb{E}[\mathcal{X}] > 0$ and $x \geq \mathbb{E}[\mathcal{X}]$, then $X^{*,x}$ remains non-negative on $[0, T]$. However, if $0 < x < \zeta \mathbb{E}[\mathcal{X}]$ where $\zeta := 1 - \exp\left(\frac{A_0 - A_0^\kappa}{2\|\eta\|} t\right) > 0$, then $X^{*,x}$ becomes negative shortly after the initial time.

Here we consider a deterministic benchmark example that can be solved explicitly. Assume that processes λ, κ, η are positive constants. The Riccati equation (5.48) can be solved:

$$A_t^\kappa = 2\eta\theta \coth\left(\theta(T - t)\right) + \frac{\kappa}{2}$$

where

$$\theta := \sqrt{\frac{\gamma}{\eta} + \frac{\kappa^2}{16\eta^2}}.$$

If all players share the same initial portfolio, then

$$X_t^* = \exp\left(-\frac{\kappa}{4\eta}t\right)\frac{\sinh(\theta(T-t))}{\sinh(\theta T)}x \tag{5.51}$$

and the optimal liquidation rate is given by

$$\begin{aligned}\xi_t^* &= \left(\theta\coth(\theta(T-t)) + \frac{\kappa}{4\eta}\right)X_t^* \\ &= \exp\left(-\frac{\kappa}{4\eta}t\right)\left(\frac{\theta\cosh(\theta(T-t))}{\sinh(\theta T)} + \frac{\kappa\sinh(\theta(T-t))}{4\eta\sinh(\theta T)}\right)x.\end{aligned}$$

When $\kappa \to 0$, then $\xi_t^* \to \frac{\widetilde{\theta}\cosh(\widetilde{\theta}(T-t))}{\sinh(\widetilde{\theta}T)}x$ with $\widetilde{\theta} = \sqrt{\frac{\gamma}{\eta}}$. This corresponds to the benchmark model in [6]. This convergence can also be seen from Figure 1. Furthermore, we see that—as in the corresponding single player models—the optimal liquidation rate is always positive, i.e., round trips are not beneficial. Moreover, we notice that the portfolio process (5.51) corresponds to the optimal portfolio process in an Almgren–Chriss model with adjusted risk aversion $\widetilde{\gamma} = \gamma + \frac{\kappa^2}{16\eta}$ and with additional exponential decay of rate $\frac{\kappa}{4\eta}$.

When $\kappa \to \infty$, then $\xi_0^* \to \infty$ while $\xi_t^* \to 0$ for $t > 0$. That is, when the impact of interaction is very strong, then the players trade very fast initially and very slowly afterwards. The intuitive reason is that in this case an individual player would benefit from trading fast slightly before his competitors start trading in order to avoid the negative drift generated by the mean-field interaction. As all the players are statistically identical, they "coordinate" on an equilibrium trading strategy as depicted in Figure 2. Thus, our model provides a possible explanation for large price increases or decreases in markets with strategically interacting players with similar preferences.

If the players hold different initial portfolios, then (5.50) shows that the optimal portfolio process is given by

$$X_t^{*,x} = (x - \mathbb{E}[\mathcal{X}])\frac{\sinh(\widetilde{\theta}(T-t))}{\sinh(\widetilde{\theta}T)} + \exp\left(-\frac{\kappa}{4\eta}t\right)\frac{\sinh(\theta(T-t))}{\sinh(\theta T)}\mathbb{E}[\mathcal{X}].$$

Figure 5.2 confirms that the sign of X^* is indeed changing when x is small.

5.4.1 Approximation by unconstrained MFGs

The solvability of the control problem with constraint in the preceding sections 5.1, 5.2 and 5.3 has been proved using a penalization scheme. Let us show a similar result here, namely the solution to our singular MFG can be approximated by the solutions

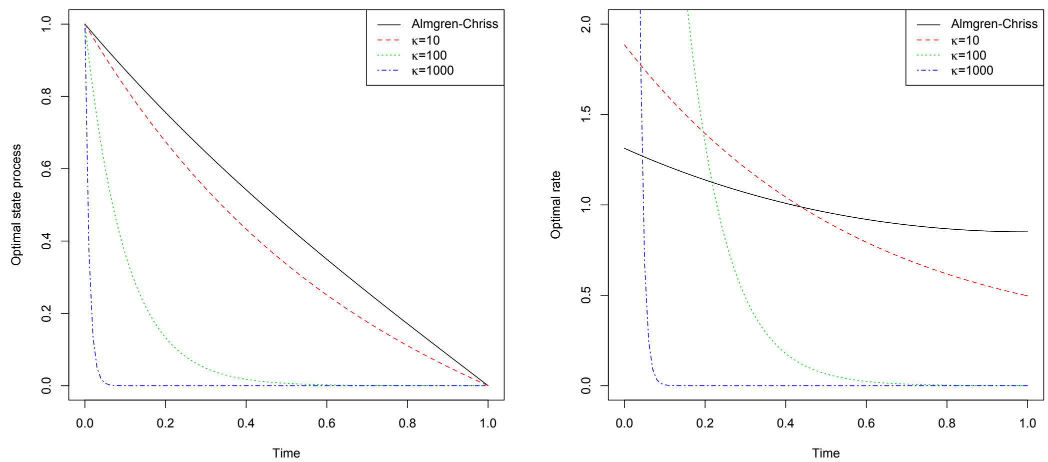


Figure 5.1: Current state X^* (left) and optimal liquidation rate α^* (right) corresponding to parameters $T = 1$, $x = 1$, $\gamma = 5$ and $\eta = 5$. The solid line corresponds to $\kappa = 0$, that is the Almgren-Chriss model with temporary impact.

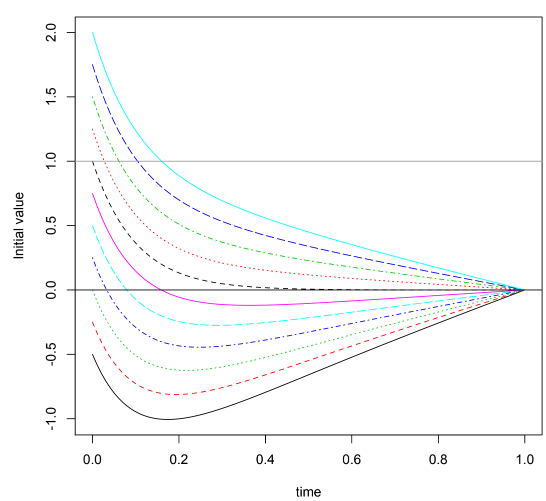


Figure 5.2: Current state $X^{*,x}$ corresponding to parameters $T = 1$, $\mathbb{E}[\mathcal{X}] = 1$, $\gamma = 5$, $\eta = 5$ and $\kappa = 100$ for different values of the initial portfolio x.

to non-singular MFGs under additional assumptions on the market impact parameter. Specifically, we consider the following unconstrained MFGs:

$$
(5.52) \qquad \begin{cases} 1. \text{ fix a process } \mu; \\ 2. \text{ solve the standard optimization problem: minimize} \\ \quad J^n(\xi;\mu) = \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \eta_t \alpha_t^2 + \gamma_t X_t^2\right) dt + n X_T^2 \middle| \mathcal{X} \right] \\ \quad \text{such that } dX_t = -\alpha_t \, dt, \quad X_0 = \mathcal{X}; \\ 3. \text{ search for the fixed point } \mu_t = \mathbb{E}[\alpha_t^{*,\mathcal{X}} | \mathcal{F}_t^0], \text{ for } a.e. \ t \in [0,T] \end{cases}
$$

We will need the following assumption on the solution $A \in \mathcal{M}_{-1}$ to the first equation in (5.41) with the terminal condition $+\infty$. It implies in particular that $X^* \in \mathcal{H}_1$.

(F4) There exists a constant C such that for any $0 \leq r \leq s < T$

$$
\exp\left(- \int_r^s \frac{A_u}{2\eta_u} du \right) \leq C \left(\frac{T-s}{T-r} \right).
$$

In [XX], we provide several conditions on η such that this condition holds.

Using the same arguments as previously, the unconstrained control problem leads to the following conditional mean field FBSDE:

$$
(5.53) \qquad \begin{cases} dX_t^n = \left(- \frac{A_t^n X_t^n + B_t^n}{2\eta_t} \right) dt, \\ -dB_t^n = \left(- \frac{A_t^n B_t^n}{2\eta_t} + \kappa_t \mathbb{E}\left[\frac{A_t^n X_t^n + B_t^n}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \right) dt - Z_t^{B^n} d\widetilde{W}_t, \\ dY_t^n = \left(-2\gamma_t X_t^n - \kappa_t \mathbb{E}\left[\frac{A_t^n X_t^n + B_t^n}{2\eta_t} \middle| \mathcal{F}_t^0 \right] \right) dt + Z_t^{Y^n} d\widetilde{W}_t, \\ X_0^n = \mathcal{X}, \\ B_T^n = 0, \\ Y_T^n = 2n X_T^n, \end{cases}
$$

where

$$
(5.54) \qquad \begin{cases} -dA_t^n = \left\{ 2\gamma_t - \frac{(A_t^n)^2}{2\eta_t} \right\} dt - Z_t^{A^n} d\widetilde{W}_t, \\ A_T^n = 2n. \end{cases}
$$

The existence of a solution (A^n, Z^{A^n}) to the BSDE (5.54) can be deduced from standard results on monotone BSDEs (for example Theorem 2.1). Sequence $(A^n, \ n \geq 1)$ is a non-decreasing sequence converging pointwise to A. Using similar arguments as in the proof of Theorem 5.6, leads to the existence and uniqueness of the solution

$(X^n, B^n, Y^n, Z^{B^n}, Z^{Y^n})$ to the following FBSDE system:

$$
\begin{cases}
dX_t^n = & -\dfrac{1}{2\eta_t}(A_t^n X_t^n + B_t^n)\,dt, \\
-dB_t^n = & \left(\kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{1}{2\eta_t}(A_t^n X_t^n + B_t^n)\middle| \mathcal{F}_t^0\right] + f_t - \dfrac{A_t^n B_t^n}{2\eta_t}\right) dt - Z_t^{B^n}\, d\widetilde{W}_t, \\
dY_t^n = & \left(-2\gamma_t X_t^n - \kappa_t \mathfrak{p}\mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle| \mathcal{F}_t^0\right] - f_t\right) dt + Z_t^{Y^n}\, d\widetilde{W}_t, \\
X_0^n = \mathcal{X} & \\
B_T^n = 0, & \\
Y_T^n = 2nX_T^n. &
\end{cases} \tag{5.55}
$$

Moreover if $f \in L^2_{\mathbb{F}}([0,T]\times\Omega;\mathbb{R})$, there exists a constant $\overline{\mathfrak{C}} > 0$ such that

$$
\|X^n\|_{n,\lambda} + \|B^n\|_{n,\zeta} + \mathbb{E}\left[\int_0^T |Y_t^n|^2\, dt\right] \le \overline{\mathfrak{C}}, \tag{5.56}
$$

for any n. The key result about the convergence of the optimal position and control is the following:

Lemma 5.7 *Under Assumptions* **(F1)**, **(F2)** *and* **(F4)**

$$
\lim_{n\to+\infty}\left\{\mathbb{E}\left[\int_0^T |X_t^n - X_t^*|^2\, dt\right] + \mathbb{E}\left[\int_0^T |B_t^n - B_t^*|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Y_t^n - Y_t^*|^2\, dt\right]\right\} = 0.
$$

Let us denote by $V^n(\mathcal{X};\mu^n)$ the value function associated with the penalized problem (5.52). The next theorem shows the convergence of $V^n(\mathcal{X};\mu^n) := V^n(\mathcal{X})$ to the value function $V(\mathcal{X};\mu) := V(\mathcal{X})$ associated with the constrained MFG.

Theorem 5.7 *Under Assumptions* **(F1)**, **(F2)** *and* **(F4)**, *the value function $V^n(\mathcal{X})$ converges to $V(\mathcal{X})$ in $L^1(\Omega)$.*

The proof of convergence of the value function simplifies substantially under the common information assumption. In particular, Assumption **(F4)** is not necessary here.

5.4.2 Approximate Nash Equilibrium

We show that an ϵ-Nash equilibrium for the N player portfolio liquidation game can be constructed from the solution to the MFG (5.35) when the number of players is large if all players share the same cost structure.

(F5) Assume for any $i = 1, \cdots, N$, κ^i, η^i and λ^i admit the following expression

$$
\kappa_t^i = \kappa(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t}), \quad \eta_t^i = \eta(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t}), \quad \gamma_t^i = \gamma(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t})
$$

for some non-negative deterministic bounded and measurable functions κ, η and λ.

Under this condition, adapting the Yamada-Watanabe argument (e.g. [60] and [15]) leads to the existence of a function ϕ independent of $(\mathcal{X}, W, W^0)$ such that

$$\alpha^* = \phi(\mathcal{X}, W, W^0),$$

where α^* is given by Theorem 5.6. Thus under Assumption **(F5)** each player's unique best response $\alpha^{*,i}$ to the mean-field equilibrium μ^* can be represented in terms of the function ϕ as

$$\alpha^{*,i} \ := \phi(\mathcal{X}^i, W^0, W^i). \tag{5.57}$$

In particular, each individual action has the same distribution as the mean-field equilibrium:

$$\mu_t^* = \mathbb{E}[\alpha_t^{*,i} | \mathcal{F}_t^0], \quad \text{a.s. a.e.} \tag{5.58}$$

The proof of Theorem 5.6 guarantees the existence of a constant C such that

$$\mathbb{E}\left[\int_0^T |\alpha_t^{*,i}|^2 \, dt\right] \leq C, \tag{5.59}$$

and the proof of (5.57) yields a real-valued function ψ, which is independent of i, such that

$$\mathbb{E}\left[\int_0^T |\alpha_t^{*,i}|^2 \, dt \middle| \mathcal{X}^i = x^i\right] = \psi(x^i), \tag{5.60}$$

Before we prove the main result of this section, we recall the cost functional $J^{N,i}\left(\vec{\xi}\right)$ from (5.32).

Theorem 5.8 *Assume that Assumption* **(F5)** *is satisfied and that the admissible control space for each player* $i = 1, \ldots, N$ *is given by*

$$\mathcal{A}^i := \left\{\alpha \in \mathcal{A}_{\mathbb{F}^i}(x^i) : \mathbb{E}\left[\int_0^T |\alpha_t|^2 \, dt \middle| \mathcal{X}^i = x^i\right] \leq M(x^i)\right\}$$

for some fixed positive function M *such that* $\psi \leq M$*. Then, for each* $1 \leq i \leq N$ *and each* $\alpha^i \in \mathcal{A}^i$*,*

$$J^{N,i}\left(\vec{\alpha^*}\right) \leq J^{N,i}(\alpha^i, \alpha^{*,-i}) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $(\alpha^i, \alpha^{*,-i}) = (\alpha^{*,1}, \cdots, \alpha^{*,i-1}, \alpha^i, \alpha^{*,i+1}, \cdots, \alpha^{*,N})$ *and* $O\left(\frac{1}{\sqrt{N}}\right)$ *is to be interpreted as* $\frac{g(x_i)}{\sqrt{N}}$ *for some real-valued function* g *independent of* i*.*

Part II

Some problems in stochastic calculus

Chapter 6

Two other works on BSDEs

6.1 Measure solution ([IV])

The generally accepted natural framework for the most efficient formulation of pricing and hedging contingent claims on complete financial markets, for instance in the classical Merton-Scholes problem, is given by martingale theory, more precisely by the elegant notion of martingale measures. Martingale measures represent a view of the world in which price dynamics do not have inherent trends. From the perspective of this world, pricing a claim amounts to taking expectations, while hedging boils down to pure conditioning and using martingale representation.

At first glance, hedging a claim is, however, a problem calling upon stochastic control: it consists in choosing strategies to steer the portfolio into a terminal random endowment the portfolio holder has to ensure. Solving stochastic backward equations (BSDE) is a technique tailor-made for this purpose. Its particular significance for the field of utility maximization in financial stochastics was clarified in El Karoui, Peng and Quenez [120]. To fix ideas, we restrict our attention to a Wiener space probabilistic environment. Recall that in this framework, the BSDE (2.1) with terminal variable ξ at time horizon T and generator f is solved by a pair of processes (Y, Z) on the interval $[0, T]$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

In the case of vanishing generator, the solution just requires an application of the martingale representation theorem in the Wiener filtration, and Z will be given as the stochastic integrand in the representation, to which we will refer as *control process* in the sequel.

Here we are looking for a notion in the context of BSDE that plays the role of the martingale measure in the context of hedging claims. Our main interest is directed to BSDE of the type (2.1) with generators that are non-Lipschitzian, and depend on the control variable z quadratically, typically $f(s, y, z) = z^2\, b(s, z), s \in [0, T], z \in \mathbb{R}$, with a bounded function b. These generators were given a through treatment in Kobylanski [199], Briand & Hu [46], and Lepeltier & San Martin [221]. While [199] and [221] consider existence and uniqueness questions for bounded terminal variables ξ, [46] goes to the limit of possible terminal variables by considering ξ for which $\exp(\gamma|\xi|)$ has finite expectation for some $\gamma > 2||b||_\infty$. All these papers employ different methods of approach following the classical pattern of arguments mentioned above. In contrast to this, we shall investigate an alternative notion of solution of BSDE, the generators of which fulfill similar conditions. In analogy with martingale measures in hedging which effectively eliminate drifts in price dynamics, we shall look for probability measures under which the generator of a given BSDE is seen as vanishing. Given such a measure $\mathbb{Q}$ which we call *measure solution* of the BSDE and supposing that $\mathbb{Q} \sim \mathbb{P}$, the processes Y and Z are the results of projection and representation respectively, i.e. $Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = Y_0 + \int_0^\cdot Z_s d\widetilde{W}_s$, where $\widetilde{W}$ is a Wiener process under $\mathbb{Q}$. The first main finding of our paper roughly states that provided the terminal variable ξ is bounded, all classical solutions can be interpreted as measure solutions. More precisely, we show that if the generator satisfies the usual continuity and quadratic boundedness conditions, classical solutions (Y, Z) exist if and only if measure solutions with $\mathbb{Q} \sim \mathbb{P}$ exist. So existence theorems obtained in the papers quoted are recovered in a more elegant and concise way in terms of measure solutions. We do not touch uniqueness questions in general. Of course, determining a measure $\mathbb{Q}$ under which the generator vanishes amounts to doing a Girsanov change of probability that eliminates it. We therefore have to look at the BSDE in the form

$$Y_t = \xi - \int_t^T Z_s \left[dW_s - \frac{f(s, Y_s, Z_s)}{Z_s} ds \right], \quad t \in [0, T], \tag{6.1}$$

define $g(s, y, z) = \dfrac{f(s, y, z)}{z}$, and study the measure

$$\mathbb{Q} = \exp\left(M - \frac{1}{2}\langle M \rangle \right) \cdot \mathbb{P}$$

for the martingale $M = \int_0^\cdot g(s, Y_s, Z_s) dW_s$. One of the fundamental problems that took some effort to solve consists in showing that $\mathbb{Q}$ is a probability measure. Here one has to dig essentially deeper than Novikov's or Kazamaki's criteria allow. We successfully employed a criterion which is based on the explosion properties of the quadratic variation $\langle M \rangle$, which we learnt from a conversation with M. Yor, and has been latent in the literature for a while, see Liptser & Shiryaev [229], or the paper by Wong & Heyde [324]. This criterion allows a simple treatment of the problem of existence of measure solutions in the case of bounded terminal variable, and a still elegant and efficient one in the borderline case of exponentially integrable terminal variable considered by Briand & Hu [46]. If ξ is only exponentially bounded, things turn essentially more complex immediately. Specializing to a very simple generator, we find a wealth of different situations looking confusing at first sight. Just to quote three basic scenarios exhibited in a series of examples of different types: in the first type we obtain one solution which is a measure solution at the same time; in the second one we find two different solutions both of which are measure solutions; in the third one we encounter two solutions one of which is a measure solution, while the other one is not. We even combine these basic examples to develop a scenario in which there exists a continuum of measure solutions, and another one in which a continuum of non-measure solutions is given.

Measure solutions: definition and first examples

Throughout the Brownian motion $(W_t)_{0\le t\le T}$ is one-dimensional, which generates the filtration $\mathbb{F} = (\mathcal{F}_t,\ t \in [0,T])$, and let $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for all $z \in \mathbb{R}$ the mapping $f(\cdot,\cdot,z)$ is predictable. If ξ is square integrable and f satisfies a Lipschitz condition **(A2')** and **(A4)**, then we already know that there exists a unique pair $(Y,Z) \in \mathbb{D}^2 \times \mathbb{H}^2$ solving the BSDE (2.1). Recall that the solution process Y has a nice representation as a conditional expectation with respect to a new probability measure if f is a linear function of the form

$$f(s,z) = b_s z, \tag{6.2}$$

where b is a predictable and bounded process. More precisely, if $D_t = \exp(\int_0^t b_s dW_s - \frac{1}{2}\int_0^t b_s^2 ds)$, and $\mathbb{Q}$ is the probability measure with density $\mathbb{Q} = D_T \cdot \mathbb{P}$, then

$$Y_t = \mathbb{E}^{\mathbb{Q}}[\xi|\mathcal{F}_t]. \tag{6.3}$$

In the following we will discuss whether Y still can be written as a conditional expectation of ξ if f does not have a representation as in (6.2) with b bounded, but satisfies only a quadratic growth condition in z. We aim at finding sufficient conditions guaranteeing that the process Y of a classical solution of a quadratic BSDE has a representation as a conditional expectation of ξ with respect to a new probability measure. For this purpose we consider the class of generators $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, satisfying for some constant $c \in \mathbb{R}_+$,

(G1) $f(s,z) = f(\cdot,s,z)$ is adapted for any $z \in \mathbb{R}$,

(G2) $g(s,z) = \frac{f(s,z)}{z}$, $z \in \mathbb{R}$, is continuous in z, for all $s \in [0,T]$,

(G3) $|f(s,z)| \le c(1+z^2)$ for any $s \in [0,T], z \in \mathbb{R}$,

(G4) there exists $\varepsilon > 0$ and a predictable process $(\psi_s)_{s\ge 0}$ such that $\int_0^\cdot \psi_s dW_s$ is a BMO-martingale and for every $|z| \le \varepsilon$, $|g(s,z)| \le \psi_s$.

We introduce for BSDEs with generators satisfying **(G1)** to **(G4)**, our concept of measure solutions.

Definition 6.1 *A triplet* $(Y,Z,\mathbb{Q})$ *is called* measure solution *of BSDE* (2.1)*, if* $\mathbb{Q}$ *is a probability measure on* $(\Omega,\mathcal{F})$*,* (Y,Z) *a pair of* $\mathbb{F}$*–predictable stochastic processes such that* $\int_0^T Z_s^2 ds < \infty$*,* $\mathbb{Q}$*-a.s. and the following conditions are satisfied:*

$$\begin{aligned}
\widetilde{W} &= W - \int_0^\cdot g(s,Z_s)ds \quad \textit{is a} \quad \mathbb{Q}- \textit{ Brownian motion},\\
\xi &\in L^1(\Omega,\mathcal{F},\mathbb{Q}),\\
Y_t &= \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t) = \xi - \int_t^T Z_s d\widetilde{W}_s, \quad t \in [0,T].
\end{aligned}$$

We want to thank Jianfeng Zhang who told us of the paper [238]. Indeed [238, Definition 2.3] with $\hat{h} = 0$ and $\hat{b} = g$, is the same as the previous definition.

It is known from the literature that if the terminal condition ξ is bounded and the generator f satisfies Assumptions **(G1)** to **(G4)**, then BSDE (2.1) has a classical solution (Y, Z) (see for example [199]). We show that in this case there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution.

Theorem 6.1 *Assume that ξ is bounded, and that f satisfies Assumptions* **(G1)** *to* **(G4)**. *Then for every classical solution (Y, Z), there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of* (2.1).

It is straightforward to see that every measure solution gives rise to a classical solution. Consequently, under the assumptions of Theorem 6.1, measure solutions exist if and if only classical solutions exist. More precisely, we obtain the following.

Corollary 6.1 *Assume that ξ is bounded, and that f satisfies Assumptions* **(G1)** *to* **(G4)**. *Then (Y, Z) is a classical solution if and only if there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of* (2.1).

We remark that the previous results can be extended to the case where W is a d-dimensional Brownian motion. Let $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a generator for which there exists a constant $c \in \mathbb{R}_+$ such that

$$|f(s, z)| \leq c(1 + |z|^2), \qquad s \in [0, T], z \in \mathbb{R}^d,$$

and assume that $g : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a function that is continuous in z and satisfies

$$\langle z, g(s, z)\rangle = f(s, z), \quad \text{for all } z \in \mathbb{R}^d \text{ and } s \in [0, T]. \tag{6.4}$$

If ξ is bounded and $\mathcal{F}_T$-measurable, then one can show with similar arguments that, starting from a classical solution (Y, Z), there exists a probability measure $\mathbb{Q}$ such that $W - \int_0^{\cdot} g(s, Z_s)\, ds$ is a $\mathbb{Q}$-Brownian motion, and $Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t)$.

Notice that the relation (6.4) may be satisfied by more than one continuous g, and consequently there may exist more than one measure solution in the multidimensional case. For example, let $d = 2$, $f(s, z) = z_1 z_2$, and observe that $|f(z)| \leq \frac{1}{2}|z|^2$. For any $a \in (0, \infty)$ let $g_a(z) = (a z_1, \frac{1}{a} z_2)$. Then, we have $\langle z, g_a(s, z)\rangle = f(s, z)$, and thus there exist more than one measure solution for a BSDE with generator f and a bounded terminal condition ξ.

6.1.1 Measure solutions of quadratic BSDEs with unbounded terminal condition

Here we discuss quadratic BSDEs with terminal conditions that are not bounded. As is known from literature, see for example Briand & Hu [46, 47], this case is by far more complex. For example, it is here that even if the generators are smooth, solutions stop to be unique. We shall exhibit examples below which complement the result shown in [47], according to which uniqueness is granted in case the generator of the BSDE possesses additional convexity properties, and the terminal variable possesses exponential moments of all orders. This fact underlines that also variations in the generator affect questions of existence and uniqueness of solutions a lot. For this reason, and also to keep better oriented on a windy track with many bifurcations, in the next section we shall choose a simpler generator, and assume that our generator is given by

$$f(s,z) = \alpha z^2.$$

We shall further assume without loss of generality that $\alpha > 0$. This can always be obtained in our BSDE by changing the signs of ξ, and the solution pair (Y, Z).

Exponentially integrable lower bounded terminal variable

Nonetheless, it turns out that positive and negative terminal variables need a separate treatment. We first show the existence of measure solutions for terminal conditions ξ bounded from below. Note that by a linear shift of Y we may assume that $\xi \geq 0$. We shall further work under exponential integrability assumptions in the spirit of [46]. According to this paper, exponential integrability of the terminal variable of the form

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{6.5}$$

for some $\gamma > 2\alpha$ is sufficient for the existence of a solution.

Under the exponential integrability assumption $\mathbb{E}(\exp(2\alpha\xi)) < \infty$, we now derive measure solutions from given classical solutions. Leaving the difficult question of uniqueness apart for a moment, we remark that with our simple generator, we obtain an explicit solution given by the formula

$$Y_t = \frac{1}{2\alpha}\ln M_t - \frac{1}{2\alpha}\ln M_0, \quad Z_t = \frac{1}{2\alpha}\frac{H_t}{M_t}, \tag{6.6}$$

where

$$M_t = \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t) = M_0 + \int_0^t H_s dW_s, \quad t \in [0,T].$$

In the sequel, we shall work with this explicit solution. In the following lemma, we prove integrability properties for the square norm of Z which will be crucial for stating the martingale property of M and other related processes later.

Lemma 6.1 *For any $p \geq 1$ we have*

$$\mathbb{E}\left[\left(\int_0^T Z_s^2 ds\right)^p\right] < \infty.$$

In particular, $\int_0^\cdot Z_s dW_s$ is a uniformly integrable martingale.

We prove that (Y, Z) gives rise to a measure solution.

Theorem 6.2 *Assume that ξ is non-negative and satisfies $\mathbb{E}(\exp(2\alpha\xi)) < \infty$. Suppose that (Y, Z) are defined as in (6.6). Then there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of (2.1).*

As a by-product of our main result, we obtain the exponential integrability of the quadratic variation of Z.

Corollary 6.2 *Under the conditions of Theorem 6.2 we have*

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha^2\int_0^T Z_s^2\,ds\right) < \infty.$$

Let us exhibit an example to show that one cannot go essentially beyond this condition without losing solvability. Let $T = 1$ and $\alpha = \frac{1}{2}$. Let us first consider

$$\xi = \frac{(W_1)^2}{2}.$$

It is immediately clear from the fact that W_1 possesses the standard normal density, that $\mathbb{E}\exp(2\alpha|\xi|) = \infty$, hence of course also for $\gamma > 2\alpha$, (6.5) is not satisfied. To find a solution (Y, Z) of (2.1) on any interval $[t, 1]$ with $t > 0$ define

$$Z_s = \frac{W_s}{s}, \quad s > 0,$$

and set for completeness $Z_0 = 0$. Let $t > 0$ and use the product formula for Itô integrals to deduce

$$\begin{aligned}\int_t^1 Z_s dW_s &= \frac{1}{2}\frac{W_s^2}{s}\Big|_t^1 + \frac{1}{2}\int_t^1 \frac{W_s^2}{s^2} ds \\ &= \xi - \frac{1}{2}\frac{W_t^2}{t} + \frac{1}{2}\int_t^1 Z_s^2 ds.\end{aligned}$$

This means that, if we set for convenience again $Y_0 = 0$, the pair of processes

$$(Y_s, Z_s) = \left(\frac{1}{2}\frac{W_s^2}{s}, \frac{W_s}{s}\right), s \in [0, 1],$$

solves the BSDE (2.1) on $[t, 1]$ for any $t > 0$. Of course, the definition of Y_0 is totally inconsistent with the BSDE. Worse than that, Z is not square integrable on $[0, 1]$, as is well known from the path behavior of Brownian motion. Hence (Y, Z) is not a solution of (2.1). To put it more strictly, there is no classical solution of (2.1) on $[0, 1]$, since, due to local Lipschitz conditions, any such solution would have to coincide with (Y, Z) on any interval $[t, 1]$ with $t > 0$.

According to Jeulin & Yor [177], transformations of this type are related to a phenomenon they call *appauvrissement de filtrations*. In fact, if $\frac{1}{2}$ is replaced with a parameter λ, they show that the natural filtration of the transformed process gets poorer than the one of the Wiener process, if and only if $\lambda > \frac{1}{2}$. Hence in the case we are interested in the Wiener filtration is preserved.

Let us now reduce the factor of $(W_1)^2$ in the definition of ξ a bit, to show that solutions exist in this setting. For $k \in \mathbb{N}$, let

$$\xi_k = \frac{W_1^2}{2(1 + 1/k)},$$

and consider the BSDE (2.1) with the generator f chosen above, and terminal condition ξ_k. In this setting, we clearly have

$$\mathbb{E}\exp(\gamma\xi_k) < \infty \quad \text{for} \quad 2\alpha \le \gamma < 2\alpha(1+1/k).$$

This shows that the condition of [46] is satisfied. It is not hard to construct the solutions of the corresponding BSDEs explicitly, in the same way as above. In fact, for $k \in \mathbb{N}$ we may define $f_k(t) = \frac{1}{k} + t, t \in [0,1]$, and set

$$Z_t^k = \frac{W_t}{f_k(t)}, \quad t \in [0,1].$$

We may then repeat the product formula for Itô integrals argument used above to obtain for $t \ge 0$

$$\begin{aligned}\int_t^1 Z_s^k dW_s &= \frac{1}{2}\frac{W_s^2}{f_k(s)}\Big|_t^1 + \frac{1}{2}\int_t^1 \frac{W_s^2 f_k'(s)}{f_k(s)^2} ds \\ &= \frac{1}{2}\frac{W_1^2}{f_k(1)} - \frac{1}{2}\frac{W_t^2}{f_k(t)} + \frac{1}{2}\int_t^1 (Z_s^k)^2 ds.\end{aligned}$$

Hence we set

$$Y_t^k = \frac{1}{2}\frac{W_t^2}{f_k(t)}, \quad t \in [0,1],$$

to identify the pair of processes (Y^k, Z^k) as a solution of the BSDE

$$Y_t^k = \xi_k - \int_t^1 Z_s^k dW_s + \frac{1}{2}\int_t^1 (Z_s^k)^2 ds, \quad t \in [0,1]. \tag{6.7}$$

We do not know at this moment whether (2.1) possesses more solutions.

Exponentially integrable upper bounded terminal variable

Sticking with the positivity of α in the generator

$$f(s,z) = \alpha z^2, \quad s \in [0,T], z \in \mathbb{R}$$

we shall now consider terminal variables ξ that fulfill the exponential integrability condition (6.5), but are bounded above by a constant. Again, by a constant shift of the solution component Y, we can assume that the upper bound is 0, i.e. $\xi \le 0$. So fix a non-positive terminal variable ξ satisfying (6.5) for some $\gamma > 2\alpha$, and denote by (Y,Z) the pair of processes given by the explicit representation of (6.6) solving our BSDE according to [46]. With respect to the following probability measure, ξ will effectively change its sign, so that we can hook up to the previous discussion. Recall $S = \int_0^\cdot Z_s dW_s$.

Lemma 6.2 *Let $V = \exp(2\alpha S - 2\alpha^2\langle S\rangle)$. Then V is a martingale of class (D), and consequently*

$$R = V_T \cdot \mathbb{P}$$

is a probability measure equivalent to $\mathbb{P}$. *Moreover,*

$$W^R = W - 2\alpha \int_0^{\cdot} Z_s ds$$

is a Brownian motion under R.

Now consider our BSDE under the perspective of the measure R with respect to the Brownian motion W^R. We may write

$$Y = \xi - \int_{\cdot}^{T} Z_s dW_s + \alpha \int_{\cdot}^{T} Z_s^2 ds = \xi - \int_{\cdot}^{T} Z_s dW_s^R - \alpha \int_{\cdot}^{T} Z_s^2 ds. \tag{6.8}$$

But this just means that by switching signs in (Y, Z), we may return, under the new measure R, to our old BSDE with ξ replaced with $-\xi$. So our measure change puts us back into the framework of the previous subsection, and we may resume our arguments there by setting

$$S^R = -\int_0^{\cdot} Z_s dW_s^R.$$

We need an analogue of Lemma 6.1 to guarantee that R is a uniformly integrable martingale.

Lemma 6.3 *For any* $p \geq 1$ *we have*

$$\mathbb{E}^R \left[\left(\int_0^T Z_s^2 ds \right)^p \right] < \infty.$$

In particular, S^R *is a uniformly integrable martingale under* R.

We are in a position to prove the main result of this subsection.

Theorem 6.3 *Assume that that* f *satisfies* $f(s, z) = \alpha z^2$, $z \in \mathbb{R}, s \in [0, T]$, *and that* ξ *is bounded above and satisfies* (6.5). *Then there is a measure solution of* (2.1) *with a measure* $\mathbb{Q}$ *that is equivalent to* $\mathbb{P}$.

Remark 6.1 *The results of the preceding two subsections clearly call for similar ones for our BSDE with exponentially integrable terminal variable that are not bounded. Due to the nonlinearity of the generator of the BSDE, it seems impossible to derive such properties by combining the results of Theorems 6.2 and 6.3.*

6.1.2 Uniqueness: quadratic BSDE with two solutions

Let us now come back to the question of uniqueness of solutions, and their measure solution property. In [46], the existence of solutions (Y, Z) in the usual sense is proved, given that (6.5) is satisfied. In a setting with more general generators the nonlinear z-part being bounded by αz^2, they provide pathwise upper and lower bounds for Y, given by the known explicit solution for this generator

$$\left(\frac{1}{2\alpha} \log \mathbb{E} \left[\exp(2\alpha\xi) \middle| \mathcal{F}_t \right] \right)_{t \in [0,T]}$$

used above, and its negative counterpart

$$\left(-\frac{1}{2\alpha}\log \mathbb{E}\left[\exp(-2\alpha\xi)\Big|\mathcal{F}_t\right]\right)_{t\in[0,T]}.$$

Briand & Hu [47] also provide a uniqueness result for the same setting, which is satisfied under the stronger integrability hypothesis

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{6.9}$$

for all $\gamma > 0$ and a convexity assumption concerning the generator. Let us start our discussion of uniqueness and the measure solution property by giving some examples.

For $b > 0$, let $\tau_b = \inf\{t \geq 0 : W_t \leq bt - 1\}$. We first consider a BSDE with random time horizon τ_b. Let the generator be further specified by $\alpha = \frac{1}{2}$. Let $\xi = 2a(b-a)\tau_b - 2a$, where $a > 0$. It will become clear along the way why this choice of terminal variable is made. In the first place, it is motivated by the striking simplicity of the solutions we shall construct. We shall in fact give two explicit solutions of the BSDE

$$Y_{t\wedge\tau_b} = \xi - \int_t^{\tau_b} Z_s dW_s + \int_t^{\tau_b} \frac{1}{2} Z_s^2 ds. \tag{6.10}$$

Appropriate choices of a and b allow for terminal variables that are bounded below as well as bounded above. The fact that the time horizon is random is not crucial. Indeed, by using a time change, any solution of Equation (6.10) can be transformed into a solution of a BSDE with the same generator and with time horizon 1. Define $\rho^{-1}(t) = \frac{t}{1-t}$, $t \in [0,1]$.

Lemma 6.4 *Let (Y_t, Z_t) be a solution of the BSDE (6.10), and let $\hat{\xi} = 2a(b-a)\frac{\hat{\tau}_b}{1-\hat{\tau}_b} - 2a$. Then $(y_t, z_t) = (Y_{\rho^{-1}(t)}, h(t)Z_{\rho^{-1}(t)})$ is a solution of the BSDE*

$$y_t = \hat{\xi} - \int_t^1 z_s d\tilde{W}_s + \int_t^1 \frac{1}{2} z_s^2 ds. \tag{6.11}$$

Let us first assess exponential integrability properties of ξ. For this, let $\gamma > 0$ be arbitrary. Then we have

$$\mathbb{E}e^{\gamma|\xi|} = \mathbb{E}e^{\gamma|2a(b-a)\tau_b - 2a|} \leq e^{2a\gamma}\mathbb{E}e^{\gamma 2a|b-a|\tau_b}.$$

Define the auxiliary stopping time

$$\sigma_b = \inf\{t \geq 0 : W_t \leq t - b\}.$$

It is well known and proved by the scaling properties of Brownian motion that the laws of τ_b and $\frac{\sigma_b}{b^2}$ are identical (see Revuz & Yor [300]). Moreover, the Laplace transform of σ_b is equally well known. According to [300] we therefore have for $\lambda > 0$

$$\mathbb{E}(\exp(-\lambda\tau_b)) = \mathbb{E}\left(\exp(-\frac{\lambda}{b^2}\sigma_b)\right) = \exp(-b[\sqrt{1 + \frac{2\lambda}{b^2}} - 1]). \tag{6.12}$$

Moreover, it is seen by analytic continuation arguments that this formula is even valid for $\lambda \geq -\frac{b^2}{2}$. Now choose $\lambda = -2a|b-a|\gamma$. Then the inequality

$$-2a|b-a|\gamma \geq -\frac{1}{2}b^2$$

amounts to

$$\gamma \leq \frac{b^2}{4a|b-a|}. \tag{6.13}$$

This in turn means that we have exponential integrability of orders bounded by $\frac{b^2}{4a|b-a|}$, in particular we may reach arbitrarily high orders by choosing a and b sufficiently close. But no combination of a and b allows exponential integrability of all orders. In the light of [47] this means that the entire field of pairs of positive a and b promises multiple solutions, and this is precisely what we will exhibit.

The first solution

It is clear from the definition that the pair (Y_t, Z_t), defined by

$$Y_t = 2aW_{t\wedge\tau_b} - 2a^2(\tau_b \wedge t), \quad \text{and} \quad Z = 2a1_{[0,\tau_b]},$$

is a solution of (6.10). To answer the question whether this defines a measure solution, we have to investigate

$$\mathbb{E}\exp\left[\int_0^{\tau_b}\frac{1}{2}Z_s dW_s - \frac{1}{8}\int_0^{\tau_b}Z_s^2 ds\right] = \mathbb{E}\exp\left[aW_{\tau_b} - \frac{a^2}{2}\tau_b\right] = \mathbb{E}(\exp(a(b-\frac{a}{2})\tau_b - a)).$$

Due to (6.12) we have

$$\mathbb{E}(\exp(a(b-\frac{a}{2})\tau_b - a)) = \exp(-b[\sqrt{1-\frac{2}{b^2}a(b-\frac{a}{2})}-1]-a) = \exp(-b[|1-\frac{a}{b}|-1]-a),$$

and the latter equals 1 in case $b \geq a$ and $\exp(2(b-a)) < 1$ in case $a > b$. This simply means that our first solution is a measure solution of (6.11) provided $b \geq a$, and it fails to be one in case $a > b$.

The second solution

We now show that the BSDE (6.10) with the same terminal variable as above possesses a second solution. By Lemma 6.4 there exists a second solution of (6.11) as well. Once this is shown, for any possible degree γ of exponential integrability we will have exhibited a negative random variable satisfying $\mathbb{E}(\exp(\gamma|\xi|)) < \infty$ for which (6.10) possesses at least two solutions. This in turn will underline that Briand & Hu's [47] uniqueness result, valid under (6.9) cannot be improved by much.

Note that the solution we will exhibit is again of the explicit form (6.6) encountered earlier. Let $M_t = \mathbb{E}[e^{\xi}|\mathcal{F}_t]$ for all $t \geq 0$. Due to the martingale representation property there exists a process H such that $M_t = M_0 + \int_0^t H_s dW_s$. We know that $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is a solution of (6.10). We show that

$$\begin{aligned}
\ln M_{\tau_b \wedge t} &= 2b - 4a + 2(b-a)W_{\tau_b\wedge t} - 2(b-a)^2(\tau_b \wedge t), \quad \text{if} \quad 2a > b,\\
\ln M_{\tau_b} &= 2aW_{\tau_b\wedge t} - 2a^2\tau_b \wedge t, \quad \text{if} \quad 2a \le b.
\end{aligned}$$

This implies that the solution $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is different from the solution $(2aW_{\tau_b\wedge t} - 2a^2(\tau_b \wedge t), 2a)$ obtained above in case $2a > b$. Note that in case $2a \le b$ we recover the solution already obtained as the first solution. Hence by Lemma 6.4 we obtain a second solution of (6.11) in this case.

Let us finally show that this second solution is in fact a measure solution for any possible combination of parameters.

Lemma 6.5 $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ *can be extended to a measure solution of* (6.10)*, hence provides a measure solution of* (6.11)*.*

Remarks:

1. We can summarize the findings of our investigations of the examples by stating that there are three basic scenarios:
   (a) for $b \ge 2a$ we obtained one solution which is a measure solution at the same time;
   (b) in the range $2a > b \ge a$ we found two different solutions both of which are measure solutions;
   (c) if $a > b$ we finally encountered two solutions one of which is a measure solution, while the other one is not.
2. Note that our examples exhibiting solutions of (6.10) that are not measure solutions are all for negative terminal variables ξ. Positive terminal variables arise in scenarios (a) or (b), and therefore only produce multiple measure solutions.

A continuum of solutions

Let us now combine the first and second solutions to obtain a continuum of solutions of our BSDE (6.10). To do this, we have to consider a still somewhat more general class of stopping times. For $c \in \mathbb{R}$, let

$$\rho_c = \inf\{t \ge 0 : W_t \le t - c\}.$$

We investigate the terminal variables

$$\xi = 2a(a-1)\rho_c + d$$

with further constants $a \ne 0, d \in \mathbb{R}$. Note first that the integrability properties of ξ are the same as those obtained before for $b = 1$. According to the preceding paragraphs, our BSDE (6.10) possesses the following two solutions

$$Z^1 = 2a1_{[0,\rho_c]}, \quad Y^1 = d_1 + 2aW_{\rho_c\wedge\cdot} - 2a^2\rho_c \wedge \cdot, \tag{6.14}$$

$$Z^2 = 2(1-a)1_{[0,\rho_c]}, \quad Y^2 = d_2 + 2(1-a)W_{\rho_c\wedge\cdot} - 2(1-a)^2\rho_c \wedge \cdot, \tag{6.15}$$

with $d_1 = -2ac$ resp. $d_2 = -2(a-1)c$. Let us now take $c = 1$ and combine the two solutions to obtain a continuum of new ones. To do this, we start with the equation

$$\rho_1 = \rho_c + \rho_{1-c} \circ \theta_{\rho_c},$$

where θ_t is the shift on Wiener space defined by

$$\theta_t(\omega) = W_{t+\cdot}(\omega) - W_t(\omega),$$

and $c \in]0,1[$. It describes the first time to reach the line with slope 1 that cuts the vertical at level -1, by decomposition with the intermediate time to reach the line with slope 1 cutting the vertical at $-c$. We mix the two solutions on the two resulting stochastic intervals, more precisely we put for $c \in]0,1[, l \in \mathbb{R}$

$$\begin{aligned} Z^c &= 2a1_{[0,\rho_c]} + 2(1-a)1_{[\rho_c,\rho_1]}, \\ Y^c &= l + 2aW_{\rho_c\wedge\cdot} - 2a^2\rho_c \wedge \cdot + 2(1-a)[W_{\rho_1\wedge\cdot} - W_{\rho_c\wedge\cdot}] \\ &\quad -2(1-a)^2[\rho_1 \wedge \cdot - \rho_c \wedge \cdot]. \end{aligned} \tag{6.16}$$

Since we have

$$\begin{aligned} Y^c_{\rho_1} &= l + 2aW_{\rho_c} - 2a^2\rho_c + 2(1-a)[W_{\rho_1} - W_{\rho_c}] - 2(1-a)^2[\rho_1 - \rho_c] \\ &= l + 2a(1-a)\rho_1 - 2ac - 2(1-a)(1-c), \end{aligned}$$

we have to set

$$l - 2ac - 2(1-a)(1-c) = d$$

in order to obtain a solution of (6.10) with $c = 1$. According to the treatment of the first and second solution, the constructed mixture is a measure solution if and only if both components of the mixture are. This is the case for $2a(1-a) > 0$, whereas for $2a(1-a) < 0$ we obtain a continuum of solutions that are no measure solutions.

Remarks:

1. This time, we may summarize our results by saying that there are two scenarios: for $2a(1-a) > 0$ there is a continuum of measure solutions of (6.10), while for $2a(1-a) < 0$ a continuum of non measure solutions is obtained.
2. Note that the initial conditions of our solutions continuum vary in a convex way between $-2a$ and $-2(1-a)$ as c varies in $]0,1[$, spanning the whole interval.

We shall now point out that the measure solution property of the second solution in case $a > b$ exhibited in the example above is not a coincidence. In fact, it will turn out that also for negative exponentially integrable ξ, solutions given by (6.6) provide measure solutions. To prove this, we will reverse the sign of ξ by looking at our BSDE from the perspective of an equivalent measure.

6.1.3 Some extensions

In the appendix (Section 9.2), we provide from scratch a construction of measure solutions for Lipschitz continuous generators f (also see [IV]).

In [130], the authors extend some results for more general functions g (given by **(G2)**). See Zhang's Diplomarbeit "Measure solutions of BSDEs and a Feynman-Kac formula" for further details.

6.2 Optimal switching ([V])

Optimal control of multiple switching models arise naturally in many applied disciplines. The pioneering work by Brennan and Schwartz [44], proposing a two-modes switching model for the life cycle of an investment in the natural resource industry, is probably first to apply this special case of stochastic impulse control to questions related to the structural profitability of an investment project or an industry whose production depends on the fluctuating market price of a number of underlying commodities or assets. Within this discipline, Carmona and Ludkosvki [64] and Deng and Xia [92] suggest a multiple switching model to price energy tolling agreements, where the commodity prices are modeled as continuous time processes, and the holder of the agreement exercises her managerial options by controlling the production modes of the assets. Target tracking in aerospace and electronic systems (see [102]) is another class of problems, where these models are very useful. These are often formulated as a hybrid state estimation problem characterized by a continuous time target state and a discrete time regime (mode) variables. All these applications seem agree that reformulating these problems in a multiple switching dynamic setting is a promising (if not the only) approach to fully capture the interplay between profitability, flexibility and uncertainty.

The optimal two-modes switching problem is probably the most extensively studied in the literature starting with above mentioned work by Brennan and Schwartz [44], and Dixit [96] who considered a similar model, but without resource extraction - see Dixit and Pindyck [97] and Trigeorgis [319] for an overview, extensions of these models and extensive reference lists. Brekke and Øksendal [42, 43], Shirakawa [305], Knudsen, Meister and Zervos [198], Duckworth and Zervos [104, 105] and Zervos [330] use the framework of generalized impulse control to solve several versions and extensions of this model, in the case where the decision to start and stop the production process is done over an infinite time horizon and the market price process of the underlying commodity is a diffusion process, while Trigeorgis [318] models the market price process of the commodity as a binomial tree. Hamadène and Jeanblanc [155] consider a finite horizon optimal two-modes switching problem in the case of Brownian filtration setting while Hamadène and Hdhiri [154] extend the set up of the latter paper to the case where the processes of the underlying commodities are adapted to a filtration generated by a Brownian motion and an independent Poisson process. Porchet et al. [291] also study the same problem, where they assume the payoff function to be given by an exponential utility function and allow the manager to trade on the commodities market. Finally, let us mention the work by Djehiche and Hamadène [98] where it is shown that including the possibility of abandonment or bankruptcy in the two-modes switching model over a finite time horizon, makes the search for an optimal strategy highly nonlinear and is not at all a trivial extension of previous results.

An example of the class of multiple switching models discussed in Carmona and Ludkovski [64] is related to the management strategies to run a power plant that converts natural gas into electricity (through a series of gas turbines) and sells it in the market. The payoff rate from running the plant is roughly given by the difference between the market price of electricity and the market price of gas needed to produce it. Suppose that besides running the plant at full capacity or keeping it completely off (the two-modes switching model), there also exists a total of $q-2$ ($q \geq 3$) intermediate operating modes, corresponding to different subsets of turbines running.

6.2.1 Our contribution

The setting is the same as in Section 2.2.1. Let $\mathcal{J} := \{1, ..., q\}$ be the set of all possible activity modes of the production of the commodity. A management strategy for the power plant is a combination of two sequences:

1. a nondecreasing sequence of stopping times $(\tau_n)_{n\geq 1}$, where, at time τ_n, the manager decides to switch the production from its current mode, say i, to another one from the set $\mathcal{J}^{-i} \subseteq \{1, \ldots, i-1, i+1, \ldots, q\}$;

2. a sequence of indicators $(\xi_n)_{n\geq 1}$ taking values in $\{1, \ldots, q\}$ of the state the production is switched to. At τ_n for $n \geq 1$, the station is switched from its current mode ξ_{n-1} to ξ_n. The value ξ_0 is deterministic and is the state of the station at time 0. Therefore, we assume that for any $n \geq 1$, ξ_n is a r.v. $\mathcal{F}_{\tau_n}$-measurable with values in $\mathcal{J}$.

For $i \in \mathcal{J}$, let $\Psi_i := (\Psi_i(t))_{0\leq t\leq T}$ be a stochastic process such that for some $p > 1$

$$\mathbb{E}\left[\int_0^T |\Psi_i(s)|^p ds\right] < \infty.$$

In the sequel, it stands for the payoff rate per unit time when the plant is in state i. On the other hand, for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{-i}$ let $\ell_{ij} := (\ell_{ij}(t))_{0\leq t\leq T}$ be a continuous process of $\mathbb{D}^p(0, T)$. It stands for the switching cost of the production at time t from its current mode i to another mode $j \in \mathcal{J}^{-i}$. For completeness we adopt the convention that $\ell_{ij} \equiv +\infty$ for any $i \in \mathcal{J}$ and $j \in \mathcal{J} - \mathcal{J}^{-i}$ ($j \neq i$). This convention is set in order to exclude the switching from the state i to another state j which does not belong to $\mathcal{J}^{-i}$. Moreover we suppose that there exists a real constant $\gamma > 0$ such that for any $i, j \in \mathcal{J}$, and any $t \leq T$, $\ell_{ij}(t) \geq \gamma$.

When the power plant is run under a strategy $\mathcal{S} = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$, over a finite horizon $[0, T]$, the total expected profit up to T for such a strategy is

$$J(\mathcal{S}, i) = \mathbb{E}\left[\int_0^T \sum_{n\geq 0} \left(\Psi_{\xi_n}(s)\mathbf{1}_{(\tau_n,\tau_{n+1}]}(s)\right) ds - \sum_{n\geq 1} \ell_{\xi_{n-1},\xi_n}(\tau_n)\mathbf{1}_{[\tau_n<T]}\right]$$

where we set $\tau_0 = 0$ and $\xi_0 = i$. The optimal switching problem we investigate is to find a management strategy $\mathcal{S}^*$ such that $J(\mathcal{S}^*, i) = \sup_{\mathcal{S}} J(\mathcal{S}, i)$.

Assume that a strategy of running the plant $\mathcal{S} := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ is given and w.l.o.g that the plant is in production mode 1 at $t = 0$. We denote by $(u_t)_{t\leq T}$ its associated indicator of the production activity mode at time $t \in [0, T]$, given by:

$$u_t = \mathbf{1}_{[0,\tau_1]}(t) + \sum_{n\geq 1} \xi_n \mathbf{1}_{(\tau_n,\tau_{n+1}]}(t).$$

Note that $\tau := (\tau_n)_{n\geq 1}$ and the sequence $\xi := (\xi_n)_{n\geq 1}$ determine uniquely u and conversely, the left continuous with right limits process u determine uniquely τ and ξ. Therefore a strategy for our multiple switching problem will be simply denoted by u. A strategy $u =: ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ will be called *admissible* if it satisfies

$$\lim_{n\to\infty} \tau_n = T \quad \mathbb{P} - \text{a.s.}$$

and the set of admissible strategies is denoted by $\mathcal{A}$. When a strategy $u := ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ is implemented the optimal yield is given by

$$J(u) = \mathbb{E}\left[\int_0^T \Psi_{u_s}(s)ds - \sum_{n\geq 1} \ell_{u_{\tau_{n-1}}, u_{\tau_n}}(\tau_n)\mathbf{1}_{[\tau_n<T]}\right].$$

We can now formulate the multi-regime starting and stopping problem as follows: find a strategy $u^* \equiv ((\tau_n^*)_{n\geq 1}, (\xi_n^*)_{n\geq 1}) \in \mathcal{A}$ such that

$$J(u^*) = \sup_{u\in\mathcal{A}} J(u).$$

An admissible strategy u is called *finite* if, during the time interval $[0, T]$, it allows the manager to make only a finite number of decisions, i.e. $\mathbb{P}[\omega, \tau_n(\omega) < T, \text{ for all } n \geq 0] = 0$. Hereafter the set of finite strategies will be denoted by $\mathcal{A}^f$. A first immediate result (due to $\ell_{ij}(t) \geq \gamma$) is that the suprema over admissible strategies and finite strategies coincide:

$$\sup_{u\in\mathcal{A}} J(u) = \sup_{u\in\mathcal{A}^f} J(u). \tag{6.17}$$

Using purely probabilistic tools such as the Snell envelop of processes and backward stochastic differential equations, inspired by the works [155], the paper [64] suggests a powerful robust numerical scheme based on Monte Carlo regressions to solve this optimal switching problem when the payoff rates are given as deterministic functions of a diffusion process. They also list a number of technical challenges, such as the continuity of the associated value function, that prevent a rigorous proof of the existence and a characterization of an optimal solution of this problem. Our objective of this work was to fill in this gap by providing a solution to the optimal multiple switching problem, using the same framework. We were able to prove existence and provide a characterization of an optimal strategy of this problem.

A verification Theorem

We first provide a Verification Theorem that shapes the problem, via the Snell envelope of processes. We show that if the verification theorem is satisfied by a vector of continuous processes $(Y^1, \ldots, Y^q)$ such that, for each $i \in \{1, \ldots, q\}$,

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau \Psi_i(s)ds + \max_{j \neq i}(-\ell_{ij}(\tau) + Y_\tau^j)1_{[\tau<T]}\big|\mathcal{F}_t\right], \tag{6.18}$$

then each Y_t^i is the value function of the optimal problem when the system is in mode i at time t:

$$Y_t^i = \operatorname*{ess\,sup}_{\mathcal{S} \in \mathcal{A}_t^i} \mathbb{E}\left[\int_t^T \sum_{n\geq 0}\left(\Psi_{\xi_n}(s)\mathbf{1}_{(\tau_n,\tau_{n+1}]}(s)\right)ds - \sum_{n\geq 1}\ell_{\xi_{n-1},\xi_n}(\tau_n)\mathbf{1}_{[\tau_n<T]}\big|\mathcal{F}_t\right].$$

where $\mathcal{A}_t^i$ is the set of admissible strategies such that $\tau_1 \geq t$ a.s. and $\xi_0 = i$. More preciselt, for τ an $\mathbb{F}$-stopping time and $(\zeta_t)_{0\leq t\leq T}$, $(\zeta'_t)_{0\leq t\leq T}$ two continuous $\mathbb{F}$-adapted and $\mathbb{R}-$valued processes let us set:

$$D_\tau(\zeta = \zeta') := \inf\{s \geq \tau, \zeta_s = \zeta'_s\} \wedge T.$$

We have the following:

Theorem 6.4 (Verification Theorem) *Assume there exist q $\mathbb{D}^p$-processes $(Y^i := (Y_t^i)_{0\leq t\leq T},\ i = 1, \ldots, q)$ that satisfy* (6.18). *Then $Y^1, \ldots, Y^q$ are unique. Furthermore :*

1. $$Y_0^1 = \sup_{v \in \mathcal{A}} J(v). \tag{6.19}$$

2. *Define the sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq 1}$ by*

$$\tau_1 = D_0\left(Y^1 = \max_{j \in \mathcal{J}^{-1}}(-\ell_{1j} + Y^j)\right) \tag{6.20}$$

and, for $n \geq 2$,

$$\tau_n = D_{\tau_{n-1}}\left(Y^{u_{\tau_{n-1}}} = \max_{k \in \mathcal{J}^{-\tau_{n-1}}}(-\ell_{\tau_{n-1}k} + Y^k)\right), \tag{6.21}$$

where

- $u_{\tau_1} = \sum_{j\in\mathcal{J}} j\mathbf{1}_{\{\max_{k\in\mathcal{J}^{-1}}(-\ell_{1k}(\tau_1)+Y^k_{\tau_1})=-\ell_{1j}(\tau_1)+Y^j_{\tau_1}\}}$;
- *for any $n \geq 1$ and $t \geq \tau_n$, $Y_t^{u_{\tau_n}} = \sum_{j\in\mathcal{J}} \mathbf{1}_{[u_{\tau_n}=j]} Y_t^j$;*

- *for $n \geq 2$, $u_{\tau_n} = l$ on the set*

$$\left\{ \max_{k \in \mathcal{J}^{-u_{\tau_{n-1}}}} (-\ell_{u_{\tau_{n-1}}k}(\tau_n) + Y^k_{\tau_n}) = -\ell_{u_{\tau_{n-1}}l}(\tau_n) + Y^l_{\tau_n} \right\},$$

where

$$\ell_{u_{\tau_{n-1}}k}(\tau_n) = \sum_{j \in \mathcal{J}} \mathbf{1}_{[\tau_{n-1}=j]} \ell_{jk}(\tau_n) \quad \textit{and} \quad \mathcal{J}^{-u_{\tau_{n-1}}} = \sum_{j \in \mathcal{J}} \mathbf{1}_{[\tau_{n-1}=j]} \mathcal{J}^{-j}.$$

Then, the strategy $u = ((\tau_n)_{n \geq 1}, (\xi_n)_{n \geq 1})$ is optimal i.e. $J(u) \geq J(v)$ for any $v \in \mathcal{A}$.

Then we prove the existence of the unique solution of the verification theorem. This solution is obtained as the limit of sequences of processes $(Y^{i,n})_{n \geq 0}$, where for any $t \leq T$, $Y^{i,n}_t$ is the value function (or the optimal yield) from t to T, when the system is in mode i at time t and only at most n switchings after t are allowed. More precisely for $i \in \mathcal{J}$, let us set, for any $0 \leq t \leq T$,

$$Y^{i,0}_t = \mathbb{E}\left[\int_0^T \Psi_i(s) ds \middle| \mathcal{F}_t \right] - \int_0^t \Psi_i(s) ds, \tag{6.22}$$

and for $n \geq 1$,

$$Y^{i,n}_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_0^\tau \Psi_i(s) ds + \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(\tau) + Y^{k,n-1}_\tau) \mathbf{1}_{[\tau < T]} \middle| \mathcal{F}_t \right] - \int_0^t \Psi_i(s) ds. \tag{6.23}$$

In the next proposition we collect some useful properties of $Y^{1,n}, \ldots, Y^{q,n}$. In particular we show that, as $n \to \infty$, the limit processes $\widetilde{Y}^i := \lim_{n \to \infty} Y^{i,n}$ exist and are only càdlàg but have the same characterization (6.18) as the Y^i's.

Proposition 6.1

1. *For each $n \geq 0$, the processes $Y^{1,n}, \ldots, Y^{q,n}$ are continuous and belong to $\mathbb{D}^p(0,T)$.*
2. *For any $i \in \mathcal{J}$, the sequence $(Y^{i,n})_{n \geq 0}$ converges increasingly and pointwisely $\mathbb{P}$-a.s. for any $0 \leq t \leq T$ and in $\mathbb{L}^p(\Omega \times [0,T])$ to a càdlàg processes $\widetilde{Y}^i$. Moreover these limit processes $\widetilde{Y}^i = (\widetilde{Y}^i_t)_{0 \leq t \leq T}$, $i = 1, \ldots, q$, satisfy*

   (a) $\mathbb{E}\left[\sup_{0 \leq t \leq T} \left| \widetilde{Y}^i_t \right|^p \right] < \infty, \quad i \in \mathcal{J}.$

   (b) For any $0 \leq t \leq T$ we have,

$$\widetilde{Y}^i_t = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau \Psi_i(s) ds + \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(\tau) + \widetilde{Y}^k_\tau) \mathbf{1}_{[\tau < T]} \middle| \mathcal{F}_t \right].$$

The existence proof of the Y^i's will consist in showing that $\widetilde{Y}^i$'s are continuous and hence satisfy the Verification Theorem.

Theorem 6.5 *The limit processes $\widetilde{Y}^1, \dots, \widetilde{Y}^q$ satisfy the Verification Theorem 6.4.*

As a consequence of the previous results we also obtain the convergence in $\mathbb{D}^p(0,T)$: for any $i \in \mathcal{J}$,

$$\mathbb{E}\left[\sup_{s\le T}\left|Y_s^{i,n} - Y_s^i\right|^p\right] \to 0 \quad \text{as} \quad n \to +\infty.$$

A penalization scheme

The previous part is based on the properties of the Snell envelop. Here we focus on a penalization scheme, which could be used for numerical simulations. First using the result by El Karoui et al. ([117], Proposition 5.1) which characterizes a Snell envelope as a solution of a one barrier reflected BSDE we deduce that the q-uplet of processes $(Y^1, \dots, Y^q)$ solution of the Verification Theorem 6.4 satisfies also a system of BSDEs with oblique reflections. Indeed for any $i \in \mathcal{J}$, there exists a pair of $\mathcal{F}_t$-adapted processes (Z^i, K^i) with value in $\mathbb{R}^d \times \mathbb{R}^+$ such that:

- Y^i, $K^i \in \mathbb{D}^p$ and $Z^i \in \mathbb{H}^p$;
- K^i is non decreasing with $K_0^i = 0$;
- for any $0 \le s \le T$

$$Y_s^i = \int_s^T \Psi_i(u)du - \int_s^T Z_u^i dB_u + K_T^i - K_s^i \; ; \tag{6.24}$$

- together with the oblique reflection: for all $0 \le s \le T$

$$\begin{cases} Y_s^i \ge \max\limits_{j\in\mathcal{J}^{-i}}\{-\ell_{ij}(s) + Y_s^j\}, \\ \\ \displaystyle\int_0^T \left(Y_u^i - \max_{j\in\mathcal{J}^{-i}}\{-\ell_{ij}(u) + Y_u^j\}\right) dK_u^i = 0. \end{cases} \tag{6.25}$$

Now we know that the solution of a reflected BSDE can be obtained as a limit of sequence of solutions of standard BSDE by approximation via penalization. Therefore, for any $n \in \mathbb{N}$, let us define the following system:

$$\forall i \in \mathcal{J},\ \forall t \in [0,T],\ Y_t^{i,n} = \int_t^T \Psi_i(s)ds + n\int_t^T (L_s^{i,n} - Y_s^{i,n})^+ ds - \int_t^T Z_s^{i,n} dB_s \tag{6.26}$$

where for every $i \in \mathcal{J}$,

$$\forall t \in [0,T],\ L_t^{i,n} = \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t) + Y_t^{k,n}).$$

Remark that if we define the generator $f := (f^1, \dots, f^q) : [0,T] \times \mathbb{R}^q \to \mathbb{R}^q$ by

$$\forall i \in \mathcal{J},\ f^i(s,y) = \Psi_i(s) + n(\max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(s) + y^k) - y^i)^+,$$

then f is a Lipschitz function w.r.t. y uniformly w.r.t. t and the $\mathbb{R}^q$-valued process $Y^n = (Y^{1,n}, \dots, Y^{q,n})$ satisfies the following BSDE:

$$\forall t \in [0,T],\ Y_t^n = \int_t^T f(s, Y_s^n) ds - \int_t^T Z_s^n dB_s. \tag{6.27}$$

Now, from Gobet et al. [143, 144], we know that the multidimensional BSDE (6.26) can be solved numerically. Therefore, if the sequence $(Y^{i,n})_n$ converges to Y^i, for any $i \in \mathcal{J}$, this provides a way to simulate the value function Y^i. Therefore we focus on this convergence. First of all we obtain

Proposition 6.2 *For every $i \in \mathcal{J}$ and all $t \in [0,T]$, the sequence $(Y_t^{i,n})_{n \in \mathbb{N}}$ is non decreasing and a.s. $Y_t^{i,n} \leq Y_t^i$.*

From this result we can prove that:

Theorem 6.6 *For every $i \in \mathcal{J}$*

$$\mathbb{E}\left[\sup_{t \leq T} |Y_t^{i,n} - Y_t^i|^p\right] \to 0 \text{ as } n \to \infty.$$

In our paper we added the next remark.

Remark 6.2 *It seems to be difficult to obtain a convergence rate in Theorem 6.6. In [155], Proposition 4.2 gives such a convergence rate because the lower barrier is constant and negative.*

Numerical results can be obtained when $q = 2$ (see [155] or [291]). For $q \geq 3$, [64] suggest a numerical scheme when the switching costs are constant. The case of non-constant switching costs seems out of reach.

Actually new results have given some answers to this remark (see Section 6.2.2).

Connection with systems of variational inequalities

Let us assume that the switching processes ℓ_{ij} are deterministic functions of the time variable. An example of such a family of switching costs is

$$\ell_{ij}(t) = e^{-rt} a_{ij},$$

where, a_{ij} are constant costs and $r > 0$ is some discounting rate. We moreover assume that the payoff rates are given by $\Psi_i(\omega, t) = \psi_i(t, X_t)$ where ψ_i are deterministic functions and $X = (X_t)_{t \geq 0}$ is a vector of stochastic processes that stands for the market price of the underlying commodities and other financial assets that influence the production of energy. When the underlying market price process X is Markov, the classical methods of solving impulse problems (cf. Brekke and Øksendal [43], Guo and Pham [152], and Tang and Yong [316]) formulates a Verification Theorem suggesting that the value function of our optimal switching problem is the unique viscosity solution of the following system of quasi-variational inequalities (QVI) with inter-connected obstacles.

$$
\begin{cases} \min\Big\{v^i(t,x) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t) + v^j(t,x)), \\ \qquad\qquad -\partial_t v^i(t,x) - \mathcal{L}v^i(t,x) - v^i(t,x)\Big\} = 0, \\ v^i(T,x) = 0, \qquad i \in \mathcal{J}, \end{cases} \tag{6.28}
$$

where $\mathcal{L}$ is the infinitesimal generator of the driving process X.

By means of yet another characterization of the Snell envelope in terms of systems of reflected backward SDEs, due to El Karoui et al. [117] (Theorems 7.1 and 8.5), we are able to show that the vector of value processes $(Y^1,\ldots,Y^q)$ of our switching problem provides a viscosity solution of the system (6.28). Actually we show that under mild assumptions on the coefficients $\psi_i(t,x)$ and $\ell_{ij}(t)$,

$$
Y_t^i = v^i(t,X_t), \quad 0 \le t \le T, \;\; i \in \mathcal{J},
$$

where $v^1(t,x),\ldots,v^q(t,x)$ are continuous deterministic functions viscosity solution of the system of QVI with inter-connected obstacles (6.28). We note that there are works which deal with the same problem in using the dynamic programming principle or/and other methods such as the stochastic target problem [40, 316]. In [316], the solution is obtained under rather stringent assumptions than ours, while in [40], Bouchard provides a solution for (6.28) in a weak sense since he faces an issue in connection with a lack of continuity of the solution. In this section, in using the well known link between BSDEs and variational inequalities, we obtain the existence of a continuous solution for (6.28) in a more general framework as e.g. the one of [316]. However we should point out that at the same time the viscosity solution approach of the switching problem is handled with weaker assumptions on the switching costs ℓ_{ij} in El Asri and Hamadène [113]. Actually, they consider the case when ℓ_{ij} depend also on x and they show existence and uniqueness of a continuous solution for (6.28). Their proof of continuity is quite technical.

Let us give the setting and the result. Now for $(t,x) \in [0,T]\times\mathbb{R}^k$, let $(X_s^{t,x})_{s\le T}$ be the solution of the following Itô diffusion (1.9)

$$
dX_s^{t,x} = b(s,X_s^{t,x})ds + \sigma(s,X_s^{t,x})dB_s, \quad t \le s \le T; \qquad X_s^{t,x} = x \;\; \text{for } s \le t,
$$

where the functions b and σ, with appropriate dimensions, satisfy the conditions **(B1)** and **(B2)**. Thus the process $X^{t,x} := (X_s^{t,x})_{0\le s\le T}$ exists, is unique and satisfies (1.11) and (1.12)

Let us now introduce the following assumption on the payoff rates ψ_i and the switching cost functions ℓ_{ij}.

- The running costs ψ_i, $i = 1,\ldots,q$, are jointly continuous and are of polynomial growth, i.e. there exist some positive constants C and δ such that for each $i \in \mathcal{J}$,

$$
|\psi_i(t,x)| \le C(1+|x|^\delta), \;\; \forall (t,x) \in [0,T]\times\mathbb{R}^k.
$$

- For any $i, j \in \mathcal{J}$, the switching costs ℓ_{ij} are deterministic continuous functions of t and there exists a real constant $\gamma > 0$ such for any $0 \leq t \leq T$, $\min\{\ell_{ij}(t), i, j \in \mathcal{J}, i \neq j\} \geq \gamma$.

Taking into account Proposition 1.1, the processes $(\psi_i(s, X_s^{t,x}))_{0 \leq s \leq T}$ belong to $\mathbb{L}^p(\Omega \times [0, T])$.

Recall the notion of viscosity solution of the system (6.28).

Definition 6.2 *Let $(v_1, \ldots, v_q)$ be a vector of continuous functions on $[0, T] \times \mathbb{R}^k$ with values in $\mathbb{R}^q$ and such that $(v_1, \ldots, v_q)(T, x) = 0$ for any $x \in \mathbb{R}^k$. The vector $(v_1, \ldots, v_q)$ is called:*

1. *a viscosity supersolution (resp. subsolution) of the system* (6.28) *if for any $(t_0, x_0) \in [0, T] \times \mathbb{R}^k$ and any q-uplet functions $(\varphi_1, \ldots, \varphi_q) \in (C^{1,2}([0, T] \times \mathbb{R}^k))^q$ such that $(\varphi_1, \ldots, \varphi_q)(t_0, x_0) = (v_1, \ldots, v_q)(t_0, x_0)$ and for any $i \in \mathcal{J}$, (t_0, x_0) is a maximum (resp. minimum) of $\varphi_i - v_i$ then we have: for any $i \in \mathcal{J}$,*

$$\min\Big\{v_i(t_0, x_0) - \max_{j \in \mathcal{J}^{-i}}(-\ell_{ij}(t_0) + v_j(t_0, x_0)),$$
$$-\partial_t \varphi_i(t_0, x_0) - \mathcal{L}\varphi_i(t_0, x_0) - \psi_i(t_0, x_0)\Big\} \geq 0 \ (resp. \leq 0)$$

2. *a viscosity solution of the system* (6.28) *if it is both a viscosity supersolution and subsolution.*

Let now $(Y_s^{1;t,x}, \ldots, Y_s^{q;t,x})_{0 \leq s \leq T}$ be the vector of value processes which satisfies the Verification Theorem 6.4 associated with $(\psi_i(s, X_s^{t,x}))_{s \leq T}$ and $(\ell_{ij}(s))_{s \leq T}$. The vector $(Y^{1;t,x}, \ldots, Y^{q;t,x})$ exists through Theorem 6.5 combined with the estimates of $X^{t,x}$ of Proposition 1.1 and our conditions on ψ_i and ℓ_{ij}.

Theorem 6.7 *Under our setting, there exist q deterministic functions $v^1(t, x), \ldots, v^q(t, x)$ defined on $[0, T] \times \mathbb{R}^k$ and $\mathbb{R}$-valued such that:*

(i) *$v^1, \ldots, v^q$ are continuous in (t, x), are of polynomial growth and satisfy, for each $t \in [0, T]$ and for every $s \in [t, T]$,*

$$Y_s^{i;t,x} = v^i(s, X_s^{t,x}), \quad \text{for every } i \in \mathcal{J}.$$

(ii) *The vector of functions $(v^1, \ldots, v^q)$ is a viscosity solution for the system of variational inequalities* (6.28).

Remark 6.3 *The viscosity solution $(v^1, \ldots, v^q)$ is unique in the class of continuous functions with polynomial growth (cf. [113], Theorem 4).*

6.2.2 Further extensions

Our paper was a first attempt to solve the optimal switching problem in a non Markovian framework with probabilistic tools when there are $q \geq 3$ modes. Note that Pham et al. [289] also solve this control problem, but using PDE arguments. These two approachs are mainly the two different ways to tackle such switching optimal problems. The first one consists in dealing with systems of obliquely reflected backward stochastic differential equations with inter-connected obstacles (Eq. (6.24) and (6.25)). The studied system becomes: for all $0 \leq t \leq T$ and $i \in \mathcal{J}$

$$(6.29) \qquad \begin{cases} Y_t^i = \xi^i + \displaystyle\int_t^T f_i(s, Y_s, Z_s^i)ds - \int_t^T Z_s^i dB_s + K_T^i - K_t^i, \\ Y_t^i \geq \displaystyle\max_{j \in \mathcal{J}^{-1}} h_{ij}(t, Y_t^j), \\ \displaystyle\int_0^T \left(Y_u^i - \max_{j \in \mathcal{J}^{-1}} h_{ij}(u, Y_u^j) \right) dK_u^i = 0. \end{cases}$$

If [155, 64, 291] and [V] are the pioneer papers, the results have been deeply extended by Hu & Tang [166], Hamadène & Zhang [158], Chassagneux et al. [69]. The second method consists in considering rather the QVI (6.28). Let us mention Lundström et al. [232]. Of course those methods are deeply related and both can be combined in the so-called Markovian context: that is when the randomness of the parameters comes from an exogenous process (which may be, for instance, the electricity or oil price in the market). In this context, (6.29) becomes

$$(6.30) \qquad \begin{cases} Y_t^i = g^i(X_T) + \displaystyle\int_t^T f_i(X_s, Y_s, Z_s^i)ds - \int_t^T Z_s^i dB_s + K_T^i - K_t^i, \\ Y_t^i \geq \displaystyle\max_{j \in \mathcal{J}^{-1}} \{Y_t^j - \ell_{ij}(X_t)\}, \\ \displaystyle\int_0^T \left(Y_u^i - \max_{j \in \mathcal{J}^{-1}} \{Y_u^j - \ell_{ij}(X_u)\} \right) dK_u^i = 0. \end{cases}$$

where X is the solution of the forward SDE (1.9). The paper of Djehiche et al. [99] successfully combines the two approaches to obtain existence and uniqueness of solutions in viscosity sense. Contrary to normally reflected BSDEs [134], existence and uniqueness result available in the literature requires structural conditions, both on the driver f and the function h or ℓ. This problem of obliquely reflected BSDEs is studied in [72].

Several directions have been also explored:

1. The case when switching costs may be non positive. It has been considered by Lundström [231].

2. The case when we only have partial information about the system (see Li et al. [225]). Such a situation may occur when the manager (or controller) cannot access

to the real data but only to a data mixed with a noise. In that case, the partial information is modelled by some new filtration and the profit index has to be computed with respect to this new filtration. Thus this problem requires the use of advanced filtering methods.

3. The case when the filtration is generated by a discontinuous process. This setting is more adapted to model unpredictable random phenomena. When the process is a Lévy one, switching problems driven by Teugels martingales have been considered (see [231] or [159]).

4. The case when the strategy of the manager may affect the dynamic is not considered. From a practical point of view, this is of great importance and needs to be tackled. This subject is partially treated in Porchet et al. [291] for example. In Elie & Kharroubi [121] the case of controlled volatility has been considered.

5. The case of switching games. The modelling of the (economic) issue of purchasing right to emit carbon can lead to some nonzero-sum switching game (see e.g. M. Ludkovski [230]). Djehiche et al. [100] have studied the systems of PDEs with two inter-connected obstacles of min-max and max-min types. Those systems are related to the value function of a zero-sum switching game. A lot of assumptions have to be imposed to ensure that the solutions of the two systems of PDEs exist and coincide.

Let us now present some results concerning numerical algorithms of the solution. Indeed there are some studies on such numerical schemes for switching problems. Let us mention Gassiat et al. [132], Chassagneux et al. [70], Aïd et al. [2], Kharroubi et al. [188], Chassagneux & Richou [71]. Roughly speaking, in [188] the idea is to solve HJB equations based on BSDEs with jump constraints and randomization of the control. And in [70, 71], the authors obtained the missing rate of convergence for Theorem 6.6 (in fact the numerical approximation is based on a discretely reflected version of (6.30)).

It would be interesting to introduce uncertainty in the model. The theory of second order BSDE is now well posed and is booming. It would be interesting to use this tool in the optimal switching context.

Chapter 7

Estimation in fractional diffusion ([VI, VIII, A])

The drift parameter estimation problem with partial observations has been given a great deal of interest over the last decades. Numerous results have been already reported in specific models, specially around Hidden Markov models (HMM). Let us mention papers [33, 37, 101, 123, 139, 176, 222, 303] where the consistency and the asymptotic normality of the Maximum Likelihood Estimator (MLE) have been discussed. Of course, this list is far from being complete (see also the references therein).

This chapter and the related papers [VI, VIII, A] are devoted to the large sample asymptotic properties of the Maximum Likelihood Estimator (MLE) for the signal drift parameter ϑ in a partially observed and possibly controlled fractional diffusion system. Actually we consider two problems. For the first statement (controlled, deterministic and partially observable signal) we establish the asymptotic (for large observation time) design problem of the input signal which gives an efficient estimator of the drift parameter. This kind of optimization problem has been treated by many authors, see *e.g.* [223, 251, 265] and references therein. Following paper [265], we can separate the initial problem in two subproblems, when the first subproblem is equivalent to the explicit computations of the first eigenvalue of a certain self-adjoint operator and the second one is devoted to the analysis of the asymptotic properties of the MLE. In contrast with the previous works, we propose to use (for the both subproblems) Laplace transform computations, in particular, the Cameron-Martin formula and the link between the Laplace transform and the eigenvalues of a covariance operator.

For the second statement we work with a linear Gaussian system, perturbed by fBm noises. We suppose that the Hurst parameter H is known and it is the same for the signal X and for the observations Y, which means that the initial observation model is not Markovian. Again, our goal is to establish the large sample asymptotic properties of the Maximum Likelihood Estimator (MLE) for the signal drift parameter ϑ. Unfortunately, the method proposed in [37, 176] can not be applied directly for continuous-time models.

To analyze the large sample asymptotic properties of the MLE, we use the program proposed in [169]. The main idea of this approach is to deduce strong properties of MLE from the weak convergence of scaled likelihoods in appropriate functional spaces, especially the convergence of moments which was not addressed even for discrete time

HMM.

The explicit expression of the likelihood can be written using the "transformation of the observation model" method proposed in [189]. Even in our particular situation, this approach is reduced to the analysis of a non homogeneous non ergodic signal. To pass this obstacle, again we proposed to use Laplace transform computations based on the Cameron-Martin formula.

7.1 The setting

We consider real-valued processes $X = (X_t,\, t \geq 0)$ and $Y = (Y_t,\, t \geq 0)$, representing the signal and the observation respectively, governed by the following linear system of stochastic differential equation interpreted as integral equation:

$$\begin{cases} dX_t &= -\vartheta X_t dt + u(t)dt + dV_t^H\,, \quad X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t^{H'}\,, \quad Y_0 = 0. \end{cases}$$

Here, $V^H = (V_t^H,\, t \geq 0)$ and $W^{H'} = (W_t^{H'}, t \geq 0)$ are independent normalized fractional Brownian motions (fBm in short) with Hurst parameters H and H' in $(0,1)$ and the coefficients $\vartheta \in \mathbb{R}_+^*$ and $\mu \neq 0$ are real constants. The unobserved signal process $X = (X_t,\, t \geq 0)$ is controlled by the real-valued function $u = (u(t),\, t \geq 0)$. Previous system has a uniquely defined solution process (X, Y) which is, due to the well known properties of the fBm, Gaussian but neither Markovian nor a semimartingale for $H \neq \frac{1}{2}$ (see, *e.g.*, [229], page 238).

Suppose that parameter $\vartheta > 0$ is unknown and is to be estimated given the observed trajectory $Y^T = (Y_t,\, 0 \leq t \leq T)$. For a fixed value of the parameter ϑ, let $\mathbb{P}_\vartheta^T$ denote the probability measure, induced by (X^T, Y^T) on the function space $\mathcal{C}_{[0,T]} \times \mathcal{C}_{[0,T]}$ and let $\mathbb{F}^Y$ be the natural filtration of Y, $\mathcal{F}_t^Y = \sigma\left(Y_s, 0 \leq s \leq t\right)$. Let $\mathcal{L}(\vartheta, Y^T)$ be the likelihood, *i.e.* the Radon-Nikodym derivative of $\mathbb{P}_\vartheta^T$, restricted to $\mathcal{F}_T^Y$ with respect to some reference measure on $\mathcal{C}_{[0,T]}$. The explicit representation of the likelihood function can be written thanks to the transformation of observation model proposed in [189].

In [VI] we focus on the model where $H = H'$:

$$\begin{cases} dX_t &= -\vartheta X_t dt + u(t)dt\,, \quad X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t^H\,, \quad Y_0 = 0. \end{cases} \tag{7.1}$$

In [VIII] we also present the results concerning the system of stochastic differential equations:

$$\begin{cases} dX_t &= -\vartheta X_t dt + dV_t^H\,, \quad X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t^H\,, \quad Y_0 = 0. \end{cases}$$

Since this model was studied by Brouste & Kleptsyna in [49], we don't discuss this model here. Finally in the working paper [A], we actually study

$$\begin{cases} dX_t &= -\vartheta X_t dt + dV_t^H\,, \quad X_0 = 0\,, \\ dY_t &= \mu X_t dt + dW_t\,, \quad Y_0 = 0. \end{cases} \tag{7.2}$$

Here $W = (W_t, t \geq 0)$ is independent Wiener process, that is $H' = 1/2$.

Remark 7.1 *Note that* (7.1) *without control u can always be transformed into* (7.2). *In other words supposing $H' = 1/2$ in* (7.2) *can be done w.l.o.g.*

7.2 Observation model (7.1)

Let $\mathcal{I}_T(\vartheta, u)$ be the Fisher information,*i.e.* $\mathcal{I}_T(\vartheta, u) = -\mathbb{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}_T(\vartheta, Y^T)$. The explicit representation of the likelihood function $\mathcal{L}_T$ will be written later (see Equation (7.6)). Let us note here that in order to have the finite Fisher information we should suppose that the admissible control u belongs to some functional space of controls $\mathcal{U}_T$. Let us therefore note

$$\mathcal{J}_T(\vartheta) = \sup_{u \in \mathcal{U}_T} \mathcal{I}_T(\vartheta, u) \ .$$

Our main goal is to find estimator $\overline{\vartheta}_T$ of the parameter ϑ which are asymptotically efficient in the sense that, for any compact $\mathbb{K} \subset \mathbb{R}^+$,

$$\sup_{\vartheta \in \mathbb{K}} \mathcal{J}_T(\vartheta) \mathbb{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta\right)^2 = 1 + o(1)\,, \tag{7.3}$$

as $T \to \infty$.

Proposition 7.1 *The asymptotical optimal input in the class of controls $\mathcal{U}_T$ is $u_{opt}(t) = \frac{\kappa_H}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$, where the constants λ and κ_H are defined in Section 7.2.*

As the optimal input does not depend on ϑ, a possible candidate is the Maximum Likelihood Estimator (MLE) $\hat{\vartheta}_T$ defined as the maximizer of the likelihood:

$$\hat{\vartheta}_T = \arg\max_{\vartheta > 0} \mathcal{L}(\vartheta, Y^T)\,,$$

with optimal input $u = u_{opt}$. MLE reaches efficiency and we deduce its large samples asymptotic properties:

Theorem 7.1 *The MLE is uniformly consistent on compacts $\mathbb{K} \subset \Theta$, i.e. for any $\nu > 0$,*

$$\lim_{T \to \infty} \sup_{\vartheta \in \mathbb{K}} \mathbb{P}_\vartheta^T \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \nu \right\} = 0\,,$$

uniformly on compacts asymptotically normal: as T tends to $+\infty$,

$$\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \Longrightarrow \mathcal{N}\left(0, \frac{\vartheta^4}{\mu^2}\right)$$

which does not depend on H *and we have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,*

$$\lim_{T \to \infty} \mathbb{E}_\vartheta \left| \sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right|^p = \mathbb{E}\left|\frac{\vartheta^2}{\mu}\zeta\right|^p$$

where $\zeta \sim \mathcal{N}(0,1)$. Finally, the MLE is efficient in the sense of (7.3).

The classical case $H = \frac{1}{2}$ have been treated in [265]. These results have been proved in [VI], for $H \geq \frac{1}{2}$ and extended to $H \in (0,1)$ in [VIII].

In the rest of this section, we give some ideas and tricks of the proof of Theorem 7.1.

Preliminaries

In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)$-adapted. Moreover the *natural filtration* of a process is understood as the $\mathbb{P}$-completion of the filtration generated by this process. Here we focus on the case $H > 1/2$. Nevertheless, the result is valid for any $H \in (0,1)$ (see Section 7.2.1).

Even if fBm are not martingales, there are simple integral transformations which change the fBm to martingales (see [257, 262]). In particular, defining for $0 < s < t$,

$$k_H(t,s) = \kappa_H^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}\,,\ K_H(t,s) = 2H\frac{d}{dt}\int_s^t r^{H-\frac{1}{2}}(r-s)^{H-\frac{1}{2}}dr,$$

$$w_H(t) = \frac{1}{2\lambda(2-2H)}t^{2-2H}$$

and the constants:

$$\kappa_H = 2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(\frac{1}{2}+H\right),\ \lambda = \lambda_H = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)}$$

and the process

$$M_t = \int_0^t k_H(t,s)dW_s^H\,,$$

then the process $M = (M_t,\ t \geq 0)$ is a Gaussian martingale, called in [257] the *fundamental martingale* whose variance function is nothing but the function w_H. Moreover, the natural filtration of the martingale M coincides with the natural filtration of the fBm W^H.

Following [189], let us introduce $Z = (Z_t,\ t \geq 0)$ the *fundamental semimartingale* associated to Y, namely

$$Z_t = \int_0^t k_H(t,s)dY_s\,.$$

Note that Y can be represented as $Y_t = \int_0^t K_H(t,s)dZ_s$ and therefore the natural filtrations of Y and Z coincide. It can be proved that the following representation holds:

$$dZ_t = \mu Q_t d\langle M\rangle_t + dM_t\,,\quad Z_0 = 0\,,$$

where

$$Q_t = \frac{d}{d\langle M\rangle_t}\int_0^t k_H(t,s)X_s ds\,.$$

Moreover, the following equation holds (see *e.g.* [189]):

$$dZ_t = \mu\lambda\ell(t)^*\zeta_t d\langle M\rangle_t + dM_t\,,\quad Z_0 = 0\,,$$

where $\zeta = (\zeta_t, t \geq 0)$ is the solution of the ordinary differential equation

$$\frac{d\zeta_t}{d\langle M\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\zeta_t + b(t)v(t),\quad \zeta_0 = 0\,, \tag{7.4}$$

where $v(t)=\dfrac{d}{d\langle M\rangle_t}\displaystyle\int_0^t k_H(t,s)u(s)ds$ and with

$$\ell(t)=\begin{pmatrix} t^{2H-1}\\ 1\end{pmatrix},\quad \mathbf{A}(t)=\begin{pmatrix} t^{2H-1} & 1\\ t^{4H-2} & t^{2H-1}\end{pmatrix}\quad\text{and}\quad b(t)=\begin{pmatrix} 1\\ t^{2H-1}\end{pmatrix}.$$

Let us note, for this problem, $\mathcal{V}_T$ the class of admissible controls

$$\mathcal{V}_T=\left\{v\quad\Big|\quad \frac{1}{T}\int_0^T |v(t)|^2 d\langle M\rangle_t\le 1\right\}.$$

Remark that the following relations between control $u(t)$ and its transformation $v(t)$ hold:

$$v(t)=\frac{d}{d\langle M\rangle_t}\int_0^t k_H(t,s)u(s)ds,\ u(t)=\frac{d}{dt}\int_0^t K_H(t,s)v(s)d\langle M\rangle_s. \tag{7.5}$$

At the first glance, we can set the admissible controls as $\mathcal{U}_T=\{u\mid v\in\mathcal{V}_T\}$. Note that these sets are non empty.

Likelihood function and the Ibragimov–Khasminskii program

Let us note $Z^T=(Z_t,\,0\le t\le T)$. We are interested in the explicit representation of the likelihood function $\mathcal{L}_T(\vartheta,Z^T)$. The classical Girsanov theorem gives the following equality

$$\mathcal{L}_T(\vartheta,Z^T)=\exp\left(\mu\lambda\int_0^T \ell(t)^*\zeta_t dZ_t-\frac{\mu^2\lambda^2}{2}\int_0^T \zeta_t^*\ell(t)\ell(t)^*\zeta_t d\langle M\rangle_t\right) \tag{7.6}$$

where $\zeta=(\zeta_t,t\ge 0)$ is the solution of the ordinary differential equation (7.4). In this case, the Fischer information stands for

$$\begin{aligned}
\mathcal{I}_T(\vartheta,v) &= -\mathbb{E}_\vartheta\frac{\partial^2}{\partial\vartheta^2}\ln\mathcal{L}_T(\vartheta,Z^T)\\
&= \mathbb{E}_\vartheta\int_0^T \mu^2\lambda^2\left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle M\rangle_t\\
&= \int_0^T \mu^2\lambda^2\left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle M\rangle_t\quad (\zeta\text{ is deterministic})\\
&= \int_0^T \left(\frac{\partial\zeta_t}{\partial\vartheta}\right)^*\mu^2\lambda^2\ell(t)\ell(t)^*\frac{\partial\zeta_t}{\partial\vartheta}d\langle M\rangle_t.
\end{aligned}$$

The proof is based on the properties of the likelihood ratio [169, Theorem I.10.1]. It is defined by:

$$\mathcal{Z}_T(\vartheta_1,\vartheta_2,Z^T)=\frac{\mathcal{L}_T(\vartheta_2,Z^T)}{\mathcal{L}_T(\vartheta_1,Z^T)}.$$

In the following, we will denote by $\mathcal{Z}_T(h,Z^T)$ the perturbation of $\mathcal{Z}_T(\vartheta,\vartheta_2,Z^T)$, when $\vartheta_2=\vartheta+\frac{h}{\sqrt{T}}$. Namely, $\mathcal{Z}_T(h,Z^T)=\mathcal{Z}_T(\vartheta,\vartheta+\frac{h}{\sqrt{T}},Z^T)$.

In order to prove both theorems, it is sufficient to check the three following conditions:

- $$\mathcal{Z}_T(h, Z^T) \xRightarrow{law} \exp\left\{h.\eta - \frac{u^2}{2}\mathcal{I}(\vartheta)\right\} \text{ with } \quad \eta \sim \mathcal{N}\left(0, \mathcal{I}(\vartheta)\right) ,$$

- for some $\chi > 0$:
$$\mathbb{E}_\vartheta \sqrt{\mathcal{Z}_T(h, Z^T)} \leq \exp\left(-\chi h^2\right) ,$$

- there exists $C > 0$ such that
$$\mathbb{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(h_1, Z^T)} - \sqrt{\mathcal{Z}_T(h_2, Z^T)}\right)^2 \leq C|h_1 - h_2|^2 ,$$

where $\mathcal{I}(\vartheta) = \dfrac{\mu^2}{\vartheta^4}$,

Uses of the Laplace transform

From Equation (7.4), this ratio can be written in the following form:

$$\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^T) = \exp\left(\mu\lambda \int_0^T \ell^* \delta_{\vartheta_1,\vartheta_2} d\nu_t^{\vartheta_1} - \frac{\mu^2\lambda^2}{2} \int_0^T \delta^*_{\vartheta_1,\vartheta_2} \ell\ell^* \delta_{\vartheta_1,\vartheta_2} d\langle M\rangle_t\right)$$

where $\delta_{\vartheta_1,\vartheta_2}(t)$ is the difference $\zeta_t^{\vartheta_2} - \zeta_t^{\vartheta_1}$ and $(\nu_t^{\vartheta_1}, t \geq 0)$ is defined by:

$$d\nu_t^{\vartheta_1} = dZ_t^O - \mu\lambda\ell(t)^*\zeta_t^{\vartheta_1} d\langle M\rangle_t , \quad \nu_0^{\vartheta_1} = 0 .$$

The behaviour of the likelihood ratio is deduced directly from the computation of the Fisher information for the optimal input (see [VI]). In the following we explain how we obtained the asymptotical Fisher information.

From (7.4), we get

$$\zeta_t = \varphi(t) \int_0^t \varphi^{-1}(s)b(s)v(s)d\langle M\rangle_s$$

where $\varphi(t)$ is the fundamental matrix, *i.e.*

$$\frac{d\varphi(t)}{d\langle M\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\varphi(t) , \quad \varphi(0) = \mathrm{Id} ,$$

and Id is the 2×2 identity matrix. Therefore

$$\begin{aligned} \mathcal{I}_T\left(\vartheta, v\right) &= \mu^2\lambda^2 \int_0^T \left(\frac{\partial \zeta_t}{\partial \vartheta}\right)^* \ell(t)\ell(t)^* \frac{\partial \zeta_t}{\partial \vartheta} d\langle M\rangle_t \\ &= \int_0^T \int_0^T K_T(s, \sigma) \frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}} v(s) \frac{\sigma^{\frac{1}{2}-H}}{\sqrt{2\lambda}} v(\sigma) ds d\sigma, \end{aligned}$$

where

$$K_T(s, \sigma) = \int_{\max(s,\sigma)}^T G(t, s)G(t, \sigma)dt ,$$

and

$$G(t,\sigma)=\frac{\partial}{\partial\vartheta}\left(\frac{\mu}{2}t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(\sigma)b(\sigma)\sigma^{\frac{1}{2}-H}\right).$$

Then

$$\begin{aligned}\mathcal{J}_T(\vartheta) &= \sup_{v\in\mathcal{V}_T}\mathcal{I}_T(\vartheta,v)\\ &= T\sup_{v\in L^2[0,T],\,||v||\leq 1}\int_0^T\int_0^T K_T(s,\sigma)v(s)v(\sigma)dsd\sigma\\ &= T\sup_{v\in L^2[0,T],\,||v||\leq 1}(K_Tv,v)\,.\end{aligned}$$

Proposition 7.2

$$\lim_{T\to+\infty}\sup_{v\in L^2[0,T],\,||v||\leq 1}(K_Tv,v)=\frac{\mu^2}{\vartheta^4}\,.$$

We obtain that $v_{opt}(t)=\sqrt{2\lambda}t^{H-\frac{1}{2}},\ 0\leq t\leq T$ is optimal in the class $\mathcal{V}_T$. As in [265],

$$\frac{1}{T}\int_0^T|v_{opt}(t)|^2d\langle M\rangle_t=1$$

Using (7.5) we have

$$u_{opt}(t)=\frac{d}{dt}\int_0^t K_H(t,s)v_{opt}(s)d\langle M\rangle_s=\frac{\kappa_H}{\sqrt{2\lambda}}t^{H-\frac{1}{2}}.$$

7.2.1 From $H>1/2$ to $H<1/2$

Thanks to [178, Corollary 5.2], for $H<1/2$, we have the relation between fBm processes of indexes H and $1-H$:

$$W_t^H=\aleph_H\int_0^t(t-s)^{2H-1}\,dW_s^{1-H}\,,\ \text{with}\ \aleph_H=\left(\frac{2H}{\Gamma(2H)\Gamma(3-2H)}\right)^{\frac{1}{2}}. \tag{7.7}$$

Using this relation, we can transform the observation model (7.1) to the following observation model:

$$\begin{cases} d\tilde{X}_t &= -\vartheta\tilde{X}_tdt+\tilde{u}(t)dt\,,\quad \tilde{X}_0=0\,,\\ d\tilde{Y}_t &= \mu\tilde{X}_tdt+dW_t^{1-H}\,,\quad \tilde{Y}_0=0\,,\end{cases} \tag{7.8}$$

with

$$\tilde{X}_t=\aleph_{1-H}\int_0^t(t-s)^{1-2H}\,dX_s,\qquad \tilde{Y}_t=\aleph_{1-H}\int_0^t(t-s)^{1-2H}\,dY_s\,,$$

and

$$\tilde{u}(t)=\aleph_{1-H}\frac{d}{dt}\int_0^t(t-r)^{1-2H}u(r)dr=(1-2H)\aleph_{1-H}\int_0^t(t-r)^{-2H}u(r)dr.$$

Then, $1-H>\frac{1}{2}$ and the results of Proposition 7.1 and Theorem 7.1 are valid for any $H\in(0,1)$. In fact we have to prove that the set of controls $\mathcal{U}_T$ remains unchanged after transformation (7.7). In other words we prove

Lemma 7.1 *If $\tilde{v}$ is defined by*

$$\tilde{v}(t) = \frac{d}{d\langle M^{1-H}\rangle_t} \int_0^t k_{1-H}(t,s)\tilde{u}(s)ds,$$

then

$$\frac{1}{T}\int_0^T \tilde{v}(s)^2 d\langle M^{1-H}\rangle_s = \frac{1}{T}\int_0^T v(s)^2 d\langle M^H\rangle_s.$$

Proof. Indeed we have

$$\begin{aligned}
\tilde{v}(t) &= \frac{2\lambda_{1-H}(1-2H)\aleph_{1-H}}{\kappa_{1-H}t^{2H-1}} \frac{d}{dt}\int_0^t s^{H-\frac{1}{2}}(t-s)^{H-\frac{1}{2}}\left(\int_0^s (s-r)^{-2H}u(r)dr\right)ds \\
&= \frac{C_H}{t^{2H-1}}\frac{d}{dt}\int_0^t u(r)F\left(\frac{r}{t-r}\right)dr = -\frac{C_H}{t^{2H-1}}\int_0^t u(r)F'\left(\frac{r}{t-r}\right)\frac{r}{(t-r)^2}dr
\end{aligned}$$

with $C_H = \frac{2\lambda_{1-H}(1-2H)\aleph_{1-H}}{\kappa_{1-H}}$. Now

$$F'(z) = \frac{(H-1/2)\Gamma(1-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}\frac{(1+z)^{2H-1}}{z^{H+\frac{1}{2}}}.$$

Hence

$$\begin{aligned}
\tilde{v}(t) &= -\frac{C_H}{t^{2H-1}}\frac{(H-1/2)\Gamma(1-2H)\Gamma(H+\frac{1}{2})}{\Gamma(\frac{3}{2}-H)}\int_0^t u(r)\frac{r}{(t-r)^2}\left(\frac{t}{t-r}\right)^{2H-1}\left(\frac{t-r}{r}\right)^{H+1/2}dr \\
&= \tilde{C}_H\int_0^t u(r)r^{1/2-H}(t-r)^{-1/2-H}dr \\
&= \frac{\tilde{C}_H\kappa_H}{1/2-H}\frac{t^{1-2H}}{2\lambda_H}\frac{d}{d\langle M^H\rangle_t}\int_0^t k_H(t,r)u(r)dr = \frac{\tilde{C}_H\kappa_H}{2\lambda_H(1/2-H)}t^{1-2H}v(t).
\end{aligned}$$

Therefore we have:

$$\tilde{v}(t)\frac{t^{H-1/2}}{\sqrt{2\lambda_{1-H}}} = \widehat{C}_H v(t)\frac{t^{1/2-H}}{\sqrt{2\lambda_H}}$$

with

$$\widehat{C}_H = \frac{\tilde{C}_H\kappa_H}{2\sqrt{\lambda_H\lambda_{1-H}}(1/2-H)}.$$

Straightforward calculus gives $\widehat{C}_H = 1$, that achieves the proof. □

For $H < 1/2$, if $u \in \mathcal{U}_T$, then $\tilde{u}$ is also in $\mathcal{U}_T$. Moreover using Proposition 7.1, we obtain that the asymptotical optimal input $\tilde{u}_{opt}$ for (7.8) will be $\tilde{u}_{opt}(t) = \frac{\kappa_{1-H}}{\sqrt{2\lambda_{1-H}}}t^{1/2-H}$. Therefore the optimal input for (7.1) will be

$$\begin{aligned}
u_{opt}(t) &= \aleph_H\frac{d}{dt}\int_0^t (t-r)^{2H-1}\tilde{u}_{opt}(r)dr = \aleph_H\frac{\kappa_{1-H}}{\sqrt{2\lambda_{1-H}}}\frac{d}{dt}\int_0^t (t-r)^{2H-1}r^{1/2-H}dr \\
&= \aleph_H\frac{\kappa_{1-H}}{\sqrt{2\lambda_{1-H}}}\frac{\Gamma(2H)\Gamma(3/2-H)}{\Gamma(1/2+H)}t^{H-1/2} = \frac{\kappa_H}{\sqrt{2\lambda_H}}t^{H-1/2}.
\end{aligned}$$

Remark that in the case $H < 1/2$, $\lim_{t\to 0} u_{opt}(t) = +\infty$ (but u_{opt} is still integrable).

7.3 Observation model (7.2)

The problem is similar to (7.1): parameter $\vartheta > 0$ is unknown and is to be estimated given the observed trajectory $Y^T = (Y_t,\, 0 \leq t \leq T)$. The sketch of the proof is similar. Using Laplace's transform and the conditional expectation $\pi_t(X) = \mathbb{E}_\vartheta\left(X_t | \mathcal{F}_t^Y\right)$, we define $L_T(a, \vartheta_1, \vartheta_2)$ the Laplace transform of the integral of the quadratic form of the difference $\delta_{\vartheta_1,\vartheta_2}(t) = \mu\pi_t^{\vartheta_2}(X) - \mu\pi_t^{\vartheta_1}(X)$:

$$L_T(a, \vartheta_1, \vartheta_2) = \mathbb{E}_{\vartheta_1} \exp\left\{-\frac{a}{2}\int_0^T \delta^2_{\vartheta_1,\vartheta_2} dt\right\}.$$

The aim is to obtain asymptotic results on this Laplace transform. Since it is difficult to work with $\pi_t(X)$, we replace it by $\pi_t^*(X)$ the stationnary approximation of $\pi_t(X)$ defined by

$$\pi_t^*(X) = \int_0^t f_1(t-s) dY_s,$$

where f_1 minimizes the filtering error

$$\arg \inf_{f_1 \in L^2(\mathbb{R}_+)} \lim_{t\to\infty} \mathbb{E}\left(X_t - \pi_t^*(X)\right)^2.$$

Up to now we are blocked by some technical issues[1]. But recently in a series of papers [74, 75, 76, 77], Chigansky et al. provide some new and deep results concerning the asymptotic of several fractional kernels. We hope that these results can help us to overcome the difficulties concerning this model.

[1]We do not develop them here.

Chapter 8

Homogenization ([IX, XXIII, B])

In this chapter we describe the results obtained with Marina Kleptsyna and Andrey Piatnitski. The goal is to characterize the rate of convergence in the homogenization problem for a second order divergence form parabolic operator with random stationary in time and periodic in spatial variables coefficients. We also aim at describing the limit behaviour of a normalized difference between solutions of the original and homogenized problems.

To avoid boundary effects we study a Cauchy problem that takes the form

$$\begin{cases} \partial_t u^\varepsilon = \mathrm{div}\Big(a\left(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^\alpha}\right)\nabla u^\varepsilon\Big), \qquad x \in \mathbb{R}^d,\ t > 0, \\ u^\varepsilon(x,0) = \imath(x). \end{cases} \tag{8.1}$$

with $\alpha > 0$. We assume that the matrix $a(z,s) = \{a^{ij}(z,s)\}$ is uniformly elliptic, $(0,1)^d$-periodic in z variable, and random stationary ergodic in s. We denote $Y = (0,1)^d$ and in what follows identify Y-periodic function with functions defined on the torus $\mathbb{T}^d$.

It is known (see [332], [191]) that under these assumptions problem (8.1) admits homogenization. More precisely, for any $\imath \in L^2(\mathbb{R}^n)$, almost surely (a.s.) solutions u^ε of problem (8.1) converge, as $\varepsilon \to 0$, to a solution of the homogenized problem

$$\begin{cases} \partial_t u^0 = \mathrm{div}\big(\mathrm{a}^{\mathrm{eff}}\nabla u^0\big) \\ u^0(x,0) = \imath(x) \end{cases} \tag{8.2}$$

with a constant (non-random) positive definite matrix $\mathrm{a}^{\mathrm{eff}}$. The convergence is in $L^2(\mathbb{R}^d \times (0,T))$. More detailed description of the existing homogenization results is given in Section 8.1.

This chapter focuses on the rate of this convergence and on higher order terms of the asymptotics of u^ε. Our goal is to study the limit behaviour of the difference $u^\varepsilon - u^0$, as ε tends to zero.

In the existing literature there is a number of works devoted to homogenization of random parabolic problems. The results obtained in [200] and [266] for random divergence form elliptic operators also apply to the parabolic case. In the presence of large lower order terms the limit dynamics might remain random and show diffusive or even more complicated behaviour. Parabolic operators with random coefficients depending both on spatial and temporal variables have been considered in [332]. The papers [53], [274], [190], [191] focus on the case of time dependent parabolic operators with periodic in spatial variables and random in time coefficients. The fully random case has been studied in [275], [16], [17], [153]. One of the important aspects of homogenization theory is estimating the rate of convergence and optimal estimates for the rate of convergence is an open issue. For random operators the first estimates have been obtained in [180]. Further important progress in this direction was achieved in the recent works [142], [141].

8.1 Setting and homogenization result

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a standard probability space equipped with a measure preserving ergodic dynamical system $\mathcal{T}_s$, $s \in \mathbb{R}$. Given a measurable matrix function $\tilde{a}(z,\omega) = \{\tilde{a}^{ij}(z,\omega)\}_{i,j=1}^d$ which is periodic in z variable with a period one in each coordinate direction, we define a random field $a(z,s)$ by

$$a(z,s) = \tilde{a}(z, \mathcal{T}_s\omega).$$

Then $a(z,s)$ is periodic in z and stationary ergodic in s. A very important particular case is the diffusion case where $a(z,s)$ has the form

$$a(z,s) = \mathrm{a}(z, \xi_s), \tag{8.3}$$

where $\mathrm{a} = \mathrm{a}(z,y)$ is a matrix periodic in z and $(\xi_s,\ s \in \mathbb{R})$ is a stationary diffusion process in $\mathbb{R}^n$.

We consider the Cauchy problem (8.1) in $\mathbb{R}^d \times (0,T]$, $T > 0$:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} & = & \mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^\alpha}\Big)\nabla u^\varepsilon\Big) \\ u^\varepsilon(x,0) & = & \imath(x) \end{cases}$$

with a small positive parameter ε, $\alpha > 0$ being a fixed number. We assume that the coefficients in (8.1) possess the following properties.

(H1) The matrix $a(z,s) = \{a^{ij}(z,s)\}_{i,j=1}^d$ is symmetric and satisfies uniform ellipticity condition, that is there exists $\lambda > 0$ such that for all (z,s) the following inequality holds:

$$\lambda|\zeta|^2 \le a(z,s)\zeta\cdot\zeta \le \lambda^{-1}|\zeta|^2, \quad \zeta \in \mathbb{R}^d.$$

Now we remind of the existing homogenization results for Problem (8.1). To this end we first introduce the so-called cell problem. For $\alpha = 2$ it reads

$$\partial_s \chi(z,s) = \mathrm{div}\big(a(z,s)(\mathbf{I} + \nabla\chi(z,s))\big), \qquad (z,s) \in \mathbb{T}^d \times (-\infty, +\infty) \tag{8.4}$$

with $\mathbf{I}$ being the unit matrix; here $\chi = \{\chi^j\}_{j=1}^d$ is a vector function. In what follows for the sake of brevity we denote $\mathrm{div}a = \mathrm{div}(a\mathbf{I}) = \frac{\partial}{\partial z^i}a^{ij}(z)$. Also, we assume summation over repeated indices. Under our setting this equation has a stationary periodic in y vector-valued solution. This solution is unique up to an additive constant. We define

$$\mathrm{a}^{\mathrm{eff}} = \mathbb{E}\int_{\mathbb{T}^d} a(z,s)\big(\mathbf{I}+\nabla\chi(z,s)\big)\,dz. \tag{8.5}$$

Notice that due to stationarity the expression on the right-hand side does not depend on s.

If $\alpha < 2$, the cell problem reads

$$\operatorname{div}\big(a(z,s)(\mathbf{I}+\nabla\chi(z,s)\big) = 0, \qquad z \in \mathbb{T}^d; \tag{8.6}$$

here s is a parameter. This equation has a unique up to a multiplicative constant solution. We then set

$$\mathrm{a}_-^{\mathrm{eff}} = \mathbb{E}\int_{\mathbb{T}^d} a(z,s)\big(\mathbf{I}+\nabla\chi(z,s)\big)\,dz. \tag{8.7}$$

For $\alpha > 2$ we first define $\overline{a}(z) = \mathbb{E}(a(z,s))$, then introduce a deterministic function $\chi(z)$ as a periodic solution to the problem

$$\operatorname{div}\big(\overline{a}(z)(\mathbf{I}+\nabla\chi(z)\big) = 0, \qquad z \in \mathbb{T}^d, \tag{8.8}$$

and finally define

$$\mathrm{a}_+^{\mathrm{eff}} = \int\limits_{\mathbb{T}^n} \overline{a}(z)\big(\mathbf{I}+\nabla\chi(z)\big)\,dz. \tag{8.9}$$

The following statement has been obtained in [332], [190] and [95].

Theorem 8.1 *Let $\imath \in L^2(\mathbb{R}^d)$, and assume that Condition* **(H1)** *holds. Then a solution u^ε of problem* (8.1) *converges a.s. in $L^2(\mathbb{R}^d\times(0,T))$ to a solution of the limit problem* (8.2) *with*

- $\mathrm{a}^{\mathrm{eff}}$ *given by* (8.5) *if $\alpha = 2$;*
- $\mathrm{a}^{\mathrm{eff}} = \mathrm{a}_-^{\mathrm{eff}}$ *defined in* (8.7) *if $\alpha < 2$;*
- $\mathrm{a}^{\mathrm{eff}} = \mathrm{a}_+^{\mathrm{eff}}$ *defined in* (8.9) *if $\alpha > 2$.*

Intuitively or roughly speaking, if $\alpha < 2$, the lower diffusive scaling implies that we first homogenize the space part (Equation (8.6)) and then we take the expectation to get $\mathrm{a}^{\mathrm{eff}}$. In the upper diffusive scaling ($\alpha > 2$), we do the converse: first we take the expectation and then we homogenize the matrix $\overline{a}$ (Equation (8.9)). For the diffusive scaling ($\alpha = 2$) the two operations are done simultaneously (Equation (8.5)).

8.1.1 The diffusion case

In this section we consider the case (8.3), namely

$$a(z,s) = \mathrm{a}(z,\xi_s),$$

and we consider the particular case of (8.1):

$$(8.10)\qquad \begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} & = & \operatorname{div}\Big(\mathrm{a}\Big(\dfrac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)u^\varepsilon\Big) \\ u^\varepsilon(0,x) & = & \imath(x) \end{cases}$$

with a diffusion process ξ_s, $s\in(-\infty,+\infty)$, with values in $\mathbb{R}^n$ (or on a compact manifold). This process is defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The corresponding Itô equation reads

$$d\xi_t = b(\xi_t)dt + \sigma(\xi_t)dB_t,$$

here B stands for a standard n-dimensional Wiener process. The infinitesimal generator of ξ is denoted by $\mathcal{L}$:

$$\mathcal{L}f(y) = q^{ij}(y)\frac{\partial^2}{\partial y^i\partial y^j}f(y) + b(y)\cdot\nabla f(y), \qquad y\in\mathbb{R}^n,$$

with a $n\times n$ matrix $q(y)=\frac{1}{2}\sigma(y)\sigma^*(y)$. We also introduce the operator

$$\mathcal{A}f(x) = \operatorname{div}_x\left(\mathrm{a}\,(x,y)\,\nabla_x f\right);$$

here y is a parameter. Applied to a function $f(z,y)$, $\mathcal{L}$ acts on the function $y\mapsto f(z,y)$ for z fixed, and $\mathcal{A}$ acts on the function $z\mapsto f(z,y)$ for y fixed.

In the diffusion case, we suppose that the following conditions hold true.

(I1) The coefficients a and q are uniformly bounded as well as their first order derivatives in all variables:

$$|\mathrm{a}(z,y)| + |\nabla_z\mathrm{a}(z,y)| + |\nabla_y\mathrm{a}(z,y)| \le C_1,$$

$$|q(y)| + |\nabla q(y)| \le C_1.$$

The function b as well as its derivatives satisfy polynomial growth condition:

$$|b(y)| + |\nabla b(y)| \le C_1(1+|y|)^{N_1}.$$

(I2) Both $\mathcal{A}$ and $\mathcal{L}$ are uniformly elliptic:

$$C_2\mathbf{I}\le \mathrm{a}(z,y), \quad C_2\mathbf{I}\le q(y), \quad \text{with } C_2>0,$$

where $\mathbf{I}$ stands for a unit matrix of the corresponding dimension.

(I3) There exist $N_2>-1$, $R>0$ and $C_3>0$ such that

$$b(y)\frac{y}{|y|} \le -C_3|y|^{N_2}$$

for all y, $|y|>R$.

Remark that **(I1)** and **(I2)** imply **(H1)**.

Let us recall that according to [275] (see also [278, 279]), under our setting, a diffusion process ξ with the generator $\mathcal{L}$ has an invariant measure in $\mathbb{R}^n$ that has a smooth density $\rho = \rho(y)$. For any $N > 0$ it holds

$$(1 + |y|)^N \rho(y) \le C_N$$

with some constant C_N. The function ρ is the unique up to a multiplicative constant bounded solution of the equation $\mathcal{L}^*\rho = 0$; here $*$ denotes the formally adjoint operator. We assume that the process ξ_t is stationary and distributed with the density ρ. Then

$$\mathbb{E} f(z, \xi_s) = \int_{\mathbb{R}^n} f(z, y)\rho(y) dy.$$

In the rest of this chapter

- $\overline{f}$ denotes the mean w.r.t. the invariant measure ρ;
- $\langle f \rangle$ is the mean on the torus.

Let us recall the result of [190] (see also [54]).

Theorem 8.2 (Theorem 8.1 for the diffusion case) *Under Assumptions* **(I1)**–**(I3)**, *the solution* u^ε *of* (8.10) *converges almost surely in the space* $L^2(\mathbb{R}^d \times (0, T))$ *to the solution of Problem* (8.2) *with*

- *for* $\alpha = 2$, *Equation* (8.5) *becomes*

$$\mathrm{a}^{\mathrm{eff}} = \int_{\mathbb{R}^n} \int_{\mathbb{T}^d} \mathrm{a}(\mathbf{I} + \nabla_z \widehat{\chi})\pi(y)\, dzdy,$$

  $\widehat{\chi}$ *being the solution of the following equation*

$$(\mathcal{A} + \mathcal{L})\widehat{\chi} = -\mathrm{div}_z \mathrm{a}(z, y). \tag{8.11}$$

- *for* $0 < \alpha < 2$, *the formula* (8.7) *for* $\mathrm{a}^{\mathrm{eff}}$ *remains unchanged, and* $\widehat{\chi}$ *satisfies:*

$$\mathcal{A}\widehat{\chi} = -\mathrm{div}_z a(z, y); \tag{8.12}$$

- *for* $\alpha > 2$, *the formula* (8.9) *becomes*

$$\mathrm{a}^{\mathrm{eff}} = \langle \overline{a}(1 + \nabla_z \widehat{\chi}) \rangle,$$

  and $\widehat{\chi}$ *is a solution of*

$$\overline{\mathcal{A}}\widehat{\chi} = \mathrm{div}\,(\overline{a}\,(z)\,\nabla\widehat{\chi}) = -\mathrm{div}\overline{a}(z). \tag{8.13}$$

Remark 8.1 *We emphasize that all formulae defining* $\mathrm{a}^{\mathrm{eff}}$ *are consistent. For example for* $\alpha = 2$ *we have:*

$$\mathbb{E} \int_{\mathbb{T}^d} \nabla_z \chi^i(z, s) \cdot \mathrm{a}(z, \xi_s)\mathbf{e}^j\, dz = \mathbb{E} \int_{\mathbb{T}^d} \nabla_z \widehat{\chi}^j(z, \xi_s) \cdot \mathrm{a}(z, \xi_s)\mathbf{e}^i\, dz,$$

where $\mathbf{e}^j$ *stands for the* j*-th coordinate vector in* $\mathbb{R}^d$.

8.2 Description of our results

The key point of Theorems 8.1 and 8.2 is that the homogenized equation (8.2) is the same for any α and is deterministic. Only the effective matrix $\mathrm{a}^{\mathrm{eff}}$ changes. Our aim is to study the difference $u^\varepsilon - u^0$. Formally we define

$$U^\varepsilon(x,t) = \frac{1}{\varepsilon^{\alpha/2}}\left[u^\varepsilon(x,t) - u^0(x,t) - \sum_{j\geq 1}\varepsilon^{\mathfrak{c}_j}\mathfrak{C}_j\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\right)\right] \tag{8.14}$$

where $\mathfrak{c}_j$ are positive constants and $\mathfrak{C}_j$ are correctors such that U^ε converges in law in $L^2(\mathbb{R}^d \times (0,T))$[1] to a solution of a SPDE with constants coefficients and an additive noise. This SPDE reads

$$\begin{cases} dU^0 = \mathrm{div}\Big(\mathrm{a}^{\mathrm{eff}}\nabla U^0 + \mu\dfrac{\partial^3}{\partial x^3}u^0\Big)dt + \Lambda^{1/2}\dfrac{\partial^2}{\partial x^2}u^0\,dW_t, \\ U^0(x,0) = 0; \end{cases} \tag{8.15}$$

where U^0 is a scalar-valued function of x and t, $\mathrm{a}^{\mathrm{eff}}$ is the homogenized coefficients matrix, u^0 is a solution of (8.2), $W_t = W_{t,ij}$ is a standard d^2-dimensional Wiener process, and $\mu = \mu^{ijk}$ and $\Lambda^{1/2} = (\Lambda^{1/2})^{ijkl}$ are constant tensors with three and four indices, respectively, so that the two driving terms in (8.15) take the form

$$\mu\frac{\partial^3}{\partial x^3}u^0 = \mu^{ijk}\frac{\partial^3 u^0}{\partial x^i\partial x^j\partial x^k}, \qquad \Lambda^{1/2}\frac{\partial^2}{\partial x^2}u^0\,dW_t = (\Lambda^{1/2})^{ijkl}\frac{\partial^2 u^0}{\partial x^i\partial x^j}\,dW_{t,kl};$$

Recall that here and in what follows we assume summation over repeated indices.

Remark that the coefficient μ in the SPDE only appears for $\alpha = 2$. The power constants $\mathfrak{c}_j$ and the correctors $\mathfrak{C}_j$ depend on the value of α.

8.2.1 First additional conditions

If Theorem 8.1 holds under the only assumption **(H1)**, the rate of convergence is proved with some additional conditions. Some of them depend on α and are precised in the related next sections. But other hypotheses hold for any α.

The first assumption concerns the initial condition. We suppose that $\imath \in C_0^\infty(\mathbb{R}^d)$. In fact, this condition can be essentially relaxed. We can only suppose that $\imath$ is J_0+1 times continuously differentiable, and for any $K>0$ there is $C_K>0$ such that

$$\sum_{|\mathbf{j}|\leq N_0}\left|\frac{\partial^{\mathbf{j}}\imath}{(\partial x^1)^{j_1}\dots(\partial x^d)^{j_d}}(x)\right| \leq C_K(1+|x|)^{-K}$$

for all $x \in \mathbb{R}^d$, where the sum is taken over all $\mathbf{j} = (j_1,\dots,j_d)$ with $\sum_{i=1}^d j_i \leq J_0$. The integer J_0 is equal to[2] 3 if $\alpha = 2$, $\lfloor\frac{\alpha}{2(2-\alpha)}\rfloor + 1$ if $\alpha < 2$ and $\lfloor\frac{\alpha}{2}\rfloor$ if $\alpha > 2$. It should be

[1] The topology depends in fact on the value of α.
[2] $\lfloor\cdot\rfloor$ stands for the integer part.

noted that under this condition on $\imath$, for any $K > 0$ and $N_0 \in \mathbb{N}$, there is $C_{K,N_0}(T) > 0$ such that a solution of problem (8.2) satisfies the estimate

$$\sum_{|\mathbf{j}|\le N_0} \Big| \frac{\partial^{\mathbf{j}}}{(\partial x^1)^{j_1}\dots(\partial x^d)^{j_d}} u^0(x,t)\Big| \le C_{K,N_0}(T)(1+|x|)^{-K} \tag{8.16}$$

for all $(x,t) \in \mathbb{R}^d \times [0,T]$. Hence in the sequel the regularity of u^0 is never discussed. But we will see that the derivatives of u^0 appear in the correctors of U^ε. Therefore this condition cannot be weaken.

The second hypothesis is a mixing condition. In the SPDE (8.15), the Brownian part follows from an invariance principle and thus from a mixing property. As the regularity condition on $\imath$, this assumption cannot be easily weaken. In order to formulate this condition we introduce $\mathcal{F}_{\le s}$ and $\mathcal{F}_{\ge s}$ the σ-algebras generated by $\{a(z,t) : z \in \mathbb{T}^d, t \le s\}$ and $\{a(z,t) : z \in \mathbb{T}^d, t \ge s\}$, respectively. We define the *strong mixing coefficient*[3] γ by:

$$\gamma(r) = \sup \big|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)\big|,$$

where the supremum is taken over all $A \in \mathcal{F}_{\le 0}$ and $B \in \mathcal{F}_{\ge r}$. We then assume that

(H2) $\gamma^{1/2}$ is integrable:

$$\int_0^\infty (\gamma(r))^{1/2} dr < +\infty.$$

Remark 8.2 *Condition* **(H2)** *is somehow implicit. In applications various sufficient conditions are often used.*

For more properties concerning mixing coefficients and the related results, see [103, 172].

Remark 8.3 *Notice that conditions* **(I1)**–**(I3)** *need not imply condition* **(H2)**. *In general, mixing properties that follow from* **(I1)**–**(I3)** *are weaker than those stated by* **(H2)**. *However, in the diffusive case these conditions are sufficient for the CLT type results used in the proofs below. This makes the diffusive case interesting. It should also be noted that in this case the conditions are given in terms of the process generator, which might be more comfortable in applications.*

The third condition concerns the smoothness of the parameters. We assume that

(H3) The realizations $a^{ij}(z,s)$ are smooth. For any $N \ge 1$ and $k \ge 1$ there exist $C_{N,k}$ such that

$$\mathbb{E}\,\|a^{ij}\|^k_{C^N(\mathbb{T}^d\times[0,T])} \le C_{N,k};$$

Let us emphasize that Condition **(H3)** cannot be satisfied in the diffusion case, since we require also regularity in time. For the diffusion case (Equation (8.3)), we add the next condition.

[3] Often called α-mixing, but here α denotes the time scaling.

(I4) The matrix a, the matrix function σ and vector-function b are smooth. Moreover, for each $N > 0$ there exists $C_N > 0$ such that

$$\|\mathrm{a}\|_{C^N(\mathbb{T}^d \times \mathbb{R}^n)} \leq C_N, \qquad \|\sigma\|_{C^N(\mathbb{R}^n)} \leq C_N, \qquad \|b\|_{C^N(\mathbb{R}^n)} \leq C_N.$$

Definition 8.1 *We say that*

- *For the dynamical system, Condition* **(H)** *(resp.* **(H*)***) holds if $\imath$ is regular and* **(H1)***,* **(H2)** *(resp.* **(H)** *and* **(H3)***) are fulfilled.*
- *For the diffusion case, Condition* **(I)** *(resp.* **(I*)***) holds if $\imath$ is regular and* **(I1)** *–* **(I3)** *(resp.* **(I)** *and* **(I4)***) are satisfied.*

8.2.2 Comparison between the three cases

But more important is the scheme to obtain the convergence of U^ε.

- For the diffusive case ($\alpha = 2$), we are able to prove convergence for a very general dynamical system, provided that **(H)** or **(I)** holds (see Theorems 8.3 and 8.4). There is only one corrector in U^ε with $\mathfrak{c}_1 = 1$ and

$$\mathfrak{C}_1\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) = \chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \cdot \nabla u^0(x,t).$$

  Note that the proof is similar in both cases (dynamical system and diffusion case) since the regularity condition is not implied.

- When $\alpha < 2$, we distinguish two sub-cases. Either the dynamical system is smooth (condition **(H*)**), especially in time and the proof of Theorem 8.6 is similar to the diffusive case ($\alpha = 2$). Or the diffusion case holds, that is $a(z,s) = \mathrm{a}(z, \xi_s)$ with assumption **(I*)**. Under this setting, we can give a complete proof when $\alpha \leq 1$ or in dimension $d = 1$.

  In any case (diffusion or not), the constants $\mathfrak{c}_j$ are equal to $j(2-\alpha)$ and the correctors take the form

$$\mathfrak{C}_j\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) = u^j(x,t),$$

  where u^j are smooth functions (defined by (8.47)). The number of correctors is equal to $J^0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$ and increases when α goes to 2. Let us mention that according to the result of Theorem 8.1, the first correctors χ coincide in both sub-cases. It is interesting to observe that the higher order correctors need not coincide[4].

[4] In (8.47), the matrices $\mathrm{a}^{k,\mathrm{eff}}$ need not to be equal.

- For $\alpha > 2$, we can prove the convergence only for the diffusion case and in dimension $d = 1$ (Theorem 8.5). The formal expansion of u^ε leads to a SPDE with a large term[5], basically of the form

$$\frac{1}{\varepsilon}\mathfrak{K}_j\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) dB_t.$$

  Unfortunately to control this term, we use a trick valid only in dimension 1. And we don't have any other hint in mind.

  In U^ε there are two different scales. In the first one, the constants are integers $\mathfrak{c}_j = j$ and the correctors are:

$$\mathfrak{C}_j\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right) = u^j(x,t) + \sum_{\ell=1}^{j} \chi^\ell\left(\frac{x}{\varepsilon}\right)\partial_x^\ell u^{j-\ell}(x,t).$$

  The number of such terms is given by $J_0 = \lfloor \frac{\alpha}{2} \rfloor$ and increases when α tends to $+\infty$. The second scale contains $J_1 = \lfloor \frac{1}{\alpha-2} + \frac{1}{2} \rfloor$ terms with powers $\mathfrak{c}_j$ equal to $j(\alpha-2)$ and with smooth correctors $v^j(x,t)$. To better understand the role of α, let us distinguish three cases:

  - $2 < \alpha < 4$: $J_0 = 1$ and $J_1 \geq 1$. J_1 tends to $+\infty$ when α goes to 2. The powers $j(\alpha-2)$ of ε are non integer (except if $\alpha = 3$).
  - $\alpha = 4$ (kind of a critical value): $J_0 = 2$ and $J_1 = 1$. There are only two correctors in U^ε with $\mathfrak{c}_j = 1$ or 2.
  - $\alpha > 4$: $J_1 = 0$. The scaling is given by the powers ε^j only.

In the rest of this chapter we give more details on those results. To understand the restrictions when $\alpha \neq 2$, we give an idea of the proof when $\alpha = 2$ for a general dynamical system. Since the time regularity is not used in this case, we deduce almost immediately the same result for the diffusion case. Then for $\alpha < 2$, if the regularity condition **(H3)** holds, a similar result is proved. However when there is no time regularity, we can prove convergence only in dimension 1. And for $\alpha > 2$, a similar issue appears, which is overcome again only in dimension 1. The key problem is due to some large martingale terms. To control them we use some trick valid only in dimension one. Finally we present a result concerning the fundamental solution of a heat SPDE. This result is new and interesting by itself. At the beginning we hoped we could use it to fill the gap for $\alpha < 2$.

8.3 The case $\alpha = 2$ ([IX])

We begin with the general dynamical system and Equation (8.1). We recall that the equation (8.4)

$$\partial_s \chi(z,s) = \mathrm{div}_z\big(a(z,s)\big(\nabla_z \chi(z,s) + \mathbf{I}\big)\big)$$

[5] For $\alpha < 2$, we have a similar term but with $\varepsilon^{1-\alpha}$ in front, instead of ε^{-1}.

has a unique up to an additive (random) constant periodic in z and stationary in s solution (see [191], [95]). Thus, the gradient $\nabla_z \chi$ is uniquely defined. The principal corrector takes the form $\varepsilon\chi\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\big) \cdot \nabla u^0(x,t)$. We study the limit behaviour of the expression

$$U^\varepsilon(x,t) := \frac{u^\varepsilon(x,t) - u^0(x,t)}{\varepsilon} - \chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \cdot \nabla u^0(x,t). \tag{8.17}$$

For generic stationary ergodic coefficients $a(z,s)$ the family $\{U^\varepsilon\}$ needs not be compact or tight in $L^2(\mathbb{R}^d \times (0,T))$. Nevertheless we prove the next result.

Theorem 8.3 *Under the assumption* **(H)***, the function U^ε converges in law, as ε goes to 0, in the space $L^2(\mathbb{R}^d \times (0,T))$ to the solution U^0 of SPDE* (8.15)

$$\begin{cases} dU^0 = \operatorname{div}\Big(\mathrm{a}^{\mathrm{eff}}\nabla U^0 + \mu\dfrac{\partial^3}{\partial x^3}u^0\Big)dt + \Lambda^{1/2}\dfrac{\partial^2}{\partial x^2}u^0\, dW_t, \\ U^0(x,0) = 0. \end{cases}$$

Notice that under proper choice of an additive constant the mean value of $\chi(z,s)$ is equal to zero. Therefore, the function $\chi\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\big)\nabla u^0(x,t)$ converges a.s. to zero weakly in $L^2(\mathbb{R}^d \times (0,T))$, as $\varepsilon \to 0$. Therefore, in the weak topology of $L^2(\mathbb{R}^d \times (0,T))$, the limit in law of the normalized difference $\varepsilon^{-1}(u^\varepsilon(x,t) - u^0(x,t))$ coincides with that of U^ε.

Let us give the explicit value of the parameters of (8.15). Recall that $\mathrm{a}^{\mathrm{eff}}$ is defined by (8.5). Letting

$$\Psi_1(s) = \int\limits_{\mathbb{T}^n} \{a(z,s)(\mathbf{I} + \nabla_z\chi(z,s)) - \mathrm{a}^{\mathrm{eff}}\}\, dz, \tag{8.18}$$

the tensor $\Lambda^{1/2}$ is defined as the square root of the $d^2 \times d^2$ symmetric and positive semi-definite matrix Λ:

$$\Lambda^{ijkl} = \int\limits_0^\infty \mathbb{E}\big(\Psi_1^{ij}(0)\Psi_1^{kl}(s) + \Psi_1^{kl}(0)\Psi_1^{ij}(s)\big)\, ds. \tag{8.19}$$

We also define

$$\Psi_2(z,s) = \{a(z,s)(\mathbf{I} + \nabla_z\chi(z,s)) - a^{\mathrm{eff}}\} - \Psi_1(s) + \operatorname{div}_z\big(a(z,s) \otimes \chi(z,s)\big) \tag{8.20}$$

with

$$\operatorname{div}_z\big(a(z,s) \otimes \chi(z,s)\big) = \big\{\frac{\partial}{\partial z^i}\big(a^{ij}(z,s)\chi^k(z,s)\big)\big\}_{i,j,k=1}^d,$$

and $\chi_2^{ij}(z,s)$ is a stationary zero mean solution of the equation

$$\partial_s\chi_2^{ij}(z,s) - \operatorname{div}_z\big(a(z,s)\nabla_z\chi_2^{ij}(z,s)\big) = \Psi_2^{ij}(z,s). \tag{8.21}$$

Then the constant tensor $\mu = \{\mu^{ijk}\}_{i,j,k=1}^d$ is defined by

$$\mu = \mathbb{E}\int_{\mathbb{T}^d} \big\{ - \mathrm{a}^{\mathrm{eff}} \otimes \chi(z,s) + a(z,s) \otimes \chi(z,s) + a(z,s)\nabla_z\chi_2(z,s)\big\}dz \tag{8.22}$$

where

$$a(z,s)\otimes\chi(z,s)=\big\{a^{ij}(z,s)\chi^k(z,s)\big\}_{i,j,k=1}^d$$

and

$$a(z,s)\nabla_z\chi_2(z,s)=\{a^{ij}(z,s)\partial_{z^\ell}\chi_2^{\ell k}(z,s)\}_{i,j,k=1}^d.$$

8.3.1 Idea of the proof

We deal with the formal asymptotic expansion of a solution of Problem (8.1). We use it in order to understand the structure of the leading terms of the difference $u^\varepsilon-u^0$. As usually in the multi-scale asymptotic expansion method we consider $z=x/\varepsilon$ and $s=t/\varepsilon^2$ as independent variables and use repeatedly the formulae

$$\frac{\partial}{\partial x^j}f\Big(x,\frac{x}{\varepsilon}\Big)=\Big(\frac{\partial}{\partial x^j}f(x,z)+\frac{1}{\varepsilon}\frac{\partial}{\partial z^j}f(x,z)\Big)_{z=\frac{x}{\varepsilon}},$$
$$\frac{\partial}{\partial t}f\Big(t,\frac{t}{\varepsilon^2}\Big)=\Big(\frac{\partial}{\partial t}f(t,s)+\frac{1}{\varepsilon^2}\frac{\partial}{\partial s}f(t,s)\Big)_{s=\frac{t}{\varepsilon}}.$$

We represent a solution u^ε as the following asymptotic series in integer powers of ε:

$$u^\varepsilon(x,t)=u^0(x,t)+\varepsilon u^1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)+\varepsilon^2u^2\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)+\dots;\tag{8.23}$$

here all the functions $u^j(x,t,z,s)$ are periodic in z. The dependence in s is not always stationary.

Substituting the expression on the right-hand side of (8.23) for u^ε in (8.1) and collecting power-like terms in (8.1) yields

$$(\varepsilon^{-1}):\qquad \partial_s u^1-\mathrm{div}_z\big(a(z,s)\nabla_z u^1\big)=-\mathrm{div}_z\big(a(z,s)\nabla_x u^0\big).$$

$$(\varepsilon^{0}):\qquad \begin{aligned}\partial_s u^2-\mathrm{div}_z\big(a(z,s)\nabla_z u^2\big)&=-\partial_t u^0+\mathrm{div}_x\big(a(z,s)\nabla_x u^0\big)\\&+\mathrm{div}_z\big(a(z,s)\nabla_x u^1\big)+\mathrm{div}_x\big(a(z,s)\nabla_z u^1\big).\end{aligned}$$

$$(\varepsilon^{1}):\qquad \begin{aligned}\partial_s u^3-\mathrm{div}_z\big(a(z,s)\nabla_z u^3\big)&=-\partial_t u^1+\mathrm{div}_x\big(a(z,s)\nabla_x u^1\big)\\&+\mathrm{div}_z\big(a(z,s)\nabla_x u^2\big)+\mathrm{div}_x\big(a(z,s)\nabla_z u^2\big).\end{aligned}$$

In equation (ε^{-1}) the variables x and t are parameters. The fact that the right-hand side of the equation is of the form $[\mathrm{div}_z\big(a(z,s)]\cdot\nabla_x u^0$ suggests that

$$u^1(x,t,z,s)=\chi(z,s)\nabla_x u^0(x,t)$$

with the vector-function $\chi=\{\chi^j(z,s)\}_{j=1}^n$ solving equation (8.4). And we obtain the formal form (8.17) of U^ε. It can be proved that $\chi\in(L^\infty(\mathbb{R}\times\mathbb{T}^d))^d\cap(L^2_{\mathrm{loc}}(\mathbb{R};H^1(\mathbb{T}^d)))^d$, and

$$\|\chi^j\|_{L^\infty(\mathbb{R}\times\mathbb{T}^d)}\le C,\qquad \|\chi^j\|_{L^2(\mathbb{R};H^1(\mathbb{T}^d))}\le C,\quad j=1,\dots,d.\tag{8.24}$$

with a deterministic constant C. For the sake of definiteness we assume from now on that

$$\int_{\mathbb{T}^d}\chi(z,s)dz=0.$$

We turn to the terms of order ε^0. Recall that u^0 satisfies problem (8.2) with $\mathrm{a}^{\mathrm{eff}}$ given by (8.5). Then using the quantities Ψ_1 and Ψ_2 defined respectively by (8.18) and (8.20), we rewrite equation (ε^0) as follows

$$\partial_s u^2 - \mathrm{div}_z\big(a(z,s)\nabla_z u^2\big) = \big(\Psi_1^{ij}(s) + \Psi_2^{ij}(z,s)\big)\frac{\partial^2}{\partial x^i \partial x^j}u^0. \tag{8.25}$$

Up to now we follow the same strategy as in the periodic case. But since the process $\int_0^s \Psi_1(r)dr$ need not be stationary, we cannot follow any more this scheme. Instead, we consider the equation

$$\begin{cases} \dfrac{\partial V^\varepsilon}{\partial t} & = & \mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla V^\varepsilon\Big) + \Psi_1^{ij}\Big(\dfrac{t}{\varepsilon^2}\Big)\dfrac{\partial^2}{\partial x^i \partial x^j}u^0(x,t) \\ V^\varepsilon(0,x) & = & 0. \end{cases} \tag{8.26}$$

This suggests to replace $\varepsilon^2 u^2$ in (8.23) by

$$\varepsilon^2 u_2(x,t) = V^\varepsilon(x,t) + \varepsilon^2\chi_2\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^2}{\partial x^2}u^0(x,t)$$

where $\chi_2^{ij}(z,s)$ is a stationary zero mean solution[6] of the equation (8.21) with

$$\|\chi_2^{ij}\|_{L^2([0,1];H^1(\mathbb{T}^d))} + \|\chi_2^{ij}\|_{L^\infty((-\infty.+\infty);L^2(\mathbb{T}^d))} \le C, \qquad i,j = 1,\dots,d, \tag{8.27}$$

By its definition, $\Psi_1(s)$ is a stationary zero mean process. Denote

$$\zeta^{ij}(s) = \int_0^s \Psi_1^{ij}(r)dr.$$

Estimates (8.24) imply that

$$\|\Psi_1^{ij}\|_{L^2(0,1)} \le C, \qquad i,j = 1,\dots,d.$$

It follows that under the mixing condition **(H2)** it holds

$$\int_0^\infty \|\mathbb{E}\{\Psi_1(s)\,|\,\mathcal{F}_{\le 0}^{\Psi_{2,1}}\}\|_{(L^2(\Omega))^{d^2}}\,ds \le C\int_0^\infty \big(e^{-\nu s/2} + \rho_{\Psi_1}(s/2)\big)dy < \infty.$$

Therefore, the invariance principle holds for this process (see [172, Theorem VIII.3.79]), that is for any $T>0$

$$\varepsilon\zeta\Big(\frac{\cdot}{\varepsilon^2}\Big) \underset{\varepsilon\to 0}{\longrightarrow} \Lambda^{1/2}W. \tag{8.28}$$

in law in the space $\big(C[0,T]\big)^{d^2}$ where Λ is given by (8.19). Here W is a standard d^2-dimensional Wiener process. The next result describes the behavior of V^ε and is a consequence of (8.28) and the fact that $u^0(x,t)$ is a smooth deterministic function that satisfies estimate (8.16).

[6]Again the existence of χ_2 can be proved as for χ.

Lemma 8.1 *The functions $\varepsilon^{-1}V^\varepsilon$ converge in law, as $\varepsilon \to 0$, in the space $C((0,T);L^2(\mathbb{R}^d))$ to the unique solution of the following SPDE with a finite dimensional additive noise:*

$$
\begin{cases} dV^0 & = & \mathrm{div}(\mathrm{a}^{\mathrm{eff}}\nabla V^0)dt + \big(\Lambda^{1/2}\big)^{ijkl}\dfrac{\partial^2}{\partial x^i\partial x^j}u^0(x,t)dW_{t,kl} \\ V^0(0,x) & = & 0. \end{cases} \tag{8.29}
$$

Now we proceed with Equation (ε^1). Its right-hand side can be rearranged as follows:

$$
\begin{aligned}
&\Psi_3(z,s)\frac{\partial^3}{\partial x^3}u^0(x,t) \\
&\quad = \big\{ -\mathrm{a}^{\mathrm{eff}}\otimes\chi(z,s) + a(z,s)\otimes\chi(z,s) + \mathrm{div}_z[a(z,s)\otimes\chi_2(z,s)] \\
&\qquad\quad + a(z,s)\nabla_z\chi_2(z,s)\big\}\frac{\partial^3}{\partial x^3}u^0(x,t)
\end{aligned}
$$

here and in what follows the symbol $\frac{\partial^3}{\partial x^3}u^0(x,t)$ stands for the tensor of third order partial derivatives of u^0, that is $\frac{\partial^3}{\partial x^3} = \big\{\frac{\partial^3}{\partial x^i\partial x^j\partial x^k}\big\}_{i,j,k=1}^d$; we have also denoted

$$
a(z,s)\otimes\chi(z,s) = \big\{a^{ij}(z,s)\chi^k(z,s)\big\}_{i,j,k=1}^d
$$

and

$$
\mathrm{div}_z[a(z,s)\otimes\chi_2(z,s)] = \big\{\partial_{z^i}[a^{ij}(z,s)\chi_2^{kl}(z,s)]\big\}_{j,k,l=1}^d.
$$

We introduce the tensor μ by (8.22) and consider the following problems:

$$
\begin{cases} \dfrac{\partial \Xi_1^\varepsilon}{\partial t} & = \mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla\Xi_1^\varepsilon\Big) + \Big(\Psi_3\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big) - \mu\Big)\dfrac{\partial^3}{\partial x^3}u^0(x,t) \\ \Xi_1^\varepsilon(x,0) & = 0, \end{cases} \tag{8.30}
$$

and

$$
\begin{cases} \dfrac{\partial \Xi_2^\varepsilon}{\partial t} & = & \mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla\Xi_2^\varepsilon\Big) + \mu\dfrac{\partial^3}{\partial x^3}u^0(x,t) \\ \Xi_2^\varepsilon(0,x) & = & 0 \end{cases} \tag{8.31}
$$

with

$$
\mu\frac{\partial^3}{\partial x^3}u^0 = \mu^{ijk}\frac{\partial^3 u^0}{\partial x^i\partial x^j\partial x^k}, \qquad \Psi_3\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^3}{\partial x^3}u^0 = \Psi_3^{ijk}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^3 u^0}{\partial x^i\partial x^j\partial x^k}.
$$

Notice that Ξ_1^ε and Ξ_2^ε are scalar-valued functions and we replace $\varepsilon^3 u^3$ in (8.23) by $\varepsilon(\Xi_1^\varepsilon + \Xi_2^\varepsilon)$.

According to [332], problem (8.31) admits homogenization. In particular, Ξ_2^ε converges a.s. in $L^2(\mathbb{R}^d\times(0,T))$ to a solution of the following problem:

$$
\begin{cases} \dfrac{\partial \Xi_0}{\partial t} & = & \mathrm{div}\big(\mathrm{a}^{\mathrm{eff}}\nabla\Xi_0\big) + \mu\dfrac{\partial^3}{\partial x^3}u^0(x,t) \\ \Xi_0(0,x) & = & 0. \end{cases} \tag{8.32}
$$

The process Ξ_1^ε does not contribute in the limit.

Lemma 8.2 *The solution of problem* (8.30) *tends to zero a.s., as* $\varepsilon \to 0$, *in* $L^2(\mathbb{R}^d \times [0,T])$. *Moreover,*

$$\lim_{\varepsilon\to 0} \mathbb{E}\big(\|\Xi_1^\varepsilon\|^2_{L^2(\mathbb{R}^d\times[0,T])}\big) = 0.$$

Now let us summarize what we have obtained up to now. Coming back to (8.23), we have

$$\begin{aligned} u^\varepsilon(x,t) &= u^0(x,t) + \varepsilon\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0(x,t) + V^\varepsilon(x,t) + \varepsilon\Xi_2^\varepsilon(x,t) \\ &+ \varepsilon^2\chi_2\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^2}{\partial x^2}u^0(x,t) + \varepsilon\Xi_1^\varepsilon(x,t) + \dots \end{aligned}$$

To finish the story with the asymptotic expansion, we need to observe the initial condition at the level ε^1. In order to fix this problem we introduce one more term of order ε^1 in the previous expansion:

$$\varepsilon\mathcal{I}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0(x,t)$$

where

$$\frac{\partial\mathcal{I}}{\partial s} = \operatorname{div}\big(a(z,s)\nabla\mathcal{I}\big), \qquad \mathcal{I}(0,z) = -\chi(0,z). \tag{8.33}$$

Lemma 8.3 *The solution of problem* (8.33) *decays exponentially as* $s\to\infty$. *We have*

$$\|\mathcal{I}(\cdot,s)\|_{L^\infty(\mathbb{T}^d)} \le Ce^{-\nu s}, \quad \|\mathcal{I}\|_{L^\infty([s,s+1];H^1(\mathbb{T}^d))} \le Ce^{-\nu s}$$

Proof. The desired statement is an immediate consequence of the fact that $\int_{\mathbb{T}^d}\mathcal{I}(z,s)dz = \int_{\mathbb{T}^d}\mathcal{I}(z,0)dz = 0$, the maximum principle and the parabolic Harnack inequality (see [191] for further details). □

To finish the proof, we define

$$\begin{aligned} \mathcal{R}^\varepsilon(x,t) &= \frac{1}{\varepsilon}\left(u^\varepsilon(x,t) - u^0(x,t) - \varepsilon\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0(x,t)\right) \\ &- \frac{1}{\varepsilon}V^\varepsilon(x,t) - \Xi_2^\varepsilon(x,t) \\ &- \mathcal{I}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0(x,t) - \varepsilon\chi_2\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^2}{\partial x^2}u^0(x,t) - \Xi_1^\varepsilon(x,t). \end{aligned}$$

We prove that $\mathcal{R}^\varepsilon$ converges to zero[7] a.s. in $L^2(\mathbb{R}^d\times(0,T))$ and with Estimate (8.27), Lemmata 8.2 and 8.3, the last three terms in the definition of $\mathcal{R}^\varepsilon$ tend to zero and from Lemma 8.1 and the convergence to Ξ_0 satisfying (8.32), we deduce the conclusion of Theorem 8.3.

Now we can formulate the same result as Theorem 8.3 under the diffusion setting (8.3):

$$a(z,s) = \mathrm{a}(z,\xi_s).$$

Recall that $\widehat{\chi}$ is the solution of (8.11).

[7]The proof does not follow directly but is somehow similar to the arguments for Lemma 8.2.

Theorem 8.4 (Theorem 8.3 for the diffusion case) *Let assumption* **(I)** *be fulfilled. Then the function* U^ε *defined by:*

$$U^\varepsilon(x,t) := \frac{u^\varepsilon(x,t) - u^0(x,t)}{\varepsilon} - \widehat{\chi}\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\right) \cdot \nabla u^0(x,t)$$

converges in law, as $\varepsilon \to 0$, *in the space* $L^2(\mathbb{R}^d \times (0,T))$ *to the solution of* (8.15).

Proof. The arguments used in the proof of Theorem 8.3 also apply in the case under consideration. We used assumption **(H2)** only once, when justified convergence (8.28). Thus, this convergence should be reproved under our standing assumptions.

Lemma 8.4 *Under assumptions* **(I1)–(I3)** *for any* $K > 0$ *there exists* C_K *such that the following estimate holds*

$$\|\mathbb{E}\{\Psi_1(s) \,|\, \mathcal{F}_{\le 0}\}\|_{L^2(\Omega)} \le C_K\big(e^{-\nu s/2} + (1+s)^{-K}\big), \qquad \nu > 0;$$

the function Ψ_1 *has been defined in* (8.18).

From the previous lemma it follows that the invariance principle holds for the process $\zeta(s)$ (see [171, Theorem VIII.3.79]), that is (8.28) holds for any $T > 0$. The rest of proof of Theorem 8.4 is exactly the same as that of Theorem 8.3. □

To summarize the diffusive case $\alpha = 2$, we need only one corrector χ to obtain the convergence of

$$U^\varepsilon(x,t) := \frac{1}{\varepsilon}\left[u^\varepsilon(x,t) - u^0(x,t) - \varepsilon\chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right) \cdot \nabla u^0(x,t)\right]$$

to a non trivial random limit, namely the solution of SPDE (8.15).

8.4 The diffusion case for $\alpha \neq 2$ ([XXIII])

We also consider the special case of diffusive dependence on time. We assume in this case that (8.3) holds, that is $a(z,s) = \mathrm{a}(z,\xi_s)$, where $(\xi_t,\ t \ge 0)$ is a stationary diffusion process in $\mathbb{R}^n$ and $\mathrm{a}(z,y)$ is a periodic in z smooth deterministic function. It should be emphasized that Theorem 8.6 does not apply because the coefficients a^{ij} do not possess the required regularity in time. That is why in the diffusive case we have to use a different approach and provide another proof of convergence.

Let us give some ideas of the proof. First we define the expansion $\mathcal{E}^\varepsilon$ of solution u^ε and the discrepancy

$$R^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\left[u^\varepsilon(x,t) - \mathcal{E}^\varepsilon(x,t)\right].$$

The goal is to prove the convergence of R^ε in a suitable space to some non trivial and random limit U^0. The convergence of U^ε can be deduced easily from R^ε (we add extra small terms in $\mathcal{E}^\varepsilon$ to ensure the proof of convergence).

This formal expansion of u^ε gives the sequences of constants $\mathrm{a}^{k,\mathrm{eff}}$ and $\underline{\mathrm{a}}^{k,\mathrm{eff}}$ and of smooth functions v^k and u^k. Note that the functions w^k in the definition of v^k will be left as *free parameters* in this first part. The rest R^ε contains with large parameters (as ε tends to zero), both in its dynamics and, for $\alpha > 2$, in its initial condition.Therefore R^ε is split into five terms $R^\varepsilon = r^\varepsilon + \check{r}^\varepsilon + \hat{r}^\varepsilon + \widetilde{r}^\varepsilon + \rho^\varepsilon$ such that:

- The dynamics of r^ε contains a large martingale term with a power $\varepsilon^{1-\alpha}$ if $\alpha < 2$ and ε^{-1} if $\alpha > 2$. This term converges to zero for $\alpha < 2$ and to U^0 for $\alpha > 2$.
- $\check{r}^\varepsilon$ appears only for $\alpha < 2$ and converges to U^0.
- $\hat{r}^\varepsilon$ exists only when $\alpha > 2$ and converges in a weak topology to zero.
- $\widetilde{r}^\varepsilon$ converges in a strong topology to zero.
- The last term ρ^ε deals with the initial condition on R^ε and contains large order terms when $\alpha > 2$. We prove that it also converges to zero.

Let us emphasize that in this section the dimension d plays no role and some terms in $\mathcal{E}^\varepsilon$ may be negligible depending on the value of α.

The main trouble comes from the convergence of r^ε. In the dynamics of R^ε, there is a martingale term with large parameters, at least when $\alpha > 1$. The free parameters w^k are used here to obtain the weak convergence to zero if $\alpha < 2$ or to the limit U^0 if $\alpha > 2$. Roughly speaking, we need w^k to obtain a uniform bound in $H^1(\mathbb{R})$ of the indefinite integral of R^ε. Here we widely use the fact that $d = 1$.

For $\alpha > 2$, a second issue comes from the initial condition on R^ε. Again we give a development of these terms and together with the properties of these expansions. Then we prove that from our particular choice of the initial condition on u^k, it is possible to define some constants $\mathcal{I}_k$ such that the initial condition of R^ε does not contribute in the limit equation, that is ρ^ε converges to zero in a strong sense. For this part, the dimension d could be any positive integer.

8.4.1 The case $\alpha < 2$

Here we assume that **(I*)** hold. Then $\chi^0 = \widehat{\chi} = \widehat{\chi}(z, y)$ is a periodic solution of the equation

$$\mathrm{div}_z\big(\mathrm{a}(z, y)\nabla_z\widehat{\chi}(z, y)\big) = -\mathrm{div}_z\mathrm{a}(z, y); \tag{8.34}$$

here $y \in \mathbb{R}^n$ is a parameter. We choose an additive constant in such a way that $\int_{\mathbb{T}^d} \widehat{\chi}(z, y)\, dz = 0$. Let us emphasize that it follows from (8.6) and (8.34) that the zero order correctors χ and $\widehat{\chi}$ coincide in both settings: $\chi(z, s) = \widehat{\chi}(z, \xi_s)$. The effective matrix is again given by (8.7):

$$\mathrm{a}^{\mathrm{eff}} = \mathbb{E}\int_{\mathbb{T}^d} \big(\mathbf{I} + \mathrm{a}(z, \xi_s)\big)\nabla_z\widehat{\chi}(z, \xi_s)\, dz.$$

Recall that the process ξ_t is stationary and distributed with the density ρ. The effective matrix can be written here as follows:

$$\mathrm{a}^{\mathrm{eff}} = \int_{\mathbb{R}^n}\int_{\mathbb{T}^d} \Big(\mathrm{a}(z, y) + \mathrm{a}(z, y)\nabla_z\chi^0(z, y)\Big)\rho(y)\, dzdy.$$

Higher order correctors are defined as periodic solutions of the equations

$$\mathrm{div}_z\big(\mathrm{a}(z, y)\nabla_z\chi^j(z, y)\big) = -\mathcal{L}_y\chi^{j-1}(z, y), \quad j = 1, 2, \ldots, J^0. \tag{8.35}$$

Notice that $\int_{\mathbb{T}^d}\chi^{j-1}(z,y)\,dz = 0$ for all $j = 1,\,2,\ldots,J^0$, thus the compatibility condition is satisfied and the equations are solvable. Here **(I4)** is used.

Remark 8.4 *The solutions χ^j defined by* (8.35) *satisfy the estimate: for any $N>0$ there exists C_N such that*

$$\|\chi^j\|_{C^N(\mathbb{T}^d\times\mathbb{R}^n)} \le C_N.$$

Remark 8.5 *We have already mentioned that according to* (8.6) *and* (8.34) *the zero order correctors coincide in both studied cases. It is interesting to compare the correctors defined in* (8.35) *with the ones given by* (8.47) *and to observe that the higher order correctors need not coincide.*

We introduce the matrices

$$\mathrm{a}^{k,\mathrm{eff}} = \int_{\mathbb{R}^n}\int_{\mathbb{T}^d}\big[\mathrm{a}(z,y)\nabla_z\chi^k(z,y)+\nabla_z\big(\mathrm{a}(z,y)\chi^k(z,y)\big)\big]\rho(y)\,dzdy, \quad k=1,\,2,\ldots,$$

and the functions $u^j = u^j(x,t)$ are defined again by (8.50) with an additional free parameter w^j: $u^j(x,0)=0$ and

$$\frac{\partial}{\partial t}u^j = \mathrm{div}(\mathrm{a}^{\mathrm{eff}}\nabla u^j) + \sum_{k=1}^{j}\{\mathrm{a}^{k,\mathrm{eff}}\}^{im}\frac{\partial^2}{\partial x_i\partial x_m}u^{j-k} + w^j$$

The functions w^j are smooth functions and defined recursively by $w^1=0$ and

$$\forall k\ge 0,\quad w^{k+2}(x,t) = -\sum_{m=0}^{k}\mathcal{C}_{k,m}u^m_{xx}(x,t) - \sum_{m=1}^{k}w^{m+1}(x,t). \tag{8.36}$$

The role of the free parameters w^j and the choice of the triangular array of constants $(\mathcal{C}_{k,m})_{0\le m\le k}$ will be precised after. All properties of Remark 8.8 still hold.

Finally, we consider the equation

$$\mathcal{L}Q^0(y) = \langle \mathrm{a}\rangle^0(y). \tag{8.37}$$

According to [279, Theorems 1 and 2], this equation has a unique up to an additive constant solution of at most polynomial growth. Denote

$$\Lambda = \{\Lambda^{ijml}\} = \int_{\mathbb{R}^n}\Big[\frac{\partial}{\partial y_{r_1}}(Q^0)^{ij}(y)\Big]q^{r_1r_2}(y)\Big[\frac{\partial}{\partial y_{r_2}}(Q^0)^{ml}(y)\Big]\rho(y)\,dy. \tag{8.38}$$

The matrix Λ is non-negative. Consequently its square root $\Lambda^{1/2}$ is well defined.

Our goal is to prove that the conclusion of Theorem 8.6 holds under this setting. We begin with the same ansatz

$$R^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\Big\{u^\varepsilon(x,t) - \sum_{k=0}^{J^0}\varepsilon^{k\delta}\Big(u^k(x,t)+\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1)}\chi^j\Big(\frac{x}{\varepsilon},\xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\Big)\Big\},$$

we substitute R^ε for u^ε in (8.1) and we obtain for R^ε a SPDE:

$$(8.39)\quad dR^\varepsilon(x,t) - \operatorname{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla R^\varepsilon(x,t)\Big]\,dt$$
$$= \Big(\varepsilon^{-\alpha/2}\sum_{j=0}^{J^0}\sum_{k=0}^{J^0-j}\varepsilon^{(k+j)\delta}\Big[\widehat{\mathrm{a}}^k\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) - \mathrm{a}^{k,\mathrm{eff}}\Big]^{im}\frac{\partial^2 u^j}{\partial x_i \partial x_m}\Big)\,dt + \mathcal{R}^\varepsilon(x,t)\,dt$$
$$+\sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t,$$

with $\mathrm{a}^{0,\mathrm{eff}} = \mathrm{a}^{\mathrm{eff}}$ and the initial condition

$$R^\varepsilon(x,0) = \sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(j\delta+1-\alpha/2)}\chi^j\Big(\frac{x}{\varepsilon}, \xi_0\Big)\nabla u^k(x,0);$$

and

$$\mathcal{R}^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\sum_{j=0}^{J^0}\varepsilon^{1+j\delta}\vartheta^j\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\Phi^j(x,t)$$

with periodic in z smooth functions $\vartheta^j = \vartheta^j(z,y)$ of at most polynomial growth in y, and smooth functions Φ^j. We represent R^ε as the sum $R^\varepsilon = r^\varepsilon + \widetilde{r}^\varepsilon + \check{r}^\varepsilon + \rho^\varepsilon$. Note that in $R^\varepsilon(x,0)$, there are only positive powers of ε. Hence we can handle these two terms as in the smooth case:

Proposition 8.1 *The solution $\check{r}^\varepsilon$ converges in law, as ε goes to 0, in $L^2(\mathbb{R}\times(0,T))$ equipped with strong topology, to the solution of* (8.15)*. The last terms $\widetilde{r}^\varepsilon$ satisfies for some $\delta > 0$:*

$$\mathbb{E}\|\widetilde{r}^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} + \mathbb{E}\|\rho^\varepsilon\|^2_{L^2(\mathbb{R}\times(0,T))} \le C\varepsilon^\delta.$$

But the last term r^ε solves the SPDE:

$$(8.40)\quad dr^\varepsilon(x,t) - \operatorname{div}\Big[\mathrm{a}\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla r^\varepsilon(x,t)\Big]\,dt$$
$$= \sum_{k=0}^{J^0}\sum_{j=0}^{J^0-k}\varepsilon^{(1-\alpha+(k+j)\delta)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^j\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^k(x,t)\,dB_t$$

with initial condition $r^\varepsilon(x,0) = 0$. In (8.40), the largest term is

$$\varepsilon^{(1-\alpha)}\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})\nabla_y\chi^0\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big)\nabla u^0(x,t)\,dB_t$$

and it creates the main trouble to obtain the convergence of R^ε (and thus of U^ε). Let us distinguish three cases:

1. If $\alpha < 1$, then there are only positive powers and thus we obtain the convergence of r^ε to zero in $L^2(\mathbb{R}^d\times(0,T))$.

2. If $\alpha = 1$, the convergence of r^ε to zero remains true, but in the weak topology of $L^2(\mathbb{R}^d \times (0,T))$.

3. If $1 < \alpha < 2$, we can prove the same result (convergence in the weak topology) if the dimension d is equal to 1. The trick is the same as for $\alpha > 2$ (see section 8.4.2). Our first attempt using the fundamental solution of the heat equation (see section 8.6) for any dimension d is not complete now.

Let us emphasize that the free parameters w^j (and the choice of $\mathcal{C}_{k,m}$) are defined exactly to obtain the convergence of zero of r^ε when $1 < \alpha < 2$. We give more details in the case $\alpha > 2$; the arguments are similar. Let us also remark that for $\alpha < 4/3$, $J_0 = 1$, that is we only need $w^1 = 0$.

8.4.2 The case $\alpha > 2$

We assume that (8.3) and **(I*)** hold. The function ρ is the density of the invariant measure of $\mathcal{L}$ and we assume that the process ξ_t is stationary and distributed with the density ρ. The effective matrix $\mathrm{a}^{\mathrm{eff}}$ is defined by (8.9) which can be written as follows:

$$\mathrm{a}^{\mathrm{eff}} = \langle\overline{\mathrm{a} + \mathrm{a}\nabla_z\chi}\rangle = \langle\overline{\mathrm{a} + (\mathrm{a}\nabla\chi^1 + \nabla(\mathrm{a}\chi))}\rangle,$$

and χ is the solution of (8.8):

$$\bar{\mathcal{A}}\chi = \mathrm{div}\ [\bar{\mathrm{a}}\nabla\chi] = -\mathrm{div}\ \bar{\mathrm{a}}_i$$

where $\bar{\mathrm{a}}$ is the mean value of a w.r.t. y:

$$\bar{\mathrm{a}}(z) = \int_{\mathbb{R}^n} \mathrm{a}(z,y)\rho(y)dy.$$

Now

- $\delta = |\alpha - 2| > 0$,
- $J_0 = \lfloor \frac{\alpha}{2\delta} \rfloor + 1$, where $\lfloor \cdot \rfloor$ stands for the integer part,
- $J_1 = \lfloor \frac{\alpha}{2} \rfloor$.

As for $\alpha < 2$, we construct a sequence of constants $\mathrm{a}^{k,\mathrm{eff}}$, $k \geq 1$, and a sequence of functions u^j, $j \geq 1$, as solutions of problems

$$\frac{\partial}{\partial t}u^j = \mathrm{div}(\mathrm{a}^{\mathrm{eff}}\nabla u^j) + \sum_{k=1}^{j} \mathrm{a}^{k,\mathrm{eff}} \frac{\partial^2}{\partial x^2} u^{j-k} + w^j \tag{8.41}$$

with initial condition $u^j(x,0) = 0$. The functions w^j are still smooth functions and defined recursively by

$$\forall k \geq 0, \quad w^{k+1}(x,t) = -\sum_{m=0}^{k} \mathcal{C}_{k,m} u^m_{xx}(x,t) - \sum_{m=1}^{k} w^m(x,t). \tag{8.42}$$

Note that these constants $\mathcal{C}_{k,m}$ are not the same if $\alpha > 2$ or if $\alpha < 2$ since the correctors used in their definition are different.

To obtain the desired convergence we need a second sequence of functions with a different scaling. We construct two other sequences of constants $(\underline{\mathrm{a}}^{k,\mathrm{eff}})_{k\geq 1}$ and $(\mathcal{I}_k)_{k\geq 1}$ such that we can define $v^0 = u^0$ and

$$v_t^j = \mathrm{a}^{\mathrm{eff}} v_{xx}^j + S^j, \quad v^j(x,0) = \mathcal{I}_j \partial_x^j u^0(x,0), \tag{8.43}$$

with for $j \geq 1$

$$S^j(x,t) = \sum_{k=1}^{j} \underline{\mathrm{a}}^{k,\mathrm{eff}} (\partial_x^{k+2} v^{j-k}) \ . \tag{8.44}$$

Indeed in the expansion of u^ε, we need to take into account the initial value of the remainder. For $\alpha < 2$, this additional term is negligible. But for $\alpha > 2$, it contains negative powers of ε and thus it should be controlled. This is the role of this sequence $\mathcal{I}_k$.

Our main result is the following.

Theorem 8.5 *Under Condition* **(I*)**, *there exists a non-negative constant* Λ *such that the normalized functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x,t) - u^0(x,t) - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) \right.$$
$$\left. - \sum_{k=1}^{J_1} \varepsilon^k \left[v^k(x,t) + \sum_{\ell=1}^{k} \chi^{\ell-1}\left(\frac{x}{\varepsilon}\right) \partial_x^\ell v^{k-\ell}(x,t) \right] \right\}$$

converge in law, as $\varepsilon \to 0$, *in* $L_w^2(\mathbb{R} \times (0,T))$ *to the unique solution of SPDE* (8.15)

$$dU^0 = \mathrm{div}(\mathrm{a}^{\mathrm{eff}} \nabla U^0)\, dt + (\Lambda^{1/2}) \left(\frac{\partial^2}{\partial x^2} u^0 \right) dW_t$$
$$U^0(x,0) = 0;$$

driven by a standard one-dimensional Brownian motion W.

Let us precise a little bit what happens for U^ε in the three cases: $2 < \alpha < 4$, $\alpha = 4$ and $\alpha > 4$.

- $2 < \alpha < 4$: $J_1 = 1$ and

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon(x,t) - u^0(x,t) - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k(x,t) \right.$$
$$\left. - \varepsilon \left[v^1(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right) \partial_x u^0(x,t) \right] \right\}$$

- $\alpha = 4$: $J_0 = 1$ and $J_1 = 2$. Thereby U^ε becomes

$$U^\varepsilon = \varepsilon^{-2}\Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \varepsilon\left[v^1(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right)\partial_x u^0(x,t)\right] \\ -\varepsilon^2\left[u^1(x,t) + v^2(x,t) + \chi^0\left(\frac{x}{\varepsilon}\right)\partial_x^2 u^0(x,t) + \chi^1\left(\frac{x}{\varepsilon}\right)\partial_x v^1(x,t)\right]\Bigg\}.$$

  $\alpha = 4$ is a kind of critical value, since here u^1 and v^2 coexist.

- $\alpha > 4$: $J_0 = 1$ and for any $m \geq 2$

$$J_1 = m \Leftrightarrow 2m \leq \alpha \leq 2(m+1).$$

  Hence

$$U^\varepsilon = \varepsilon^{-\alpha/2}\Bigg\{ u^\varepsilon(x,t) - u^0(x,t) - \varepsilon^\delta u^1(x,t) \\ - \sum_{k=1}^{J_1}\varepsilon^k\left[v^k(x,t) + \sum_{\ell=1}^{k}\chi^\ell\left(\frac{x}{\varepsilon}\right)\partial_x^\ell v^{k-\ell}(x,t)\right]\Bigg\}.$$

Remark 8.6 (When $J_0 = 1$) *For $\alpha > 4$ or $\alpha < 4/3$, we have $\delta > \alpha/2$ and $J_0 = 1$. Thus we may remove u^1 in the quantity U^ε: $\varepsilon^{\delta-\alpha/2}u^1$ tends to zero for the strong topology and thus does not contribute directly to the limit U^0 of U^ε. Nevertheless we emphasize that u^1 and w^1 are used to obtain the weak convergence of U^ε.*

Let us give some ideas of the proof. The beginning is very similar to the case $\alpha < 2$. We define the expansion $\mathcal{E}^\varepsilon$ of solution u^ε and the discrepancy

$$R^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\left[u^\varepsilon(x,t) - \mathcal{E}^\varepsilon(x,t)\right].$$

R^ε is decomposed into four terms $R^\varepsilon = r^\varepsilon + \hat{r}^\varepsilon + \widetilde{r}^\varepsilon + \rho^\varepsilon +$ such that:

- $\hat{r}^\varepsilon$ (resp. $\widetilde{r}^\varepsilon$) converges in a weak (resp. strong) topology to zero.
- The dynamics of r^ε contains a large martingale term with a power ε^{-1}.
- The last term ρ^ε deals with the initial condition on R^ε and contains large order terms.

Let us detail the properties of r^ε. We have: $r^\varepsilon(x,0) = 0$ and its dynamics contains the terms with large parameters:

$$dr^\varepsilon = (\mathcal{A}^\varepsilon r^\varepsilon)dt + \sum_{k=1}^{N_0}\varepsilon^{k\delta-\alpha/2}w^k(x,t)dt - \frac{1}{\varepsilon}\Bigg[\kappa_y^1\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha}\right)u_x^0(x,t) \\ + \sum_{k=1}^{N_0}\varepsilon^{k\delta}\Theta^k\left(\frac{x}{\varepsilon},\xi_{t/\varepsilon^\alpha},x,t\right)\Bigg]\sigma(\xi_{t/\varepsilon^\alpha})dB_t. \tag{8.45}$$

The precise definitions of N_0, Θ^k and κ^1 are not given here. The crucial point is the power ε^{-1}. Recall that for $\alpha < 2$, we only have $\varepsilon^{-\alpha+1}$, hence the contribution of these large martingale terms is zero when ε tends to zero. However for $\alpha > 2$, the limit of r^ε is equal to U^0, if we carefully choose the free parameters w^k (see [XXIII, Section 4]).

To obtain this convergence, we use the next trick. We define v^ε as the solution of

$$\begin{aligned} dv^\varepsilon &= a\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) v^\varepsilon_{xx} dt - \sum_{k=1}^{N_0} \varepsilon^{k\delta-\alpha/2} \widetilde{w}^k(x,t)dt \\ &- \sum_{k=0}^{N_0} \varepsilon^{k\delta} \widetilde{\Upsilon}^k \left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) u^k_x(x,t)\sigma(\xi_{\frac{t}{\varepsilon^\alpha}})dB_t, \end{aligned}$$

such that $v^\varepsilon_x = r^\varepsilon + \check{v}^\varepsilon$. This can be done *only in dimension 1*. Then it is not difficult to prove that $\check{v}^\varepsilon$ tends to 0 in $\mathbb{L}^2(\mathbb{R} \times (0,T))$ and in probability. We prove that we can choose $\mathcal{C}_{k,m}$ such that v^ε is bounded in $H^1(\mathbb{R})$, together with a tightness result. Hence the sequence v^ε weakly converges to the unique solution $\widetilde{r}^0$ of the SPDE:

$$d\widetilde{r}^0 = \langle \overline{P^0 a} \rangle \widetilde{r}^0_{xx} dt + \left(\overline{\| \langle P^0 \widetilde{\Upsilon}^0 \rangle \sigma \|^2} \right)^{1/2} u^0_x dW_t.$$

Thereby we conclude that $r^\varepsilon = v^\varepsilon_x$ converges to U^0.

When $\alpha > 2$, the second issue comes from the initial condition of the discrepancy R^ε. Indeed ρ^ε satisfies

$$d\rho^\varepsilon = (\mathcal{A}^\varepsilon \rho^\varepsilon)dt,$$

and has the initial condition

$$\rho^\varepsilon(x,0) = -\sum_{k=1}^{J_1} \varepsilon^{k-\alpha/2} \left[\mathcal{I}_k + \sum_{\ell=1}^{k} \mathcal{I}_{k-\ell} \chi^{\ell-1}\left(\frac{x}{\varepsilon}\right) \right] \partial^k_x u^0(x,0).$$

Contrary to the case $\alpha > 2$, this initial condition may have large terms when ε goes to zero. However in [XXIII, Section 5], we show that a particular choice of the constants $\mathcal{I}_k$ leads to the convergence of ρ^ε to zero. Let us emphasize that the proof is true even if the dimension d is not equal to one.

8.5 The smooth case for $\alpha < 2$ ([B])

In the present section we consider the case $0 < \alpha < 2$ for Problem (8.1). In other words, bearing in mind the diffusive scaling, we assume that the oscillation in spatial variables is faster than that in time. In this case the principal part of the asymptotics of $u^\varepsilon - u^0$ consists of a finite number of correctors, the oscillating part of each of them being a solution of an elliptic PDE with periodic in spatial variable coefficients. The number of correctors increases as α approaches 2. After subtracting these correctors, the resulting expression divided by $\varepsilon^{\alpha/2}$ converges in law to a solution of the limit SPDE (8.15) (Theorem 8.6).

In contrast with the diffusive scaling, for $\alpha < 2$ the interplay between the scalings in spatial variables and time and the necessity to construct higher order correctors results in additional regularity assumptions on the coefficients. The result mentioned in the previous section 8.3 holds if the coefficients $a^{ij}(z,s)$ in (8.1) are smooth enough functions. Hence we assume that Conditions **(H*)** holds: regularity of the initial condition ϕ, **(H1)** (uniform ellipticity), **(H2)** (mixing condition) and **(H3)** (regularity of the dynamical system).

Due to ellipticity of the matrix a, Equation (8.6) has a unique, up to an additive constant vector, periodic solution, $\chi \in \big(L^\infty(\mathbb{T}^d) \cap H^1(\mathbb{T}^d)\big)^d$. This constant vector is chosen in such a way that

$$\int_{\mathbb{T}^d} \chi(z,s)\,dz = 0 \qquad \text{for all } s \text{ and } \omega. \tag{8.46}$$

We then set

$$\mathrm{a}^{\mathrm{eff}} = \mathbb{E}\int_{\mathbb{T}^d} a(z,s)\big(\mathbf{I} + \nabla\chi(z,s)\big)\,dz$$

(see Equation (8.7)).

In order to formulate the main result we need a number of auxiliary functions and quantities. Denote $\chi^0 = \chi$ and for $j = 1, 2, \ldots, J^0$ with $J^0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$, the higher order correctors are introduced as periodic solutions to the equations

$$\mathrm{div}\big(a(z,s)\nabla\chi^j(z,s)\big) = \partial_s\chi^{j-1}(z,s), \tag{8.47}$$

where $\lfloor\cdot\rfloor$ stands for the integer part. Due to (8.46) for $j = 1$ this equation is solvable in the space of periodic in z functions. A solution χ^1 is uniquely defined up to an additive constant vector. Choosing the constant vector in a proper way yields

$$\int_{\mathbb{T}^d} \chi^1(z,s)\,dz = 0 \qquad \text{for all } s \text{ and } \omega$$

and thus the solvability of the equation for χ^2. Iterating this procedure, we define all χ^j, $j = 1, 2, \ldots, J^0$. Note that **(H3)** is used several times here.

Remark 8.7 *Since $\chi^0(\cdot,s)$ only depends on $a(\cdot,s)$, the solution with zero average is stationary and the strong mixing coefficient of the pair $(a(\cdot,s), \chi^0(\cdot,s))$ coincides with that for $a(\cdot,s)$. The same statement is valid for any finite collection $(a(\cdot,s), \chi^0(\cdot,s), \chi^1(\cdot,s), \ldots)$. By the classical elliptic estimates, under our standing assumptions we have*

$$\|\chi^0\|_{L^\infty(\mathbb{T}^d\times[0,T])} \le C, \qquad \mathbb{E}\|\chi^0\|^N_{C^k(\mathbb{T}^d\times[0,T])} \le C_{k,N}. \tag{8.48}$$

Indeed, multiplying equation (8.6) by χ^0, using the Schwartz and Poincare inequalities and considering (8.46), we conclude that $\|\chi^0(\cdot,s)\|_{H^1(\mathbb{T}^d)} \le C$ for all $s \in \mathbb{R}$. The first estimate in (8.48) then follows from [140, Theorem 8.4]. The second estimate follows from the Schauder estimates, see [140, Chapter 6]

By the similar arguments, the solutions χ^j of equations (8.47) are stationary, satisfy strong mixing condition with the same coefficient $\gamma(r)$, and the following estimates hold: for any $N \ge 1$ and $k \ge 0$

$$\mathbb{E}\|\chi^j\|^N_{C^k(\mathbb{T}^d\times[0,T])} \le C_{k,N}, \quad j = 0, 1, \ldots, J_0. \tag{8.49}$$

Next, we introduce the functions $u^j = u^j(x,t)$, $j = 1, \dots, J^0$. They solve the following problems: $u^j(x,0) = 0$ and

$$\frac{\partial}{\partial t}u^j = \operatorname{div}(\mathrm{a}^{\mathrm{eff}}\nabla u^j) + \sum_{k=1}^{j}\{\mathrm{a}^{k,\mathrm{eff}}\}^{im}\frac{\partial^2}{\partial x_i \partial x_m}u^{j-k} \tag{8.50}$$

with

$$\mathrm{a}^{k,\mathrm{eff}} = \mathbb{E}\int_{\mathbb{T}^d} a(z,s)\nabla\chi^k(z,s)\,dz. \tag{8.51}$$

Again we assume summation from 1 to d over repeated indices.

Remark 8.8 *Solutions u^0 and u^j of Problems* (8.2), (8.50) *are smooth functions. Moreover, for any $\mathbf{k} = (k_0, k_1, \dots, k_d)$ and $N > 0$ there exists a constant $C_{\mathbf{k},N}$ such that*

$$|D^{\mathbf{k}}u^j| \le C_{\mathbf{k},N}(1+|x|)^{-N}, \tag{8.52}$$

where $D^{\mathbf{k}}f(x,t) = \dfrac{\partial^{k_0}}{\partial t^{k_0}}\dfrac{\partial^{k_1}}{\partial x_1^{k_1}}\dots\dfrac{\partial^{k_d}}{\partial x_d^{k_d}}f(x,t).$

To characterize the diffusive term in the limit equation we introduce the matrix

$$\Xi(s) = \int_{\mathbb{T}^d}\Big[\big(a(z,s) + a(z,s)\nabla\chi^0(z,s)\big) - \mathbb{E}\big\{a(z,s) + a(z,s)\nabla\chi^0(z,s)\big\}\Big]dz.$$

By construction the matrix function Ξ is stationary and its entries satisfy condition **(H2)** (mixing condition). Denote

$$\Lambda = \frac{1}{2}\int_0^\infty \mathbb{E}\Big(\Xi(s)\otimes\Xi(0) + \Xi(0)\otimes\Xi(s)\Big)\,ds, \qquad \Lambda = \{\Lambda^{ijkl}\},$$

where $(\Xi(s)\otimes\Xi(0))^{ijkl} = \Xi^{ij}(s)\Xi^{kl}(0)$. Under condition **(H2)** the integral on the right-hand side converges.

The first main result of this section is

Theorem 8.6 *Let Condition* **(H*)** *be fulfilled, and assume that $\alpha < 2$. Then the functions*

$$U^\varepsilon = \varepsilon^{-\alpha/2}\Big(u^\varepsilon - u^0 - \sum_{j=1}^{J_0}\varepsilon^{j(2-\alpha)}u^j\Big)$$

converge in law, as $\varepsilon \to 0$, in $L^2(\mathbb{R}^d \times (0,T))$ to the unique solution of the SPDE (8.15)

$$dU^0 = \operatorname{div}(\mathrm{a}^{\mathrm{eff}}\nabla U^0)\,dt + (\Lambda^{1/2})^{ijkl}\frac{\partial^2}{\partial x_i\partial x_j}u^0\,dW_t^{kl}$$
$$U^0(x,0) = 0;$$

where $W_\cdot = \{W_\cdot^{ij}\}$ is the standard d^2-dimensional Brownian motion.

Remark again that the number of correctors $J_0 = \lfloor \frac{\alpha}{2(2-\alpha)} \rfloor + 1$ tends to $+\infty$ when α tends to 2.

Proof. The proof is similar to the ideas of Section 8.3.1. We give only some tricks used in the complete proof. We write down the following ansatz

$$\mathcal{V}^\varepsilon(x,t) = \varepsilon^{-\alpha/2}\Big\{u^\varepsilon(x,t) - \sum_{k=0}^{J^0} \varepsilon^{k\delta}\Big(u^k(x,t) + \sum_{j=0}^{J^0-k} \varepsilon^{(j\delta+1)}\chi^j\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^\alpha}\Big)\nabla u^k(x,t)\Big)\Big\},$$

here and in what follows the symbol δ stands for $2-\alpha$. Then we substitute $\mathcal{V}^\varepsilon$ for u^ε in (8.1) and we obtain for $\mathcal{V}^\varepsilon$ a PDE with random coefficients. We prove that $\mathcal{V}^\varepsilon$ converges in law in the suitable functional space to the solution of (8.15). We combine the definition of correctors, formula (8.50) and the Cental Limit Theorem for stationary mixing processes. □

8.6 A result on the heat SPDE ([C])

In the proof of our convergence result, we obtain two SPDEs (8.40) ($\alpha < 2$) and (8.45) ($\alpha > 2$) of the form

$$dv(x,t) - \operatorname{div}\Big[a\Big(x,t\Big)\nabla v\Big]\,dt = G\,(x,t)\;dB_t \tag{8.53}$$

with the initial condition $v(x,0) = 0$. The matrix a is supposed to be measurable from $\mathbb{R}^d \times [0,+\infty[\times\Omega$ into $\mathbb{R}^{d\times d}$ and for each $(x,t) \in \mathbb{R}^d \times [0,+\infty[$, $a(x,t)$ is $\mathcal{F}_t$-measurable. Our aim is to prove that the solution v is given by:

$$v(x,t) \;=\; \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s)G\,(y,s)\,dydB_s, \tag{8.54}$$

where Γ is the fundamental solution of the PDE:

$$\frac{\partial u}{\partial t}(x,t) = \operatorname{div}\Big[a\Big(x,t\Big)\nabla u\Big]. \tag{8.55}$$

The stochastic integral in (8.54) has to be defined properly since $\Gamma(x,t,y,s)$ is measurable w.r.t. the σ-field $\mathcal{F}_t$ generated by the random variables B_u with $u \le t$.

The existence of the fundamental solution Γ and the description of its properties is an old story that has given rise to a vast literature (see among others [127, 212, 292, 109, 206] and the references therein). One of the most famous result in this field is the Aronson estimate (see [12, Theorem 7] or [109]). Under **(H1)** (boundedness and uniform ellipticity of a), there exist two constants ς and ϖ depending only on the constant λ in Assumption **(H1)** and the dimension d, such that

$$0 \le \Gamma(x,t,y,s) \le g_{\varsigma,\varpi}(x-y,t-s). \tag{8.56}$$

Here and in the sequel of this section, for two positive constants c and C, the function $g_{c,C}(x,t)$ is defined as follows:

$$g_{c,C}(x,t) = \frac{c}{t^{\frac{d}{2}}}\exp\left(-\frac{C|x|^2}{t}\right), \qquad t>0, \quad x \in \mathbb{R}^d.$$

This inequality is called the **Aronson estimate**[8]. Let us empahsize that no regularity assumption on the coefficients of a is required.

To obtain similar estimates on the spatial derivatives of Γ, it is usually assumed in the existing literature that the matrix a is Hölder continuous w.r.t. both x and t (see [212], Chapter IV, sections 11 to 13 or [127], Chapter I): for some $\hbar \in (0,1)$

$$|a(x,t) - a(x',t')| \leq K_a \left(|x-x'|^{\hbar} + |t-t'|^{\hbar/2} \right).$$

Notice that this setting is not well adapted to the stochastic framework, for example if $a(x,t) = \mathrm{a}(x,\xi_t)$ where ξ is a diffusion process. Indeed, in this case the constant K_a depends on the continuity properties of ξ and is random (see for example [32] for details). Hence the constants in the estimate of $\nabla_x \Gamma$ need not be uniformly bounded if we follow directly this construction.

Our first goal is to obtain Aronson type estimates for the spatial derivatives of Γ, without any regularity assumption on the dependence $t \mapsto a(x,t)$. We impose only a uniform Lipschitz continuity condition on the dependence $x \mapsto a(x,t)$. Then the upper bounds only depend on the ellipticity constants and L^∞ norm of the gradient of the coefficients (see Theorem 8.7).

When our paper was submitted we learnt that a number of results closely related to that of Theorem 8.7 have been obtained in the recent work [73]. In this work, for parabolic operators in non-divergence form with time dependent coefficient, the regularity of heat kernel and solutions w.r.t. spatial variables is studied. In particular, the result of our Theorem 8.7 can be derived from the results of this work. However, the approach used in [73] is rather different.

In our homogenization problem, we deal with the stochastic heat equation (8.53). with the initial condition $v(x,0) = 0$ (see Remark 8.11 for more general initial value). B is a standard Brownian motion, generating the filtration $\mathbb{F} = (\mathcal{F}_t,\ t \geq 0)$. The matrix a is supposed to be a measurable function from $\mathbb{R}^d \times [0,+\infty[\times \Omega$ into $\mathbb{R}^{d\times d}$ and for each $(x,t) \in \mathbb{R}^d \times [0,+\infty[$, $a(x,t)$ is $\mathcal{F}_t$-measurable. This stochastic partial differential equation (SPDE in short) in divergence form is somehow classical and among many other we refer to the books [127, 212] on PDE in divergence form, [85, 86, 209, 268, 322] and the references therein on SPDE. The results of these works have than been extended in several directions, among them are: Hörmander's condition [205, 207], Hölder spaces [78, 254], L^p-spaces [93, 204, 253, 255], Laplace-Beltrami operator [311].

Our aim is to prove that the SPDE in (8.53) admits a mild solution v given by (8.54), where Γ is the fundamental solution of equation (8.55). If the matrix a is deterministic, Γ is also deterministic and the existence of a mild solution v given by (8.54) is well known (see [322, Chapter 5]). However, when a is random, the stochastic integral in (8.54) has to be defined properly since $\Gamma(x,t,y,s)$ is measurable w.r.t. the σ-field $\mathcal{F}_t$ generated by the random variables B_u with $u \leq t$. In other words Equation (8.54) involves an anticipating integral. To our best knowledge, there is only one work on this topic by

[8]The function Γ has a lower bound similar to the upper bound (see [12, Theorem 7]).

Alos et al. [7]. Compared to our setting, the authors in [7] consider a space-time Wiener process, but the matrix a is Hölder continuous in time[9] (condition (A3) in [7]).

We already know that Γ and its spatial derivative admit Aronson's type upper bounds and we extend these bounds to the Malliavin derivatives of Γ, again without regularity assumption on a w.r.t. t (see Theorem 8.8 and, in the diffusion case, Corollary 8.1). Finally, since our noise is a one parameter Brownian motion, we also want to obtain a regular mild solution v on $\mathbb{R}^d \times (0,T)$ in the sense of Definition 8.2 of Equation (8.53). Compared to [7], since we have no space noise, we do not impose any condition on the dimension d and our solution is derivable w.r.t. x (see Theorem 8.9 and Corollary 8.2).

In a recent paper paper [280] a similar subject is handled with a parametrix construction. However, since the studied operator is not in the divergence form, the authors have to impose more regularity assumptions on the diffusion matrix a. Also, the SPDEs investigated in this paper are rearranged in such a way that the anticipating stochastic calculus can be avoided.

Let us remark that our initial motivation was that for the two SPDEs (8.40) or (8.45), we can show that the solution is of the form

$$v^\varepsilon(x,t) = \int_0^t \left[\int_{\mathbb{R}^d} \Gamma^\varepsilon(x,t,y,s) G\left(\frac{y}{\varepsilon}, \xi_{\frac{s}{\varepsilon^\alpha}}, y, s\right) dy \right] dB_s, \tag{8.57}$$

where Γ^ε is the fundamental solution of the following parabolic equation:

$$\frac{\partial u^\varepsilon}{\partial t}(x,t) = \operatorname{div}\Big[a\Big(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}}\Big) \nabla u^\varepsilon \Big]. \tag{8.58}$$

Then using integration by parts and some uniform estimates on Γ^ε, we could obtain the desired convergence result in the case $\alpha < 2$. However our estimation (of the derivative on Γ^ε) still depends on ε and is not sufficient to get the result with this method.

8.6.1 Estimate for the spatial and Malliavin derivative of the fundamental solution

Surprisingly if there are a lot of works on fundamental solution, the question studied in our paper has not been raised, except in the recent work [73]. In this work, for parabolic operators in non-divergence form with time dependent coefficient, the regularity of heat kernel and solutions w.r.t. spatial variables is studied. In particular, our result on Γ can be derived from the results of this work. However, the approach used in [73] is rather different.

Let us assume that **(H1)** holds, that is for any $(t,x,\zeta) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$

$$\lambda^{-1}|\zeta|^2 \le a(x,t)\zeta \cdot \zeta \le \lambda |\zeta|^2.$$

We also suppose that

[9] At the end of [7, section 5], the authors make a remark and give an example on this time regularity assumption.

(J1) The matrix a is measurable on $\mathbb{R}^d \times \mathbb{R}_+$, and for any $t \geq 0$ the function $a(\cdot, t)$ is of class C^1 w.r.t. $x \in \mathbb{R}^d$. Moreover, there is a constant K_a such that for all t and x

$$|\nabla a(x,t)| \leq K_a.$$

Theorem 8.7 *If the matrix $a = a(x,t)$ satisfies the uniform ellipticity condition* **(H1)** *and the above regularity condition* **(J1)***, then the (weak) fundamental solution Γ of equation (8.55) admits the following estimate: there exist two constants $\varrho > 0$ and $\varpi > 0$ such that*

$$|\nabla_x \Gamma(x,t,y,s)| \leq \frac{1}{\sqrt{t-s}}\, g_{\varrho,\varpi}(x-y, t-s); \tag{8.59}$$

here ϖ depends only on the uniform ellipticity constant λ and the dimension d, while ϱ might also depend on K_a and on T.

Weak fundamental solution is defined in [109, Definition VI.6]. Let us emphasize that these estimates are coherent with [109, Theorem VI.4]. The novelty is that the regularity of a w.r.t. t is not required.

Now assume that $a = a(x,t)$ are random fields defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that carries a d-dimensional Brownian motion B and that the filtration $\mathbb{F} = (\mathcal{F}_t,\ t \geq 0)$ is generated by B, augmented with the $\mathbb{P}$-null sets. The matrix $a : \mathbb{R}^d \times [0,+\infty) \times \Omega \to \mathbb{R}^{d\times d}$ depends[10] also on ω and we assume that conditions **(H1)** and **(J1)** are fulfilled uniformly w.r.t. ω. In particular the ellipticity constant λ and the bound K_a do not depend on ω. Since **(H1)** and **(J1)** hold, by Theorem 8.7 the fundamental solution Γ of (8.55) and its spatial derivatives satisfy estimates (8.56) and (8.59).

For the Malliavin differentiability property of Γ, we use the approach developed in Alòs et al. [7]. We assume that, in addition to **(H1)** and **(J1)**, the matrix a possesses the following properties:

(J2) For each $(x,t) \in \mathbb{R}^d \times [0,+\infty)$, $a(x,t)$ is a $\mathcal{F}_t$-measurable random variable.

(J3) For each $(x,t) \in \mathbb{R}^d \times [0,+\infty)$ the random variable $a(x,t)$ belongs to $\mathbb{D}^{1,2}_M$.

(J4) There exists a non negative process ψ such that for any $t \in [0,T]$ and any $x \in \mathbb{R}^d$,

$$|D_r a(x,t)| + |D_r \nabla a(x,t)| \leq \psi(r).$$

Moreover, ψ satisfies the integrability condition: for some $p > 1$

$$\mathbb{E}\left(\int_0^T \psi(r)^{2p} dr\right) < +\infty.$$

[10] Note that here and in the sequel we follow the usual convention and omit the function argument ω.

Note that if **(J4)** holds, then for all $(x, x', t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$|D_r a(x,t) - D_r a(x',t)| \leq \psi(r)|x - x'|.$$

Indeed

$$a_{ij}(x,t) - a_{ij}(x',t) = \int_0^1 \nabla a_{ij}(x' + \theta(x-x'), t) d\theta (x - x').$$

We differentiate both sides in the Malliavin sense and we use the estimate on $D_r \nabla a$. It is important to remark that we don't require any regularity condition w.r.t. the time variable. In particular we can use the results in the case (8.3): $a(z,s) = a(z, \xi_s)$. Our second main result is

Theorem 8.8 *Under conditions* **(H1)**, **(J1)**–**(J4)**, *the fundamental solution* Γ *of* (8.55) *and its spatial derivatives belong to* $\mathbb{D}^{1,2}_M$ *for every* $(t,s) \in [0,T]^2$, $s < t$ *and* $(x,y) \in (\mathbb{R}^d)^2$. *Moreover, there exist two constants* ϱ *and* ϖ *that depend only on the uniform ellipticity constant* λ, *the dimension* d, *on* K_a *and on* T, *such that*

$$|D_r \Gamma(x,t,y,s)| \leq \psi(r) g_{\varrho,\varpi}(x-y, t-s), \tag{8.60}$$

and

$$|D_r \nabla_x \Gamma(x,t,y,s)| \leq \frac{\psi(r)}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s). \tag{8.61}$$

Finally Γ *and* $D_r\Gamma$ *are continuous w.r.t.* $(x,y) \in \mathbb{R}^{2d}$ *and* $0 \leq s < t \leq T$.

Let us emphasize that the constant ϖ depends only on the uniform ellipticity constant λ and the dimension d, whereas the constant ϱ also depends on K_a and T.

Remark 8.9 *Estimate* (8.59) *holds under assumptions* **(H1)** *and* **(J1)**. *The other derivatives of* a *in* **(J3)** *and* **(J4)** *are used to control the Malliavin derivatives.*

Here we consider the special case for any $(x,t) \in \mathbb{R}^d \times [0,+\infty)$ $a(x,t) = \mathrm{a}(x, \xi_t)$, with a matrix-valued function a defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that **(I1)** and **(I2)** hold, that is

- a is uniformly elliptic: for any $(x,y,\zeta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$

$$\lambda^{-1}|\zeta|^2 \leq \mathrm{a}(x,y)\zeta \cdot \zeta \leq \lambda |\zeta|^2.$$

- a is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and of class C^1 w.r.t. x with a bounded derivative: for any (x,y)

$$|\nabla_x \mathrm{a}(x,y)| \leq K_{\mathrm{a}}.$$

The process ξ is given as the solution of the SDE (1.9)

$$d\xi_t = b(t,\xi_t)dt + \sigma(t,\xi_t)dB_t.$$

We assume that the matrix-function σ and vector-function b satisfy **(B1)**, are globally Lipschitz continuous (condition **(B2)**) and are at least two times differentiable w.r.t. x

with uniformly bounded derivatives (assumption **(B6)**). From Proposition 1.2, ξ is the unique strong solution of the SDE (1.9) and for any $T \geq 0$ and any $p \geq 2$

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |\xi_t|^p\Big) \leq C.$$

Moreover the process ξ belongs to $\mathbb{D}_M^{2,\infty}$ and the second derivatives $D_r^i D_s^j \xi_t^k$ satisfy also a linear stochastic differential equation with bounded coefficients. For any $r \in [0,T]$ we define

$$\psi(r) = \sup_{t\in[r,T]} \|D_r \xi_t\|. \tag{8.62}$$

From Lemma 1.2 we have for any $p \geq 2$

$$\sup_{r\in[0,T]} \mathbb{E}\left(\psi(r)^p\right) < +\infty. \tag{8.63}$$

Our setting implies that Conditions **(H1)**, **(J1)** and **(J2)** hold. Moreover let us assume that the matrix a is smooth w.r.t. y and satisfies the following regularity condition **(I4)**: for any $1 \leq j, k \leq d$

$$|\nabla_y \mathrm{a}(x,y)| + \left|\frac{\partial^2}{\partial x_j \partial y_k}\mathrm{a}(x,y)\right| \leq K_a.$$

Then we obtain that

$$D_r^j a_{i,\ell}(x,t) = \sum_k \frac{\partial \mathrm{a}_{i,\ell}}{\partial y_k}(x,\xi_t) D_r^j \xi_t^k.$$

Thus $D_r a(x,t) = 0$ if $r > t$, while for $r \leq t$ we have

$$|D_r^k a_{ij}(x,t)| \leq \left|\frac{\partial \mathrm{a}_{ij}}{\partial y_\ell}\right| |D_r(\xi_t)| \leq K_a \psi(r). \tag{8.64}$$

The same computation shows that

$$D_r^k \frac{\partial a_{ij}}{\partial x_\ell}(x,t) = \sum_k \frac{\partial^2 a_{i,\ell}}{\partial x_\ell \partial y_k}(x,\xi_t) D_r^j \xi_t^k.$$

Hence

$$\left|D_r^k \frac{\partial a_{ij}}{\partial x_\ell}(x,t)\right| \leq K_a \psi(r).$$

We deduce that $a(x,t)$ belongs to $\mathbb{D}_M^{1,2}$, and that **(J3)** and **(J4)** hold. From Theorems 8.7 and 8.8 we deduce immediately the following result.

Corollary 8.1 *If $a(x,t) = \mathrm{a}(x,\xi_t)$, then the fundamental solution Γ of equation* (8.55) *and its spatial derivatives belong to $\mathbb{D}^{1,2}$ and satisfy Estimates* (8.56), (8.59), (8.60) *and* (8.61).

8.6.2 Mild solution of the heat SPDE (8.53)

Now we deal with the stochastic heat equation (8.53):

$$dv(x,t) - \operatorname{div}\Big[a\Big(x,t\Big)\nabla v(x,t)\Big]\,dt = G\,(x,t)\;dB_t$$

with the initial condition $v(x,0)=0$ (see Remark 8.11 for more general initial value). The next definition is very closed to [7, Definition 5.9].

Definition 8.2 *Let $v=\{v(x,t),\ (x,t)\in\mathbb{R}^d\times[0,+\infty)\}$ be a random field. We say that v is a weak solution of Equation* (8.53) *if*

- *v is continuous on $\mathbb{R}^d\times(0,+\infty)$ with a.s. for any $x\in\mathbb{R}^d$,*

$$\lim_{t\downarrow 0} v(x,t) = 0\ ;$$

- *v has all first derivatives w.r.t. x on $\mathbb{R}^d\times(0,+\infty)$;*
- *for any test function $\phi\in C_0^\infty(\mathbb{R}^d)$ and for all $t\in[0,T]$ we have*

$$\int_{\mathbb{R}^d} v(x,t)\phi(x)dx + \int_0^t\int_{\mathbb{R}^d} a(x,s)\nabla\phi(x)\nabla v(x,s)dx$$
$$= \int_0^t\int_{\mathbb{R}^d} G(x,s)\phi(x)dxdB_s.$$

Our aim is to prove that the SPDE (8.53) admits a weak solution v given by (8.54), where Γ is the fundamental solution of the equation in (8.55). The stochastic integral in (8.54) has to be defined properly since $\Gamma(x,t,y,s)$ is measurable w.r.t. the σ-field $\mathcal{F}_t$ generated by the random variables B_u with $u\le t$. The correct definition can be found in [259] and is based on Malliavin's calculus. Let us assume the following conditions.

(K1) The function $G:\mathbb{R}^d\times[0,+\infty)\times\Omega\to\mathbb{R}^d$ is a progressively measurable function such that

$$(1+|x|)^N|G(x,t)|\le\mathfrak{G}(t)$$

for some constant $N>d/2$ and some adapted process $\mathfrak{G}$ s.t. for some $q>1$

$$\mathbb{E}\left(\int_0^T \mathfrak{G}(t)^{2q}dt\right)<+\infty.$$

(K2) For each $(x,t)\in\mathbb{R}^d\times[0,+\infty)$, the random variable $G(x,t)$ belongs to $\mathbb{D}^{1,2}_M$ and for any $t\in[0,T]$ and any $x\in\mathbb{R}^d$,

$$|D_rG(x,t)|\le\widetilde{G}(x,t)\psi(r).$$

The process ψ is the same as in Condition **(J4)** and $\widetilde{G}$ verifies the growth assumption

$$(1+|x|)^N|\widetilde{G}(x,t)|\le\mathfrak{G}(t).$$

(K3) The constants p of **(J4)** and q satisfy the relation

$$p > q > 2d + 4.$$

(K4) The process $\mathfrak{G}$ verifies

$$\mathbb{P}\left(\sup_{t\in[0,T]} \mathfrak{G}(t) < +\infty\right) = 1.$$

Remark 8.10 *Note that considering the same process ψ in* **(J4)** *and* **(K2)** *and the same growth estimate for G and $\widetilde{G}$ can be assumed without loss of generality.*

Under this setting we prove that

Theorem 8.9 *If Conditions* **(H1)**, **(J1)**–**(J3)** *and* **(K1)** – **(K4)** *hold, then on $\mathbb{R}^d \times (0,+\infty)$, the random field v is continuous w.r.t. (x,t) and has first derivatives w.r.t. x such that*

$$\mathbb{E}\left[\sup_{x,t}\left(|v(x,t)|^{\frac{2pq}{p+q}} + |\nabla v(x,t)|^{\frac{2pq}{p+q}}\right)\right] < +\infty.$$

Moreover a.s. for any $x \in \mathbb{R}^d$

$$\lim_{t\downarrow 0} v(x,t) = 0.$$

And v is a weak solution of the SPDE (8.53).

In the first step of the proof, using [259], we prove that for any $(t,x) \in [0,T] \times \mathbb{R}^d$, the stochastic integral

$$v(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s) G\left(y,s\right) dy dB_s$$

is well defined and

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (v(x,t))^2 dx dt\right] < +\infty.$$

Then we show that $(x,t) \mapsto v(x,t)$ is continuous and $x \mapsto v(x,t)$ is differentiable. We cannot directly use [259, Theorem 5.2] since Γ also depends on t. Even if Γ is continuous on $\{0 \le s < t \le T\}$, the singularity at time t should be handled carefully. We follow some ideas contained in [7, Section 3] and the regularity results concerning the volume potential. The main trick is to transform the anticipating stochastic integral v into a Lebesgue integral.

Remark 8.11 *If the initial condition for v is given by a function $\imath$, then by linearity of the SPDE, we should add in* (8.54) *one term:*

$$v(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s) G\left(y,s\right) dy dB_s + \int_{\mathbb{R}^d} \Gamma(x,t,y,0)\imath(y) dy$$

Under the setting of Theorem 8.7, this additional term is well defined provided that the function $\imath$ increases no faster than a function $\exp(cx^2)$ *(see [127, Theorem I.7.12]).*

Remark 8.12 *Under* **(K3)***, we have the weaker condition* $\frac{1}{p} + \frac{1}{q} \leq 1$*. From the proofs, we are aware that this condition* **(K3)** *is a little bit too strong. But a relation between* p*,* q *and* d *is needed with our arguments. In [7], this relation is implicitly given: for example in Theorem 3.5, the authors impose* $p > 8$ *(for* $d = 1$*).* **(K1)** *and* **(K4)** *is a little bit more general than in [7] where* G *is bounded with respect to* (x,t)*.*

If we assume that $a(x,t) = \mathrm{a}(x, \xi_t)$ where ξ is the solution of the SDE in (**??**). Let us fix a measurable function $g : \mathbb{R}^d \times [0,+\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that g is of class C^1 w.r.t. the last component and

$$G\,(x,t) = g(x,t,\xi_t)$$

Then the Malliavin derivative of G can be computed by a chain rule argument: $D_r G\,(x,t) = \nabla_y g(x,t,\xi_t) D_r \xi_t$. Hence

$$|D_r G\,(x,t)\,| \leq |\nabla_y g(x,t,\xi_t)| \psi(r).$$

Let us assume that for some $N > d/2$:

$$|g(x,t,y)| + |\nabla_y g(x,t,y)| \leq C \frac{|y|}{(1+|x|)^N}.$$

Then $\mathfrak{G}(t) = |\xi_t|$ is continuous w.r.t. t, thus **(K4)** holds. And, for any $q > 1$,

$$\mathbb{E}\left(\sup_{t \in [0,T]} |\xi_t|^{2q} \right) \leq C.$$

Therefore, **(K1)** and **(K2)** are also satisfied. From Theorem 8.9 we get

Corollary 8.2 *If the previous assumptions are satisfied, then the conclusion of Theorem 8.9 holds in the diffusion case.*

Chapter 9

Appendix

9.1 On the constants δ^* and m^* of (C6) and (C7)

In the proof of Lemma 3.7, the existence and uniqueness of the solution is guaranteed by Theorem 2.2. We only need to check the assumptions **(A1.1")** and **(A1.2")** of this result. Recall that $\nu = \nu(r)$ is defined by (2.7):

$$\nu = \nu(r) := \begin{cases} \chi + K^2 & \text{if } r \geq 2, \\ \chi + \frac{K^2}{r-1} + \frac{K_{f,u}^2}{\varepsilon(r, K_{f,u})} & \text{if } r < 2. \end{cases}$$

Here we correct some computations of [XI] for $r < 2$.

First for some $r > 1$ and $\rho > \nu(r)$ such that $r\rho \leq \delta$, from **(C5)** and **(C6)**

$$\mathbb{E}\left[e^{r\rho\tau}|\xi \wedge L|^r + \int_0^\tau e^{\delta t}|f_t^0|^r dt \right] \leq C\mathbb{E}\left[e^{r\rho\tau}\right] < +\infty.$$

In other words **(A1.1")** holds with $p = r$.

If $\xi_t^{(L)} = \mathbb{E}(\xi \wedge L|\mathcal{F}_t)$, $(\eta^{(L)}, \gamma^{(L)}, N^{(L)})$ are given by the martingale representation:

$$\xi \wedge L = \mathbb{E}(\xi \wedge L) + \int_0^\infty \eta_s^{(L)} dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s^{(L)}(e)\widetilde{\pi}(de, ds) + N_\tau^{(L)}$$

with for any $p > 1$ and ϱ such that $p\varrho \leq \delta$

$$\mathbb{E}\left[\left(\int_0^\infty e^{2\varrho s}|\eta_s^{(L)}|^2 ds + \int_0^\infty e^{2\varrho s} \int_{\mathcal{E}} |\gamma_s^{(L)}(e)|^2 \pi(de, ds) + \int_0^\infty e^{2\varrho s} d[N^{(L)}]_s \right)^{p/2} \right] < +\infty.$$

We consider

$$\mathbb{E}\left[\int_0^\tau e^{r\rho t}|f(t, \xi_t^{(L)}, \eta_t^{(L)}, \gamma_t^{(L)})|^r dt \right].$$

From **(A3)** and **(A4)**, we obtain

$$\begin{aligned} |f(t, \xi_t^{(L)}, \eta_t^{(L)}, \gamma_t^{(L)})| &\leq K_{f,z}|\eta_t^{(L)}| + K_{f,u}\|\gamma_t^{(L)}\|_{\mathfrak{B}_\mu^2} + |f_t^0| + |f(t, \xi_t^{(L)}, 0, 0) - f_t^0| \\ &\leq K_{f,z}|\eta_t^{(L)}| + K_{f,u}\|\gamma_t^{(L)}\|_{\mathfrak{B}_\mu^2} + |f_t^0| + U_t(L). \end{aligned}$$

Using Hölder's inequality with $r \leq 2$ and ρ such that $r\rho \leq \delta$ yields to

$$\mathbb{E}\left[\int_0^\tau e^{r\rho t}|f(t,\xi_t^{(L)},\eta_t^{(L)},\gamma_t^{(L)})|^r dt\right] \leq C\mathbb{E}\left[\int_0^\tau e^{r\rho t}|\eta_t^{(L)}|^r dt\right] + C\mathbb{E}\left[\int_0^\tau e^{r\rho t}\|\gamma_t^{(L)}\|^r_{\mathfrak{B}^2_\mu} dt\right]$$
$$+ C\mathbb{E}\left[\int_0^\tau e^{r\rho t}|f_t^0|^r dt\right] + C\mathbb{E}\left[\int_0^\tau e^{r\rho t}|U_t(L)|^r dt\right]$$
$$\leq C\mathbb{E}\left[\tau^{\frac{2-r}{2}}\right]\mathbb{E}\left[\left(\int_0^\tau e^{2\rho t}|\eta_t^{(L)}|^2 dt + \int_0^\tau e^{2\rho t}\int_{\mathcal{E}}|\gamma_t^{(L)}(e)|^2\pi(de,dt)\right)^{\frac{r}{2}}\right]$$
$$+ C\mathbb{E}\left[e^{r\rho\tau}\right] + C\mathbb{E}\left[\left(e^{\bar{h}r\rho\tau}\right)^{\frac{1}{\bar{h}}}\right]\mathbb{E}\left[\left(\int_0^\tau |U_t(L)|^{rh}dt\right)^{1/h}\right]$$

Hence if we summarize, we need to find three parameters $1 < r \leq 2$, $h > 1$ and $\rho > \nu(r)$ such that if $\bar{h}$ is the Hölder conjugate of h, $rh \leq m$ and $\bar{h}r\rho \leq \delta$. δ^* and m^* are chosen such that we can solve all these conditions. Thus **(A1.2")** holds (again with $p = r$) and we can apply Theorem 2.2 to get Lemma 3.7.

To satisfy all wanted conditions, we need in particular that $r\nu(r) < \delta$. Let us denote

$$f(r) = r\nu(r) = \begin{cases} r(\chi + K^2) & \text{if } r \geq 2, \\ r\left(\chi + \frac{K^2}{r-1} + \frac{L^2_{f,u}}{\varepsilon(r,L_{f,u})}\right) & \text{if } r < 2, \end{cases}$$

Remark that $\nu(r) \geq \chi + K^2$ for any $r > 1$ and that $f(1) = +\infty$. Hence if $\chi \leq -K^2$ (**Case 1**), then

$$\min f = \delta^* = 2(\chi + K^2), \quad \text{argmin } (f) = r^* = 2.$$

Moreover if $\chi < -K^2$, Condition **(C6)** is satisfied for any stopping time τ (including $\tau = +\infty$ a.s.) since one can choose $\delta < 0$ in this case. Conversely, if $\chi + K^2 > 0$, then $\nu(r) > 0$, hence $\delta^* > 0$.

Let us distinguish the two cases, $K_{f,u} = 0$ and $K_{f,u} > 0$. The first case is developed in [XI, Lemma 10].

9.1.1 If f does not depend on u.

Then the constant $K_{f,u}$ is equal to zero. Generator $g(t,y) = -y|y|^{q-1}/\eta_t$ in **(C2)** gives us an example. This case was studied in Lemma 10 in the appendix of [XI]. Note that $K^2 = K^2_{f,z}/2$ in this case.

Proposition 9.1 *If $K_{f,u} = 0$, then δ^* is equal to:*

$$\delta^* = \begin{cases} 2(\chi + K^2) & \text{if } \chi \leq K^2 \\ \chi\left(1 + \dfrac{K}{\sqrt{\chi}}\right)^2 & \text{if } \chi > K^2. \end{cases}$$

And m^* verifies $m^* < 0$ if $\chi < -K^2$, and

$$m^* = \begin{cases} \dfrac{2\delta}{\delta - 2(K^2+\chi) + (\sqrt{\delta} - 2K)^2 \mathbf{1}_{\delta > 4K^2}} & \text{if } |\chi| \le K^2, \\ \dfrac{\delta}{\sqrt{\delta} + \sqrt{\chi} - K} \times \dfrac{1}{\sqrt{\delta} - \sqrt{\delta^*}} & \text{if } \chi > K^2. \end{cases}$$

The interesting point is that if $\chi < -K^2$, then $m^* < 0$. Thus taking $m = 0$ in Condition **(C7)**, this assumption becomes: $\mathbb{E}(\tau) < +\infty$; which is true if $\delta > 0$ in **(C6)**.

If f is non-increasing w.r.t. y, we have:

Corollary 9.1 *If $\chi = K_{f,u} = 0$, then $\delta^* = 2K^2$ and $m^* = \dfrac{2\delta}{\delta - \delta^* + (\sqrt{\delta} - 2K)^2 \mathbf{1}_{\delta > 4K^2}}$.*

Hence δ^* and m^* in **(C6)** and **(C7)** should be greater than these values.

If $K^2 = 0$, that is if f only depends on y, we have:

$$\delta^* = \begin{cases} 2\chi & \text{if } \chi \le 0 \\ \chi & \text{if } \chi > 0. \end{cases}$$

And m^* verifies $m^* < 0$ if $\chi < 0$, and for $\chi \ge 0$, $m^* = \dfrac{\delta}{\delta - \chi} \ge 1$.

Corollary 9.2 *If f depends only on y, with $\chi = 0$ (non-increasing function), then $\delta^* = 0$ and $m^* = 1$.*

9.1.2 If f depends on u, that is $K_{f,u} > 0$

This case was not written in [XI]. The computations for $K_{f,u} > 0$ are more tedious. We use the values defined by (2.8):

$$\varepsilon(r) = \frac{r-1}{\left(C_u^{\frac{1}{r-1}} + 1\right)^{2-r}}, \qquad C_u = (4(2K_{f,u} + 2) + 1).$$

Hence for $r < 2$

$$f(r) = r\chi + \frac{r}{r-1}\left(K^2 + K_{f,u}^2 \zeta(r)\right), \quad \text{with} \quad \zeta(r) = \left(C_u^{\frac{1}{r-1}} + 1\right)^{2-r}.$$

Note that for $r \in (1, 2)$, $f(r) \ge \chi + K^2 + K_{f,u}^2$.

Proposition 9.2 *Value δ^* is equal to*

$$\delta^* = \begin{cases} 2(\chi + K^2) & \text{if } \chi \le \kappa(K^2, K_{f,u}), \\ f(r^*) & \text{if } \chi > \kappa(K^2, K_{f,u}) \end{cases}$$

where $r^* \in (1,2)$ satisfies $f'(r^*) = 0$ and $\kappa(K^2, K_{f,u})$ is a threshold such that $f(r^*) < f(2)$ if $\chi > \kappa(K^2, K_{f,u})$. This threshold can be defined by:

$$\kappa(K^2, K_{f,u}) = \frac{1}{(r^\dagger - 1)^2}\left(K^2 + K_{f,u}^2 g(r^\dagger)\zeta(r^\dagger)\right), \tag{9.1}$$

where $r^\dagger$ is the unique root of the equation:

$$(2 - r^\dagger)^2 K^2 + K_{f,u}^2 \zeta(r^\dagger)\left[g(r^\dagger)(2 - r^\dagger) - r^\dagger(r^\dagger - 1)\right] = 0. \tag{9.2}$$

Proof. Indeed the derivative of f is given by:

$$f'(r) = \chi - \frac{1}{(r-1)^2}\left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right)$$

where

$$g(r) = \left(1 + r(r-1)\ln\left(1 + C_u^{\frac{1}{r-1}}\right) + r(2-r)\ln\left(C_u^{\frac{1}{r-1}}\right)\frac{C_u^{\frac{1}{r-1}}}{C_u^{\frac{1}{r-1}} + 1}\right).$$

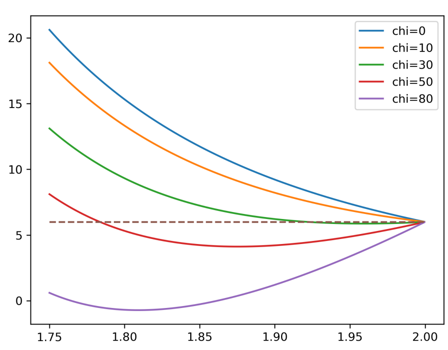
with $g(2) = 1 + 2\ln(1 + C_u)$. Moreover $g\zeta$ is a non-increasing function, thus $g(r)\zeta(r) \geq g(2) > 0$ on the interval $(1,2]$. Thereby

$$f''(r) = \frac{2}{(r-1)^3}\left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) - \frac{K_{f,u}^2}{(r-1)^2}(g\zeta)'(r) > 0.$$

The function f is convex on $(1,2)$.

In particular if $\chi \leq K^2 + K_{f,u}^2 g(2)$, again $\delta^* = 2(\chi + K^2)$ and $r^* = 2$ (**Case 2**). But if $\chi > K^2 + K_{f,u}^2 g(2)$, then the minimum of f on $(1,2)$ is attained at a unique point $r^\sharp \in (1,2)$.

On the next graph, we draw $r \mapsto f(r) - f(2)$ with $L_z^2 = 1$ and $K_{f,u}^2 = 3$ for different values of χ. Note that $\lim_{r\to 2} f(r) - f(2) = 2K_{f,u}^2$ and that here $K^2 + K_{f,u}^2 g(2) = 5 + 6\ln(1 + C_u) \approx 24.03$.



Note that

$$\chi = \frac{1}{(r^\sharp - 1)^2}\left(K^2 + K_{f,u}^2 g(r^\sharp)\zeta(r^\sharp)\right) \iff r^\sharp = \phi(\chi).$$

When χ goes on $+\infty$, then $r^\sharp$ tends to one. Moreover the implicit function theorem shows that ϕ is a decreasing function. Now

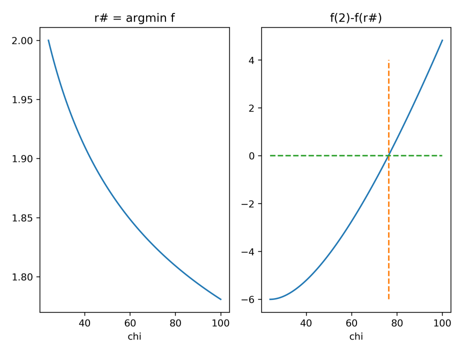
$$f(r^\sharp) = \phi(\chi)\left[\chi + \frac{1}{\phi(\chi) - 1}\left(K^2 + K_{f,u}^2 \zeta(\phi(\chi))\right)\right] = \widetilde{f}(\chi).$$

Hence

$$\begin{aligned}
\widetilde{f}'(\chi) &= \phi'(\chi)\left[\chi + \frac{1}{\phi(\chi)-1}\left(K^2 + K_{f,u}^2\zeta(\chi)\right)\right] \\
&\quad + \phi(\chi)\left[1 - \frac{\phi'(\chi)}{(\phi(\chi)-1)^2}\left(K^2 + K_{f,u}^2\zeta(\chi)\right)\right] + \phi(\chi)\left[\frac{\phi'(\chi)}{(\phi(\chi)-1)}\left(K_{f,u}^2\zeta'(\phi(\chi))\right)\right] \\
&= \phi(\chi) + \phi'(\chi)\left[\chi + \frac{1}{\phi(\chi)-1}\left(K^2 + K_{f,u}^2\zeta(\chi)\right) - \frac{\phi(\chi)}{(\phi(\chi)-1)^2}\left(K^2 + K_{f,u}^2\zeta(\chi)\right)\right. \\
&\qquad\qquad \left. + \frac{\phi(\chi)}{(\phi(\chi)-1)}\left(K_{f,u}^2\zeta'(\phi(\chi))\right)\right] = \phi(\chi).
\end{aligned}$$

Therefore if we compare $f(2) - f(r^\sharp) = 2(\chi + K^2) - \widetilde{f}(\chi) = \check{f}(\chi)$, the derivative of this function is equal to $2 - \phi(\chi) > 0$ and its second derivative is equal to $-\phi'(\chi)$. Hence the function $\check{f}$ is an increasing and convex function. Thus there exists a unique threshold $\kappa(K^2, K_{f,u}) > K^2 + K_{f,u}^2 g(2)$ such that for any $\chi > \kappa(K^2, K_{f,u})$, $f(r^\sharp) < f(2)$.

The next graph illustrates these properties with $K_{f,z}^2 = 1$ and $K_{f,u}^2 = 3$. On the left side, we draw $\chi \mapsto r^\sharp = \phi(\chi)$, and on the right side, the function $\chi \mapsto \check{f}(\chi) = f(2) - f(r^\sharp)$, both of them for $\chi \geq K^2 + K_{f,u}^2 g(2)$.



To summarize we have three cases.

- **Case 1** if $\chi \leq -K^2$. Then

$$\delta^* = 2(\chi + K^2), \quad \text{argmin } (f) = r^* = 2.$$

- **Case 2** if $-K^2 < \chi \leq \kappa(K^2, K_{f,u})$. Then

$$0 < \delta^* = 2(\chi + K^2), \quad \text{argmin } (f) = r^* = 2.$$

- **Case 3** if $\kappa(K^2, K_{f,u}) < \chi$. f attains a minimum at some point $r^* \in (1,2)$ such that $f(r^*) < f(2)$.

Let us remark that $\kappa(K^2, 0) = K^2$. And note that in the **Case 2**, there are two different situations: $\chi \leq K^2 + K_{f,u}^2 g(2)$ (f is decreasing on $(1,2]$), and $\chi > K^2 + K_{f,u}^2 g(2)$ (f has a minimum on $(1,2)$ but this minimum is greater than $f(2)$).

For $\chi = \kappa(K^2, K_{f,u})$, we denote by $r^\dagger \in (1,2)$ the value such that $\min f = f(r^\dagger) = f(2)$. Then $r^\dagger$ verifies: $f'(r^\dagger) = 0$, that is

$$\kappa(K^2, K_{f,u}) = \chi = \frac{1}{(r^\dagger - 1)^2}\left(K^2 + K_{f,u}^2 g(r^\dagger)\zeta(r^\dagger)\right),$$

and $f(r^\dagger) = f(2)$, namely:

$$r^\dagger\chi + \frac{r^\dagger}{r^\dagger - 1}\left(K^2 + K_{f,u}^2\zeta(r^\dagger)\right) = 2(\chi + K^2).$$

Hence we deduce the implicit definition (9.2) of $r^\dagger$, and the value (9.1) of $\kappa(K^2, K_{f,u})$:

$$\kappa(K^2, K_{f,u}) = \frac{1}{(r^\dagger - 1)^2}\left(K^2 + K_{f,u}^2 g(r^\dagger)\zeta(r^\dagger)\right) = \frac{K^2}{r^\dagger - 1} + K_{f,u}^2 \zeta(r^\dagger)\frac{r^\dagger}{(r^\dagger - 1)(2 - r^\dagger)}.$$

□

We already know that

$$\kappa(K^2, K_{f,u}) > K^2 + K_{f,u}^2 g(2) = K^2 + K_{f,u}^2 + 2K_{f,u}^2 \ln(1 + C_u).$$

The following table gives some values of $\kappa(K^2, K_{f,u})$.

$K_{f,u}^2$ \ $K_{f,z}^2$	0	0.5	1	1.5	2	2.5	5	10
0	0	0.25	0.5	0.75	1	1.25	2.5	5
0.5	11.39	11.71	12.02	12.33	12.64	12.95	14.51	17.59
1	23.59	23.90	24.21	24.52	24.83	25.14	2§.69	29.78
1.5	36.20	36.52	36.82	37.13	37.44	37.75	39.29	42.37
2	49.12	49.43	49.74	50.04	50.35	50.66	52.2	55.27
3	75.61	75.91	76.22	76.53	76.83	77.14	78.67	81.73
5	130.44	130.75	131.05	131.36	131.66	131.97	133.49	136.53
10	274.18	274.49	274.79	275.09	275.39	275.69	277.20	280.22

Using the convexity of $\check{f}$, we can prove that

$$\kappa(K^2, K_{f,u}) \le \frac{1}{0.9}K^2 + \frac{19}{0.9}K_{f,u}^2\zeta(1.9) = \frac{1}{0.9}K^2 + \frac{19}{0.9}K_{f,u}^2\left(1 + C_u^{\frac{1}{0.9}}\right)^{0.1}.,$$

and

$$\kappa(K^2, K_{f,u}) \ge K^2 + K_{f,u}^2 g(2) + K_{f,u}^2. \tag{9.3}$$

Indeed let us consider the straight line going from $(K^2 + K_{f,u}^2 g(2), -2K_{f,u}^2)$ to $(\kappa(K^2, K_{f,u}), 0)$. This line is crossing zero at some point $\chi^\dagger$ such that by convexity of $\check{f}$, $\chi^\dagger \le \kappa(K^2, K_{f,u})$. Solving the equation we have:

$$\chi^\dagger = K^2 + K_{f,u}^2 g(2) + \frac{2K_{f,u}^2}{p}$$

where slope p is given by:

$$p = \frac{\check{f}(\kappa(K^2, K_{f,u})) - \check{f}(K^2 - K_{f,u}^2 g(2))}{\kappa(K^2, K_{f,u}) - K^2 - K_{f,u}^2 g(2)} = \frac{2K_{f,u}^2}{\kappa(K^2, K_{f,u}) - K^2 - K_{f,u}^2 g(2)}.$$

Since $0 \le \check{f}' \le 2$, the slope is bounded by 2. Thus

$$\chi^\dagger \ge K^2 + K_{f,u}^2 g(2) + K_{f,u}^2.$$

Nonetheless the explicit value of threshold $\kappa(K^2, K_{f,u})$ is not attainable.

As for $K_{f,u} = 0$, there is an interval (R_1, R_2) such that for any $r \in (R_1, R_2)$, $\delta^* \leq f(r) = r\nu(r) \leq \delta$ and on this interval we need to study the function

$$h(r) = \frac{r\delta}{\delta - r\nu(r)} = \frac{r\delta}{\delta - f(r)}$$

and we try to obtain its minimum m^*. In the **Case 1**, $R_1 > 1$ et $R_2 = 2$. We can choose $\delta < 0$, such that $h(r) < 0$. And $m^* < 0 \leq m$.

Lemma 9.1 *The quantity m^* verifies again $m^* < 0$ if $\chi < -K^2$, and in the* **Case 2** *for χ,*

$$m^* = \begin{cases} \dfrac{2\delta}{\delta - 2(K^2 + \chi)} & \text{if } \delta \leq \varpi(K^2, K_{f,u}), \\ h(r^\natural) & \text{if } \delta > \varpi(K^2, K_{f,u}), \end{cases}$$

where $r^\natural \in (1, 2)$ satisfies $h'(r^\natural) = 0$ and

$$\varpi(K^2, K_{f,u}) = 4K^2 + 4K_{f,u}^2 + 4K_{f,u}^2 \ln(1 + C_u).$$

In **Case 3**, *we always have $m^* = h(r^\natural)$.*

Proof.

For $1 < r < 2$, the derivative of h (expect for $r = 2$) is equal to

$$\begin{aligned} h'(r) &= \frac{\delta}{(\delta - f(r))^2} (\delta - f(r) + rf'(r)) \\ &= \frac{\delta}{(\delta - f(r))^2} \left(\delta - \frac{r^2}{(r-1)^2} K^2 - \frac{rK_{f,u}^2}{(r-1)^2} (r - 1 + g(r)) \zeta(r) \right) \\ &= \frac{\delta}{(\delta - f(r))^2} \left(\delta - \frac{r^2}{(r-1)^2} \left(K^2 + K_{f,u}^2 \left(C_u^{\frac{1}{r-1}} + 1 \right)^{2-r} \widetilde{g}(r) \right) \right) \end{aligned}$$

with

$$\widetilde{g}(r) = 1 + (r-1) \ln \left(1 + C_u^{\frac{1}{r-1}} \right) + (2 - r) \ln \left(C_u^{\frac{1}{r-1}} \right) \frac{C_u^{\frac{1}{r-1}}}{C_u^{\frac{1}{r-1}} + 1}.$$

And since

$$h''(r) = \frac{2\delta}{(\delta - f(r))^3} r \left(f'(r)\right)^2 + \frac{\delta}{(\delta - f(r))^2} \left(rf''(r) + 2f'(r)\right),$$

with

$$\begin{aligned} 2f'(r) + rf''(r) &= 2\chi - \frac{2}{(r-1)^2} \left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) \\ &\quad + \frac{2r}{(r-1)^3} \left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) - r\frac{K_{f,u}^2}{(r-1)^2} (g\zeta)'(r) \\ &= 2\chi - \frac{2}{(r-1)^2} \left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) \\ &\quad + \frac{2(r-1)+2}{(r-1)^3} \left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) - r\frac{K_{f,u}^2}{(r-1)^2} (g\zeta)'(r) \\ &= 2\chi + \frac{2}{(r-1)^3} \left(K^2 + K_{f,u}^2 g(r)\zeta(r)\right) - r\frac{K_{f,u}^2}{(r-1)^2} (g\zeta)'(r) \end{aligned}$$

we deduce that h is still a convex function.

The structure of h' is similar to the expression of f', since the sign of h' only depends on

$$\delta - \frac{r^2}{(r-1)^2}\left(K^2 + K_{f,u}^2\left(C_u^{\frac{1}{r-1}}+1\right)^{2-r}\widetilde{g}(r)\right).$$

Note that $g(2) = 1 + 2\ln(1+C_u)$ and $\widetilde{g}(2) = 1 + \ln(1+C_u)$; thus $2\widetilde{g}(2) = g(2)+1$. Moreover if

$$\delta \leq 4(K^2 + K_{f,u}^2\widetilde{g}(2)) = 4K^2 + 2K_{f,u}^2 g(2) + 2K_{f,u}^2 = \varpi(K^2, K_{f,u}),$$

then $h'(r) \leq 0$ and thus $m^* = h(2)$. In the case $\delta > 4(K^2 + K_{f,u}^2\widetilde{g}(2))$, there exists a unique $r^\natural \in (1,2)$ such that $h'(r^\natural) = 0$.

- **Case 2** ($\chi \leq \kappa(K^2, K_{f,u})$). Then $\delta^* = 2(\chi + K^2)$. Thus we can have $\delta \leq 4(K^2 + K_{f,u}^2\widetilde{g}(2))$ if and only if

$$2(\chi + K^2) \leq 4K^2 + 2K_{f,u}^2 g(2) + 2K_{f,u}^2 \Longrightarrow \chi \leq K^2 + K_{f,u}^2 g(2) + K_{f,u}^2.$$

  Thus we have two subcases:

  - If $\chi \leq K^2 + K_{f,u}^2 g(2) + K_{f,u}^2$ and if $\delta \leq 4(K^2 + K_{f,u}^2\widetilde{g}(2))$, then $m^* = h(2)$.
  - Else $m^* = h(r^\natural)$.

- **Case 3** ($\chi > \kappa(K^2, K_{f,u})$). Recall that the map $\chi \mapsto \delta^* = f(r^\sharp)$ is non decreasing. Then $\delta^* = f(r^*) \geq f(\kappa(K^2, K_{f,u})) = 2(\kappa(K^2, K_{f,u}) + K^2)$. Thereby from (9.3)

$$\delta > \delta^* \geq 2(\kappa(K^2, K_{f,u}) + K^2) \geq \varpi(K^2, K_{f,u}).$$

This achieves the proof of the Lemma. □

The following table gives some values of

$$\varpi(K^2, K_{f,u}) = 2L_z^2 + 6K_{f,u}^2 + 4K_{f,u}^2\ln(1+C_u).$$

$K_{f,u}^2$ \ $K_{f,z}^2$	0	1	2	5	10
0	0	2	4	10	20
0.5	8.50	10.50	12.50	18.50	28.50
1	17.56	19.56	21.56	27.56	37.56
2	36.47	38.47	40.47	46.47	56.47
3	56.06	58.06	60.06	66.06	76.06
5	96.56	98.56	100.56	106.56	116.56
10	202.55	204.55	206.55	212.55	222.55

9.2 The existence of measure solutions in the Lipschitz case

In Section 6.1, the notion of measure solution for a BSDE is introduced (see Definition 6.1).

We shall now construct measure solutions from first principles. In particular, we shall not assume any knowledge about strong solutions. We shall give a complete self-contained construction for measure solutions with Lipschitz continuous generator for which the Lipschitz constant may be time dependent. Our construction provides the measure solution along an algorithm which just iterates the procedures of projecting the terminal variable by a given measure. The martingale representation theorem with respect to the measure $\mathbb{Q}^n$ in step n will produce a control process Z^n which is then plugged into the generator of the BSDE. The resulting drift is taken off by applying Girsanov's theorem which produces a new measure $\mathbb{Q}^{n+1}$ with which we continue along the lines just sketched in step $n+1$. The sequence $(\mathbb{Q}^n)_{n\in\mathbf{N}}$ thus produced has to be shown to possess at least an accumulation point in the weak topology. This is seen by a simple argument using the Lipschitz and boundedness properties. The extension to a continuous or quadratic generator and bounded terminal condition is straightforward from this perspective, and uses monotone approximations following the scheme in [221]. But this result is already contained in the results of [199] and Theorem 6.1. Hence we do not write the details here. The extension of our intrinsic construction of measure solutions to unbounded terminal conditions is left for future research.

In order to obtain a self-contained theory that is not using any knowledge on classical solutions, we first construct measure solutions in a setting for which they have been studied mostly: for generators that increase at most linearly and possess Lipschitz properties with time dependent and random Lipschitz constants. More formally, in this section we consider the following class of generators. Let

$$f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$$

such that $f(s,z) = f(\cdot,s,z)$ is adapted for any $z \in \mathbb{R}$ and satisfy **(A1)** and a weaker version of **(A4)**:

- for some $\gamma \geq 1$
$$\mathbb{E}|\xi|^\gamma + \mathbb{E}\left(\int_0^T |f(s,0)|^\gamma ds\right) < \infty.$$
- for some non-negative process ϕ,
$$|f(s,z) - f(s,z')| \leq \phi_s |z - z'|, \quad \forall s \in [0,T],\ (z,z') \in \mathbb{R}^2.$$

We add the next condition:

(G5) The set $\{s \in [0,T],\ f(s,.) \text{ is not continuous}\}$ is of Lebesgue measure zero;

We shall assume in the following that $f(s,0) = 0$ for all $s \in [0,T]$. This can be done without loss of generality, since we may replace ξ with the γ-integrable random variable

$$\tilde{\xi} = \xi + \int_0^T f(s,0)ds.$$

Now we define the function $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ by the requirement that for all $s \in [0,T],\ z \in \mathbb{R}$:

$$g(s,z) = \begin{cases} \dfrac{f(s,z)}{z}, \text{ if } z \neq 0, \\ 0, \text{ if } z = 0. \end{cases}$$

Therefore we have defined the function g with values in $\mathbb{R}$ and g is bounded by the process ϕ.

The process ϕ verifies either

$$\exists \kappa > 1,\ \mathbb{E}\left[\exp\left(\frac{\kappa}{2}\int_0^T \phi_r^2 dr\right)\right] < +\infty \tag{9.4}$$

or

$$\text{the martingale } \left(L_t = \int_0^t \phi_r dW_r\right)_{t\in[0,T]} \text{ is BMO.} \tag{9.5}$$

We denote by $\|L\|$ the BMO_2-norm of L. From [184, Theorem 2.2], (9.5) implies (9.4), with $1/\kappa = 2\|L\|^2$. Remark that (9.4) is a stronger Novikov condition. From these assumptions (see [184, Theorem 2.3]), we know that for $0 \leq t \leq T$,

$$\mathcal{E}(\phi W)_t = \exp\left(\int_0^t \phi_r dW_r - \frac{1}{2}\int_0^t \phi_r^2 dr\right)$$

is a uniformly integrable martingale.

We define the process Φ by

$$\forall t \in [0,T], \quad \Phi_t = \int_0^t \phi_s^2 ds,$$

and we assume that there exists two constants $\alpha > \Psi$ and $\delta > \Psi$ such that

$$\mathbb{E}(e^{\alpha \Phi_T}|\xi|^\delta) < +\infty. \tag{9.6}$$

The constant $\Psi > 1$ is given for (9.4) by:

$$\Psi(\kappa) = \Psi_{(9.4)}(\kappa) = 1 + 4\frac{\sqrt{\kappa}}{(\sqrt{\kappa}-1)^2} = \left(1 + \frac{2\sqrt{\kappa}+1}{\kappa}\right)\frac{\kappa}{(\sqrt{\kappa}-1)^2},$$

and for (9.5) by:

$$\Psi(\|L\|) = \Psi_{(9.5)}(\|L\|) = \left(1 + \frac{\|L\|}{2}\right)\frac{\theta^{-1}(\|L\|)}{\theta^{-1}(\|L\|) - 1}.$$

The function $\theta :]1,+\infty[\to \mathbb{R}_+^*$ is the continuous decreasing function given by

$$\forall q \in]1,+\infty[,\ \theta(q) = \left\{1 + \frac{1}{q^2}\ln\frac{2q-1}{2(q-1)}\right\}^{\frac{1}{2}} - 1.$$

We can check that $\Psi_{(9.5)} :]0,+\infty[\to]1,+\infty[$ is an increasing function such that $\Psi(0) = 1$ and $\Psi(\infty) = \infty$.

Remark 9.1 *If* **(A4)** *holds, that is f is a Lipschitz continuous function:*

$$|f(t,z) - f(t,z')| \leq K|z - z'|,$$

then ϕ is the constant K. Then (9.4) *is satisfied for all $\kappa > 1$, and* (9.6) *holds if $\gamma > 1$.*

We are now in a position to state our existence Theorem.

Theorem 9.1 *Suppose Assumptions* **(A1)**, **(A4)** *and* **(G5)**, (9.4) *or* (9.5), *and* (9.6) *hold. There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an adapted process Z such that $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$ such that, setting*

$$R_T = \exp\left(\int_0^T g(s, Z_s) dW_s - \frac{1}{2}\int_0^T g(s, Z_s)^2 ds\right), \quad W^{\mathbb{Q}} = W - \int_0^{\cdot} g(s, Z_s) ds,$$

we have

$$\mathbb{Q} = R_T \cdot \mathbb{P},$$

and such that the pair (Y, Z) defined by

$$Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^{\mathbb{Q}}(\xi) + \int_0^{\cdot} Z_s dW_s^{\mathbb{Q}}$$

solves BSDE (2.1).

Let us give some parts of the proof (for details, see [IV]). The solution algorithm for our BSDE (2.1)

$$Y_t = \xi + \int_t^T f(s, Z_s) ds - \int_t^T Z_s dW_s$$

is based on a recursively defined change of measure. Let $\mathbb{Q}^0 = \mathbb{P}$, and $W^0 = W$, the coordinate process which is a Wiener process under $\mathbb{Q}^0$. Set

$$Y^1 = \mathbb{E}(\xi|\mathcal{F}_\cdot) = \mathbb{E}(\xi) + \int_0^{\cdot} Z_s^1 dW_s^0,$$

and

$$\mathbb{Q}^1 = \exp\left(\int_0^T g(s, Z_s^1) dW_s - \frac{1}{2}\int_0^T g(s, Z_s^1)^2 ds\right) \cdot \mathbb{P} = R_T^1 \cdot \mathbb{P}.$$

Then

$$W^1 = W - \int_0^{\cdot} g(s, Z_s^1) ds$$

is a Wiener process under $\mathbb{Q}^1$. Indeed under (9.4), the Novikov condition is satisfied, and under (9.5), the martingale

$$M_t^1 = \int_0^t g(s, Z_s^1) dW_s$$

is BMO. Now since $(\mathbb{Q}^1, \mathbb{Q}^0)$ is a Girsanov pair, it is well known that the predictable representation property is inherited from the Brownian motion W^0 to the Brownian motion W^1. See for example [300], p. 335. Hence there exists a pair (Y^2, Z^2) of processes such that for all $t \in [0,T]$

$$Y_t^2 = \mathbb{E}^{\mathbb{Q}^1}(\xi|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}^1}(\xi) + \int_0^t Z_s^2 dW_s^1.$$

Assume that $\mathbb{Q}^n$ is recursively defined, along with the Brownian motion

$$W^n = W - \int_0^\cdot g(s, Z_s^n) ds$$

under $\mathbb{Q}^n$. Then the results in [300] may be applied to obtain two processes (Y^{n+1}, Z^{n+1}) such that

$$Y^{n+1} = \mathbb{E}^{\mathbb{Q}^n}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^n(\xi|\mathcal{F}_\cdot) = \mathbb{E}^n(\xi) + \int_0^\cdot Z_s^{n+1} dW_s^n.$$

Now set

$$\mathbb{Q}^{n+1} = \exp\left[\int_0^T g(s, Z_s^{n+1}) dW_s - \int_0^T g(s, Z_s^{n+1})^2 ds\right] \cdot \mathbb{P} = R_T^{n+1} \cdot \mathbb{P}$$

to complete the recursion step. Then from our assumptions on ϕ, and from the boundedness of g by ϕ, the sequence of probability measures $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ is well defined and consists of measures equivalent with $\mathbb{P}$. It is not hard to show tightness for this sequence.

Lemma 9.2 *Under* (9.4) *or* (9.5), *the sequence* $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ *is tight.*

In a second step, we shall now establish the boundedness in L^2 of the control sequence $(Z^n)_{n\in\mathbb{N}}$ obtained by the algorithm. Before let us give some estimates.

Lemma 9.3 *If* $\delta > \Psi$ *and* (9.6) *holds, there exist two constants* $\beta > 0$ *and* $p > 1$ *such that*

$$\forall n \in \mathbb{N}^*, \quad \mathbb{E}^{n-1}\left(e^{\beta\Phi_T}|\xi|^p\right) < +\infty. \tag{9.7}$$

Moreover there exists a constant C *such that for every* $n \in \mathbb{N}^*$,

$$\mathbb{E}^{n-1}\left[\sup_{t\in[0,T]}(e^{\beta\Phi_t}|Y_t^n|^p) + \left(\int_0^T e^{\beta\Phi_t}|Z_t^n|^2 dt\right)^{p/2}\right] \\ \leq C\mathbb{E}^{n-1}\left[\exp\left(\beta\Phi_T \max\left(\frac{p}{2}, 1\right)\right)|\xi|^p\right].$$

And

$$\mathbb{E}\left(\sup_{t\in[0,T]} e^{\beta\Phi_t}(Y_t^n)^p\right) \quad \textit{and} \quad \mathbb{E}\left[\left(\int_0^T e^{\beta\Phi_s}(Z_s^n)^2 ds\right)^{\frac{p}{2}}\right]$$

are bounded sequences.

Corollary 9.3 *There exists a subsequence of* Z^n *(still denoted* Z^n*) which converges* $\mathbb{P}\otimes\lambda$*-a.e. to some process* Z.

The sequence R_T^n *converges also* $\mathbb{P}$*-a.s. to*

$$R_T = \exp\left(\int_0^T g(s, Z_s) dW_s - \frac{1}{2}\int_0^T (g(s, Z_s))^2 ds\right).$$

The conclusion of the theorem follows.

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