A Mean Field Game Of Optimal Portfolio Liquidation.

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Outline

1. Introduction: optimal position closure for a single player
   - Solving the linear quadratic case.

2. MFGs of optimal portfolio liquidation
   - Conditional mean-field type FBSDE
   - Common information environments
   - General case

3. Nash Equilibrium and approximation by unconstrained MFGs

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Unwinding large positions is part of day-to-day business.

... of banks, funds, insurance companies, energy companies, ...

- Sell x shares of $\cdots$ within T minutes using market orders.

Symb	WKN	Name	Bid Anz	Bid Vol in Stck	Bid	Ask	Ask Vol in Stck	Ask Anz	Preis	Letzter Umsatz	Zeit Preis	Ph	Vortag
ADS	A1EWWW	adidas AG							83,680	133	12:33:29	CO	83,140
	Bid/Ask Orders												
			2	505	83,650	83,680	162	2					
			5	586	83,640	83,690	275	2					
			9	925	83,630	83,700	670	7					
			7	869	83,620	83,710	1.125	10					
			5	566	83,610	83,720	1.062	8					
			6	676	83,600	83,730	1.085	8					
			7	583	83,590	83,740	405	4					
			5	790	83,580	83,750	952	9					
			7	776	83,570	83,760	246	4					
			2	117	83,560	83,770	888	6					

- Limited market liquidity leads to a price impact.
- Aim: Optimize trading strategies to minimize execution costs.

Price impact model of Almgren & Chriss (2000).

- Fix an initial position $x \in \mathbb{R}$ and a time horizon T.
- Execution strategy X with control ξ:

$$X_t = x - \int_0^t \xi_s ds$$

s.t. $X_T = 0$.

- Unaffected price process S^0: to disentangle investment from execution strategies, S^0 is often assumed to be a martingale.
- A price impact model assigns to each execution strategy X a realized price process S^X.

$$S_t^X = S_t^0 + \underbrace{\int_0^t g(\xi_s) ds}_{\text{permanent}} + \underbrace{h(\xi_t)}_{\text{temporary}} .$$

- Gatheral (2010): Take $g(x) = -\kappa x$ to rule out price manipulation.

Expected Revenues.

Assume

$$S_t^X = S_t^0 - \int_0^t \kappa_s \xi_s ds - \eta_t \xi_t.$$

Revenues obtained from following X (with $X_T = 0$)

$$R_T(X) = -\int_0^T S_t^X dX_t.$$

Integrating by parts $\rightsquigarrow$ decomposition of expected revenues

$$\mathbb{E}\left[R_T(X)\right] = \underbrace{xS_0^0}_{\text{naive book value}} - \underbrace{\mathbb{E}\left[\int_0^T \kappa_s \xi_s X_s ds\right]}_{\text{costs entailed by perm impact}} - \underbrace{\mathbb{E}\left[\int_0^T \eta_s (\xi_s)^2 ds\right]}_{\text{costs entailed by temp impact}}$$

(Non exhaustive) literature review.

- Mean-variance optimization: Almgren & Chriss (1999, 2000), Almgren (2003), Lorenz & Almgren (2011), ...
- Expected-Utility maximization: Schied & Schöneborn (2009), Schied, Schöneborn & Tehranchi (2010), Schöneborn (2011), ...
- Time-averaged Risk Measures: Gatheral & Schied (2011), Forsyth, Kennedy, Tse & Windcliff (2012), Ankirchner & Kruse (2012), ...

- Overview : Guéant (2016): The Financial Mathematics of Market Liquidity: From Optimal Execution to Market Making.

Extensions

- Models with transient impact.
- Models with non aggresive strategies.
- Including a dark pool.
- ...

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Linear quadratic control problem.

Admissible controls: $\xi \in \mathcal{A}$ iff

$$X_s = x - \int_0^s \xi_u du, \; s \in [0, T]$$

with the terminal state constraint: $X_T = 0$.

Random cost parameters η, λ and κ: non negative and bounded.

- Expected running execution costs

$$\mathcal{J}(\xi) = \mathbb{E}\left[\int_0^T \left(\eta_s(\xi_s)^2 + \kappa_s \mu_s X_s + \underbrace{\lambda_s (X_s)^2}_{\text{risk aversion}}\right) ds\right]$$

- Value function

$$v(x) = \inf_{\xi \in \mathcal{A}} \mathcal{J}(\xi)$$

How to solve the linear quadratic problem ?

Stochastic maximum principle (B. Djehiche's lectures)

$$\begin{cases} \xi_u = \dfrac{Y_u}{2\eta_u}, \\ X_t = x - \displaystyle\int_0^t \xi_u \, du \qquad \text{(forward dynamics)}, \\ Y_t = Y_\tau + \displaystyle\int_t^\tau (\kappa_u \mu_u + 2\lambda_u X_u) \, du - \int_t^\tau Z_u \, dW_u, \\ \qquad \qquad \text{("backward" dynamics)}, \\ X_T = 0 \qquad \text{(terminal constraint)}. \end{cases}$$

with $0 \leq t \leq \tau < T$

Remarks:

- Y_T cannot be determined a priori. It is implicitly encoded in the FBSDE.
- The first equation holds on $[0, T]$, the second equation holds on $[0, T)$.

Decoupling field.

Ansatz: $Y = AX + B$ + Itô's formula:

- A unique solution of the singular BSDE (AJK-2014 and GHS-2017)

$$\begin{cases} -dA_s = \left(2\lambda_s - \dfrac{A_s^2}{2\eta_s}\right) ds - Z_s^A \, dW_s, \\ A_T = \infty \end{cases} \qquad (1)$$

  if $1/\eta$ is also bounded and $(T - \cdot)A$ is non-negative and bounded.

- B satisfies the linear BSDE:

$$\begin{aligned} B_t &= \int_t^T \left(\kappa_s \mu_s - \frac{A_s B_s}{2\eta_s} \right) ds - \int_t^T Z_s^B dW_s \\ &= \mathbb{E}\left[\int_t^T \kappa_s \mu_s \exp\left(- \int_t^s \frac{A_u}{2\eta_u} du \right) ds \middle| \mathcal{F}_t \right]. \end{aligned}$$

Outline

Game of optimal liquidation between N players.

Transaction price for each player $i = 1, \dots, N$

$$S_t^i = S_t^0 - \int_0^t \kappa_s^i \left(\frac{1}{N} \sum_{j=1}^N \xi_s^j \right) ds - \eta_t^i \xi_t^i.$$

Optimization problem of player $i = 1, \dots, N$: minimize

$$J^{N,i}\left(\vec{\xi}\right) = \mathbb{E}\left[\int_0^T \left(\kappa_t^i \left(\frac{1}{N} \sum_{j=1}^N \xi_t^j \right) X_t^i + \eta_t^i (\xi_t^i)^2 + \lambda_t^i (X_t^i)^2 \right) dt \middle| \mathcal{X}^i = x^i \right]$$

subject to the state dynamics

$$dX_t^i = -\xi_t^i \, dt, \; X_0^i = \mathcal{X}^i \quad \text{and} \quad X_T^i = 0.$$

$\vec{\xi} = (\xi^1, \cdots, \xi^N)$: vector of strategies of each player.

Game with asymmetric information.

Probabilistic setting:

- $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\}, \mathbb{P})$ be a probability space.
- Carries independent standard Brownian motions $W^0, W^1, ..., W^N$
- and i.i.d. random variables $\mathcal{X}^1, ..., \mathcal{X}^N$ with law ν, independent of the Brownian motions.

Filtrations:

$$\mathbb{F}^i := (\mathcal{F}^i_t, 0 \leq t \leq T), \quad \text{with} \quad \mathcal{F}^i_t := \sigma(\mathcal{X}^i, W^0_s, W^i_s, 0 \leq s \leq t).$$

Assumptions on the processes $(\kappa^i, \eta^i, \lambda^i)$

- Progressively measurable with respect to the augmented σ-field $\mathbb{F}^i$.
- Conditionally independent and identically distributed, given W^0.

Literature.

Probabilistic approach for MFGs:

- R. Carmona & F. Delarue (2013): stochastic maximum principle and McKean-Vlasov FBSDEs.
- R. Carmona, F. Delarue & D. Lacker (2016): MFGs with common noise.
- R. Carmona, F. Delarue (2018): Probabilistic Theory of Mean Field Games with Applications I-II.

Closest papers:

- R. Carmona & D. Lacker (2015).
- X. Huang, S. Jaimungal & M. Nourian (2015).
- P. Cardaliaguet & C. Lehalle (2017).

Novelty

- Private information and common noise.
- Interaction through the impact of their strategies.
- Terminal constraint.

Outline

Formal problem.

1. Fix a $\mathbb{F}^0$ progressively measurable process μ (in some suitable space).
   - $\mathbb{F}^0 := (\mathcal{F}^0_t, 0 \leq t \leq T)$ with $\mathcal{F}^0_t = \sigma(W^0_s, 0 \leq s \leq t)$.
2. Solve the parameterized constrained optimization problem:

$$\inf_{\xi} \mathbb{E}\left[\int_0^T \left(\kappa_s \mu_s X_s + \eta_s \xi_s^2 + \lambda_s X_s^2\right) ds\right]$$

s.t.

$$dX_t = -\xi_t \, dt, \; X_0 = \mathcal{X} \quad \text{and} \quad X_T = 0.$$

   - W^0 and W are independent.
   - $\mathbb{F} := (\mathcal{F}_t, 0 \leq t \leq T)$ with $\mathcal{F}_t := \sigma(\mathcal{X}, W^0_s, W_s, 0 \leq s \leq t)$.
   - κ, η and λ are $\mathbb{F}$ progressively measurable.
3. Search for the fixed point

$$\mu_t = \mathbb{E}[\xi^*_t | \mathcal{F}^0_t], \quad \text{for a.e. } t \in [0, T],$$

where ξ^* is an optimal strategy of the second step.

Conditional mean-field type FBSDE.

Standard approach yields the candidate optimal control $\xi_s^* = \frac{Y_s}{2\eta_s}$ and MFG

$\longrightarrow$ conditional mean-field type FBSDE (with $\widetilde{W} = (W^0, W)$)

$$\begin{cases} dX_s = -\dfrac{Y_s}{2\eta_s}\, ds, \\ -dY_s = \left(\kappa_s \mathbb{E}\left[\dfrac{Y_s}{2\eta_s} \middle| \mathcal{F}_s^0 \right] + 2\lambda_s X_s \right) ds - Z_s\, d\widetilde{W}_s, \\ X_0 = \mathcal{X}, \quad X_T = 0. \end{cases}$$

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Setting on the cost coefficients.

Assumption: κ, λ, $\frac{1}{\lambda}$, η and $\frac{1}{\eta}$ belong to $L^\infty_{\mathbb{F}^0}([0,T] \times \Omega; [0,\infty))$.

Conditional-MF-FBSDE becomes:

$$\begin{cases} dX_s = -\dfrac{Y_s}{2\eta_s}\, ds, \\ -dY_s = \left(\dfrac{\kappa_s}{2\eta_s} \mathbb{E}\left[Y_s | \mathcal{F}^0_s \right] + 2\lambda_s X_s \right) ds - Z_s\, dW^0_s, \\ X_0 = \mathcal{X}, \quad X_T = 0. \end{cases}$$

Common information and initial position.

Common initial portfolio $\mathcal{X} = x$. Conditional-MF-FBSDE $\rightsquigarrow$ FBSDE

$$\begin{cases} dX_s = -\dfrac{Y_s}{2\eta_s}\, ds, \\ -dY_s = \left(\dfrac{\kappa_s}{2\eta_s} Y_s + 2\lambda_s X_s \right) ds - Z_s\, dW^0_s, \\ X_0 = x, \quad X_T = 0. \end{cases}$$

Linear decoupling field $Y = A^\kappa X$ yields

$$-dA^\kappa_s = \left(2\lambda_s + \frac{\kappa_s A^\kappa_s}{2\eta_s} - \frac{(A^\kappa_s)^2}{2\eta_s} \right) ds - Z^{A^\kappa}_s\, dW^0_s, \quad A^\kappa_T = \infty.$$

and optimal state process $X^\dagger$

$$X^\dagger_t = x \exp\left(- \int_0^t \frac{A^\kappa_r}{2\eta_r}\, dr \right).$$

Private initial position...

but common environment. From the preceding case:

- MF-equilibrium

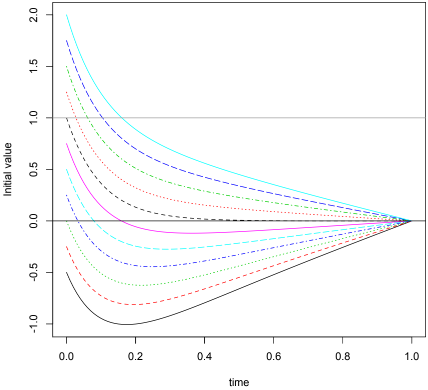
$$\mu_t^* = \frac{1}{2\eta_t}\mathbb{E}(Y_t \big| \mathcal{F}_t^0) = \frac{1}{2\eta_t}\mathbb{E}(Y_t^\dagger \big| \mathcal{F}_t^0) = \frac{\mathbb{E}[\mathcal{X}]}{2\eta_t} A_t^\kappa e^{-\int_0^t \frac{A_r^\kappa}{2\eta_r} dr}.$$

- With $A = A^0$, optimal state process for a given initial position $\mathcal{X} = x \in \mathbb{R}$:

$$X_t^{*,x} = (x - \mathbb{E}[\mathcal{X}]) \exp\left(- \int_0^t \frac{A_r}{2\eta_r} dr \right) + \mathbb{E}[\mathcal{X}] \exp\left(- \int_0^t \frac{A_r^\kappa}{2\eta_r} dr \right).$$

Contrary to the previous case, the sign of the optimal portfolio process X^* may change on the interval $[0, T]$ (if $0 < x < \zeta\mathbb{E}[\mathcal{X}]$).

Common information but private initial position.



Current state $X^{*,x}$ corresponding to parameters $T = 1$, $\mathbb{E}[\mathcal{X}] = 1$, $\lambda = 5$, $\eta = 5$ and $\kappa = 100$ for different values of the initial portfolio x.

Outline

Solving the conditional mean-field type FBSDE.

Conditional mean-field type FBSDE

$$\begin{cases} dX_s = -\dfrac{Y_s}{2\eta_s}\,ds, \\ -dY_s = \left(\kappa_s \mathbb{E}\left[\dfrac{Y_s}{2\eta_s}\middle|\,\mathcal{F}_s^0\right] + 2\lambda_s X_s\right) ds - Z_s\,d\widetilde{W}_s, \\ X_0 = \mathcal{X}, \quad X_T = 0. \end{cases} \tag{2}$$

Partial decoupling field: $Y = AX + B \rightsquigarrow$

$$\begin{cases} dX_s = -\dfrac{1}{2\eta_t}(A_s X_s + B_s)\,ds, \\ -dB_s = \left(\kappa_s \mathbb{E}\left[\dfrac{1}{2\eta_s}(A_s X_s + B_s)\middle|\,\mathcal{F}_s^0\right] - \dfrac{A_s B_s}{2\eta_s}\right) ds - Z_s^B\,d\widetilde{W}_s, \\ X_0 = \mathcal{X}, \quad B_T = 0. \end{cases} \tag{3}$$

General case.

Assumption: κ, λ, $\frac{1}{\lambda}$, η and $\frac{1}{\eta}$ belong to $L^\infty_{\mathbb{F}}([0,T] \times \Omega; [0,\infty))$ with technical condition:

$$16\eta_\star\lambda_\star > \|\kappa\|^2.$$

Theorem

There exists a unique solution (X, B, Y, Z^B, Z^Y) to the FBSDEs (2) and (3) s.t.

- Weighted spaces for X and B:

$$\mathbb{E}\left[\sup_{0\le s\le T}\left|\frac{X_s}{(T-s)^\alpha}\right|^2\right] + \mathbb{E}\left[\sup_{0\le s\le T}\left|\frac{B_s}{(T-s)^\gamma}\right|^2\right] < +\infty$$

  where $\alpha := \frac{\eta_\star}{\|\eta\|} \in (0,1]$ and $0 < \gamma < (1/2) \wedge \alpha$.
- $Y \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}) \cap S^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R})$
- $(Z^B, Z^Y) \in L^2_{\mathbb{F}}([0,T] \times \Omega; \mathbb{R}^m) \times L^2_{\mathbb{F}}([0,T-] \times \Omega; \mathbb{R}^m)$.

Proof based on continuation method.

Optimal liquidation strategy & equilibrium for the MFG.

Candidates for the optimal portfolio process and the optimal trading strategy:

$$X_t^* = \mathcal{X} e^{-\int_0^t \frac{A_r}{2\eta_r} dr} - \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r} dr} ds,$$

$$\xi_t^* = \mathcal{X} e^{-\int_0^t \frac{A_r}{2\eta_r} dr} \frac{A_t}{2\eta_t} + \frac{B_t}{2\eta_t} - \frac{A_t}{2\eta_t} \int_0^t \frac{B_s}{2\eta_s} e^{-\int_s^t \frac{A_r}{2\eta_r} dr} ds.$$

Theorem

The process ξ^* is an optimal control. Hence $\mu^* = \mathbb{E}[\xi^* | \mathcal{F}^0]$ is the solution to the MFG. Moreover, the value function is given by

$$V(\mathcal{X}; \mu^*) = \frac{1}{2} A_0 \mathcal{X}^2 + \frac{1}{2} B_0 \mathcal{X} + \frac{1}{2} \mathbb{E}\left[\int_0^T \kappa_s X_s^* \xi_s^* \, ds \middle| \mathcal{X} \right].$$

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A Yamada-Watanabe result.

Assumption: for any $i = 1, \cdots, N$

$$\kappa_t^i = \kappa(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t}), \quad \eta_t^i = \eta(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t}), \quad \lambda_t^i = \lambda(t, \mathcal{X}^i, W^i_{\cdot\wedge t}, W^0_{\cdot\wedge t})$$

for some non-negative deterministic and bounded functions κ, η and λ.

Proposition (MFG equilibrium)

There exists a measurable function $\Phi : \mathbb{R}\times(\mathcal{C}[0,T])^2 \to \mathcal{H}_\alpha \times (\mathcal{C}[0,T-])^2$ *s.t.*

$$\left(X_t^i, Y_t^i, \int_0^t Z^i ds\right)_{0\leq t\leq T} = \Phi(\mathcal{X}^i, W^i, W^0),$$

where (X^i, Y^i, Z^i) *is the solution to FBSDE* (2) *associated with* $(W^0, \mathcal{X}^i, W^i, \kappa^i, \eta^i, \lambda^i)$. *In particular*

$$\xi^* = \phi(\mathcal{X}, W, W^0) \rightsquigarrow \mu^* = \mathbb{E}[\xi^* | \mathcal{F}^0].$$

It holds for each $i = 1, ..., N$ *that a.s. a.e.* $\mu_t^{*,i} = \mu_t^*$.

ε-Nash equilibrium.

Theorem

If the admissible control space for each player $i = 1, \ldots, N$ is given by

$$\mathcal{A}^i := \left\{ \xi \in \mathcal{A}_{\mathbb{F}^i}(x^i) : \mathbb{E}\left[\int_0^T |\xi_t|^2 \, dt \middle| \mathcal{X}^i = x^i \right] \leq M(x^i) \right\}$$

for some fixed positive function M such that $\psi \leq M$. Then, for each $1 \leq i \leq N$ and each $\xi^i \in \mathcal{A}^i$,

$$J^{N,i}\left(\vec{\xi^*}\right) \leq J^{N,i}(\xi^i, \xi^{*,-i}) + O\left(\frac{1}{\sqrt{N}}\right),$$

where $(\xi^i, \xi^{*,-i}) = (\xi^{*,1}, \cdots, \xi^{*,i-1}, \xi^i, \xi^{*,i+1}, \cdots, \xi^{*,N})$ and $O\left(\frac{1}{\sqrt{N}}\right)$ is to be interpreted as $\frac{g(x_i)}{\sqrt{N}}$ for some real-valued function g independent of i.

Unconstrained MFGs.

For a given integer n

1. Fix a process μ;
2. Solve the standard optimization problem: minimize

$$J^n(\xi;\mu) = \mathbb{E}\left[\int_0^T \left(\kappa_t \mu_t X_t + \eta_t \xi_t^2 + \lambda_t X_t^2\right) dt + nX_T^2\right]$$

   such that

$$dX_t = -\xi_t \, dt \quad X_0 = x;$$

3. Solve the fixed point equation :

$$\mu_t^* = \mathbb{E}[\xi_t^* | \mathcal{F}_t^0] \; a.e. \; t \in [0, T],$$

   where ξ^* is the optimal strategy from step 2.

Related conditional mean field FBSDE.

Conditional MF-FBSDE:

$$\begin{cases} dX_t^n = \left(-\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\right) dt, \\ -dB_t^n = \left(-\dfrac{A_t^n B_t^n}{2\eta_t} + \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle| \mathcal{F}_t^0\right]\right) dt - Z_t^{B^n} d\widetilde{W}_t, \\ dY_t^n = \left(-2\lambda_t X_t^n - \kappa_t \mathbb{E}\left[\dfrac{A_t^n X_t^n + B_t^n}{2\eta_t}\middle| \mathcal{F}_t^0\right]\right) dt + Z_t^{Y^n} d\widetilde{W}_t, \\ X_0^n = x, \quad B_T^n = 0, \quad Y_T^n = 2nX_T^n, \end{cases}$$

where

$$-dA_t^n = \left\{2\lambda_t - \frac{(A_t^n)^2}{2\eta_t}\right\} dt - Z_t^{A^n} d\widetilde{W}_t, \quad A_T^n = 2n.$$

Existence and estimates in some suitable spaces.

Approximation result

Assumption: there exists a constant C such that for any $0 \leq r \leq s < T$

$$\exp\left(-\int_r^s \frac{A_u}{2\eta_u} du\right) \leq C\left(\frac{T-s}{T-r}\right).$$

Proposition

The value function $V^n(x)$ converges to $V(x)$ and

$$\lim_{n\to+\infty}\left\{\mathbb{E}\left[\int_0^T |X_t^n - X_t^*|^2\, dt\right] + \mathbb{E}\left[\int_0^T |B_t^n - B_t^*|^2\, dt\right] + \mathbb{E}\left[\int_0^T |Y_t^n - Y_t^*|^2\, dt\right]\right\} = 0.$$

Remark: no direct proof that (X^n, B^n, Y^n) is a Cauchy sequence.

Thank you for your attention !