Optimal Liquidation with Conditions on Minimum Price[*]

Mervan Aksu[†], Alexandre Popier[‡], Ali Devin Sezer[§]

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Abstract

The classical optimal trading problem is the closure of an initial position in a financial asset over a fixed time interval; the trader tries to maximize an expected utility under the constraint that the position be fully closed by terminal time. Given that the asset price is stochastic, the liquidation constraint may be too restrictive; the trader may want to relax the full liquidation constraint or slow down/stop trading depending on price behavior. We consider two additional parameters that serve these purposes within the Almgren-Chriss liquidation framework: a binary valued process I that prescribes when trading takes place and a measurable set $\boldsymbol{S}$ that prescribes when full liquidation is required. We give four examples for $\boldsymbol{S}$ and I which are all based on a lower bound specified for the price process. The terminal cost of the stochastic optimal control problem is ∞ over $\boldsymbol{S}$; this represents the liquidation constraint. The permanent price impact defines the negative part of the terminal cost over the complement of $\boldsymbol{S}$. The I parameter enters the stochastic optimal control problem as a multiplier of the running cost. Except for quadratic liquidation costs the problem turns out to be non-convex. A terminal cost that can take negative values implies 1) the backward stochastic differential equation (BSDE) associated with the value function of the control problem can explode to $-\infty$ backward in time and 2) the existence results on minimal supersolutions of BSDE with singular terminal values and monotone drivers are not directly applicable. To tackle these we introduce an assumption that balances the market volume process and the permanent price impact in the model over the trading horizon. In the quadratic case, assuming only that the noise driving the asset price is a martingale, we show that the minimal supersolution of the BSDE gives both the value function and the optimal control of the stochastic optimal control problem. A key point in our arguments is that the negative part of the terminal value of the BSDE (arising from permanent price impact) is bounded. For the non-quadratic case, we assume a Brownian motion driven stochastic volatility model and focus on choices of I and $\boldsymbol{S}$ that are either Markovian or can be broken into Markovian pieces. These assumptions allow us to represent the value functions as solutions of PDE or PDE systems. The PDE arguments are based on the smoothness of the value functions and do not require convexity.

We quantify the financial performance of the resulting liquidation algorithms by the percentage difference between the initial stock price and the average price at which the position is (partially) closed in the time interval $[0, T]$. We note that this difference can be divided into three pieces: one corresponding to permanent price impact (A_1),

[†]Middle East Technical University, Institute of Applied Mathematics, Ankara, Turkey, and Mardin Artuklu University, Faculty of Economics and Administrative Sciences, Mardin, Turkey, mervanaksu@artuklu.edu.tr

[‡]Laboratoire Manceau de Mathématiques, Le Mans Université, France, apopier@univ-lemans.fr

[§]Middle East Technical University, Institute of Applied Mathematics, Ankara, Turkey, devinsezer@gmail.com

one corresponding to random fluctuations in the price (A_2) and one corresponding to transaction/bid-ask spread costs (A_3). A_1 turns out to be a linear function of the portion of the portfolio that is closed; therefore, its distribution is fully determined by the distribution of that portion. We provide a numerical study of the distribution of the closed portion and the conditional distributions of A_2 and A_3 given the closed portion under the assumption that the price process is Brownian for $I = 1$ and a $\boldsymbol{S}$ corresponding to a lowerbound on terminal price.

Contents

1 Introduction

There is a range of order types available to an investor to close a position on an asset; the book [14] presents the following: implementation shortfall (IS), target close and volume weighted average price. Given a trading horizon $[0, T]$, all of these order types are constrained to close an initial position at a terminal time T. Mathematically this is expressed as the constraint $Q_T = 0$, where Q_t denotes the position of the investor at time t (we assume $Q_0 > 0$, i.e., an initial long position, for a more brief presentation; everything below applies to a short position $Q_0 < 0$). Given that the price process is stochastic, this constraint can be too restrictive. For example, in IS orders the goal is to close an initial position near the initial price S_0; it may happen that the price drops substantially during the trading interval and the investor holding the position may no longer wish to be strict about closing the position. The present work studies algorithms that offer this type of flexibility in execution. We focus on IS

type orders targeting the initial price S_0 because this is the most basic and most commonly studied order in the current literature but similar ideas can be considered for other types of orders and pursued in future research. The work [16] already presents an optimal liquidation algorithm in which the full liquidation constraint is relaxed; the algorithms presented in the current work extend those in [16] in two directions: 1) the model in the present work involves a permanent market impact component and 2) we allow the model to impose constraints on when trading takes place (for example, the model can be set up so that trading is allowed only when the current price is above a given threshold). The resulting stochastic optimal control problem leads to a backward stochastic differential equation of the form

$$dY_t = \text{vol}_t |Y_t|^p + dM_t, \quad Y_T = \xi, \quad p > 1,$$

with a singular terminal condition (i.e., ξ can take the value ∞). When compared to BSDE with singular terminal values arising from optimal liquidation problems studied in [16] and other works on optimal liquidation, this BSDE has three new features: its generator/driver is not monotone in Y (see below for the precise definition), vol is allowed to be 0, and its terminal value can take negative values. For several versions of the problem with a Markovian formulation we also treat the partial differential equation (PDE) versions of this equation. As we discuss in detail below, the new features have nontrivial implications for the analysis of the BSDE and the PDE.

The IS order is commonly formulated as a stochastic optimal control problem optimizing the expected utility of the cash position that the order generates [14, Chapter 4]. Section 2 presents the stochastic optimal control formulation of the modified IS orders that the present work focuses on. A great deal of the literature on optimal liquidation, including the model presented in section 2, is based on a model proposed by Almgren and Chriss in [3]. This model assumes that the price consists of three pieces: a random fluctuations term $\bar{S}$, a transaction cost term and a permanent market impact term. The permanent price impact is a term $\kappa(Q'_t)dt$ added to $d\bar{S}_t$ where Q'_t is the time derivative of Q at time t; together with the random fluctuation term they make up the midprice process S. The transaction cost term corresponds to trading commissions and the bid-ask spread; this is modeled using a so-called execution cost function L. The common choice for L is a power function $L(\rho) = \eta|\rho|^{\hat{p}}$, $\hat{p} > 1$ (we reserve the letter p for the Hölder conjugate of $\hat{p}$, which arises in the solution of the problem). In most liquidation models the permanent price impact is assumed to be linear in Q'_t (see [12] for more comments on this assumption). This and the constraint $Q_T = 0$ of the standard IS order lead to an interesting situation for this type of order: the permanent price impact ends up having no role in the stochastic optimal control formulation of the standard IS order (see (5) and (7)). These assumptions have another important mathematical implication for the standard IS order: the position variable can be factored out of the value function and out of the backward stochastic differential and partial differential equations that the value function satisfies (see Proposition 3.3)- we will refer to this property of the value function as homogeneity. When we relax the constraint $Q_T = 0$ the permanent price impact enters directly into the stochastic optimal control problem as a part of the terminal cost. In the modifications of the IS order we treat in the present work we would like to keep the homogeneity property of the value function as this greatly influences the analysis of the problem. This turns out to be possible if the permanent price price impact function is chosen compatible with the execution cost function L as follows:

$$\kappa(Q'_t, Q_t) = kQ'_t|Q_t|^{\hat{p}-2}, k > 0,$$

κ remains linear in Q'_t; for $\hat{p} > 2$ ($\hat{p} < 2$) decays (increases) with the size of the remaining position.

As we have already noted, the permanent price impact has no role in the continuous time stochastic optimal control formulation of the standard IS order in the Almgren-Chriss framework. We see a reflection of this fact in the optimal liquidation literature that is based on this framework, as follows: some works introduce a permanent price impact parameter in the model (see e.g., [10, 24] and [14, Chapter 3]), but it ends up having no role in the stochastic optimal control problem, while others drop the permanent price impact parameter altogether from the model assumptions, see, e.g., [4, 16]. The seminal work [3] includes a permanent price impact parameter in a discrete time framework, which ends up having a role in the optimal controls but this role disappears as the discrete time step size converges to 0. A key feature of the problem and the analysis presented in the current work is how the permanent price impact enters and changes the analysis when the full liquidation constraint is relaxed. We discuss this in detail below.

The output of the standard IS order is the cash position generated by the trading algorithm. When full liquidation is no longer required, i.e., when we don't have the constraint $Q_T = 0$, the output of the trading process at time T will be (X_T, Q_T) where X_T is the cash generated by the trading process and Q_T is the position remaining in the asset being traded. There are several choices available to formulate an expected utility maximization problem based on this output. One option is to use a general utility function whose input is the pair (X_T, Q_T); a simpler option is to first assign a monetary value $m(Q_T)$ to the position Q_T, use a utility function whose only input is a monetary value and apply it to $X_T + m(Q_T)$. In the present work we use the latter approach. For $m(Q_T)$ the present work focuses on $m(Q_T) = Q_T S_T$; this is the market value of the position at terminal time T ignoring trading costs. The technical advantage of this choice is that it preserves the homogeneity of the value function; a $|q|^{\hat{p}}$ term can also be added that preserves homogeneity. For the utility function we focus on the identity function, i.e., we consider the problem of maximizing expected terminal position; an additional risk term can be included in the final stochastic optimal control problem with minor modifications to the analysis. We further comment on these points in the conclusion (Section 7).

In subsection 2.1 we discuss two ways the stochastic optimal control problem modeling the IS order can be modified to delay/stop liquidation depending on price behavior 1) by relaxing the full liquidation constraint if the price is too low (which was first proposed in [16]) 2) stopping/pausing trade if the price is too low (proposed in the present work). These two modifications are parameterized in the stochastic optimal control problem by a measurable set $\boldsymbol{S}$ and a process I taking values in $\{0, 1\}$. The set $\boldsymbol{S}$ prescribes when full liquidation is required and enters the stochastic optimal control problem as a part of the terminal cost; the process I prescribes when trading takes place and enters the stochastic optimal control problem by multiplying the volume process (see (13)). We give four examples for $\boldsymbol{S}$ and I in subsection 2.1 which are all based on a lower bound specified for the price process: $(I^{(1)}, \boldsymbol{S}^{(1)})$ puts no constraints on trading, the position is constrained to be fully closed if the closing price is above a given threshold; $(I^{(2)}, \boldsymbol{S}^{(2)})$ allows trading until the price goes below a given threshold, the position is constrained to be fully closed if the price remains above the given threshold across the whole trading interval; $(I^{(3)}, \boldsymbol{S}^{(3)})$ allows trading only when the price is above a given threshold, the position is constrained to be fully closed if the closing price is above the given threshold; $(I^{(4)}, \boldsymbol{S}^{(4)})$ is the same as previous except that for trading to restart the price process first has to upcross a higher threshold. The resulting stochastic optimal control problem for the modified IS order is given in (13). This problem has the

same structure as the one studied in [16] except for the following differences: 1) inclusion of the permanent market impact in the model implies that the terminal cost can take negative values 2) the presence of the I_t term in the running cost (for $k = 0$ and $I_t = 1$ the problem in fact reduces to the one studied in [16]). The solution method in [16] is the derivation and the analysis of a BSDE associated with the value function of the control problem. This is also one of the solutions approaches we will pursue in the present work. We next discuss how the above new features of the problem impact the associated BSDE and its analysis.

The set $\boldsymbol{S}$ (specifying conditions for full liquidation) defines the singular component $\infty \cdot \mathbf{1}_{\boldsymbol{S}}$ of the terminal condition of the BSDE; allowing a nontrivial permanent price impact term introduces an additional negative term $-\frac{k}{p}\mathbf{1}_{\boldsymbol{S}^c}$ in the terminal condition of the BSDE. The driver $y \mapsto f_t(y), y \in \mathbb{R}$ of a BSDE is said to be monotone if there exists $\chi \in \mathbb{R}$ such that $(f(t, y, \omega) - f(t, y', \omega))(y - y') \leqslant \chi(y - y')^2$ for any $t \geqslant 0$ and $y, y' \in \mathbb{R}$ almost surely. The monotonicity of the driver is a key property in establishing the existence of solutions to BSDE. The work [16] focuses on non-negative terminal costs; a non-negative terminal cost corresponds to a nonnegative terminal condition for the associated BSDE. This and the dynamics and cost structure of the control problem lead in [16] to a BSDE with the monotone driver $(t, y) \mapsto -\lambda_t y|y|^{p-1}$ (for $y \geqslant 0$, $y|y|^{p-1} = |y|^p = y^p$). In the case of the stochastic optimal control problem (13), the terminal condition is allowed to take negative values and this forces us to work with the non-monotone convex driver $(t, y) \mapsto I_t \mathrm{Vol}_t |y|^p$. Finally, as already noted, the process I (specifying when trading is allowed) enters the BSDE by multiplying the driver of the BSDE. The resulting BSDE is given in (25) and (23).

Note that although the terminal condition (23) can take negative values, the negative component is bounded above by a constant. For this reason we focus on terminal conditions with bounded negative parts. The challenges/new points that arise in the analysis of (25), (23) as a result of these new features are as follows: the currently available literature doesn't contain existence results on the minimal supersolution of a BSDE such as (25) with a singular terminal condition (23) and with a driver involving the function $y \mapsto |y|^p$. Secondly, a terminal condition that can take negative values and the superlinearity of the driver imply that the solution of the BSDE can explode to $-\infty$ backwards in time.

We handle both of these issues in Section 3 by deriving an apriori lower bound process z on any supersolution of the BSDE with a terminal condition whose negative part is bounded by a constant K ; z is obtained by solving (25) with the terminal condition $\xi = -K$ (for which (25) reduces to an ordinary differential equation). The interval over which z is defined depends on model parameters. We introduce an assumption on the permanent price impact parameter and market volume that guarantees the existence and boundedness of the lower bound process over the interval $[0, T]$ (Assumption 1). The lower bound process is increasing in t; therefore, the value z_0 provides a lower bound on supersolutions on the whole interval $[0, T]$. We then deal with the non-monotonicity of the driver by replacing the portion of $y \mapsto |y|^p$ over $(-\infty, z_0]$ with its linear approximation over the same interval, which results in a monotone driver. Hence existence and uniqueness of the solution for BSDE for integrable terminal condition is obtained in Proposition 3.1. The only way the I term impacts the analysis of the BSDE is by entering Assumption 2 that guarantees that the supersolution can attain the singular value ∞ at terminal time T; in particular Assumption 2 requires $I_t = 1$ for $t \in [T - \epsilon, T]$ for some $\epsilon > 0$. Under these conditions, Proposition 3.2 provides a minimal supersolution for the BSDE with singular terminal condition.

The verification argument presented in subsection 3.1 connects the minimal supersolution of the BSDE (25), (22) to the value function and optimal control of the stochastic optimal

control problem (13) (Proposition 3.3). This argument is based on the convexity of the cost in Q'; under Assumption 2, this turns out to be the case only for the quadratic case (Lemma 2), therefore this verification argument assumes $\widehat{p} = p = 2$.

Assumption 2 is a stringent condition and $I^{(j)}$, $j = 2, 3, 4$, proposed in subsection 2.1 (see (15), (16) and (17)) which depend on price behavior don't satisfy it. In subsections 3.2 and 3.3, we break up the stochastic optimal control problem and the BSDE corresponding to these choices of I into pieces where each piece involving a singular terminal condition satisfies Assumption 2 (Proposition 3.5 and Lemma 4).

The main advantage of the BSDE arguments in Section 3 is that we can work with a general filtration, without any further assumption except the standard ones (completeness and right-continuity) and the left-continuity at time T. The drawback is the restriction to the quadratic case $\widehat{p} = p = 2$.

We call the problem Markovian if the price process is Markovian and the cost structure of the problem is a function of the price process. In Section 4 we explore the case $p \neq 2$ and a PDE representation of the value function and the optimal control when the problem is Markovian or can be broken into Markovian pieces. A popular choice for price dynamics in finance applications is the stochastic volatility model. To the best of our knowledge it is rarely treated in the context of optimal liquidation; in Section 4 we assume the price dynamics to follow this model. For a direct PDE representation (i.e., identifying the value function as a supersolution of a related PDE), the process I and the measurable set $\boldsymbol{S}$ must also be functions of the price process. Of the four possible choices for I and $\boldsymbol{S}$ given in (14)-(17), only $(I^{(1)}, \boldsymbol{S}^{(1)})$ is given as a function of the price process. In subsection 4.1 we present the PDE representation of this case. For $(I^{(j)}, \boldsymbol{S}^{(j)})$, $j = 2, 3, 4$, the decompositions/reductions given in subsections 3.2 and 3.3 yield Markovian subproblems. The resulting PDE problems are treated in subsections 4.2-4.4. The PDE representation of $(I^{(4)}, \boldsymbol{S}^{(4)})$ involves two coupled PDE (one for active trading and one when trading is paused). To compute a solution we introduce an additional parameter n, which is the number of switches allowed between trading and no trading. We solve this problem recursively, letting $\mathcal{N} \to \infty$ gives the solution to the coupled PDE. To the best of our knowledge, there exists no readily available results in the current literature for the existence and smoothness of solutions of PDE that arise in the analysis presented in Section 4. We obtain solutions to these PDE as follows: we use the BSDE results of Section 3 to first obtain viscosity supersolutions. We then use regularization bootstrapping, parameter smoothing and the regularity of the underlying price process to obtain the smoothness of these supersolutions. Once smoothness is proved classical verification arguments from stochastic optimal control can be constructed to relate the solutions to the stochastic optimal control problems. These arguments do not require $p = 2$ as opposed to the verification arguments given in Section 3 which are directly based on the BSDE representation.

The output of the standard IS order is the cash position X_T at time T generated by the trading algorithm; under the assumption that the price process is a Brownian motion, X_T turns out to be normally distributed whose mean and variance have simple formulas in terms of the model parameters. When we relax the IS order so that full liquidation is no longer required at terminal time, the output of the IS order consists of the pair of real random variables (Q_T, X_T) where X_T is, as before, the total cash generated by the trading process and Q_T is the remaining position at terminal time in the asset being traded. For the relaxed/modified IS orders, X_T is not normally distributed even when the price process is

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taken to be Brownian and the joint distribution of (X_T, Q_T) doesn't have an explicit form. Define

$$A = \frac{X_T - (Q_0 - Q_T)S_0}{(Q_0 - Q_T)S_0} \tag{1}$$

A is the percentage deviation from the target price S_0 of the average price at which the position is (partially) closed in the time interval $[0, T]$. In Section 5 we study the joint distribution of $(Q_T/Q_0, A)$. We note that A can be divided into three pieces: one corresponding to permanent price impact (A_1), one corresponding to transaction/bid-ask spread costs (A_2) and one corresponding to random fluctuations in the price (A_3). A_1 turns out to be a linear function of $1 - Q_T/Q_0$; therefore, its distribution is fully determined by that of Q_T/Q_0. We provide a numerical study of the distribution of Q_T/Q_0 and the conditional distributions of A_2 and A_3 given Q_T/Q_0 under the assumption that $\bar{S}_t = \sigma W_t$ for the case $(I = 1, \boldsymbol{S} = \{W_T \geqslant \ell\})$. The same section also provides numerical examples of the sample path behavior of the optimal controls of this modified IS order. Section 7 comments further on the models presented in this work and on possible future research.

2 Definitions

The following model is based on the Almgren Chriss framework for liquidation with price impact (see, e.g., [14, Chapter 3]). Everything is assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$, which satisfies the usual assumptions: completeness and right-continuity. The market volume at time t is denoted by Vol_t, which is a positive process adapted to the filtration $\mathbb{F}$. The initial position is denoted by $q_0 > 0$. The position of the investor at time t is Q_t, in particular $Q_0 = q_0$. The process Q is assumed to be absolutely continuous in the time variable, let Q'_t denote its derivative at time t; Q and Q' are adapted to $\mathbb{F}$.

We will be working with two positive real numbers p and $\hat{p}$ that are Hölder conjugates of each other $1/p + 1/\hat{p} = 1$; we use $\hat{p}$ in the problem formulation, p appears in the associated backward stochastic and partial differential equations.

We suppose that the permanent price impact function κ is

$$\kappa : \mathbb{R}^2 \mapsto \mathbb{R}, \quad \kappa(q', q) = k\, |q|^{\hat{p}-2}\, q'; \tag{2}$$

where $k > 0$ is a real constant. For $\hat{p} > 2$ ($\hat{p} < 2$), κ decreases (increases) with position size and for $\hat{p} = 2$ it is independent of position size. For $\hat{p} = 2$, κ reduces to kq' which is the standard choice for permanent price impact (see [12] or [14, Chapter 3]). The midprice process S is

$$S_t = S_0 + \bar{S}_t + \int_0^t \kappa(Q'_s, Q_s)ds \tag{3}$$

where $\bar{S}$ is a martingale adapted to $\mathbb{F}$.

The actual trading price at time t is $S_t + g_t(Q'_t/\mathrm{Vol}_t)$ where g_t models transaction costs and the bid-ask spread (g depends on t, ω and Q'_t/Vol_t). The process g is often specified via the so-called execution cost function l_t: $l_t(\rho) = \rho g_t(\rho)$. The actual trading price at time t, expressed in terms of l is

$$S_t + \frac{\mathrm{Vol}_t}{Q'_t} L_t \left(\frac{Q'_t}{\mathrm{Vol}_t} \right).$$

The cash position that Q generates is

$$X_T = -\int_0^T \left(S_t + \frac{\text{Vol}_t}{Q_t'} L_t\left(\frac{Q_t'}{\text{Vol}_t}\right)\right) Q_t' dt = -\int_0^T S_t Q_t' dt - \int_0^T \text{Vol}_t L_t\left(\frac{Q_t'}{\text{Vol}_t}\right) dt \quad (4)$$

($Q_t' < 0$ corresponds to selling, hence an increase in X, and $Q_t' > 0$ corresponds to buying, a decrease in X). Note that $d(|x|^{\widehat{p}})/dx = \widehat{p}x|x|^{\widehat{p}-2}$; this and integrating the first term by parts give:

$$\int_0^T S_t Q_t' dt = S_T Q_T - S_0 Q_0 - \int_0^T Q_t dS_t$$

$$= S_T Q_T - S_0 Q_0 - \int_0^T Q_t d\bar{S}_t - \int_0^T k Q_t Q_t' |Q_t|^{\widehat{p}-2} dt$$

$$= S_T Q_T - S_0 Q_0 - \int_0^T Q_t d\bar{S}_t - \frac{k}{\widehat{p}}(|Q_T|^{\widehat{p}} - |Q_0|^{\widehat{p}})$$

Then

$$X_T = Q_0 S_0 - Q_T S_T + K\left(|Q_T|^{\widehat{p}} - |Q_0|^{\widehat{p}}\right) + \int_0^T Q_t d\bar{S}_t - \int_0^T \text{Vol}_t L_t\left(\frac{Q_t'}{\text{Vol}_t}\right) dt,$$

where

$$K = \frac{k}{\widehat{p}}.$$

Let us first review the classical IS order, in which the position is required to be closed fully at terminal time, i.e., we impose the constraint $Q_T = 0$ on the problem. The terminal cash position under this constraint is

$$X_T = Q_0 S_0 - K Q_0^{\widehat{p}} + \int_0^T Q_t d\bar{S}_t - \int_0^T \text{Vol}_t L\left(\frac{Q_t'}{\text{Vol}_t}\right) dt. \quad (5)$$

To identify the optimal liquidation strategy Q^* one maximizes the expected utility $\mathbb{E}[U(X_T)]$, over the admissible strategies Q, where U is the utility function of the trader. The standard choice for the utility function in optimal liquidation literature is

$$U(x) = -e^{-\gamma x}, \quad (6)$$

where γ is the risk aversion parameter of the investor [14]. In the current work we will be focusing on the case $\gamma \to 0$, for which the problem becomes[1]

$$\sup_{Q \in \mathcal{A}_{1,\Omega}} \mathbb{E}[X_T], \quad (7)$$

with

$$\mathcal{A}_{1,\Omega} = \left\{Q : \text{absolutely continuous}, Q' \text{ progressively measurable},\right.$$

$$\left.\mathbb{E}\left[\int_0^T \text{Vol}_t L\left(\frac{Q_t'}{\text{Vol}_t}\right) dt\right] < +\infty \text{ and } Q_0 = q_0, Q_T = 0 \text{ a.s.}\right\},$$

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[1]Using $(-e^{-\gamma x} + 1)/\gamma \approx x$ when γ goes to zero.

see Definition 2.1 below. The term $\int_0^\cdot Q_t d\bar{S}_t$ is a martingale[2] ; taking the expectation of X_T in (5), (7) reduces to

$$\inf_{Q \in \mathcal{A}_{1,\Omega}} \mathbb{E}\left[\int_0^T \text{Vol}_t L\left(\frac{Q'_t}{\text{Vol}_t} \right) dt \right].$$

This is the standard version of the stochastic optimal control formulation of the IS order in the Almgren Chriss framework for $\gamma = 0$. This problem and its generalization where $\gamma > 0$ is a well studied problem in the current literature, as in [13, 24, 25]. In the next subsection we propose several modifications to this problem relaxing the constraint $Q_T = 0$ and introducing constraints on when trading takes place.

2.1 Modifications

When full liquidation is no longer required, i.e., when we don't have the constraint $Q_T = 0$, the output of the trading process at time T will be (X_T, Q_T) where X_T is the cash generated by the trading process and Q_T is the position remaining in the asset being traded. As discussed in the introduction, to formulate a utility maximization problem similar to (7), we assign a monetary value $m(Q_T)$ to the position Q_T and add it to X_T. In the present work we mainly focus on the following simple choice for m: the market value of the position at terminal time T ignoring trading costs, i.e., $Q_T S_T$. With this choice, the monetary value of the position (X_T, Q_T) is

$$\tilde{X}_T = X_T + Q_T S_T = Q_0 S_0 - K|Q_0|^{\hat{p}} + K|Q_T|^{\hat{p}} + \int_0^T Q_t d\bar{S}_t - \int_0^T \text{Vol}_t L_t\left(\frac{Q'_t}{\text{Vol}_t} \right) dt. \quad (8)$$

Recall that our goal is to modify the IS order to not liquidate depending on price behavior in two ways 1) by relaxing the full liquidation constraint if the price is too low 2) stopping/pausing trade if the price is too low. The following formulation allows both of these possibilities. Let $t \mapsto I_t$ be an adapted process taking values in $\{0,1\}$. Let $\boldsymbol{S} \in \mathcal{F}_T$ be a measurable set. The trading set is defined by

$$\mathcal{I} = \{t \in [0,T], \ I_t = 1\}.$$

Definition 2.1. *Define $\mathcal{A}_{I,\boldsymbol{S}}$ as the set of processes Q that satisfy:*

- Q *is absolutely continuous in t and $t \mapsto Q'_t$ is progressively measurable ;*

- $Q_0 = q_0$;

- $Q_T(\omega) = 0$ *if $\omega \in \boldsymbol{S}$ (liquidation constraint) ;*

- *for $\lambda \otimes \mathbb{P}$-almost every $(t,\omega) \in [0,T] \times \Omega$, $I_t(\omega) = 0 \Rightarrow Q'_t(\omega) = 0$ (if $t \notin \mathcal{I}$, $Q'_t = 0$) ;*

- *the cost*

$$\int_0^T \text{Vol}_t L\left(\frac{Q'_t}{\text{Vol}_t} \right) dt - K|Q_T|^{\hat{p}}$$

  belongs to $L^\varrho(\Omega)$ for some $\varrho > 1$.

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[2]In general it is only a local martingale. However we will show that for the optimal strategy, Q is bounded. Hence there is no harm in assuming that this process is indeed a martingale.

For ease of notation set

$$\mathcal{A} = \mathcal{A}_{1,\varnothing}, \tag{9}$$

i.e., when $I = 1$ and $\boldsymbol{S} = \varnothing$ we simply write $\mathcal{A}$ instead of $\mathcal{A}_{I,\boldsymbol{S}}$.

We modify (7) to

$$\sup_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}[\tilde{X}_T].$$

The formula (8) implies that this control problem is equivalent to:

$$\inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}\left[\int_0^T \mathrm{Vol}_t L\left(\frac{Q'_t}{\mathrm{Vol}_t} \right) dt - K|Q_T|^{\widehat{p}} \right]$$

As noted in the introduction we work with the execution cost function

$$L_t(\rho) = \eta_t |\rho|^{\widehat{p}},$$

where η is an adapted and positive valued process. This choice of L reduces the problem to

$$\inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}\left[\int_0^T \frac{\eta_t}{\mathrm{Vol}_t^{\widehat{p}-1}} |Q'_t|^{\widehat{p}} dt - K|Q_T|^{\widehat{p}} \right]. \tag{10}$$

Let us next note that η_t can always be assumed to be 1 by appropriately modifying the volume process, i.e., if we set

$$\widetilde{\mathrm{Vol}}_t = \frac{\mathrm{Vol}_t}{\eta_t^{\frac{1}{\widehat{p}-1}}} = \frac{\mathrm{Vol}_t}{\eta_t^{p-1}} \tag{11}$$

(10) can be written as

$$\inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}\left[\int_0^T \frac{1}{\widetilde{\mathrm{Vol}}_t^{\widehat{p}-1}} |Q'_t|^{\widehat{p}} dt - K|Q_T|^{\widehat{p}} \right];$$

in what follows we always assume $\eta_t = 1$ and that Vol is modified to $\widetilde{\mathrm{Vol}}$ if the original η process is not identically 1. A commonly used convention in the prior literature is

$$\infty \cdot 0 = 0, \infty \cdot c = \begin{cases} \infty, & \text{if } c > 0, \\ -\infty, & \text{if } c < 0. \end{cases} \tag{12}$$

In addition to this we will also set $0/0 = 1$ and $c/0 = \infty$ for $c > 0$. With these conventions (10) can be written as

$$\inf_{Q \in \mathcal{A}} \mathbb{E}\left[\int_0^T \frac{1}{I_t \mathrm{Vol}_t^{\widehat{p}-1}} |Q'_t|^{\widehat{p}} dt + (-K\mathbf{1}_{\boldsymbol{S}^c} + \infty \cdot \mathbf{1}_{\boldsymbol{S}}) |Q_T|^{\widehat{p}} \right], \tag{13}$$

where we use the convention (9). Note that the I process controls when trading takes place (by effectively switching the volume process on and off) and the event $\boldsymbol{S}$ controls when full liquidation is required.

In the next section we obtain a representation of the value function and the optimal control of the problem (13) via the minimal supersolution of an associated BSDE. Before that let us give several examples for the process I and the event $\boldsymbol{S}$. The midprice process S consists of two components: $\bar{S}_t$ and $\int_0^\cdot \kappa(Q'_s, Q_s) ds$. The first component, $\bar{S}$, is the random

component of the change in the midprice; a large and unpredictable drop in price that the investor may fear can only arise from this component. Given this observation, a reasonable approach in choosing I and $\boldsymbol{S}$ is by putting a lower bound ℓ on this component. For this, define:

$$\tau_\ell = \inf\{s \geqslant 0 : \bar{S}_s < \ell\},$$
$$\tau_{t,\ell} = \inf\{s \geqslant t : \bar{S}_s < \ell\}.$$

Then some possible choices for I and $\boldsymbol{S}$ are:

$$I_t = I_t^{(1)} \doteq 1, \boldsymbol{S} = \boldsymbol{S}^{(1)} \doteq \{\bar{S}_T \geqslant \ell\} : \tag{14}$$

trading is allowed at all times, full liquidation is forced only when the terminal price $\bar{S}_T$ is above ℓ.

$$I_t = I_t^{(2)} \doteq \mathbf{1}_{\{t \leqslant \tau_\ell\}}, \boldsymbol{S} = \boldsymbol{S}^{(2)} \doteq \{\tau_\ell > T\} : \tag{15}$$

trading stops once $\bar{S}$ hits the lower bound ℓ; full liquidation takes place if $\bar{S}$ remains above ℓ throughout $[0, T]$.

$$I_t = I_t^{(3)} \doteq \mathbf{1}_{[\ell,\infty)}(\bar{S}_t)\mathbf{1}_{\{t \leqslant \tau_{T-\delta,\ell}\}}, \boldsymbol{S} = \boldsymbol{S}^{(3)} = \{\tau_{T-\delta,\ell} \geqslant T\} : \tag{16}$$

trading pauses when the price $\bar{S}$ is below ℓ, full liquidation takes place if the price process $\bar{S}$ remains above ℓ in the time interval $[T - \delta, T]$ and trading stops if $\bar{S}$ goes below ℓ in the same interval; $\delta > 0$ is a small fixed constant.

Let us comment on the $\delta > 0$ parameter in this formulation: essentially we would like to continue with the liquidation when the price is not too below our target price S_0 and close the position fully if the terminal price is also near our target price. However, allowing trading (re)start arbitrarily close to T and forcing a full liquidation implies high transaction costs (in fact, ∞ transaction costs under the current model). This is the reason for the $\delta > 0$ parameter: full liquidation is forced only if the price remains above ℓ in the time interval $[T - \delta, T]$.

In the last formulation trading pauses once $\bar{S}$ hits ℓ; if $\bar{S}$ is a continuous diffusion process, once it hits ℓ, it will hit ℓ infinitely often and the trading process will switch on and off infinitely often as $\bar{S}$ crosses ℓ. One can get a discrete sequence of on and off trading intervals by putting a buffer of size $b > 0$ above ℓ between trading and no trading; once trading pauses, it is turned back on once $\bar{S}$ goes above $b+\ell$. The corresponding I and S are expressed through the following sequence of hitting times:

$$\tau_{\ell,0} = \tau_\ell, \quad \tau_{b,-1} = 0$$
$$\tau_{b,0} = \inf\{t : t > \tau_{\ell,0}, \bar{S}_t \geqslant b + \ell\},$$
$$\tau_{\ell,k} = \inf\{t : t > \tau_{b,k-1}, \bar{S}_t < \ell\},$$
$$\tau_{b,k} = \inf\{t : t > \tau_{\ell,k}, \bar{S}_t \geqslant b + \ell\}.$$
$$\bar{\tau}_{b,k} = \begin{cases} \tau_{b,k}, & \text{if } \tau_{b,k} + \delta < T, \\ T, & \text{otherwise.} \end{cases}$$

Adding a buffer of size $b > 0$ between no-trading and trading in (16) amounts to the following definitions:

$$I_t = I_t^{(4)} \doteq \sum_{k=-1}^{\infty} \mathbf{1}_{[\tau_{b,k}, \tau_{\ell,k+1}]}(t), \quad \boldsymbol{S} = \boldsymbol{S}^{(4)} \doteq \{I_T = 1\}. \tag{17}$$

3 BSDE Analysis

In our arguments the concepts of monotonicity (of the driver of a BSDE) and the minimal supersolution of a BSDE play a key role, let us begin by giving a precise definition of these terms.

Definition 3.1. *The function $(t, y, \omega) \mapsto f(t, y, \omega)$ is said to be monotone if there exists $\chi \in \mathbb{R}$ such that a.s. and for any $t \geqslant 0$ and $y, y' \in \mathbb{R}$,*

$$(f(t, y, \omega) - f(t, y', \omega))(y - y') \leqslant \chi(y - y')^2.$$

Let ξ be an $\mathcal{F}_T$-measurable real valued random variable. A pair (Y, M) is said to be a supersolution of the BSDE

$$dY_t = -f(t, Y_t)dt + dM_t, \quad Y_T = \xi,$$

if

1. *Y is adapted to $\mathbb{F}$;*

2. *M is a martingale[3] ;*

3. *$Y_s = Y_t + \int_s^t f(u, Y_u)du + (M_t - M_s)$ for $0 \leqslant s < t < T$;*

4. *the inequality*

$$\liminf_{t \to T} Y_t \geqslant \xi \tag{18}$$

   holds a.s..

It is said to be minimal if a.s. for any t, $Y_t' \geqslant Y_t$ for any other supersolution (Y', M').

The work [16] studies the following stochastic optimal control problem:

$$Q_t = Q_t^C + Q_t^J, \quad \min_{Q \in \mathcal{A}'} \mathbb{E}\left[\int_0^T \left(\eta_t \left|\frac{dQ_t^C}{dt}\right|^{\hat{p}} + \gamma_t|Q_t|^{\hat{p}} + C(Q^J)_t\right)dt + \xi|Q_T|^{\hat{p}}\right], \tag{19}$$

where: $\xi \in \mathcal{F}_T$ is a non-negative random variable that is allowed to take the value $+\infty$, Q^C is the absolutely continuous part of Q, Q^J is the jumping part of Q, $C(Q^J)$ a running cost associated with Q^J, and $\mathcal{A}'$ an appropriate modification of $\mathcal{A}$. On the set $\{\xi = +\infty\}$, the constraint $Q_T = 0$ is necessary to ensure a finite cost. Compared to (13) this problem has an additional term Q^J in its dynamics and two additional terms ($\gamma_t|Q_t|^{\hat{p}}$ and $C(Q^J)_t$) in its cost structure. To focus on the novelties associated with the terminal cost we will assume these terms to be 0. The work [16] identifies the value function of (19) as $Y_t|Q_t|^{\hat{p}}$ where Y is the minimal supersolution of the BSDE

$$dY_t = (\hat{p} - 1)\frac{Y_t|Y_t|^{p-1}}{\eta_t^{p-1}} + dM_t, \qquad Y_T = \xi \geqslant 0. \tag{20}$$

The generator of this BSDE is monotone and establishing the existence of the minimal supersolution makes use of this property of the generator. The reason that [16] is able to use this monotone generator is the assumption $\xi \geqslant 0$. In the present work we allow ξ to

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[3]In this paper, we always consider a right-continuous with left limits modification of the martingale

take negative values therefore it is no longer possible to work with a generator involving the function $y \mapsto y|y|^{p-1}$. The BSDE corresponding to the stochastic optimal control problem (13) turns out to be

$$dY_t = (\hat{p} - 1)I_t \mathrm{Vol}_t |Y_t|^p dt + dM_t \tag{21}$$

with terminal condition

$$Y_T = \xi, \tag{22}$$

$$\xi = -K\mathbf{1}_{\boldsymbol{S}^c} + \infty \cdot \mathbf{1}_{\boldsymbol{S}} \tag{23}$$

(a rigorous link between this BSDE and (13) will be established in subsection 3.1 via a verification argument).

Define

$$\mathrm{vol}_t = (\hat{p} - 1)I_t \mathrm{Vol}_t; \tag{24}$$

vol is the restricted volume process available to the trader (up to some constant). Then we can write (21) also as

$$dY_t = \mathrm{vol}_t |Y_t|^p dt + dM_t. \tag{25}$$

Compared to (20) the novel features of (25), (22) are the following: its terminal condition is allowed to take negative values, its driver is convex in Y and not monotone and its generator can take the value 0 (because of the presence of the I term). We introduce several assumptions to deal with these new features when obtaining the existence of a minimal supersolution to this BSDE. First, note that the terminal condition ξ of (23) is bounded below by $-K$; for this reason for our purposes it suffices to focus on terminal conditions whose negative parts are bounded above by a constant, i.e., all of the terminal conditions we consider satisfy

$$\xi^- \leqslant K.$$

The generator of (25) is defined in terms of the function $y \mapsto |y|^p$ and its terminal value can take negative values: a consequence of these facts is that any solution to (25) and (22) can explode to $-\infty$ backward in time (see Lemma 1 below). To deal with this, we introduce the following assumption that ensures that an explosion doesn't happen in $[0, T]$:

Assumption 1. vol *is non-negative and one of the next two conditions holds:*

- vol *is deterministic and satisfies*

$$K^{p-1} \int_0^T \mathrm{vol}_t dt < \frac{1}{p-1}, \tag{26}$$

- vol *is bounded by a constant* $\overline{\mathrm{vol}} > 0$ *such that*

$$K^{p-1}T\overline{\mathrm{vol}} < \frac{1}{p-1}. \tag{27}$$

We will assume throughout that either (26) or (27) holds. Note that when $(\hat{p} - 1)\mathrm{Vol}$ satisfies one of these, vol $= (\hat{p} - 1)I$ Vol also does because $I \in \{0, 1\}$. This assumption balances the negative part of the terminal condition (determined by the permanent price impact parameter k) with the trading volume available to the trader.

If $\mathbb{P}(\boldsymbol{S}) > 0$, ξ can take the value ∞ with positive probability and the terminal condition is said to be singular (terminal condition is said to be singular also when ξ doesn't belong

to $L^{\varrho}(\Omega)$ for some ϱ). We need a further assumption to deal with this possible singularity. To ensure that there exists a solution which is finite on $[0, T)$, the generator should not be equal to zero close to time T. The corresponding assumption in [16] is [16, (A.6)], which is: $\mathbb{E} \int_0^T \eta_s ds < +\infty$. In the present work, to guarantee the existence of a minimal supersolution, we make the following assumption on $\text{vol} = I\,\text{Vol}$:

Assumption 2. *There exists some $\varsigma > 1$ and some $\epsilon > 0$ such that*

$$\mathbb{E}\left[\int_{T-\epsilon}^T \frac{1}{(\text{vol}_s)^{(\hat{p}-1)\varsigma}} ds \right] < +\infty. \tag{28}$$

Evoke that $\hat{p}$ is the Hölder conjugate of p.

Assumption 2 can be interpreted as the availability of liquidity (through Vol and I) at terminal time. In particular it means that $I_t = 1$ on $[T - \epsilon, T]$.

We begin our analysis by deriving a lower bound process z which we will use to guarantee that the minimal supersolution to (20) doesn't explode to $-\infty$ in $[0, T]$. Under Assumption 1 the lower bound process z is defined as follows:

$$z_t = -\left(\frac{1}{K^{1-p} - (p-1) \int_t^T \text{vol}_s ds} \right)^{\frac{1}{p-1}}, \tag{29}$$

if (26) holds; and

$$z_t = -\left(\frac{1}{K^{1-p} - (p-1)\overline{\text{vol}}(T-t)} \right)^{\frac{1}{p-1}}, \tag{30}$$

if (27) holds.

Lemma 1. *z of (29) satisfies*

$$\frac{dz}{dt} - \text{vol}_t |z|^p = 0, \tag{31}$$

and z of (30) satisfies

$$\frac{dz}{dt} - \overline{\text{vol}}\, |z|^p = 0. \tag{32}$$

Both z satisfy $z_T = -K$. Under Assumption 1, z is increasing on $[0, T]$ and satisfies $-\infty < z_0 \leqslant z_t \leqslant -K$ for any $t \in [0, T]$.

Proof. Assumption 1 implies

$$K^{1-p} - (p-1) \int_t^T \text{vol}_s ds > 0 \quad \text{or} \quad K^{1-p} - (p-1)\overline{\text{vol}}(T-t) > 0,$$

for $t \in [0, T]$. Therefore, $z_t < 0$ on $[0, T]$. Non-negativity of vol and $\overline{\text{vol}}$ imply that z is increasing. One can check by differentiation that z of (29) satisfies (31) and z of (30) satisfies (32). □

The standard way to obtain the minimal supersolution of a BSDE with a singular terminal condition is approximation from below, i.e., we truncate the terminal condition ξ to $\xi \wedge n$, solve the resulting BSDE and let $n \nearrow \infty$. Therefore, the treatment of singular terminal values requires the solution of the same BSDE with bounded/integrable terminal values. The next proposition addresses such terminal values:

Proposition 3.1. *Suppose that Assumption 1 holds. Furthermore assume:*

- $\xi^+ \in L^\varrho(\Omega)$ *for some* $\varrho > 1$,

- $\xi^- \leqslant K$ *where K is the constant appearing in Assumption 1.*

Then BSDE (25) has a unique solution (Y, M) such that Y^- is bounded and

$$\mathbb{E}\left(\sup_{t \in [0,T]} |Y_t^+|^\varrho + [M]_T^{\varrho/2}\right) < +\infty.$$

Moreover if ξ^+ is bounded, Y is also bounded.

In the Brownian setting, M is replaced by $\int_0^\cdot Z_s dW_s$ and

$$[M]_T = \int_0^T |Z_s|^2 ds.$$

Proof. Recall that the generator $y \mapsto -\mathrm{vol}_t|y|^p$ is not monotone. However, if the negative part Y^- of the solution is bounded by some constant $c_1 > 0$, that is, if Y is bounded from below by $-c_1$, then we can replace the generator $(t, y) \mapsto -\mathrm{vol}_t|y|^p$ by a monotone continuous generator defined as follows:

$$\widetilde{f}_{-c_1}(s, y) = -\mathrm{vol}_t|y|^p \mathbf{1}_{y \geqslant -c_1} + \mathrm{vol}_t c_1^{p-1}(py + (p-1)c_1)\mathbf{1}_{y < -c_1}. \tag{33}$$

This generator is indeed monotone since

$$(y - y')(\widetilde{f}_{-c_1}(t, y) - \widetilde{f}_{-c_1}(t, y')) \leqslant p\mathrm{vol}_t c_1^{p-1}(y - y')^2$$

for any t, y, y'. Since $Y \geqslant -c_1$,

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - (M_T - M_t) = \xi + \int_t^T \widetilde{f}_{-c_1}(s, Y_s)ds - (M_T - M_t).$$

Furthermore, by Assumption 1, vol belongs to $L^1(0, T)$ almost surely. These imply that the uniqueness result for BSDE driven by a monotone generator ([19, Proposition 5.24] or [15, Lemma 5]) apply in our current setting; therefore, if it exists and if its negative part is bounded, the solution of (25) is unique.

We know from Lemma 1 that the process z is bounded from below by $z_0 < 0$. Consider the generator $\widetilde{f}_{z_0}$ and the BSDE

$$Y_t = \xi + \int_t^T \widetilde{f}_{z_0}(s, Y_s)ds - (M_T - M_t). \tag{34}$$

Since $\widetilde{f}_{z_0}$ is monotone with respect to y and since vol $\in L^1(0, T)$, BSDE (34) has a unique solution (Y, M) (see again [19, Proposition 5.24] or [15, Theorems 1 and 2]). Note that for $\xi = -K$, the solution is $(z, 0)$. Since $\xi \geqslant -K$, the comparison principle ([15, Proposition 4] or [19, Proposition 5.33]) states that $Y_t \geqslant z_t \geqslant z_0$ a.s. for any t. In other words (Y, M) is a solution of the BSDE (25), and this achieves the proof of the proposition. $\square$

Remark 1 (On the negative part of Y). *Itô-Tanaka formula (applied to $y \mapsto y^-$ and Y) implies that Y^- is a subsolution of the BSDE*

$$U_t = \xi^- + \int_t^T \mathrm{vol}_s |U_s|^p \mathbf{1}_{U_s \geqslant 0} ds - \int_t^T dN_s \tag{35}$$

The generator $\widetilde{f} : (t, y) \mapsto \mathrm{vol}_t y^p \mathbf{1}_{y \geqslant 0}$ is not monotone. However, it is increasing and positive. From Lemma 1, $(U^*, V^*) = (-z, 0)$ is a bounded supersolution of this BSDE. Following [8], we deduce the existence of a minimal bounded supersolution (U, N) which is also bounded and non-negative (see [8, Theorems 3.3 and 4.1]). Using again $\widetilde{f}_{z_0}$, we deduce that (U, N) in fact is the unique solution of the BSDE (35). Therefore, assumptions in the previous proposition can be replaced by the existence of a supersolution to the BSDE (35). In other words, these assumptions are sufficient to obtain a supersolution, but not necessary. As a by-product of these calculations, we obtain a better bound: $0 \leqslant Y_t^- \leqslant U_t \leqslant -z_t$. almost surely for any $t \in [0, T]$.

Remark 2 (On the positive part of Y). *Consider the BSDE*

$$\Upsilon_t = \xi^+ + \int_t^T \mathrm{vol}_s \left[-(\Upsilon_s - U_s)^p + (U_s)^p \right] \mathbf{1}_{\Upsilon_s \geqslant 0} ds - (\mathcal{M}_T - \mathcal{M}_t)$$

$$= \xi^+ + \int_t^T \hat{f}(s, \Upsilon_s) ds - (\mathcal{M}_T - \mathcal{M}_t),$$

where U is the solution of (35). U is bounded by K and $\partial_y \hat{f}(s, y) \leqslant p \mathrm{vol}_s K^{p-1}$ imply that the driver $\hat{f}$ is monotone:

$$(y - y')(\hat{f}(t, y) - \hat{f}(t, y')) \leqslant p \mathrm{vol}_t K^{p-1} (y - y')^2$$

and the existence and uniqueness of the solution holds, if ξ^+ belongs to some space $L^\varrho(\Omega)$. Define $Y = \Upsilon - U$ and $M = \mathcal{M} - N$:

$$Y_t = \Upsilon_t - U_t = \xi^+ - \xi^- + \int_t^T (\hat{f}(s, \Upsilon_s) - \widetilde{f}(s, U_s)) ds$$

$$- (\mathcal{M}_T - N_T - \mathcal{M}_t + N_t)$$

$$= \xi - \int_t^T \mathrm{vol}_s (\Upsilon_s - U_s)^p ds - (M_T - M_t).$$

Hence, (Y, M) solves the BSDE (25) and $0 \leqslant Y_t^+ \leqslant \Upsilon_t$ holds almost surely for $t \in [0, T]$.
Another estimate can be obtained using the Itô-Tanaka formula since

$$(Y_t)^+ = \xi^+ - \int_t^T \mathrm{vol}_s (Y_s)^p \mathbf{1}_{Y_s \geqslant 0} ds - \int_t^T \mathbf{1}_{Y_s \geqslant 0} dM_s - \frac{1}{2} \int_t^T dL_s^Y.$$

If $\widetilde{\Upsilon}$ solves the BSDE with monotone generator

$$\widetilde{\Upsilon}_t = \xi^+ - \int_t^T \mathrm{vol}_s (\widetilde{\Upsilon}_s)^p \mathbf{1}_{\widetilde{\Upsilon}_s \geqslant 0} ds - \int_t^T d\widetilde{\mathcal{M}}_s,$$

the comparison principle implies that a.s. for any t, $Y_t^+ \leqslant \widetilde{\Upsilon}_t$.

Our main result on the BSDE (25) when its terminal condition is singular is the following:

Proposition 3.2. *Suppose $\xi^- \leqslant K$ and Assumptions 1 and 2 hold. Assume that the filtration $\mathbb{F}$ is left-continuous at time T. Then there exists a minimal[4] supersolution $(Y^{\min}, M^{\min})$ to the BSDE (25) with terminal condition ξ such that $Y^{\min}$ has a left-limit at time T and the negative part of this minimal supersolution is bounded.*

Several points: there is no condition on ξ^+, the only condition on ξ^- is $\xi^- \leqslant K$, in particular, the terminal condition (23) arising from the stochastic optimal control problem satisfies the conditions of this proposition. Condition (28) implies that vol should remain away from zero close to T.

Proof. The proof proceeds parallel to that of [16, Proposition 3]. Let us consider for any $n \geqslant 0$

$$\xi^n = \xi \wedge n.$$

The solution (Y^n, M^n) of

$$Y_t^n = \xi^n - \int_t^T \text{vol}_s |Y_s^n|^p ds - (M_T^n - M_t^n),$$

has the same upper bound $U = -z$ for the negative part $(Y^n)^-$ for any n. Using the comparison principle for monotone BSDE, arguing as in [16] gives

$$Y^n \nearrow Y.$$

From Remark 2, $Y^n \leqslant \Upsilon^n$ with

$$\Upsilon_t^n = (\xi^+ \wedge n) - \int_t^T \text{vol}_s (\Upsilon_s^n)^p \mathbf{1}_{\Upsilon_s^n \geqslant 0} ds - (N_T^n - N_t^n).$$

From [16, Lemma 1], we have

$$\Upsilon_t^n \leqslant \frac{1}{(T-t)^{\widehat{p}}} \mathbb{E}\left[\int_t^T \left(\frac{\widehat{p}-1}{\text{vol}_s} \right)^{\widehat{p}-1} ds \bigg| \mathcal{F}_t \right]$$

for any $t \in [T - \epsilon, T]$, almost surely. Therefore, from (28), Y_t^n, $T - \epsilon \leqslant t < T$, is finite and bounded in $L^\varsigma(\Omega)$ uniformly with respect to n. In particular, for any $\epsilon' < \epsilon$, there exists a constant C such that for any n,

$$\mathbb{E}\left(|Y_{T-\epsilon'}^n|^\varsigma \right) \leqslant C.$$

Stability result for BSDE ([19, Theorem 5.10]) shows that (Y^n, M^n) converges to (Y, M):

$$\lim_{n \to +\infty} \mathbb{E}\left(\sup_{t \in [0, T-\epsilon']} |Y_t^n - Y_t|^\varsigma + [M^n - M]_{T-\epsilon'}^{\varsigma/2} \right) = 0.$$

Thus (Y, M) solves (25): for any $0 \leqslant t \leqslant r < T$

$$Y_t = Y_r - \int_t^r \text{vol}_s |Y_s|^p ds - (M_r - M_t).$$

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[4]among all supersolutions with bounded negative part

Moreover $Y^- \leqslant U$. And since the filtration is left-continuous at time T, we obtain that a.s.

$$\liminf_{t \to T} Y_t \geqslant \xi.$$

Finally, minimality can be obtained as in the proof of [16, Proposition 4]. If $(\widetilde{Y}, \widetilde{M})$ is another supersolution, we add to both solutions $\widetilde{Y}$ and Y^n the quantity $-z$ and the same arguments on $\widetilde{Y} - z$ and $Y^n - z$ lead to a.s. $\widetilde{Y} - z \geqslant Y^n - z$.

The only remaining problem is the existence of a limit at time T. Compared to [21, Theorem 2.1], the novelty is the negative part of $Y^{\min}$ or of Y^n, which approximates $Y^{\min}$. To deal with the negative part, we can apply the arguments of the proof of [21, Theorem 2.1] using the function

$$\Theta(y) = \int_y^\infty \frac{1}{1 + |w|^p} dw$$

(Θ can be defined in terms of the hypergeometric functions):

$$\Theta(Y_t) = \mathbb{E}\left[\Theta(\xi)|\mathcal{F}_t\right] + \psi_t^+ - \psi_t^-$$

where ψ^+ and ψ^- are two non-negative supermartingales such that ψ^+ converges a.s. to zero. To obtain this result, we crucially use that the negative part of Y^n is bounded uniformly with respect to n and also that the martingale $N^n = \int \mathbf{1}_{Y^n \leqslant 0} dM^n$ is uniformly bounded in the sense that there exists a constant C such that for any n

$$\mathbb{E}\left[N^n\right]_T^2 \leqslant C.$$

To obtain this last inequality, the Itô-Tanaka formula for $(Y^n)^-$ is applied (see Remark 1 and Equation (35)). Since the negative part of ξ is bounded, a apriori estimate for BSDE ([19, Proposition 5.7] or [15, Proposition 2]) leads to this uniform estimate on N^n. The other arguments can be copied from the proof of [21, Theorem 2.1] with straightforward modifications. This achieves the proof. $\qquad\square$

Our next task is to relate the solution/minimal supersolution of the BSDE (25) to the solution of the stochastic optimal control problem (13).

3.1 Solution of the quadratic stochastic optimal control problem

The goal of this subsection is to relate the minimal supersolution of the BSDE (25) with terminal condition (22) to the solution of the stochastic optimal control problem (13). This will be achieved through a verification argument based on the convexity of the cost structure of (13). The convex structure holds only for $\widehat{p} = 2$, for this reason in this subsection we assume $\widehat{p} = p = 2$. We deal with the case $\widehat{p} \neq 2$ in the next section in a Markovian framework.

For ease of reference let us restate our stochastic optimal control problem (13):

$$\inf_{Q \in \mathcal{A}_{I,S}} \mathbb{E}\left[\int_0^T \frac{|Q_t'|^2}{\text{vol}_t} dt + \xi |Q_T|^2\right] = \inf_{Q \in \mathcal{A}} \mathbb{E}\left[\int_0^T \frac{|Q_t'|^2}{\text{vol}_t} dt + \xi |Q_T|^2\right]; \qquad (36)$$

we remind the reader that in the second formulation we are using the convention (12); the definitions of $\mathcal{A}_{I,S}$, $\mathcal{A}$ and vol are given Definition (2.1), (9) and (24).

Remark 3. *Under Assumption 1 on* vol *(with $p = 2$) the Cauchy-Schwarz inequality implies*

$$\sup_{t \in [0,T]} |Q_t - Q_0|^2 = \sup_{t \in [0,T]} \left| \int_0^t Q_s' ds \right|^2 \leqslant \left(\int_0^T \mathrm{vol}_s ds \right) \left(\int_0^T \frac{|Q_s'|^2}{\mathrm{vol}_s} ds \right)$$

$$\leqslant \frac{1}{K} \left(\int_0^T \frac{|Q_s'|^2}{\mathrm{vol}_s} ds \right).$$

Hence if $Q \in \mathcal{A}_{I,\boldsymbol{S}}$, $\mathbb{E} \sup_{t \in [0,T]} |Q_t - Q_0|^{2\varrho} < +\infty$. In particular

$$|Q_T|^2 \leqslant 2|Q_0|^2 + 2\frac{1}{K} \left(\int_0^T \frac{|Q_s'|^2}{\mathrm{vol}_s} ds \right).$$

Hence for any bounded ξ, the integrability condition in Definition 2.1 is equivalent to

$$\mathbb{E} \left[\left(\int_0^T \frac{|Q_s'|^2}{\mathrm{vol}_s} ds \right)^\varrho \right] < +\infty.$$

Our goal is to prove the following result:

Proposition 3.3. *Suppose $p = \widehat{p} = 2$. Suppose Assumptions 1 and 2 hold, suppose $\xi^- \leqslant K$ and let $(Y^{\min}, M^{\min})$ be the minimal supersolution of (25), (22). Then*

$$Q_t^* = Q_0 \exp \left(-\int_0^t Y_s^{\min} \mathrm{vol}_s ds \right), t \in [0, T), \tag{37}$$

(equivalently, $(Q^*)_t' = -Y_t^{\min} \mathrm{vol}_t Q_t^*$) is the optimal state process for the stochastic optimal control problem (36). Moreover the value function of (36) at time t, namely

$$\inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}(t)} \mathbb{E} \left[\int_t^T \frac{|Q_s'|^2}{\mathrm{vol}_s} ds + \xi |Q_T|^2 \Big| \mathcal{F}_t \right],$$

is given by $V(t, q) = q^2 Y_t^{\min}$ ($\mathcal{A}_{I,\boldsymbol{S}}(t)$ is defined by Definition 2.1, but the process Q starts at time t from the deterministic position q).

The proof directly follows from the next two lemmas and is given at the end of this subsection. Let's call J the expression inside the min in (36): for $v = Q'$

$$J(v) = \int_0^T \frac{v_t^2}{\mathrm{vol}_t} dt + \xi \left(Q_0 + \int_0^T v_t dt \right)^2.$$

We start with the following observation:

Lemma 2. *If*

$$\xi^- \int_0^T \mathrm{vol}_t dt < 1 \tag{38}$$

almost surely, then the functional $v \mapsto J(v)$ is strictly convex. The Gâteaux derivative of J at point v in direction w, is given by

$$\langle DJ(v), w \rangle = 2 \int_0^T \frac{v_t w_t}{\mathrm{vol}_t} dt + 2\xi \left(Q_0 + \int_0^T v_t dt \right) \left(\int_0^T w_t dt \right).$$

Remark 4. *Assumption 1 with $p = \hat{p} = 2$ and $\xi^- \leqslant K$ imply* (38).

Proof. Taking v and $\tilde{v}$ and $\theta \in [0, 1]$, we have

$$J(\theta v + (1 - \theta)\tilde{v}) - \theta J(v) - (1 - \theta)J(\tilde{v})$$

$$= -\theta(1 - \theta)\left[\int_0^T \frac{(v_t - \tilde{v}_t)^2}{\text{vol}_t}dt + \xi\left(\int_0^T (v_t - \tilde{v}_t)dt\right)^2\right]$$

$$\leqslant -\theta(1 - \theta)\left[\int_0^T \frac{(v_t - \tilde{v}_t)^2}{\text{vol}_t}dt - \xi^-\left(\int_0^T (v_t - \tilde{v}_t)dt\right)^2\right]$$

$$\leqslant \theta(1 - \theta)\left[-1 + \xi^-\int_0^T \text{vol}_t dt\right]\int_0^T \frac{(v_t - \tilde{v}_t)^2}{\text{vol}_t}dt \leqslant 0.$$

We use the Cauchy-Schwarz inequality for the inequality. Now for any $\epsilon > 0$ and v and w

$$\frac{1}{\epsilon}(J(v + \epsilon w) - J(v))$$

$$= 2\int_0^T \frac{v_t w_t}{\text{vol}_t}dt + \epsilon\int_0^T \frac{w_t^2}{\text{vol}_t}dt$$

$$+ 2\xi\left(Q_0 + \int_0^T v_t dt\right)\left(\int_0^T w_t dt\right) - \epsilon\xi\left(\int_0^T w_t dt\right)^2.$$

Letting ϵ to zero gives the desired formula. $\qquad\square$

The last intermediate result we need is a version of Proposition 3.3 where ξ is bounded.

Lemma 3. *Suppose ξ is bounded. If (Y, M) is the solution of (25) with terminal condition $Y_T = \xi$, then the optimal state process Q^* (resp. optimal control $v^* = (Q^*)'$) of (36) is given by*

$$Q_t^* = Q_0 - \int_0^t (Y_s \text{vol}_s)Q_s^* ds \quad (resp. \; -Y_s \text{vol}_s Q_s^*).$$

Moreover, $Y_0(Q_0)^2$ is the value function of the control problem.

Proof. Note that for bounded ξ, the BSDE (25) with terminal condition $Y_T = \xi$ has a unique solution and it equals the minimal supersolution: $Y = Y^{\min}$, $M = M^{\min}$. Since ξ is bounded, from Proposition 3.1, Y is bounded and for any $\varrho > 1$

$$\mathbb{E}\left([M]_T^{\frac{\varrho}{\varrho-1}}\right) < +\infty.$$

Thus

$$Q_t^* = Q_0 \exp\left(-\int_0^t Y_s \text{vol}_s ds\right)$$

is also bounded and from Assumption 1

$$\int_0^T \frac{(v_s^*)^2}{\text{vol}_s}ds = \int_0^T \text{vol}_s(Y_s Q_s^*)^2 ds$$

is bounded. Thus Q^* is in $\mathcal{A}_{I,\boldsymbol{S}}$.

Let Q be in $\mathcal{A}_{I,\boldsymbol{S}}$ with L^ϱ-integrability, and define $v = Q'$, $w = v^* - v$ and $\widehat{Q} = \int_0 w_s ds$. If

$$N_t \doteq 2Y_t Q_t^* = -2\frac{v_t^*}{\mathrm{vol}_t},$$

integration by parts gives:

$$N_t = 2Y_0 Q_0 + 2\int_0^t Q_s^* \mathrm{vol}_s (Y_s)^2 ds - \int_0^t 2Y_s Y_s \mathrm{vol}_s Q_s^* ds + 2\int_0^t Q_s^* dM_s = 2\int_0^t Q_s^* dM_s.$$

Hence N and $\int (Q^*)^2 dM$ are martingales (since Q^* is bounded). Moreover

$$\int_0^T (\widehat{Q}_s Q_s^*)^2 d[M]_s \leqslant C \sup_{t\in[0,T]} (\widehat{Q}_s)^2 [M]_T;$$

this and 3 imply

$$\mathbb{E}\int_0^T (\widehat{Q}_s Q_s^*) d[M]_s \leqslant C \left[\mathbb{E} \sup_{t\in[0,T]} (\widehat{Q}_s)^{2\varrho} \right]^{\frac{1}{\varrho}} \left[\mathbb{E}\left[M\right]_T^{\frac{\varrho}{\varrho-1}} \right]^{\frac{\varrho-1}{\varrho}} < +\infty.$$

Hence $\int \widehat{Q} Q^* dM$ is also a martingale.

Now integration by parts implies:

$$\int_0^T w_t \frac{2v_t^*}{\mathrm{vol}_t} dt = \int_0^T \widehat{Q}_t dN_t - N_T \widehat{Q}_T = \int_0^T \widehat{Q}_t dN_t - 2\xi q_T^* \widehat{Q}_T.$$

This and Lemma 2 give

$$\langle DJ(v^*), w \rangle = \int_0^T \widehat{Q}_t dN_t = 2\int_0^T \widehat{Q}_t Q_t^* dM_t,$$

which is a martingale. With the convexity of J we obtain

$$\mathbb{E}(J(v^*) - J(v)) \leqslant \mathbb{E}\langle DJ(v^*), v^* - v \rangle = 0.$$

Therefore, v^* is the optimal control (unique from the strict convexity of J). Itô's formula applied to $Y_t(Q_t^*)^2$ gives

$$d(Y_t(Q_t^*)^2) = (Q_t^*)^2 \mathrm{vol}_t (Y_t)^2 dt + (Q_t^*)^2 dM_t + 2Y_t q_t^* (-Y_t \mathrm{vol}_t q_t^*) dt$$

$$= -\mathrm{vol}_t (Q_t^* Y_t)^2 dt + (Q_t^*)^2 dM_t = -\frac{(v_t^*)^2}{\mathrm{vol}_t} dt + (Q_t^*)^2 dM_t.$$

Since $\int (Q^*)^2 dM$ is a martingale,

$$Y_t(Q_t^*)^2 = \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\mathrm{vol}_s} ds + \xi(Q_T^*)^2 \Big| \mathcal{F}_t \right].$$

In other words $Y_t(Q_t^*)^2$ is the value function of the control problem. $\qquad \square$

We now give

Proof of Proposition 3.3. For $N = 2Y^{\min}Q^*$, one can use arguments parallel to those in the proof of the previous lemma to show that N is a martingale on $[0, T)$. Since $(Y^{\min})^-$ is bounded (by K) we have

$$\exp\left(-\int_0^t Y_s^{\min}\mathrm{vol}_s ds\right) \leqslant \exp\left(\int_0^t (Y_s^{\min})^-\mathrm{vol}_s ds\right) \leqslant \exp\left(K\int_0^t \mathrm{vol}_s ds\right).$$

This and Assumption 1 imply that Q^* is also bounded. Since $(Y^{\min})^-$ is also bounded, the martingale N is bounded from below. Therefore, the limit at time T of N exists in $\mathbb{R}$ and

$$Q_t^* = \frac{N_t}{2Y_t^{\min}}$$

tends to zero a.s. on the set $\boldsymbol{S} = \{\xi = +\infty\}$, since $\lim_{t\to T} Y_t^{\min}\boldsymbol{1}_{\boldsymbol{S}} = +\infty$.

Now we apply Itô's formula to $Y^{\min}(Q^*)^2$: for any $0 \leqslant t \leqslant r < T$

$$Y_t^{\min}(Q_t^*)^2 = Y_r^{\min}(Q_r^*)^2 - \int_t^r (Q_s^*)^2\mathrm{vol}_s(Y_s^{\min})^2 ds$$

$$- \int_t^r Y_s^{\min}2(Q_s^*)(-Y_s^{\min}\mathrm{vol}_s Q_s^*)ds - \int_t^r (Q_s^*)^2 dM_s^{\min}$$

$$= Y_r^{\min}(Q_r^*)^2 + \int_t^r \frac{(v_s^*)^2}{\mathrm{vol}_s}ds - \int_t^r (Q_s^*)^2 dM_s^{\min}$$

with $v_s^* = -\mathrm{vol}_s Y_s^{\min}Q_s^*$. Taking the conditional expectation we get

$$Y_t^{\min}(Q_t^*)^2 = \mathbb{E}\left[Y_r^{\min}(Q_r^*)^2 + \int_t^r \frac{(v_s^*)^2}{\mathrm{vol}_s}ds\Big|\mathcal{F}_t\right].$$

By the the monotone convergence theorem

$$\liminf_{r\to T} \mathbb{E}\left[\int_t^r \frac{(v_s^*)^2}{\mathrm{vol}_s}ds\Big|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\mathrm{vol}_s}ds\Big|\mathcal{F}_t\right].$$

And by Fatou's lemma ($(Y^{\min})^-$ is bounded)

$$\liminf_{r\to T} \mathbb{E}\left[Y_r^{\min}(Q_r^*)^2\Big|\mathcal{F}_t\right] \geqslant \mathbb{E}\left[\liminf_{r\to T}(Y_r^{\min}(Q_r^*)^2)\Big|\mathcal{F}_t\right].$$

Recall the definition of $N = 2Y^{\min}Q^*$ and that the limit of N at time T exists in $\mathbb{R}$. Moreover $\lim_{r\to T} Q_r^* = 0 = Q_T^*$, when $\lim_{r\to T} Y_r^{\min} = +\infty$. Therefore, if $\lim_{r\to T} Y_r^{\min} = +\infty$, then

$$\liminf_{r\to T}(Y_r^{\min}(Q_r^*)^2) = \liminf_{r\to T}(N_r)\lim_{r\to T} Q_r^* = 0 = \xi(Q_T^*)^2$$

(with the convention $\infty \cdot 0 = 0$). If $\liminf_{r\to T} Y_r^{\min} < +\infty$, then

$$\liminf_{r\to T}(Y_r^{\min}(Q_r^*)^2) \geqslant \xi(Q_T^*)^2.$$

In both cases we obtain

$$Y_t^{\min}(Q_t^*)^2 \geqslant \mathbb{E}\left[\int_t^T \frac{(v_s^*)^2}{\mathrm{vol}_s}ds + \xi(Q_T^*)^2\Big|\mathcal{F}_t\right].$$

Thus $Y^{\min}q^2$ dominates the value function $V(\cdot, q)$ of the constrained control problem.

Now if Q is in $\mathcal{A}_{I,\boldsymbol{S}}$, it is in $\mathcal{A}_{I,\varnothing}$ (no terminal constraint on Q_T). Therefore, the value function V dominates the value function of the unconstrained control problem with terminal penalty $\xi \wedge n$, for any n. Denote by Y^n the solution of the BSDE (25) with bounded terminal value $Y_T^n = \xi \wedge n$. From Lemma (3), $Y^n q^2$ is the value function of the unconstrained control problem. We deduce that for any n

$$Y_t^n q^2 \leqslant V(t, q) \leqslant Y_t^{\min} q^2.$$

Since Y^n converges to $Y^{\min}$, we obtain that $Y^{\min}q^2$ is the value function of the constrained control problem and that Q^* is the optimal state process.

Note that the value function is finite at time 0, that is

$$\mathbb{E}\left[\int_0^T \frac{(v_s^*)^2}{\mathrm{vol}_s}ds + \xi(Q_T^*)^2\right] \leqslant Y_0^{\min}(Q_0^*)^2 < +\infty.$$

Using (28) and the proof of Proposition 3.2, for $t < T$, $Y_t^{\min}$ belongs to $L^\varsigma(\Omega)$ and we can also deduce that

$$\mathbb{E}\left[\left(\int_0^T \frac{(v_s^*)^2}{\mathrm{vol}_s}ds + \xi(Q_T^*)^2\right)^\varsigma\right] < +\infty.$$

Therefore Q^* belongs to $\mathcal{A}_{I,\boldsymbol{S}}$ and this achieves the proof. $\qquad\square$

3.2 Reduction to random time interval $[\![0, \tau_\ell \wedge T]\!]$ for $I_t = 1_{\{t \leqslant \tau_\ell\}}$

In general, $\mathbb{P}(T - \epsilon < \tau_\ell \leqslant T) > 0$ for any $\epsilon > 0$. therefore, for $I = I^{(2)} = \mathbf{1}_{\{t \leqslant \tau_\ell\}}$, Assumption 2 in general doesn't hold regardless of what Vol is. For $I^{(2)}$ a natural way to deal with this is to consider the problem on the random interval $[\![0, \tau_\ell \wedge T]\!]$: for $Q \in \mathcal{A}_{I^{(2)},\boldsymbol{S}}$ we have $Q_t = Q_{\tau_\ell}$, $Q_t' = 0$ for $t > \tau_\ell$. Therefore, the stochastic optimal control problem (13) can also be expressed as

$$\min_{Q \in \mathcal{A}} \mathbb{E}\left[\int_0^{\tau_\ell \wedge T} \frac{(Q_t')^{\widehat{p}}}{\mathrm{Vol}_t}dt + \left(-K\mathbf{1}_{\{\tau_\ell < T\}} + \infty \cdot \mathbf{1}_{\{\tau_\ell \geqslant T\}}\right)Q_T^{\widehat{p}}\right]. \tag{39}$$

The corresponding BSDE is again (25) but with terminal condition

$$Y_{\tau_l \wedge T} = \xi^{(2)} = -K\mathbf{1}_{\{\tau_\ell < T\}} + \infty \cdot \mathbf{1}_{\{\tau_\ell \geqslant T\}}, \tag{40}$$

and we have a BSDE with random terminal time $\tau_\ell \wedge T$. Note that now the dynamics of Y is considered only on the random interval $[\![0, \tau_\ell \wedge T]\!]$ and $I_t^{(2)} = 1$ on this random interval.

The next proposition (formulated in terms of a general stopping time τ) states that the BSDE (25) has a minimal supersolution for terminal conditions of the form

$$Y_{\tau \wedge T} = \xi = \zeta\mathbf{1}_{\{\tau \geqslant T\}} + \psi_\tau\mathbf{1}_{\{\tau < T\}}, \tag{41}$$

where τ is a stopping time and ζ, ψ are bounded from below by $-K$ and ψ is also bounded from above; (40) is a special case of (41).

Proposition 3.4. *Assumption 1 holds. Let ξ be as in (41). If for some $\varrho > 1$*

$$\mathbb{E}\int_0^T \mathrm{vol}_s(\mathbb{E}[\zeta|\mathcal{F}_s])^\varrho ds < +\infty, \tag{42}$$

then the BSDE (25) with terminal condition (41) has a unique solution (Y, M) such that

- $\mathbb{P}$-*a.s., on the set* $\{t \geqslant \tau \wedge T\}$, $Y_t = \xi$ *and* $M_t = 0$,

- $\mathbb{P}$-*a.s., for all* $0 \leqslant t \leqslant r \leqslant T$,

$$Y_{t \wedge \tau} = Y_{r \wedge \tau} + \int_{t \wedge \tau}^{r \wedge \tau} \mathrm{vol}_s |Y_s|^p ds - \int_{t \wedge \tau}^{r \wedge \tau} dM_s,$$

- *and for some constant* C *depending on* vol, K *and* ϱ

$$\mathbb{E}\left[|Y_{\tau \wedge T}|^\varrho + \int_0^{\tau \wedge T} |Y_s|^\varrho ds + [M]_{\tau \wedge T}^{\varrho/2} \right] \leqslant C\mathbb{E}|\xi|^\varrho.$$

If, instead of the integrability condition (42) *on* ζ, *the following modified version of condition* (28) *holds*

$$\mathbb{E}\left[\int_{\tau \wedge (T-\epsilon)}^{\tau \wedge T} \frac{1}{(\mathrm{vol}_s)^{(\widehat{p}-1)\varsigma}} ds \right] < +\infty, \tag{43}$$

then there exists a minimal supersolution $(Y^{\min}, Z^{\min})$ *for the BSDE* (25) *with terminal condition* (41).

Proof. The first part of the claim comes from [15, Proposition 6] on BSDEs with random terminal time. Note that our terminal time $\tau \wedge T$ is bounded. As in the proof of Proposition 3.1, since the negative part of the data is bounded, we can modify the generator in order to have a monotone generator (see Equation (33)).

In the singular case, we proceed by truncation. For n sufficiently large:

$$\xi \wedge n = (\zeta \wedge n)\mathbf{1}_{\tau \geqslant T} + \psi_\tau \mathbf{1}_{\tau < T}.$$

Again we construct a sequence of solutions (Y^n, M^n), such that (Y^n) is non-decreasing. It only remains to control Y^n, uniformly in n.

We define $(\widehat{U}, \widehat{V})$ as the solution with terminal condition $\psi_\tau \mathbf{1}_{\tau < T} - \zeta^- \mathbf{1}_{\tau \geqslant T}$. Note that $\widehat{U}$ is bounded. Now for any $t \leqslant s \leqslant T$ we have

$$Y_{t \wedge \tau}^n - \widehat{U}_{t \wedge \tau} = Y_{s \wedge \tau}^n - \widehat{U}_{s \wedge \tau} - \int_{t \wedge \tau}^{s \wedge \tau} \mathrm{vol}_r \left[\left| Y_r^n - \widehat{U}_r + \widehat{U}_r \right|^p - \left| \widehat{U}_r \right|^p \right] dr$$
$$- \int_{t \wedge \tau}^{s \wedge \tau} (dM_r^n - d\widehat{V}_r).$$

Hence $\Upsilon^n = Y^n - \widehat{U}$ solves the BSDE with terminal condition $(\zeta^+ \wedge L)\mathbf{1}_{\tau \geqslant T}$ and generator

$$g(s, y) = -\mathrm{vol}_s \left[\left| y + \widehat{U}_s \right|^p - |\widehat{U}_s|^p \right].$$

Once again, the map $y \mapsto g(s, y)$ is not monotone on $\mathbb{R}$. But since the negative part of the data is bounded, we can modify g on $(-\infty, -c_1)$, such that g becomes monotone (see Equation (33)). And again by the comparison principle, Υ^n is non-negative.

To control Υ^n, we again follow the arguments of [16, Lemma 1]. Young's inequality implies that for any $y \geqslant 0$ and $c \geqslant 0$

$$\left| y + \widehat{U}_s \right|^p \geqslant \frac{\widehat{p}}{\widehat{p}-1} c^{p-1}(y + \widehat{U}_s) - \frac{c^p}{\widehat{p}-1}.$$

Recall that $\widehat{p}$ is the Hölder conjugate of p. Thus with $c^{p-1} = (\widehat{p} - 1)/(\text{vol}_s(T-s))$

$$g(s,y) \leqslant -\frac{\widehat{p}}{T-s}y + \left(\frac{\widehat{p}-1}{\text{vol}_s}\right)^{\widehat{p}-1}\left(\frac{1}{T-s}\right)^{\widehat{p}} - \frac{\widehat{p}}{T-s}\widehat{U}_s + \text{vol}_s|\widehat{U}_s|^p.$$

Since $\Upsilon_{T\wedge\tau}^n = 0$ if $\tau < T$, explicit solution for linear BSDE and comparison principle imply that for $t < T$

$$\Upsilon_{\tau\wedge t}^n \leqslant \frac{1}{(T-\tau\wedge t)^{\widehat{p}}}\mathbb{E}\left[\int_{t\wedge\tau}^{T\wedge\tau}\left(\frac{\widehat{p}-1}{\text{vol}_s}\right)^{\widehat{p}-1}ds\Bigg|\mathcal{F}_{\tau\wedge t}\right]$$

$$+ \frac{1}{(T-\tau\wedge t)^{\widehat{p}}}\mathbb{E}\left[\int_{t\wedge\tau}^{T\wedge\tau}(T-s)^{\widehat{p}-1}[\text{vol}_s(T-s)|\widehat{U}_s|^p - \widehat{p}\widehat{U}_s]ds\Bigg|\mathcal{F}_{\tau\wedge t}\right].$$

This uniform bound on Υ^n, thus on Y^n, allows us to define the solution of the BSDE with terminal time $\tau\wedge T$ and a singular terminal condition. $\qquad\square$

For $\tau = \tau_\ell$ we have vol $=$ Vol and for the existence of a minimal supersolution Vol must satisfy (43). The next proposition connects the value function of (39) to the minimal supersolution $(Y^{\min}, M^{\min})$ whose existence was derived above.

Proposition 3.5. *Suppose* Vol *satisfies* (43) *with* $\widehat{p} = 2$. *Let* $(Y^{\min}, M^{\min})$ *be the minimal supersolution of* (25) *and* (40). *For any* $t \in [0, T)$,

$$Q_t^* = Q_0\exp\left(-\int_0^t Y_s^{\min}\text{vol}_s ds\right),$$

(equivalently, $(Q^*)_t' = -Y_t^{\min}\text{vol}_t Q_t^*$*) is the optimal control for the stochastic optimal control problem* (39)*. Moreover the value function of the same control problem at time t equals $Y_t^{\min}q^2$.*

Proof. Following the arguments of the proof of Proposition 3.3, we have that for any $0 \leqslant t \leqslant r < T$:

$$Y_{\tau_\ell\wedge t}^{\min}(Q_{\tau_\ell\wedge t}^*)^2 = Y_{\tau_\ell\wedge r}^{\min}(Q_{\tau_\ell\wedge r}^*)^2 + \int_{\tau_\ell\wedge t}^{\tau_\ell\wedge r}\frac{(v_s^*)^2}{\text{vol}_s}ds - \int_{\tau_\ell\wedge t}^{\tau_\ell\wedge r}(Q_s^*)^2 dM_s^{\min}$$

with $v^* = (Q^*)'$. Taking the conditional expectation and passing to the limit on r, we obtain

$$Y_{\tau_\ell\wedge t}^{\min}(Q_{\tau_\ell\wedge t}^*)^2 \geqslant \mathbb{E}\left[\int_{\tau_\ell\wedge t}^{\tau_\ell\wedge T}\frac{(v_s^*)^2}{\text{vol}_s}ds + \xi(Q_{\tau_\ell\wedge T}^*)^2\Bigg|\mathcal{F}_t\right].$$

The rest of the proof continues as that of Proposition 3.3. $\qquad\square$

3.3 Reduction to time interval $[0, T-\delta]$ for $(I, \boldsymbol{S}) = (I^{(3)}, \boldsymbol{S}^{(3)})$ and $(I^{(4)}, \boldsymbol{S}^{(4)})$

As already noted, for $I = I^{(3)}$ and $I = I^{(4)}$, Assumption 2 doesn't hold in general since both $I_t^{(3)}$ and $I_t^{(4)}$ can be zero for t arbitrarily close to T. However, in both of these cases the problem can be reduced to the time interval $[0, T-\delta]$ where this assumption is no longer needed.

Both $(I, \boldsymbol{S}) = (I^{(3)}, \boldsymbol{S}^{(3)})$ and $(I, \boldsymbol{S}) = (I^{(4)}, \boldsymbol{S}^{(4)})$ consist of two phases: before and after time $T-\delta$, the reason for this was explained in the paragraph following (16). In both cases

the trading process for $I^{(3)}$ and $I^{(4)}$ proceeds exactly as in $I^{(2)}$ after time $T - \delta$: if the algorithm is in trading mode at time $T - \delta$, the position is fully closed only when the price remains above ℓ throughout the interval $[T - \delta, T]$; trading stops (and doesn't restart) if the price hits ℓ. This implies that the stochastic optimal control problem (36) can be written as

$$\inf_{Q \in \mathcal{A}} \mathbb{E} \left[\int_0^{T-\delta} \frac{|Q_t'|^{\widehat{p}}}{\text{vol}_t} dt + I_{T-\delta}^{(j)} V_{T-\delta}^\infty - (1 - I_{T-\delta}^{(j)}) K |Q_{T-\delta}|^{\widehat{p}} \right] \tag{44}$$

where $V_{T-\delta}^\infty$ is the value function of the stochastic optimal control problem corresponding to $(I^2, \boldsymbol{S}^{(2)})$ on the time interval $[\![T - \delta, T \wedge \tau_{T-\delta,\ell}]\!]$ with initial position $Q_{T-\delta}$. For $\hat{p} = 2$, we know by Proposition 3.5 that $V_{T-\delta}^\infty = Q_{T-\delta}^{\widehat{p}} Y_{T-\delta}^{\infty,T-\delta}$ where $Y^{\infty,T-\delta}$ is the minimal supersolution of the BSDE (25) on the time interval $[\![T - \delta, \tau_{T-\delta,\ell} \wedge T]\!]$ with terminal condition

$$\zeta = -K \mathbf{1}_{\{\tau_{T-\delta,\ell} < T\}} + \infty \cdot \mathbf{1}_{\{\tau_{T-\delta,\ell} \geqslant T\}}.$$

Existence of $Y^{\infty,T-\delta}$ follows from Proposition 3.4. Then for $\widehat{p} = 2$, (44) can be written as

$$\inf_{Q \in \mathcal{A}} \mathbb{E} \left[\int_0^{T-\delta} \frac{|Q_t'|^{\widehat{p}}}{\text{vol}_t} dt + \xi |Q_{T-\delta}|^{\widehat{p}} \right]$$

with terminal cost factor

$$\xi = I_{T-\delta}^{(j)} Y_{T-\delta}^{\infty,T-\delta} - (1 - I_{T-\delta}^{(j)}) K. \tag{45}$$

$Y_{T-\delta}^{\infty,T-\delta}$ is $\mathcal{F}_{T-\delta}$-measurable and belongs to $L^\varsigma(\Omega)$. Now consider the solution (Y, M) of the BSDE (25) on the time interval $[0, T - \delta]$ with the $\mathcal{F}_{T-\delta}$-measurable terminal condition ξ of (45). Define

$$Y_t^{\min} = \begin{cases} Y_t & \text{if } t < T - \delta \\ Y_t^{\infty,T-\delta} I_{T-\delta} & \text{if } t \geqslant T - \delta. \end{cases}$$

Lemma 4. *Suppose $\widehat{p} = p = 2$. For any $t \in [0, T)$,*

$$Q_t^* = Q_0 \exp \left(- \int_0^t Y_s^{\min} \text{vol}_s ds \right),$$

(equivalently, $v_t^* = -Y_t^{\min} \text{vol}_t Q_t^*$) is the optimal control for the stochastic optimal control problem corresponding to $(I, S) = (I^{(3)}, S^{(3)})$ or $(I, S) = (I^{(4)}, S^{(4)})$. Moreover, the value function of the same control problem at time t equals $q^2 Y_t^{\min}$.

Proof. From our previous arguments, we know that if $v_t^* = -Y_t \text{vol}_t Q_t^*$ is the derivative of Q_t^*, with starting point Q_0, then for any other strategy v we have

$$Y_0(Q_0)^2 = \mathbb{E} \left[\int_0^{T-\delta} \frac{(v_s^*)^2}{\text{vol}_s} ds + \xi (Q_{T-\delta}^*)^2 \right] \leqslant \mathbb{E} \left[\int_0^{T-\delta} \frac{(v_s)^2}{\text{vol}_s} ds + \xi (Q_{T-\delta}^v)^2 \right]. \tag{46}$$

Now let us define for $t \geqslant T - \delta$

$$Q_t^{\infty,T-\delta} = Q_{T-\delta}^* - \int_{T-\delta}^t Y_s^{\infty,T-\delta} \text{vol}_s Q_s^{\infty,T-\delta} ds$$

and $v^{\infty,T-\delta} = -Y^{\infty,T-\delta}\text{vol}Q^{\infty,T-\delta}$. From Proposition 3.5, $Y^{\infty,T-\delta}(Q^{\infty,T-\delta})^2$ is the value function of the related control problem starting at time $T-\delta$ from $Q^*_{T-\delta}$:

$$Y^{\infty,T-\delta}_{T-\delta}(Q^*_{T-\delta})^2 = \mathbb{E}\left[\int_{T-\delta}^{\tau_{T-\delta,\ell}\wedge T}\frac{(v^{\infty,T-\delta}_s)^2}{\text{vol}_s}ds + \zeta(Q^{\infty,T-\delta}_{\tau_{T-\delta,\ell}\wedge T})^2\bigg|\mathcal{F}_{T-\delta}\right]$$

$$\leqslant \mathbb{E}\left[\int_{T-\delta}^{\tau_{T-\delta,\ell}\wedge T}\frac{(v^{T-\delta}_s)^2}{\text{vol}_s}ds + \zeta(Q^{v,T-\delta}_{\tau_{T-\delta,\ell}\wedge T})^2\bigg|\mathcal{F}_{T-\delta}\right] \qquad (47)$$

for any process $Q^{v,T-\delta}$ with derivative $v^{T-\delta}$ starting at time $T-\delta$ from $Q^*_{T-\delta}$. Multiplying this equality by $I_{T-\delta}$, we obtain

$$\xi(Q^*_{T-\delta})^2 = I_{T-\delta}Y^{\infty,T-\delta}_{T-\delta}(Q^*_{T-\delta})^2 - (1-I_{T-\delta})k/2(Q^*_{T-\delta})^2$$

$$= \mathbb{E}\left[\int_{T-\delta}^{\tau_{T-\delta,\ell}\wedge T}\frac{(I_{T-\delta}v^{\infty,T-\delta}_s)^2}{\text{vol}_s}ds + I_{T-\delta}\zeta(Q^{\infty,T-\delta}_{\tau_{T-\delta,\ell}\wedge T})^2\bigg|\mathcal{F}_{T-\delta}\right] - K(1-I_{T-\delta})(Q^*_{T-\delta})^2.$$

If for $t \geqslant T-\delta$

$$Q^*_t = Q^*_{T-\delta} - \int_{T-\delta}^t I_{T-\delta}Y^{\infty,T-\delta}_s\text{vol}_sQ^*_s ds,$$

we have

$$Q^*_t = \begin{cases} Q^{\infty,T-\delta}_t & \text{if } I_{T-\delta}=1 \\ Q^*_{T-\delta} & \text{if } I_{T-\delta}=0 \end{cases}$$

and therefore

$$\xi(Q^*_{T-\delta})^2 = \mathbb{E}\left[\int_{T-\delta}^{\tau_{T-\delta,\ell}\wedge T}\frac{(v^*_s)^2}{\text{vol}_s}ds + \zeta(Q^*_{\tau_{T-\delta,\ell}\wedge T})^2\bigg|\mathcal{F}_{T-\delta}\right].$$

From (46) and (47), we deduce the optimality of the defined process Q^*. $\qquad\square$

4 PDE Analysis

In this section, we will assume the price process to be Markovian and the cost structure to be a function of the price process; under these assumptions, our goal is to relate the value function of the stochastic optimal control problem (13) to a PDE version of the BSDE for the four choices of I and $\boldsymbol{S}$ given in (14)-(17).

As noted in the introduction, we assume the price process $\bar{S}$ to be driven by a stochastic volatility model:

$$\bar{S}_t = \int_0^t \sqrt{\nu_t}dW^{(1)}_t,$$

where ν_t is the stochastic volatility process:

$$d\nu_t = \alpha(\theta - \nu_t)dt + c\sqrt{\nu_t}dW^{(2)}_t; \qquad (48)$$

$W = (W^{(1)}, W^{(2)})$ is a Brownian motion in $\mathbb{R}^2$ ($W^{(i)}_1$, $i = 1, 2$, have unit variance, but they are correlated with coefficient ρ). We assume that the Feller condition, $2\alpha\theta > c^2$, ensuring a positive process ν holds. Let $\mathcal{L}$ denote the second-order differential operator corresponding to these dynamics:

$$\mathcal{L}u = \frac{1}{2}\nu\partial^2_{ss}u + \frac{1}{2}\nu c^2\partial^2_{\nu\nu}u + \alpha(\theta-\nu)\partial_\nu u + c\nu\rho\partial^2_{s\nu}u. \qquad (49)$$

The variables (ν, s) belong to $D = (0, +\infty) \times \mathbb{R}$.

To get a PDE representation we take η and Vol processes to be functions of $(\nu, \bar{S})$. With a slight abuse of notation, we assume the market volume process to be $t \mapsto \text{Vol}(t, \nu_t, \bar{S}_t)$ where $\text{Vol} : [0, T] \times D \to \mathbb{R}_+$ is a non negative valued function; similarly the transaction cost process is $t \mapsto \eta(t, \nu_t, \bar{S}_t)$ where $\eta : [0, T] \times D \to (0, +\infty)$ is a strictly positive valued function. In this section we still use (11) and (24), we denote by Vol the quantity $\widetilde{\text{Vol}}$, and will use the assumption (27) on vol:

$$0 \leqslant \text{vol}(t, s, \nu) \leqslant \overline{\text{vol}}, \qquad K^{p-1} T \overline{\text{vol}} < \frac{1}{p-1} \tag{50}$$

where $K = k/\widehat{p}$ and k is the constant in the permanent impact given by (2).

4.1 PDE representation for $I = 1, \boldsymbol{S} = \{\bar{S}_T \geqslant \ell\}$

In this subsection we assume $I = 1$, i.e., $\text{vol}_t = (\widehat{p} - 1)\text{Vol}_t$. Let us consider terminal values of the form

$$\xi = \Phi(\nu_T, \bar{S}_T); \tag{51}$$

where

$$\Phi : D \to [-K, \infty] \tag{52}$$

is a measurable function. By (23), the choice $\boldsymbol{S} = \{\bar{S}_T \geqslant \ell\}$ corresponds to the Φ function

$$\Phi = \Phi^{(1)}(s) = -\frac{k}{\widehat{p}} 1_{(-\infty, \ell)}(s) + \infty \cdot 1_{[\ell, \infty)}(s).$$

Under the Markovian assumptions of the present section, and for $I = 1$ the BSDE (25) and the terminal condition (51) correspond to the following PDE: for any $(\nu, s) \in D$ and $t \in [0, T)$

$$\partial_t u + \mathcal{L}u - \text{vol}_t |u|^p = 0. \tag{53}$$

with the terminal constraint

$$u(T, \cdot, \cdot) = \Phi. \tag{54}$$

Our goal, under the Markovian assumptions of the present section, is to prove that the value function of the stochastic optimal control problem (13) with $I = 1$ and $\boldsymbol{S} = \boldsymbol{S}^{(1)}$ can be expressed as a multiple of the unique solution of this PDE with terminal condition $\Phi = \Phi^{(1)}$. To express the value function as a solution to the above PDE, we first extend the stochastic optimal control problem (13) to allow it to start from any time point t. Accordingly, we define

$$\bar{S}_r^{t, \nu, s} = s + \int_t^r \sqrt{\nu_u^{t, \nu, s}} dW_u^{(1)},$$

$$\nu_r^{t, \nu, s} = \nu + \int_t^r \alpha(\theta - \nu_u^{t, \nu, s}) du + \int_t^r c \sqrt{\nu_u^{t, \nu, s}} dW_u^{(2)},$$

$$\xi = \Phi(\nu_T^{t, \nu, s}, \bar{S}_T^{t, \nu, s}), \qquad \text{vol}_u^{t, \nu, s} = \text{vol}(u, \nu_u^{t, \nu, s}, \bar{S}_u^{t, \nu, s}).$$

Under our assumptions on Vol and Φ, Proposition 3.2 implies that the following BSDE has a unique minimal supersolution:

$$Y_r^{t, \nu, s} = \xi - \int_r^T \text{vol}_v^{t, \nu, s} \left| Y_v^{t, \nu, s} \right|^p dv - \int_r^T Z_v^{t, \nu, s} dW_v;$$

set

$$u^\Phi(t, \nu, s) := Y_t^{t, \nu, s}. \tag{55}$$

As we already noted, our goal in this subsection is to prove that u^Φ is the minimal supersolution (or the unique solution if Φ is finite) of the PDE (53) for any Φ of the form (52) (and in particular for $\Phi = \Phi^{(1)}$). Our first step in this direction is the following:

Lemma 5. *If Φ is continuous and with polynomial growth on D and if vol is also continuous on $[0, T] \times D$, then u^Φ is a continuous function of $(t, \nu, s) \in [0, T] \times D$ and is the unique viscosity solution of the PDE (53) with polynomial growth on D.*

Proof. See [19, Theorem 5.37] or [5, Theorems 3.4 and 3.5] (see also [22]). If Φ is bounded, the solution $Y^{t, \nu, s}$ and thus u are also bounded. Hence our generator is Lipschitz continuous.

If Φ satisfies $-k/\widehat{p} = -K \leqslant \Phi(s, \nu) \leqslant K(1 + |\nu|^m + |s|^m)$, then ξ^+ satisfies the condition imposed in Proposition 3.1. Thus $Y^{t, \nu, s}$ and thus u are bounded from below, and we can modify our generator such that it becomes monotone (see the proof of Proposition 3.1). Then existence and uniqueness follow from [19, Theorem 5.37] (this result is stated for $(\nu, s) \in \mathbb{R}^2$ in [19] but all arguments continue to work when $(\nu, s) \in D$). $\square$

Now suppose that Φ is a continuous function from D to $[-k/\widehat{p}, +\infty]$. We use the proof of Proposition 3.2. For any $n \geqslant 0$, we consider the bounded function $\Phi^n = \Phi \wedge n$. By the previous lemma there exists a unique bounded viscosity solution $u^{\Phi \wedge n}$ and by comparison principle,

$$u^\Phi(t, \nu, s) = \lim_{n \to +\infty} u^{\Phi \wedge n}(t, \nu, s)$$

is well-defined with a bounded negative part. Suppose now that for some $m \geqslant 1$ and some $\epsilon > 0$:

$$\forall (t, \nu, s) \in [T - \epsilon, T] \times D, \quad \frac{1}{\text{vol}(t, \nu, s)} \leqslant C(1 + |\nu|^m + |s|^m).$$

Then Condition (28) holds and we have on $[T - \epsilon, T] \times D$,

$$u^{\Phi \wedge n}(t, \nu, s) \leqslant \frac{1}{(T-t)^{\widehat{p}}} \mathbb{E}\left[\int_t^T \left(\frac{\widehat{p} - 1}{\text{vol}_\rho^{t, \nu, s}} \right)^{\widehat{p}-1} d\rho \right]$$

$$\leqslant \frac{C}{(T-t)^{\widehat{p}-1}} (1 + |\nu|^m + |s|^m)^{\widehat{p}-1}. \tag{56}$$

On the rest of the interval $[0, T]$, the bound of the solution $u^{\Phi \wedge n}$ is controlled by the previous estimate with $t = T - \epsilon$. In other words we have a bound on $u^{\Phi \wedge n}$ which does not depend on n. Hence u is lower semi-continuous on $[0, T] \times D$ and finite (even locally bounded) on $[0, T) \times D$. These considerations give us the following result:

Lemma 6. *Suppose Φ is a continuous function from D to $[-k/\widehat{p}, +\infty]$; then u^Φ is the minimal viscosity solution of the PDE (53) on $[0, T) \times D$ (among all viscosity solutions with bounded negative part).*

Proof. See [22, Theorem 1]. $\square$

The concept of a viscosity solution allows even a discontinuous solution and doesn't address the issue of smoothness/regularity of the solution; therefore the previous result doesn't say anything about the regularity of u^Φ. The properties of the operator $\mathcal{L}$ (and

smoothness assumptions on Vol and η) allow us to establish the smoothness of u^Φ, with a regularization bootstrap argument for parabolic PDE. (see [26, Lemma 5] for a similar argument).

Lemma 7. *Suppose that* vol *is continuously differentiable with respect to all of its arguments. Assume that Φ_n is a sequence of continuous functions, converging to Φ, such that the related (viscosity) solutions u_n of the PDE* (53) *converge pointwise to u. Then u belongs to $C^{1,2}([0,T] \times D)$ and is a classical solution of the PDE* (53).

Proof. Fix some $\epsilon > 0$ and $\mathfrak{K}$ a compact subset of D. First note that from (56), the bound of u_n on $[0, T - \epsilon] \times \mathfrak{K}$ does not depend on the terminal value, that is on n, but only on ϵ and $\mathfrak{K}$.

Moreover, the operator $\mathcal{L}$ can be written as follows:

$$\mathcal{L}u = \frac{1}{2}\nu\partial_{ss}^2 u + \frac{1}{2}\nu c^2\partial_{\nu\nu}^2 u + \alpha(\theta - \nu)\partial_\nu u + c\nu\rho\partial_{s\nu}^2 u$$
$$= \frac{1}{2}\text{div}\,(a(\nu, s)\nabla u) + b(\nu, s)\nabla u,$$

with

$$\nabla u(\nu, s) = \begin{pmatrix} \partial_s u \\ \partial_\nu u \end{pmatrix}$$

and

$$a(\nu, s) = \nu \begin{pmatrix} 1 & c\rho \\ c\rho & c^2 \end{pmatrix}, \quad b(\nu, s) = \begin{pmatrix} \dfrac{1}{2} + \dfrac{c\rho}{2} \\ \alpha(\theta - \nu) + \dfrac{c^2}{2} + \dfrac{c\rho}{2} \end{pmatrix}.$$

Our coefficients a and b are bounded on $\mathfrak{K}$, and a is uniformly elliptic on $\mathfrak{K}$. Since vol is also continuously differentiable with respect to all of their arguments, then we can easily check that all conditions called a) (uniform ellipticity and boundedness condition), b) (growth condition on the derivatives, take $m = 2$) and c) (regularity condition) of [17, Theorem VI.4.4] hold. From this theorem, if there exists a function ψ continuous on $[0, T - \epsilon] \times \mathfrak{K}$ and is of class $H^{1+\beta/2, 2+\beta}(]0, T - \epsilon[\times \overset{\circ}{\mathfrak{K}})$ for some $\beta > 0$ (space of functions which are C^2 with β-Hölder continuous second derivatives in the space variable x and C^1 with $\beta/2$-Hölder continuous in the times variable t), then the PDE

$$\partial_t v + \mathcal{L}v - \text{vol}_t |v|^p = 0$$

with the boundary condition $u = \psi$, has a unique solution v with the same regularity as ψ. Now our viscosity solutions u_n are continuous and bounded on $[0, T - \epsilon] \times \mathfrak{K}$. Let us consider a sequence of smooth mollifiers ζ_m and define $\psi_m = u_n \star \zeta_m$. There exists a classical smooth solution $u_{n,m}$ of the PDE (53) with boundary condition ψ_m and pointwise $u_{n,m}$ converges to u_n as m goes to $+\infty$. As u_n, the bound of $u_{n,m}$ on $[0, T - \epsilon] \times \mathfrak{K}$ does not depend on n and m.

In the next step, we prove that $u_{n,m}$ belongs to $H^{\beta/2, \beta}([0, T - \epsilon] \times \mathfrak{K})$ (space of functions which β-Hölder continuous in the space variable x and $\beta/2$-Hölder continuous in the times variable t) and that the norm of $u_{n,m}$ in this space does not depend on n and m. Note that Conditions (1.2) and (7.1) of [17, Chapter 3] are satisfied. Suppose that v is a smooth solution of the PDE (53) on $[0, T] \times D$: for any $(t, \nu, s) \in [0, T) \times D$

$$\partial_t v + \mathcal{L}v - \text{vol}_t |v|^p = 0,$$

such that for any $\epsilon > 0$ and any compact subset $\mathfrak{K}$ of D , v is bounded on $[0, T - \epsilon] \times \mathfrak{K}$. Thus v solves on $[0, T - \epsilon] \times \mathfrak{K}$ the PDE

$$\partial_t v + \mathcal{L} v = \mathrm{vol}_t |v|^p = f,$$

where f is a bounded function. We can apply [17, Theorem III.10.1]. Hence v is in $H^{\beta/2,\beta}([0, T - \epsilon] \times \mathfrak{K})$. The value of $\beta > 0$ and the Hölder norm of v depend on ϵ, $\mathfrak{K}$ and the bound on v. In other words β does not depend on the terminal value.

In the last step, we get the desired regularity. Now we know that $u_{n,m}$ belongs to $H^{\beta/2,\beta}([0, T - \epsilon] \times \mathfrak{K})$ and solves the PDE

$$\partial_t v + \mathcal{L} v = \mathrm{vol}_t |u_{n,m}|^p = f_{n,m}.$$

Since vol is also in $C^1(D)$, then from [17, Theorem IV.10.1], $u_{n,m}$ is in $H^{1+\beta/2,2+\beta}([0, T - \epsilon'] \times \mathfrak{K}')$ for any $\epsilon' > \epsilon$ and $\mathfrak{K}' \subset \mathfrak{K}$, and the norm depends only on the H^β-norm of $f_{n,m}$. Therefore, u_n, and thus u, belong to the same space[5], that is on any subset $[0, T - \epsilon] \times \mathfrak{K}$, u_n and u are in $C^{1,2}$. $\qquad\square$

The proof also shows that the regularity of any solution does not depend on the terminal value. In other words, far from $t = T$ and $\nu = 0$, the solutions are smooth and classical solutions.

Let us summarize the foregoing results:

Proposition 4.1. *Suppose that* vol *is continuously differentiable on* $[0, T] \times D$, *that* Φ *is bounded from below by* $-k/\widehat{p}$ *and that (27) holds. If one of the next conditions holds:*

- Φ *is continuous and with polynomial growth on* D,

- Φ *is continuous from* D *to* $[-k/\widehat{p}, +\infty]$ *and for some* $m \geqslant 1$ *and* $\epsilon > 0$

$$\forall (t, \nu, s) \in [T - \epsilon, T] \times D, \quad \frac{1}{\mathrm{vol}(t, \nu, s)} \leqslant C(1 + |\nu|^m + |s|^m),$$

then there exists a viscosity solution u *of the PDE (53) with terminal value* $\Phi(\cdot, \cdot)$. *Moreover* u *is of class* $C^{1,2}([0, T) \times D)$ *and is the minimal viscosity solution (among all viscosity solutions with bounded negative part).*

Remark 5. *The particular dynamics for* $\bar{S}$ *is not important, as soon as the related operator* $\mathcal{L}$ *regularizes the solution of the PDE.*

Remark 6. *Here we use the existence of a solution for the BSDE to deduce the existence of a viscosity solution for the PDE and then we prove the regularity of the solution. The existence of a solution for the PDE could be directly proved (without probabilistic arguments) starting from [17, Theorem V.8.1]. The proof would involve arguments similar to those given above because of the lack of monotonicity (see Assumption b) in [17, Theorem V.8.1] or Condition (4.17) in [11, Chapter 7]). Hence the proof of existence of solutions using PDE arguments will be as involved as the BSDE based approach given above.*

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[5]The Arzela-Ascoli theorem implies that $u_{n,m}$ (up to a subsequence) converges to some function $\widetilde{u}_n \in H^{\beta/2,\beta}([0, T - \epsilon] \times \mathfrak{K})$. Here $\widetilde{u}_n = u_n$ since pointwise convergence has been proved before.

Let us next comment on the smoothness of u on the boundary $\nu = 0$, called the hyperbolic part of the boundary. Recall that our operator $\mathcal{L}$ is defined by (49) and that the dynamics of ν is given by (48). The Feller condition ensuring a positive process ν is $2\alpha\theta > c^2$. Under this condition, the Fichera function

$$b(\nu) = \alpha(\theta - \nu) - \frac{1}{2}c^2$$

is positive when ν goes to zero. Hence no boundary condition has to be supplied on $\nu = 0$ (see for example [6]).

For the control problem, we have to consider the terminal condition

$$\Phi_\ell(T, \nu, \cdot) = \Phi_\ell(\cdot) = \infty \cdot \mathbf{1}_{[\ell,\infty)}(\cdot) - \frac{k}{\hat{p}}\mathbf{1}_{(-\infty,\ell)}(\cdot).$$

Note that we cannot directly apply Proposition 4.1, since Φ_ℓ is not continuous. Nonetheless

Lemma 8. *There exists a minimal viscosity solution u_ℓ, which is of class $C^{1,2}$ on $[0,T) \times D$.*

Proof. Indeed let us define

$$\Phi^n(\cdot) = n \cdot \mathbf{1}_{[\ell,\infty)}(\cdot) - \frac{k}{\hat{p}}\mathbf{1}_{(-\infty,\ell)}(\cdot)$$

and

$$\begin{aligned}
\Phi^{n,m}(\cdot) = {} & n \cdot \mathbf{1}_{[\ell,\infty)}(\cdot) - \frac{k}{\hat{p}}\mathbf{1}_{(-\infty,\ell-1/m]}(\cdot) \\
& + [(n + k/\hat{p})m(\cdot - \ell + 1/m) - k/\hat{p}]\,\mathbf{1}_{(\ell-1/m,\ell)}(\cdot).
\end{aligned}$$

$\Phi^{n,m}$ is continuous and non-decreasing with respect to m and converges to Φ^n. Therefore, the related continuous viscosity solutions $u^{n,m}$ converge to u^n. Arguing as in the proof of Proposition 4.1, we obtain a uniform norm of u^n in the space $H^{1+\beta,2+\beta}([0,T-\epsilon] \times \mathfrak{K})$, for any compact subset $\mathfrak{K}$ of D. Then we pass on the limit on n to obtain the desired result. Minimality can be obtained as for Proposition 4.1. $\qquad\square$

Next we connect the solutions of PDE obtained above to the value function of the control problem.

Extended control problem

The L function corresponding to the permanent impact κ given by (2) is $L_t(\rho) = \eta_t|\rho|^{\hat{p}}$ and the control problem (13) is

$$\inf_{Q \in \mathcal{A}_{I,S}} \mathbb{E}\left[\int_0^T \frac{1}{\mathrm{Vol}_t^{\hat{p}-1}}|Q_t'|^{\hat{p}}dt + \Phi(\nu_T, \bar{S}_T)|Q_T|^{\hat{p}}\right]. \tag{57}$$

Here $I \equiv 1$ on $[0,T]$, with (11) and (24) with the notation $\mathrm{Vol} = \widetilde{\mathrm{Vol}}$.

For $p = 2$, by Proposition 3.3, $u^\Phi(t,\nu,s)q^2$ is the value function of the extended version of the stochastic optimal control problem. With a slight abuse of language, we will refer to u^Φ simply as the value function of the extended stochastic optimal control problem with terminal cost $\Phi|Q_T|^{\hat{p}}$.

These give us the main result of this subsection:

Proposition 4.2. *Suppose* (50) *and the conditions of Proposition 4.1 hold.* u^Φ *is the solution of the PDE* (53) *with terminal condition* (54)*. If* Φ *is bounded and continuous or if* $\Phi = \Phi^{(1)}$*, then the value function of* (57) *is given by* $|q|^{\widehat{p}} u^\Phi$ *and an optimal control is given by:*

$$v(r, \nu, s, q) = -(p-1)\mathrm{vol}(r, \nu, s)|u(r, \nu, s)|^{p-1}\mathrm{sgn}(u(r, \nu, s))q. \tag{58}$$

Proof. Since the proof is quite standard, let us provide an outline of the main arguments. First standard computations show that for any x:

$$\inf_v \left[xv + \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v|^{\widehat{p}} \right] = -(\widehat{p}-1)\mathrm{Vol}_t\left(\frac{|x|}{\widehat{p}}\right)^p$$

and that the infimum is attained at

$$v^* = -\frac{\mathrm{Vol}_t}{(\widehat{p})^{p-1}}|x|^{p-1}\mathrm{sgn}\,(x).$$

If Φ is continuous and bounded, by Proposition 4.1, $u = u^\Phi$ is of class $C^{1,2}([0,T] \times D)$ and bounded and continuous on $[0,T] \times D$. Now define $V(t, \nu, s, q) = |q|^{\widehat{p}}u(t, \nu, s)$. This function has the same regularity as u with respect to (t, ν, s) and is of class C^1 with respect to $q \in \mathbb{R}$. Take $Q = Q^w \in \mathcal{A}_{I,\boldsymbol{S}}$ and $w = (Q^w)'$ is its derivative. Then Itô's formula leads to: for any $t < T$

$$V(0, \nu, s, Q_0) = V(t, \nu_t, \bar{S}_t, Q_t^w) - \int_0^t (\partial_t V + \mathcal{L}V)(r, \nu_r, \bar{S}_r, Q_r^w) + \partial_q V(r, \nu_r, \bar{S}_r, Q_r^w)w_r dr$$

$$- \int_0^t \sqrt{\nu_r}\left[\partial_s V(r, \nu_r, \bar{S}_r, Q_r^w)dW_r^{(1)} + \partial_\nu V(r, \nu_r \bar{S}_r, Q_r^w)cdW_r^{(2)} \right]$$

$$= V(t, \nu_t, \bar{S}_t, Q_t^w) + \int_0^t \left[-\mathrm{vol}_r|u(r, \nu_r \bar{S}_r)|^p|Q_r^w|^{\widehat{p}} - \partial_q V(r, \nu_r, \bar{S}_r, Q_r^w)w_r \right] dr$$

$$- \int_0^t \sqrt{\nu_r}\left[\partial_s V(r, \nu_r, \bar{S}_r, Q_r^w)dW_r^{(1)} + \partial_\nu V(r, \nu_r \bar{S}_r, Q_r^w)cdW_r^{(2)} \right].$$

Note that $\partial_q V(r, \nu, s, q) = u(r, \nu, s)\widehat{p}|q|^{\widehat{p}-1}\mathrm{sgn}\,(q)$. Thus

$$-\mathrm{vol}_r|u(r, \nu_r \bar{S}_r)|^p|Q_r^w|^{\widehat{p}} = -\frac{\mathrm{vol}_r}{(\widehat{p})^p}\left|\partial_q V(r, \nu_r \bar{S}_r, Q_r^w)\right|^p$$

$$= -(\widehat{p}-1)\mathrm{Vol}_r\frac{|\partial_q V(r, \nu_r \bar{S}_r, Q_r^w)|^p}{(\widehat{p})^p}$$

$$= \inf_v \left[\partial_q V(r, \nu_r \bar{S}_r, Q_r^w)v + \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v|^{\widehat{p}} \right]$$

$$\leqslant \partial_q V(r, \nu_r \bar{S}_r, Q_r^w)v + \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v|^{\widehat{p}}.$$

Hence we obtain

$$V(0, \nu, s, Q_0) \leqslant V(t, \nu_t, \bar{S}_t, Q_t^w) + \int_0^t \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|w_r|^{\widehat{p}}dr$$

$$- \int_0^t \sqrt{\nu_r}\left[\partial_s V(r, \nu_r, \bar{S}_r, Q_r^w)dW_r^{(1)} + \partial_\nu V(r, \nu_r \bar{S}_r, Q_r^w)cdW_r^{(2)} \right].$$

Letting t tend to T and taking the expectation, we have

$$V(0, \nu, s, Q_0) \leqslant \mathbb{E}\left[|Q_T^w|^{\widehat{p}}\Phi(\nu_T, \bar{S}_T) + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|w_r|^{\widehat{p}}dr\right].$$

Moreover if we take

$$v_r^* = -(p-1)\mathrm{vol}_r|u(r, \nu, s)|^{p-1}\mathrm{sgn}\ (u(r, \nu, s))Q_r^*,$$

we have an equality. The first part of the proposition is proved.

If $\Phi = \Phi^{(1)}$, then $u = u^{\Phi^{(1)}}$ is not continuous at time T, but $C^{1,2}$ on $[0, T) \times D$. Hence for any $t < T$:

$$V(0, \nu, s, Q_0) = V(t, \nu_t, \bar{S}_t, Q_t^*) + \int_0^t \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v_r^*|^{\widehat{p}}dr$$

$$- \int_0^t \sqrt{\nu_r}\left[\partial_s V(r, \nu_r, \bar{S}_r, Q_r^*)dW_r^{(1)} + \partial_\nu V(r, \nu_r\bar{S}_r, Q_r^*)cdW_r^{(2)}\right].$$

We take the expectation and we use Fatou's lemma:

$$V(0, \nu, s, Q_0) \geqslant \mathbb{E}\left[\liminf_{t \to T} V(t, \nu_t, \bar{S}_t, Q_t^*) + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v_r^*|^{\widehat{p}}dr\right]$$

$$= \mathbb{E}\left[\liminf_{t \to T}\left(u(t, \nu_t, \bar{S}_t)|Q_t^*|^{\widehat{p}}\right) + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v_r^*|^{\widehat{p}}dr\right].$$

A direct computation shows that $N_t = Y_t^{\min}|Q_t^*|^{\widehat{p}-1} = u(t, \nu_t, \bar{S}_t)|Q_t^*|^{\widehat{p}-1}$ is a martingale on $[0, T)$. This martingale being bounded from below, it has a limit at time T. Arguing as at the end of the proof of Proposition 3.3, we deduce that

$$V(0, \nu, s, Q_0) \geqslant \mathbb{E}\left[\Phi(\nu_T, \bar{S}_T)|Q_T^*|^{\widehat{p}} + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|v_r^*|^{\widehat{p}}dr\right].$$

Using the proof of Lemma 8, we obtain a sequence of smooth functions $u^{n,m}$ such that $|q|^{\widehat{p}}u^{n,m}$ is the value function of (57) with $\Phi^{n,m}$ as terminal condition. Hence

$$|Q_0|^{\widehat{p}}u^{n,m}(0, \nu, s) \leqslant \inf_{Q \in \mathcal{A}_{I,S}} \mathbb{E}\left[\Phi(\nu_T, \bar{S}_T)|Q_T|^{\widehat{p}} + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|Q_r'|^{\widehat{p}}dr\right]$$

$$\leqslant V(0, \nu, s, Q_0) = |Q_0|^{\widehat{p}-1}u(0, \nu, s).$$

Passing through the limit on $m \to +\infty$ and then on $n \to +\infty$ achieves the proof of this proposition. $\square$

An optimal state process Q^* given by (58) can be written:

$$(Q_r^*)' = v_r^* = -(p-1)\mathrm{vol}_r|Y_r^{\min}|^{p-1}\mathrm{sgn}\ (Y_r^{\min})Q_r^*.$$

And our proof shows that this control is admissible and

$$\inf_{Q \in \mathcal{A}_{I,S}} \mathbb{E}\left[\xi|Q_T|^{\widehat{p}} + \int_0^T \frac{1}{\mathrm{Vol}_t^{\widehat{p}-1}}|Q_r'|^{\widehat{p}}dr\right] \leqslant |Q_0|^{\widehat{p}}Y_0^{\min}.$$

However, even for bounded ξ, and except in the case $p = \widehat{p} = 2$, the lack of convexity prevents us proving equality. Therefore the Markovian setting and the fact that the solution u of the HJB equation is smooth, are crucial to extend our result to any $\widehat{p} > 1$.

4.2 PDE representation for $I_t = \mathbf{1}_{\{t \leqslant \tau_\ell\}}$ and $\boldsymbol{S} = \{\tau_\ell \geqslant T\}$

To get a PDE representation of the BSDE (25), (22) for $I_t = I_t^{(2)} = \mathbf{1}_{\{t \leqslant \tau_\ell\}}$, we consider the problem in the interval $[\![0, \tau_\ell \wedge T]\!]$. As discussed in subsection 3.2, the corresponding BSDE is again (25) but with terminal condition (40). This reduced formulation of the problem is indeed Markovian. The PDE is the same as before (53) but solved over the domain $D = (0, \infty) \times [\ell, \infty)$ and with boundary conditions

$$u|_{[0,T) \times (0,\infty) \times \{\ell\}} = -k/\widehat{p}, \qquad u|_{\{T\} \times (0,\infty) \times [\ell,\infty)} = \infty. \tag{59}$$

The value function $u^{(2)}$ of the extended version of the stochastic control problem is again defined through (55) and we have:

Proposition 4.3. $u^{(2)}$ *is the minimal viscosity solution of* (53) *and boundary conditions* (59). *Moreover* $u^{(2)}$ *is of class* $C^{1,2}$ *on* $[0, T) \times (0, \infty) \times (\ell, \infty)$.

The proof proceeds parallel to the arguments given in the previous section. We therefore provide an outline. We begin by considering the case where the boundary condition is given by

$$u|_{[0,T) \times (0,\infty) \times \{\ell\}} = \psi, \qquad u|_{\{T\} \times (0,\infty) \times [\ell,\infty)} = \Phi \tag{60}$$

where ψ is a continuous and bounded function. If Φ is bounded and if the compatibility constraint $\Phi(\nu, \ell) = \psi(T, \nu)$ is verified, we can directly apply [19, Theorem 5.41] to obtain the existence of a unique viscosity bounded and continuous solution u of the PDE (53) with the boundary condition (60). The regularity inside the domain can be obtained by the arguments of Lemma 7. If the compatibility condition does not hold, the solution still exists but is not continuous up to the boundary. Finally, the ∞ terminal condition can be handled via approximation from below (as was done in the previous section as well as in Section 3 in the treatment of the BSDE (25) and the singular terminal condition (22)).

Using Proposition 3.5, the value function of the control problem (39) is: $V(t, \nu, s, q) = |q|^2 u(t, \nu, s)$ for any $s \geqslant \ell$. But the regularity of $u^{(2)}$ also allows us to solve the control problem (57) up to $\tau_\ell \wedge T$:

$$|Q_0|^{\widehat{p}} u(0, \nu, s) = \min_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}\left[\int_0^{\tau_\ell \wedge T} \frac{1}{\text{Vol}_t^{\widehat{p}-1}} |Q_t'|^{\widehat{p}} dt + \Phi(\nu_{\tau_\ell \wedge T}, \bar{S}_{\tau_\ell \wedge T}) |Q_{\tau_\ell \wedge T}|^{\widehat{p}} \right].$$

The arguments are the same as in the proof of Proposition (4.2), where T is replaced by $\tau_\ell \wedge T$.

4.3 PDE representation for $I_t = \mathbf{1}_{\{\bar{S}_t \geqslant \ell\}}$ and $\boldsymbol{S} = \{\tau_{T-\delta, \ell} \geqslant T\}$

In Section 3.3, we already explain how to reduce the problem on the interval $[0, T - \delta]$. Let us consider the PDE

$$\partial_t u + \mathcal{L}u - \text{vol}_t |u|^p = 0$$

on the set $[0, T] \times (0, +\infty) \times [\ell, +\infty)$, with terminal condition $+\infty$ at time T and $-k/\widehat{p}$ on the lateral boundary $\{s = \ell\}$. From the previous section, there exists a unique solution u^∞ defined on $[0, T] \times (0, +\infty) \times [\ell, +\infty)$.

Following the representation given by (45), we define

$$r(t, s, v) = |v|^p \text{vol}_t \mathbf{1}_{s > \ell},$$

and solve

$$\partial_t u + \mathcal{L}u - r(t, s, u) = 0 \tag{61}$$

over $[0, T - \delta] \times (0, +\infty) \times \mathbb{R}$ with terminal boundary condition

$$g(\nu, s) = u^{\infty}(T - \delta, \nu, s)\mathbf{1}_{s > \ell} - (k/\widehat{p})\mathbf{1}_{s \leqslant \ell}.$$

Note that the terminal boundary condition g is bounded and continuous. Nonetheless the free term r is not continuous at $s = \ell$.

Lemma 9. *There exists a function v such that v is bounded and continuous on $[0, T - \delta] \times (0, +\infty) \times \mathbb{R}$ and is a solution of class $C^{1,2}$ of the PDE (61) on $[0, T - \delta) \times (0, +\infty) \times (\mathbb{R} \backslash \{\ell\})$.*

Proof. To circumvent the discontinuity of r, let us introduce

$$\phi^{\epsilon}(s) = \left(\mathbf{1}_{s > \ell + \epsilon} + \frac{(s - \ell)}{\epsilon}\mathbf{1}_{\ell < s \leqslant \ell + \epsilon} \right), \quad r^{\epsilon}(t, s, v) = |v|^{p}\mathrm{vol}_t\phi^{\epsilon}(s).$$

This function is Lipschitz continuous with respect to s, satisfies $r^{\epsilon} \leqslant r$ and converges increasingly and pointwise to r when ϵ tends to zero.

From standard arguments (see [19, Theorem 5.37]), there exists a unique bounded and continuous viscosity solution v^{ϵ} of the PDE

$$\partial_t u + \mathcal{L}u - r^{\epsilon}(t, s, u) = 0$$

with the same terminal condition g as v. Note that the bounds on v^{ϵ} do not depend on ϵ. Thus arguing as in Lemma 7, we can prove that v^{ϵ} is of class $C^{1,2}$ on $[0, T - \delta) \times (0, +\infty) \times (\mathbb{R} \backslash \{\ell\})$ with a norm independent of ϵ.

The comparison principle shows that v^{ϵ} is a decreasing sequence and thus we can define $v^{\star}$ as the decreasing limit of v^{ϵ} as ϵ tends to zero. We obtain immediately that $v^{\star}$ is bounded and upper semi-continuous and is a viscosity subsolution of PDE (61) (well-known result on stability for viscosity solutions [7]).

The only remaining point concerns the continuity of $v^{\star}$ on the set $\{s = \ell\}$. Let us define another approximating sequence w^{ϵ} defined as the solution of PDE (61) where r is replaced by $\widetilde{r}^{\epsilon}$:

$$\psi^{\epsilon}(s) = \left(\mathbf{1}_{s > \ell} + \frac{(s - \ell)}{\epsilon}\mathbf{1}_{\ell - \epsilon < s \leqslant \ell} \right), \quad \widetilde{r}^{\epsilon}(t, s, v) = |v|^{p}\mathrm{vol}_t\psi^{\epsilon}(s).$$

As v^{ϵ}, w^{ϵ} converges to $v_{\star}$, which is lower semi-continuous and is a viscosity supersolution of PDE (61). Moreover by comparison principle, $w^{\epsilon} \leqslant v_{\star} \leqslant v^{\star} \leqslant v^{\epsilon}$. Comparing sub- and supersolution implies that $v_{\star} = v^{\star}$ (standard result for viscosity solution). Let us prove this statement in our case. For any (ν, s) we have

$$v^{\epsilon}(t, \nu, s) - w^{\epsilon}(t, \nu, s) = Y_t^{\epsilon, t, \nu, s} - \widetilde{Y}_t^{\epsilon, t, \nu, s}$$

$$= Y_{T-\delta}^{\epsilon, t, \nu, s} - \widetilde{Y}_{T-\delta}^{\epsilon, t, \nu, s} - \int_t^{T-\delta} r^{\epsilon}(u, \bar{S}_u^{t, \nu, s}, Y_u^{\epsilon, t, \nu, s}) - \widetilde{r}^{\epsilon}(u, \bar{S}_u^{t, \nu, s}, \widetilde{Y}_u^{\epsilon, t, \nu, s})du$$

$$- \int_t^{T-\delta} (Z_u^{\epsilon, t, \nu, s} - \widetilde{Z}_u^{\epsilon, t, \nu, s})dW_u$$

$$= - \int_t^{T-\delta} \mathrm{vol}(u, \nu_u^{t, \nu, s}, \bar{S}_u^{t, \nu, s})\phi^{\epsilon}(\bar{S}_u^{t, \nu, s})(Y_u^{\epsilon, t, \nu, s} - \widetilde{Y}_u^{\epsilon, t, \nu, s})h(Y_u^{\epsilon, t, \nu, s}, \widetilde{Y}_u^{\epsilon, t, \nu, s})du$$

$$- \int_t^{T-\delta} \mathrm{vol}(u, \nu_u^{t, \nu, s}, \bar{S}_u^{t, \nu, s})\left|\widetilde{Y}_u^{\epsilon, t, \nu, s}\right|^{p}\left(\phi^{\epsilon}(\bar{S}_u^{t, \nu, s}) - \widetilde{\phi}^{\epsilon}(\bar{S}_u^{t, \nu, s})\right)du$$

$$- \int_t^{T-\delta} (Z_u^{\epsilon, t, \nu, s} - \widetilde{Z}_u^{\epsilon, t, \nu, s})dW_u.$$

Here

$$h(y, \widetilde{y}) = p \int_0^1 |\widetilde{y} + \alpha(y - \widetilde{y})|^{p-1}\mathrm{sgn}\ (\widetilde{y} + \alpha(y - \widetilde{y}))d\alpha.$$

Using the boundedness of $Y^{\epsilon,\cdot,\cdot,\cdot}$ and $\widetilde{Y}^{\epsilon,\cdot,\cdot,\cdot}$ (uniformly with respect to ϵ) and standard stability result for BSDEs, we obtain the existence of a constant C independent of ϵ such that

$$|v^\epsilon(t, \nu, s) - w^\epsilon(t, \nu, s)|^2 \leqslant C\mathbb{E}\left[\left[\int_t^{T-\delta}\left(\phi^\epsilon(\bar{S}_u^{t,\nu,s}) - \widetilde{\phi}^\epsilon(\bar{S}_u^{t,\nu,s})\right)^2 du\right]\right.$$

$$\leqslant 2C\int_t^{T-\delta}\mathbb{P}\left(\ell - \epsilon \leqslant \bar{S}_u^{t,\nu,s} \leqslant \ell + \epsilon\right)du$$

Fix some $\eta > 0$. The uniform ellipticity of $\mathcal{L}$ implies that there exists ϵ_0 such that for any $\epsilon < \epsilon_0$,

$$\mathbb{P}\left(\ell - \epsilon \leqslant \bar{S}_u^{t,\nu,s} \leqslant \ell + \epsilon\right) \leqslant \eta^2/(2C).$$

Hence letting ϵ go to zero, we get for any $\eta > 0$

$$|v^\star(t, \nu, s) - v_\star(t, \nu, s)| \leqslant \eta.$$

Thus $v^\star = v_\star$ and thus v is continuous.

Finally, by a regularization argument, it is a classical solution of the PDE (61) on $[0, T - \delta) \times (0, +\infty) \times (\mathbb{R}\backslash\{\ell\})$.

$\square$

If we define $Y_t = v(t, \nu_t, \bar{S}_t)$, using Itô's formula (allowed since $\bar{S}_t \neq \ell$ a.s.) we check that $Y = Y^{\min}$ is the solution of BSDE (25) with terminal condition (45).

Now since v is a smooth function, $|q|^{\widehat{p}}v(t, \nu, s)$ is the value function of the control problem (57) with $I_t = 1_{\{\bar{S}_t \geqslant \ell\}}$ and $\boldsymbol{S} = \{\tau_{T-\delta,\ell} \geqslant T\}$ and

$$Q_t^* = Q_0 \exp\left(-\int_0^t (p-1)\mathrm{vol}(s, \nu_s, \bar{S}_s)|v(s, \nu_s, \bar{S}_s)|^{p-1}\mathrm{sgn}\ (v(s, \nu_s, \bar{S}_s))ds\right)$$

$$= Q_0 \exp\left(-\int_0^t (p-1)\mathrm{vol}(s, \nu_s, \bar{S}_s)|Y_s^{\min}|^{p-1}\mathrm{sgn}\ (Y_s^{\min})ds\right).$$

The proof is a direct adaptation of the proof of Proposition 4.2 to the current setting.

4.4 PDE representation for $I^{(4)}$ and $\boldsymbol{S}^{(4)}$

Finally, let us derive a PDE representation for the control problem (57) with $I = I^{(4)}$ and $\boldsymbol{S} = \boldsymbol{S}^{(4)}$. From Section 3.3, we know that the problem can be reduced to the time interval $[0, T - \delta]$; as opposed to what happens with $I = I^{(3)}$ and $\boldsymbol{S} = \boldsymbol{S}^{(3)}$, the problem is not Markovian after this reduction. This is because, the choice $I = I^{(4)}$ and $\boldsymbol{S} = \boldsymbol{S}^{(4)}$ introduces an additional state variable, which is $I^{(4)}$ itself; $I^{(4)}$ keeps track of whether the system is in the trading state or in the waiting state (for $I_t = I_t^{(3)} = \boldsymbol{1}_{[\ell,\infty)}(W_t)$ I is directly a function of W and W serves as the state of the system). Correspondingly, we expect a value function of the form $V = u_i(t, \nu, s)|q|^{\widehat{p}}$ where the additional variable i (shown as a subscript) takes values in $\{1, 0\}$ (1 for the trading state and 0 for no trading) and u satisfies the coupled PDE system

$$\partial_t u_0 + \mathcal{L}u_0 = 0 \tag{62}$$

where $\mathcal{L}$ is defined by (49) solved in the region $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b)$ with boundary conditions

$$u_0(T - \delta, \nu, s) = -\frac{k}{\widehat{p}}, \quad u_0(t, \nu, \ell + b) = u_1(t, \nu, \ell + b)$$

and

$$\partial_t u_1 + \mathcal{L} u_1 - \mathrm{vol}_t |u_1|^P = 0, \tag{63}$$

solved in the region $[0, T] \times (0, \infty) \times (\ell, \infty)$ with boundary conditions

$$u_1(T, \nu, s) = \infty,$$
$$u_1(t, \nu, \ell) = u_0(t, \nu, \ell), \ t < T - \delta,$$
$$u_1(t, \nu, \ell) = -\frac{k}{\widehat{p}}, \ t \geqslant T - \delta.$$

Note that the first equation (corresponding to the waiting state) is linear since in this state no control is applied and the PDE is determined only by the underlying diffusion. We think that a solution to this system can be obtained from the minimal supersolution of the corresponding BSDE as we did in earlier sections. In the following subsection we provide an alternative solution based on the number of switches between trading and waiting states. The sequence of control problems where $\mathcal{N}$ is bounded by n can be solved recursively, letting $n \to \infty$ gives a solution to the above system. An advantage of this approach is that it also gives a numerical algorithm to compute the value function. The difficulty in trying to numerically solve the above system directly is that u_1 and u_0 appear as boundary conditions in the equations that the other satisfies.

Finite trading approximation

Define

$$\mathcal{N}_t \doteq (\sup\{k : \bar{\tau}_{b,k} < t\} \vee (-1)) + \mathbf{1}_{\tau_\ell > 0}$$

The number of trading intervals realized up to terminal time t is equal to $\mathcal{N}_t + 1$. Recall that for $I = I^{(4)}$ the set of all trading times before terminal time T is

$$\mathcal{I} = \{t \in [0, T], \ I_t = 1\} = \bigcup_{k=1}^{\mathcal{N}_T + 1} [\![\underline{\rho}_k, \overline{\rho}_k]\!],$$

and $\boldsymbol{S}^{(4)} = \{T \in \mathcal{I}\}$. Define

$$\mathcal{I}_n = \bigcup_{k=1}^{n} [\![\underline{\rho}_k, \overline{\rho}_k]\!], \quad \varpi_n = \sup \mathcal{I}_n, \qquad n = 1, 2, 3, \ldots$$

Note that ϖ_n is one of the stopping times $\tau_{\ell,k}$. The control problem with a limit on the number of active intervals is:

$$V_n(Q_0, \nu, s) = \inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}^n} \mathbb{E}\left[\int_0^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}} |Q_T|^{\widehat{p}}\right],$$

where

$$\mathcal{A}_{I,\boldsymbol{S}}^n = \{Q : \mathbb{F}\text{-adapted, absolutely continuous,}$$
$$Q_T = 0 \text{ if } T \in \mathcal{I}_n, \quad Q_t' = 0, \ \text{if } t \notin \mathcal{I} \text{ or } t > \varpi_n\}.$$

Lemma 10. *The sequence $V_n(Q_0, \nu, s)$ is non-increasing and tends to $V(Q_0, \nu, s)$, the value function of the control problem (57) with $(I, \boldsymbol{S}) = (I^{(4)}, \boldsymbol{S}^{(4)})$, as n tends to ∞.*

Proof. Note that $\mathcal{A}_{I,\boldsymbol{S}}^n$ is included in $\mathcal{A}_{I,\boldsymbol{S}}^{n+1}$. Hence V_n is a non-increasing sequence. Moreover taking $Q \in \mathcal{A}_{I,\boldsymbol{S}}$ and defining $\widetilde{Q}$ equal to Q on the random interval $[\![0, \varpi_n]\!]$ and $\widetilde{Q}'$ equal to zero after ϖ_n, the strategy $\widetilde{Q}$ belongs to $\mathcal{A}_{I,\boldsymbol{S}}^n$ and

$$\int_0^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}} = \int_0^{\varpi_n} \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds + \int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}$$

$$= \int_0^{\varpi_n} \frac{|\widetilde{Q}_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|\widetilde{Q}_T|^{\widehat{p}} + \int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}(|Q_T|^{\widehat{p}} - |\widetilde{Q}_T|^{\widehat{p}}).$$

Taking the expectation, we have

$$\mathbb{E}\left[\int_0^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}\right] = \mathbb{E}\left[\int_0^{\varpi_n} \frac{|\widetilde{Q}_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|\widetilde{Q}_T|^{\widehat{p}}\right]$$

$$+ \mathbb{E}\left[\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}(|Q_T|^{\widehat{p}} - |\widetilde{Q}_T|^{\widehat{p}})\right]$$

$$\geqslant V_n(Q_0, \nu, s) + \mathbb{E}\left[\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}(|Q_T|^{\widehat{p}} - |\widetilde{Q}_T|^{\widehat{p}})\right].$$

Since $\widetilde{Q}_T = Q_{\varpi_n}$, we have

$$\mathbb{E}\left[|Q_T|^{\widehat{p}} - |\widetilde{Q}_T|^{\widehat{p}}\right] = \mathbb{E}\left[\left(\int_{\varpi_n}^T Q_s' ds\right) \int_0^1 \widehat{p}|\widetilde{Q}_T + \alpha(Q_T - \widetilde{Q}_T)|^{\widehat{p}-1}\mathrm{sgn}\,(\widetilde{Q}_T + \alpha(Q_T - \tilde{q}_T))d\alpha\right]$$

$$\leqslant \left[\mathbb{E}\left(\int_0^1 \widehat{p}|\widetilde{Q}_T + \alpha(Q_T - \widetilde{Q}_T)|^{\widehat{p}-1}d\alpha\right)^p\right]^{\frac{1}{p}} \left[\mathbb{E}\left(\int_{\varpi_n}^T Q_s' ds\right)^{\widehat{p}}\right]^{\frac{1}{\widehat{p}}}$$

$$\leqslant CT^{\frac{1}{p}} \left[\mathbb{E}\left(|\widetilde{Q}_T|^{\widehat{p}} + |Q_T|^{\widehat{p}}\right)\right]^{\frac{1}{p}} \left[\mathbb{E}\left(\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{\widehat{p}}}$$

under Assumption 1 on vol. Let us notice that

$$\mathbb{E}\left(|\widetilde{Q}_T|^{\widehat{p}} + |Q_T|^{\widehat{p}}\right) \leqslant 2^{\widehat{p}}\mathbb{E}\left(|Q_0|^{\widehat{p}} + \int_0^T |Q_s'|^{\widehat{p}} ds\right).$$

Thereby we obtain for any strategy $Q \in \mathcal{A}_{I,\boldsymbol{S}}$ and any n:

$$\mathbb{E}\left[\int_0^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}\right] \geqslant V_n(Q_0, \nu, s) + \mathbb{E}\left[\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds\right]$$

$$- kCT^{\frac{1}{\widehat{p}}} \left[\mathbb{E}\left(|Q_0|^{\widehat{p}} + \int_0^T |Q_s'|^{\widehat{p}} ds\right)\right]^{\frac{1}{p}} \left[\mathbb{E}\left(\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{\widehat{p}}}$$

$$\geqslant V(Q_0, \nu, s) + \mathbb{E}\left[\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds\right]$$

$$- kCT^{\frac{1}{\widehat{p}}} \left[\mathbb{E}\left(|Q_0|^{\widehat{p}} + \int_0^T |Q_s'|^{\widehat{p}} ds\right)\right]^{\frac{1}{p}} \left[\mathbb{E}\left(\int_{\varpi_n}^T \frac{|Q_s'|^{\widehat{p}}}{\mathrm{vol}_s} ds\right)\right]^{\frac{1}{\widehat{p}}}.$$

From the monotone convergence theorem we deduce

$$\mathbb{E}\left[\int_0^T \frac{|Q'_s|^{\widehat{p}}}{\mathrm{vol}_s}ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}\right] \geqslant \lim_n V_n(Q_0, \nu, s) \geqslant V(Q_0, \nu, s).$$

Since this holds for any $Q \in \mathcal{A}_{I,\boldsymbol{S}}$, we obtain the desired result. □

Now we consider the dynamical version of the control problem with n trading intervals: for any $t \in [0, T]$:

$$V_{i,n}(t, q, \nu, s) = \inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}^{n,t}} \mathbb{E}\left[\int_t^T \frac{|Q'_s|^{\widehat{p}}}{\mathrm{vol}_s}ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}\right],$$

the value function of the control problem starting at an arbitrary time $t < T$. The additional variable i indicates the starting value of the process I; $I_t = 1$ means that the problem starts from a trading state and $I_t = 0$ means that the problem starts from a waiting state. The definition of $\mathcal{A}^{n,t}$ is the same as $\mathcal{A}^n$: there are at most n trading intervals during the time interval $[t, T]$.

Considering the problem until the first transition from one state to the other (from trading to no trading or vice versa) we can write the above optimal control problems recursively as follows:

$$V_{0,n}(t, q, \nu, s) = \mathbb{E}\left(V_{1,n}(\bar{\tau}_{b,j_t}, q, \nu_{\bar{\tau}_{b,j_t}}, \ell + b)\mathbf{1}_{\{\bar{\tau}_{b,j_t} < T\}} - \frac{k}{\widehat{p}}|q|^{\widehat{p}}\mathbf{1}_{\{\bar{\tau}_{b,j_t} \geqslant T\}}\right) \tag{64}$$

where $j_t = \inf\{j, \ \bar{\tau}_{b,j} > t\}$ and

$$V_{1,n}(t, q, \nu, s) = \inf_{Q \in \mathcal{A}_{I,\boldsymbol{S}}^{n,t}} \mathbb{E}\left(\int_t^{\tau_{\ell,j_t} \wedge T} \frac{|Q'_s|^{\widehat{p}}}{\mathrm{vol}_s}ds - \frac{k}{\widehat{p}}|Q_T|^{\widehat{p}}\mathbf{1}_{\{\tau_{\ell,j_t} \geqslant T\}}\right.$$
$$\left. + V_{0,n-1}(\tau_{\ell,j_t}, Q_{\tau_{\ell,j_t}}, \nu_{\tau_{\ell,j_t}}, \ell)\mathbf{1}_{\{\tau_{\ell,j_t} < T\}}\right) \tag{65}$$

where τ_{ℓ,j_t} is the first time after t when the price $\bar{S}$ goes below ℓ. In (64) the problem starts in a waiting interval; the controller waits until the first time after t when $\bar{S}$ goes above $\ell + b$ ($\bar{\tau}_{b,j_t}$) or T, whichever comes first. If it is $\bar{\tau}_{b,j_t}$, trading starts; if it is T, the controller pays the terminal cost. Note that the recursion (64) involves no control since the liquidation process starts in the waiting state (i.e., no trading, $Q' = 0$) and remains in that state until $\bar{\tau}_{b,j_t}$. In (65) the agent already uses one trading possibility, after τ_{ℓ,j_t}, there are at most $n-1$ trading intervals.

As before the homogeneous cost structure suggests

$$V_{i,n}(t, q, \nu, s) = |q|^{\widehat{p}}u_{i,n}(t, \nu, s),$$

and we define $u_{0,0}$ to be the constant function $-\frac{k}{\widehat{p}}$: there is no trading (i.e., no control and $Q' = 0$) and at terminal time the trader pays the terminal cost. The above recursions imply the following sequence of PDE to compute $u_{0,n}$ and $u_{1,n}$ for $n \geqslant 1$:

$$\partial_t u_{0,n} + \mathcal{L}u_{0,n} = 0, \tag{66}$$

where $\mathcal{L}$ is defined by (49) solved in the region $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b)$ with boundary conditions

$$u_{0,n}(T - \delta, \nu, s) = -\frac{k}{\widehat{p}}, \quad u_{0,n}(t, \nu, \ell + b) = u_{1,n}(t, \nu, \ell + b). \tag{67}$$

And

$$\partial_t u_{1,n} + \mathcal{L} u_{1,n} - \text{vol}_t |u_{1,n}|^p = 0, \tag{68}$$

solved in the region $[0, T] \times (0, \infty) \times (\ell, \infty)$ with boundary conditions

$$
\begin{aligned}
u_{1,n}(T, \nu, s) &= \infty, \\
u_{1,n}(t, \nu, \ell) &= u_{0,n-1}(t, \nu, \ell), \ t < T - \delta, \\
u_{1,n}(t, \nu, \ell) &= -\frac{k}{\widehat{p}}, \ t \geqslant T - \delta.
\end{aligned}
$$

The PDE for $u_{1,n}$ and its boundary conditions on the set $[T - \delta, T] \times (0, +\infty) \times [\ell, \infty)$ do not depend on n. Hence on this set, we have $u_{1,n} = u^\infty$ (see Section 3.2 and the beginning of Section 4.3).

For $n = 1$ we have: $u_{1,1} = u^{(2)}$ where $u^{(2)}$ is the value function in Section 4.2, Proposition 4.3. The arguments of Lemma 7 show that $u_{1,1}$ is of class $C^{1,2}$ on $[0, T - \delta] \times (0, \infty) \times (\ell, \infty)$ and continuous on $[0, T - \delta] \times (0, \infty) \times [\ell, \infty)$. Once $u_{1,1}$ is available, the rest of the value functions can be computed recursively by solving (66) and (68) in the following order:

$$u_{1,1} \to u_{0,1} \to u_{1,2} \to u_{0,2} \cdots$$

Note that $u_{0,1}$ solves the linear PDE (66) with smooth and bounded boundary conditions $-k/\widehat{p}$ and $u_{1,1}(\cdot, \cdot, \ell + b)$. Therefore, $u_{0,1}$ is also of class $C^{1,2}$ on $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b)$ and continuous on $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b]$. In particular $u_{0,1}(T - \delta, \nu, \ell) = -k/\widehat{p}$. Hence the boundary condition for $u_{1,2}$ is continuous. Recursively all functions $u_{1,n}$ (resp. $u_{0,n}$) are well-defined and bounded on $[0, T - \delta] \times (0, \infty) \times [\ell, \infty)$ (resp. $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b]$), continuous on $[0, T - \delta] \times (0, \infty) \times [\ell, \infty)$ (resp. $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b]$) and of class $C^{1,2}$ on $[0, T - \delta] \times (0, \infty) \times (\ell, \infty)$ (resp. $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b)$).

Recall from (24) that

$$\text{vol}_t = (\widehat{p} - 1) I_t \text{Vol}_t.$$

To conclude we give

Lemma 11 (Verification)**.** *For $n \geqslant 1$, the representation $V_{i,n}(q, \nu, s) = |q|^{\widehat{p}} u_{i,n}(0, \nu, s)$, $i \in \{0, 1\}$ holds, and the optimal strategy is given by*

$$Q_t^* = Q_0 \exp\left(-\int_0^t I_s^{(4,n)} \, \text{Vol}_s |Y_s^{\min}|^{p-1} \text{sgn} \, (Y_s^{\min}) ds \right), t \in [0, T],$$

where

$$Y_t^{\min} = u_{I_t, (n - \mathcal{N}_t)^+}(t, \nu_t, \bar{S}_t), \qquad I_t^{(4,n)} = I_t^{(4)} 1_{(-\infty, n]}(\mathcal{N}_t + 1)$$

Note that $(n - \mathcal{N}_t)^+$ is the number of remaining trading intervals at time t and $1_{(-\infty, n]}(\mathcal{N}_t + 1)$ indicates whether n trading intervals have been used by time t.

Proof. The definition of vol implies $\text{vol}(s, \nu_s, \bar{S}_s) = 0$ when $s \notin \mathcal{I}$. Hence the optimal control $(Q^*)'$ is equal to zero when $I = 0$. For $n = 0$, there is no trading, $\varpi_0 = 0$, $Q_t = Q_0$ and

$$V_{0,0}(Q_0, 0, \nu, s) = -(k/\widehat{p})|Q_0|^{\widehat{p}} = |Q_0|^{\widehat{p}} u_{0,0}(0, \nu, s).$$

For $n = 1$, there are two cases. For $\tau_\ell > 0$ ($I_0 = 1$) and $\varpi_1 = \tau_\ell$: the control problem starts from the trading state and the result follows from Section 4.2: $V_{1,1}(q, t, \nu, s) = u_{1,1}(t, \nu, s)|q|^{\widehat{p}}$. Moreover for any $n \geqslant 2$, $V_{1,n}(q, t, \nu, s) \leqslant u_{1,1}(t, \nu, s)|q|^{\widehat{p}}$: indeed after ϖ_1,

there could be some opportunity for trading. The second case is when the trader starts from a waiting state ($I_0 = 0$): the trader has to wait until $\tau_{b,0}$ before trading and $\varpi_1 = \tau_{\ell,1}$. Hence for any strategy,

$$Q_t = Q_0, \quad t \in [\![0, \tau_{b,0}]\!].$$

Then due to the Markovian structure, for $t \in [\![\tau_{b,0}, \varpi_1]\!]$

$$Q_t = Q_0 \exp\left(-\int_{\tau_{b,0}}^{t \wedge \varpi_1} (p-1)\mathrm{vol}(s, \nu_s, \bar{S}_s)|u_{1,1}(s, \nu_s, \bar{S}_s)|^{p-1}\mathrm{sgn}\,(u_{1,1}(s, \nu_s, \bar{S}_s))ds\right),$$

is the optimal state process on the interval $[\![\tau_{b,0}, T]\!]$ and

$$\begin{aligned}
V_{1,1}(Q_0, \tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) &= |Q_0|^{\widehat{p}}u_{1,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) \\
&= |Q_0|^{\widehat{p}}u_{1,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \ell + b) \\
&= |Q_0|^{\widehat{p}}u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \ell + b),
\end{aligned}$$

where the last equality comes from the boundary condition (67) of $u_{0,n}$. The value function at time 0 is thus given by:

$$V_{0,1}(Q_0, 0, \nu, s) = \mathbb{E}V_{1,1}(Q_0, \tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) = |Q_0|^{\widehat{p}}\mathbb{E}u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \ell + b).$$

The PDE that $u_{0,1}$ satisfies and Itô's formula give

$$\mathbb{E}u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \ell + b) = \mathbb{E}u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) = u_{0,1}(0, \nu, s).$$

Thus we get for any $n \geqslant 2$

$$V_{0,1}(Q_0, 0, \nu, s) = |Q_0|^{\widehat{p}}u_{0,1}(0, \nu, s),$$

which achieves the proof for the case $n = 1$.

The rest of the proof proceeds by induction on n. Let us detail the case $n = 2$ when $I_0 = 1$. The trader starts by following the strategy $u_{1,2}$. Itô's formula gives:

$$\begin{aligned}
|Q_{\tau_\ell}|^{\widehat{p}}u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell}) &= |Q_0|^{\widehat{p}}u_{1,2}(0, \nu, s) + \int_0^{\tau_\ell} \widehat{p}Q'_r|Q_r|^{\widehat{p}-1}\mathrm{sgn}\,(Q_r)u_{1,2}(r, \nu_r, \bar{S}_r)dr \\
&\quad + \int_0^{\tau_\ell} |Q_r|^{\widehat{p}}\left(\frac{\partial}{\partial t} + \mathcal{L}\right)(u_{1,2}(r, \nu_r, \bar{S}_r)dr + \int_0^{\tau_\ell} |Q_r|^{\widehat{p}}\nabla u_{1,2}(r, \nu_r, \bar{S}_r)dW_r \\
&= |Q_0|^{\widehat{p}}u_{1,2}(0, \nu, s) + \int_0^{\tau_\ell} |Q_r|^{\widehat{p}}\nabla u_{1,2}(r, \nu_r, \bar{S}_r)dW_r \\
&\quad + \int_0^{\tau_\ell} \left[\widehat{p}Q'_r|Q_r|^{\widehat{p}-1}\mathrm{sgn}\,(Q_r)u_{1,2}(r, \nu_r, \bar{S}_r) + |Q_r|^{\widehat{p}}\mathrm{vol}_r|u_{1,2}(r, \nu_r, \bar{S}_r)|^p\right]dr.
\end{aligned}$$

If $Q'_r = -(p-1)\mathrm{vol}_r Q_r|u_{1,2}(r, \nu_r, \bar{S}_r)|^{p-1}\mathrm{sgn}\,(u_{1,2}(r, \nu_r, \bar{S}_r))$, we obtain that

$$\begin{aligned}
|Q_{\tau_\ell}|^{\widehat{p}}u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell}) &= |Q_0|^{\widehat{p}}u_{1,2}(0, \nu, s) + \int_0^{\tau_\ell} |Q_r|^{\widehat{p}}\nabla u_{1,2}(r, \nu_r, \bar{S}_r)dW_r \\
&\quad - \int_0^{\tau_\ell} \frac{(\widehat{p}-1)^{\widehat{p}-1}}{(\mathrm{vol}_r)^{\widehat{p}-1}}|Q'_r|^{\widehat{p}}dr \\
&= |Q_0|^{\widehat{p}}u_{1,2}(0, \nu, s) - \int_0^{\tau_\ell} \frac{1}{\mathrm{Vol}_r^{\widehat{p}-1}}|Q'_r|^{\widehat{p}}dr \\
&\quad + \int_0^{\tau_\ell} |Q_r|^{\widehat{p}}\sqrt{\nu_r}\left[\partial_s u_{1,2}(r, \nu_r, \bar{S}_r)dW_r^{(1)} + c\partial_\nu u_{1,2}(r, \nu_r, \bar{S}_r)dW_r^{(2)}\right]
\end{aligned}$$

from the definition of vol.

If $\tau_\ell \geqslant T$, the problem stops and since the used strategy has the same dynamics as $u_{1,1}$ with the same terminal condition at time T, from the case $n = 1$, we know that it is the best strategy on $[0, T]$.

If $\tau_\ell < T$, then the boundary condition connecting $u_{1,2}$ and $u_{0,1}$ over $\bar{S} = \ell$, the PDE satisfied by $u_{0,1}$ and Itô's formula give

$$u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell}) = u_{0,1}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell})$$
$$= u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) - \int_{\tau_\ell}^{\tau_{b,0}} \sqrt{\nu_r} \left[\partial_s u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(1)} + c\partial_\nu u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(2)} \right].$$

And since there is no trading between the times τ_ℓ and $\tau_{b,0}$, $Q_t = Q_{\tau_\ell}$ for any $t \in [\![\tau_\ell, \tau_{b,0}]\!]$. The boundary conditions and $Q'_t = 0$ for $t \in [\![\tau_l, \tau_{b,0}]\!]$ imply

$$|Q_{\tau_{b,0}}|^{\hat{p}} u_{1,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) = |Q_{\tau_\ell}|^{\hat{p}} u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}})$$
$$= |Q_{\tau_\ell}|^{\hat{p}} u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell})$$
$$+ \int_{\tau_\ell}^{\tau_{b,0}} |Q_{\tau_\ell}|^{\hat{p}} \sqrt{\nu_r} \left[\partial_s u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(1)} + c\partial_\nu u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(2)} \right].$$

Again the strategy is optimal and there are again two cases. If $\tau_{b,0} \geqslant T$, the trading is finished and the agent has traded only one time. If $\tau_{b,0} < T$, then the agent starts again to trade until $\tau_{\ell,1} \wedge T = \varpi_2$. We use the step $n = 1$, starting at time $\tau_{b,0}$ from the value $Q_{\tau_{b,0}}$. The best strategy is to follow $u_{1,1}$ and the value function of the trader is given by:

$$V_{1,1}(Q_{\tau_{b,0}}, \tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) = |Q_{\tau_{b,0}}|^{\hat{p}} u_{1,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}).$$

Gathering all steps together leads to

$$|Q_0|^{\hat{p}} u_{1,2}(0, \nu, s) = |Q_{\tau_\ell}|^{\hat{p}} u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell}) + \int_0^{\tau_\ell} \frac{1}{\text{Vol}_r^{\hat{p}-1}} |Q'_r|^{\hat{p}} dr$$
$$+ \int_0^{\tau_\ell} |Q_r|^{\hat{p}} \sqrt{\nu_r} \left[\partial_s u_{1,2}(r, \nu_r, \bar{S}_r) dW_r^{(1)} + c\partial_\nu u_{1,2}(r, \nu_r, \bar{S}_r) dW_r^{(2)} \right]$$
$$= \left[|Q_{\tau_\ell}|^{\hat{p}} u_{1,2}(\tau_\ell, \nu_{\tau_\ell}, \bar{S}_{\tau_\ell}) + \int_0^{\tau_\ell} \frac{1}{\text{Vol}_r^{\hat{p}-1}} |Q'_r|^{\hat{p}} dr \right] \mathbf{1}_{\tau_\ell \geqslant T}$$
$$+ \left[\int_0^{\tau_\ell} \frac{1}{\text{Vol}_r^{\hat{p}-1}} |Q'_r|^{\hat{p}} dr \right] \mathbf{1}_{\tau_\ell < T} + |Q_{\tau_{b,0}}|^{\hat{p}} u_{0,1}(\tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) \mathbf{1}_{\tau_\ell < T, \tau_{b,0} \geqslant T}$$
$$+ V_{1,1}(Q_{\tau_{b,0}}, \tau_{b,0}, \nu_{\tau_{b,0}}, \bar{S}_{\tau_{b,0}}) \mathbf{1}_{\tau_\ell \leqslant \tau_{b,0} < T}$$
$$+ \int_0^{\tau_\ell \wedge T} |Q_r|^{\hat{p}} \sqrt{\nu_r} \left[\partial_s u_{1,2}(r, \nu_r, \bar{S}_r) dW_r^{(1)} + c\partial_\nu u_{1,2}(r, \nu_r, \bar{S}_r) dW_r^{(2)} \right]$$
$$- \int_{\tau_\ell \wedge T}^{\tau_{b,0} \wedge T} |Q_{\tau_\ell}|^{\hat{p}} \sqrt{\nu_r} \left[\partial_s u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(1)} + c\partial_\nu u_{0,1}(r, \nu_r, \bar{S}_r) dW_r^{(2)} \right].$$

Since we cover all possible scenarios and since the strategies are optimal on each (random) time intervals, taking the expectation, we conclude that

$$V_{1,2}(0, Q_0, \nu, s) = |Q_0|^{\hat{p}} u_{1,2}(0, \nu, s),$$

and that the optimal strategy is $u_{1,2} \to u_{0,1} \to u_{1,1}$, which achieves the proof of this particular case.

The proof for the other scenarios uses very similar arguments. $\qquad\square$

Remark 7. *Our sequence of solutions $u_{1,n}$ and $u_{0,n}$ satisfies for any $b > 0$ and n*

$$\forall s < \ell, \ u_{0,n}(T - \delta, \nu, s) = -k/\widehat{p}, \qquad \forall t > T - \delta, \ u_{1,n}(t, \nu, \ell) = -k/\widehat{p}.$$

The boundary condition imposed on v in (61) is coherent with these previous properties. Moreover $u_{0,n}$ (resp. $u_{1,n}$) solves the PDE (66) on $[0, T - \delta) \times (0, +\infty) \times (-\infty, \ell)$ (resp. (68) on $[0, T - \delta) \times (0, +\infty) \times (\ell, +\infty)$). In other words they solve (61) on their own domain of definition.

A natural but non-trivial question is: do $u_{1,n}$ and $u_{0,n}$ provide some approximation of v given by Lemma 9, when b tends to zero and n to ∞ ? This question is left for further research.

When the numbers of active intervals tends to ∞

With Lemma 10 we saw that the value function of the control problem with a finite number of trading intervals converge to the value function of the same control problem with no bounds on the number of trading intervals. We now argue that 1) a similar result holds for the u_n 2) the limit determines the value function of the control problem with unbounded number of trading intervals and 3) the limit is a solution of the coupled PDE system (62) and (63). Since the arguments follow the same lines as those above we only provide an outline.

The argument of the previous verification lemma gives

$$V_{i,n}(t, q, \nu, s) = |q|^{\widehat{p}} u_{i,n}(t, \nu, s).$$

Together with Lemma 10, this shows that $u_{i,n}$ is a non-increasing sequence. Hence it converges pointwise to some function u_i defined on $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b]$ for $i = 0$ and $[0, T - \delta] \times (0, \infty) \times [\ell, \infty)$ for $i = 1$. Moreover we have the relation:

$$V_i(t, q, \nu, s) = |q|^{\widehat{p}} u_i(t, \nu, s), i \in \{0, 1\}.$$

Since u_i is bounded from below and bounded from above by $u_{i,1}$ and is upper semi-continuous, u_0 (resp. u_1) is a viscosity subsolution of (66) (resp. (68)) with terminal condition $u_0(\cdot, \cdot, T - \delta) = -k/\widehat{p}$ (resp. $u_1(\cdot, \cdot, T - \delta) = u^\infty(\cdot, \cdot, T - \delta)$). Since the terminal conditions are bounded and continuous, the arguments of Lemma 7 imply that u_0 and u_1 are smooth solution of the PDE (66) and (68) on the set $[0, T - \delta] \times (0, \infty) \times (-\infty, \ell + b)$ and $[0, T - \delta] \times (0, \infty) \times (\ell, \infty)$.

Let us consider the lateral boundary conditions. For $s = \ell + b$, u_0 is equal to $\psi = u_1(\cdot, \cdot, \ell + b)$, which is a smooth function. From [17], the PDE (66) with terminal condition $-k/\widehat{p}$ at time $T - \delta$ and lateral condition ψ has a unique smooth solution. A similar argument shows that u_1 is the unique smooth solution of (68) with terminal condition u^∞ at time $T - \delta$ and lateral condition $\phi = u_0(\cdot, \cdot, \ell)$. These results give us the required regularity to proceed as in Lemma 11 and to prove that for $t \in [0, T]$

$$Q_t = Q_0 \exp\left(-\int_0^t (p-1)\mathrm{vol}(s, \nu_s, \bar{S}_s)|u_{I_t}(s, \nu_s, \bar{S}_s)|^{p-1}\mathrm{sgn}\,(u_{I_t}(s, \nu_s, \bar{S}_s))ds\right)$$

is an optimal control for the control problem (57) with $I = I^{(4)}$ and $\boldsymbol{S} = \boldsymbol{S}^{(4)}$.

5 A partial analysis of the output of the algorithm and numerical examples

Recall A of (1), which is the percentage deviation from the target price S_0 of the average price at which the position is (partially) closed in the time interval $[0, T]$. The actual output of the trading algorithm defined by the optimal control Q^* of (37) or (58), is the random pair (Q_T^*, A), where A is computed for $Q = Q^*$. An important question is the distribution of this pair and the dependence of this distribution on model parameters. In this section we would like to give a partial analysis of this question including some numerical examples. Compared to the original IS order, the modified IS order considered in the present work has two additional parameters: the process I that determines when trading takes place and the event $\boldsymbol{S}$ that determines when full liquidation takes place. In the numerical examples we will limit ourselves to $I = I^{(1)} = 1$, $\boldsymbol{S} = \boldsymbol{S}^{(1)} = \{\bar{S}_T \geqslant \ell\}$. To further simplify the presentation and the calculations we take $p = 2$, $\bar{S} = \sigma W$, where W is a standard Brownian motion, $\sigma > 0$ a constant and $\mathrm{Vol}_t = V > 0$; these are also the choices made for these parameters in the standard Almgren-Chriss framework [14, Chapter 3]. Under these assumptions we will compute Q^* by discretizing and numerically solving the corresponding PDE, which becomes:

$$u_t + \frac{1}{2}\sigma^2 u_{xx} - \frac{V}{\eta} I_t u^2 = 0, \tag{69}$$

where the domain of the equation and its boundary conditions depend on $\boldsymbol{S}$ and I.

To better understand how Q^* and $(Q_T^*/q_0, A)$ change with the model parameters, we factor out as many parameters as possible from the calculations. If we let

$$v(t, x) = \frac{V}{\eta} u(t, \sigma x) \tag{70}$$

the equation (69) reduces to

$$v_t + \frac{1}{2}v_{xx} - I_t v^2 = 0. \tag{71}$$

To see how A depends on model parameters let us reduce the expression (1) as much as possible (remember that $Q_0 = q_0$ is the initial position size):

$$A = \frac{X_T - (Q_0 - Q_T)S_0}{(Q_0 - Q_T)S_0}$$

By (4) (the expression for X_T) and the assumption $L(v) = \eta v^2$:

$$= \frac{-\int_0^T S_t Q_t' dt - \frac{\eta}{V}\int_0^T (Q_t')^2 dt - S_0(Q_0 - Q_T)}{S_0(Q_0 - Q_T)}$$

By the definition (3) of S_t and the assumptions $\kappa(v) = kv$, $\bar{S}_t = \sigma W_t$:

$$= \frac{-k/2(Q_T - Q_0)^2 - \sigma \int_0^T W_t Q_t' dt - \frac{\eta}{V}\int_0^T (Q_t')^2 dt}{(Q_0 - Q_T)S_0}.$$

Simplifying the last expression we get

$$A = -\frac{kQ_0}{2S_0}\left(1 - \frac{Q_T}{Q_0}\right) - \frac{\sigma}{S_0}\frac{1}{(Q_0 - Q_T)}\int_0^T W_t Q_t' dt - \frac{\eta}{VS_0}\frac{1}{(Q_0 - Q_T)}\int_0^T (Q_t')^2 dt. \tag{72}$$

From this expression we see that A consists of three components: 1) one due to the permanent price impact 2) one due to random fluctuations in price and 3) one due to transaction costs. All components consist of a coefficient term and a term depending on Q or its derivative Q':

$$\text{Permanent market impact term: } A_1 = 1 - \frac{Q_T}{Q_0}, \qquad \text{coefficient: } -\frac{kQ_0}{2S_0},$$

$$\text{Random fluctuations term: } A_2 = \frac{1}{(Q_0 - Q_T)} \int_0^T W_t Q'_t dt, \text{ coefficient: } \frac{\sigma}{S_0},$$

$$\text{Transaction costs term: } A_3 = \frac{1}{1 - \mathfrak{q}_T} \int_0^T (Q'_t/Q_0)^2 dt, \qquad \text{coefficient: } \frac{\eta Q_0}{V S_0},$$

where

$$\mathfrak{q}_t = Q_t/Q_0.$$

The permanent impact term $1 - Q_T/Q_0 = 1 - \mathfrak{q}_T$ is the portion of the initial position that is closed; A depends linearly on this portion with coefficient $\frac{kQ_0}{2S_0}$. Secondly note that if S_0, k and η are parameterized as multiples of σ then none of the coefficients appearing in A depend on σ. We will comment on the behavior of the other two terms below.

Before we move on let us note the following for comparison. The case $I = 1$ and $\boldsymbol{S} = \Omega$ corresponds to the standard Almgren Chriss liquidation algorithm for which the optimal control is known to be

$$Q_t^{*,S} = q_0 \frac{T - t}{T}, \tag{73}$$

i.e., closing the position with uniform speed over the time interval $[0, T]$. Then $(Q')_t^{*,S}/q_0 = -1/T$ and $\mathfrak{q}_T = 0$. These reduce A_3 to

$$A_3^S = \frac{1}{T}$$

for the standard IS algorithm. Similarly, for A_2 we have

$$A_2^S = -\frac{1}{T} \int_0^T W_t dt, \tag{74}$$

which is normally distributed with 0 mean by the iid normal increments of W.

We continue our analysis with the choices $I = I^{(1)} = 1$ and $\boldsymbol{S} = \boldsymbol{S}^{(1)} = \{\bar{S}_T \geqslant \ell\}$ for I and $\boldsymbol{S}$ given in (14); these choices correspond to: no restriction on trading and closing the position fully is required only when the terminal price $\bar{S}_T$ is above a given threshold ℓ. Parallel to the change of variable in (70) we assume ℓ is given as a multiple of $\sigma > 0$; with this convention and the assumption $\bar{S}_t = \sigma W_t$, $\boldsymbol{S}$ becomes $\boldsymbol{S} = \{W_T \geqslant \ell\}$. For $I_t = 1$, the PDE (71) is

$$v_t + \frac{1}{2} v_{xx} - v^2 = 0; \tag{75}$$

for $\boldsymbol{S} = \{W_T \geqslant \ell\}$ the domain and the boundary conditions for this PDE are: $(t, x) \in [0, T] \times \mathbb{R}$ and

$$v(T, x) = \infty \cdot \mathbf{1}_{[\ell, \infty)}(x) - \frac{kV}{2\eta} \cdot \mathbf{1}_{(-\infty, \ell)}(x), \tag{76}$$

$x \in \mathbb{R}$, where we again use the scaling (70).

Recall our convention that k and η are specified as multiples of σ; it follows that PDE (75) and its boundary condition (76) are independent of σ. The optimal control Q^* is computed from v via the formula (58)

$$Q_t^* = q_0 \mathfrak{q}_t = q_0 \exp\left(-\int_0^T v(t, W_t)dt\right) \tag{77}$$

we note that Q^* is independent of σ. We have already noted that the coefficients in (70) are independent of σ. We have observed above that the same is true also for Q^*, therefore all of A_1, A_2 and A_3 are independent of σ as well. This gives us the following result:

Lemma 12. *Suppose all of S_0, k, η and ℓ are parameterized as multiples of σ. Then Q^* and A do not depend on σ.*

Note that the same analysis in fact holds for all of $I = I^{(i)}$, $\boldsymbol{S} = \boldsymbol{S}^{(i)}$, $i = 2, 3, 4$ treated in the previous sections.

As already noted Q^* is computed via the solution of the PDE (75) which obviously doesn't have an explicit solution. To see how Q^* behaves we will solve (75) numerically; for the parameter values we begin by considering those used in [14, Chapter 3]: $T = 1$, $\eta = 0.1$, $V = 4 \times 10^6$, $S_0 = 45$, $\sigma = 0.6$. Recall that the permanent price impact parameter k doesn't appear in the control problem corresponding to the original IS order, so no value for k is specified in [14, Chapter 3]. A k value of $k = 2 \times 10^{-7}$ accompanying these parameter values is given in [14, Chapter 8] in the context of block trade pricing. The assumption (1) in the present case reduces to

$$kV/2\eta < 1;$$

for the above parameter values we have $kV/2\eta = 4$, therefore the above parameter values do not satisfy Assumption 1. To continue with our numerical example, we take $\eta = 0.3$, $V = 4 \times 10^6$ and $k = 10^{-7}$ for these values we have $kV/2\eta = 2/3$ which satisfies (1). In addition to these, we need to provide a value for the ℓ parameter, which we choose as $\ell = 1.4 \times \sigma$. The graph of u for the parameter values above are shown in Figures 1 and 2.
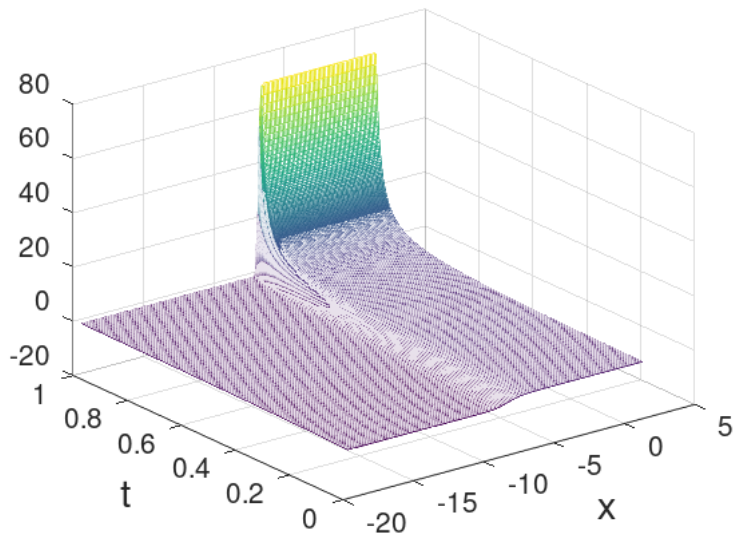

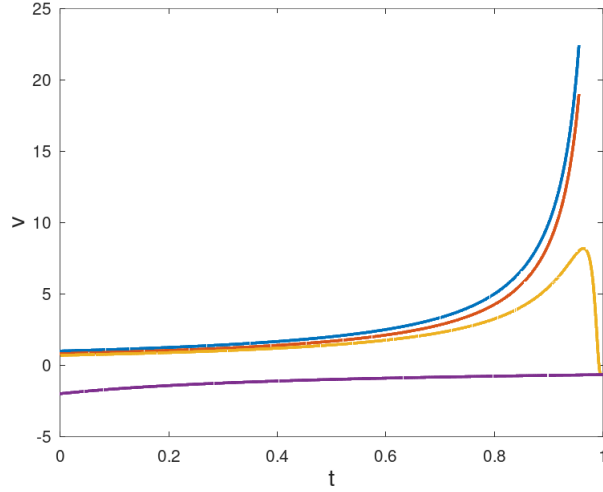Figure 1: Graph of u for $I = 1$ and $\boldsymbol{S} = \{W_T \geqslant \ell\}$

Figure 2: Graph of $u(x, \cdot)$ for $x \in \{0, \ell, 1.2\ell, 5\ell\}$

We note that for $x > \ell$ and x away from ℓ, $u(x, \cdot)$ behaves like $t \mapsto y_t = 1/(T - t)$ (the solution of (69) with terminal condition $y_T = \infty$). and for $x < \ell$ and x away from ℓ, $u(x, \cdot)$ behaves like $t \mapsto z_t$ (given by (29)) The negative boundary condition for u means that $u(x, t)$ takes negative values for $x < \ell$; (4) implies that whenever u is negative, the corresponding Q^* is actually buying the underlying stock.
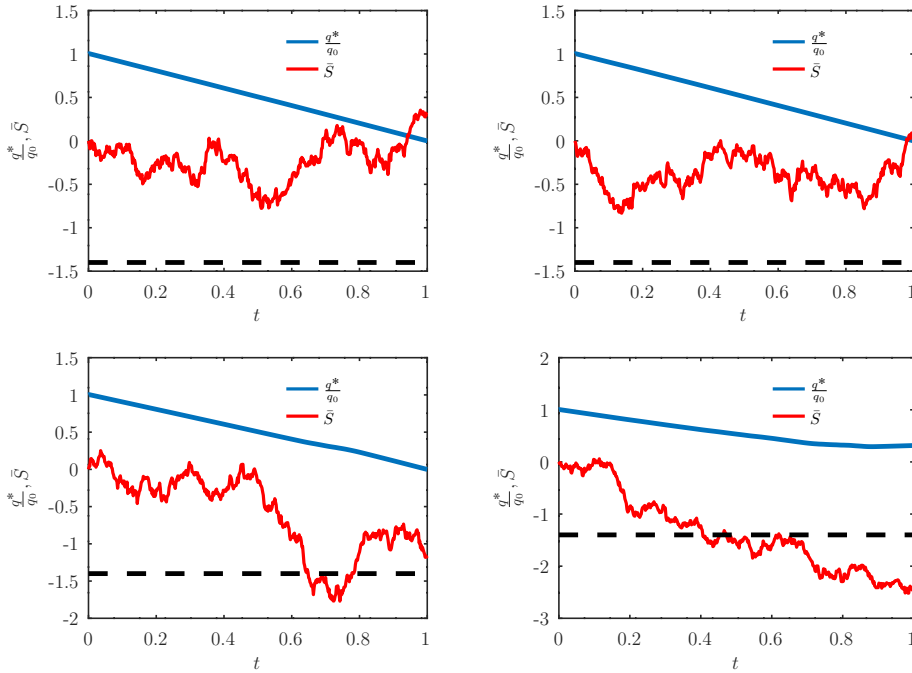
Figure 3: Sample paths of $\bar{S}$ and Q^* for $I = 1$ and $\boldsymbol{S} = \{\bar{S}_T \geqslant \ell\}$; the dashed line shows $\ell = -1.4\sigma$

Figure 3 shows four sample paths of $\bar{S}$ and Q^*. In the first two examples $\bar{S}$ stays above ℓ

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at all times and the corresponding Q^* goes parallel to $Q^{*,S}$ of (73), the optimal liquidation path for the standard IS order. In the third example $\bar{S}$ is below ℓ approximately in the time interval $[0.6, 0.8]$ when trading slows down, it goes above ℓ around 0.8 and closes above ℓ; correspondingly Q^* speeds up trading after 0.8 and closes the position at terminal time. In the fourth example, $\bar{S}$ hits ℓ around the middle of the trading interval and remains below ℓ till the end; correspondingly Q^* slows down and stops trading and the position is only partially closed at terminal time. In the last example Q^* is in fact slightly increasing near $t = T = 1$ (i.e., Q^* buying the underlying asset) ; this is due to the negative value that the terminal value takes for $x < \ell$. These examples suggest that Q^* behaves approximately as follows: when $\bar{S}$ is above ℓ, it behaves like the standard IS algorithm $Q^{*,S}$, linearly closing the remaining position; when $\bar{S}$ goes below ℓ, Q^* slows down/ stops trading. The negative boundary condition implies that the algorithm can in fact execute buy trades especially when the price is below ℓ near terminal time T.

Distribution of $(Q_T^*/q_0, A)$

For $\boldsymbol{S} = \{\bar{S}_T \geqslant \ell\}$, the position fully closes when the closing price is above the lower-bound l, therefore, the probability that the algorithm closes the position at terminal time is:

$$\mathbb{P}(Q_T^* = 0) = \mathbb{P}(W_T \geqslant \ell) = 1 - N_{0,1}(\ell/\sqrt{T})$$

where $N_{0,1}$ denotes the standard normal distribution.

A random variable E is said to be exponentially distributed with rate λ if $\mathbb{P}(E > x) = e^{-\lambda x}$, i.e.,

$$-\log(\mathbb{P}(E > x)) = \lambda x; \tag{78}$$

a well known fact is

$$\mathbb{E}[E] = \frac{1}{\lambda}. \tag{79}$$

The distribution of $\mathfrak{q}_T = Q_T^*/q_0$ over $(0, \infty)$ depends on u via (77) (or (58)) and it obviously doesn't have an explicit formula. Figure 4 shows graphs of $x \mapsto \mathbb{P}(\mathfrak{q}_T > x | \mathfrak{q}_T > 0)$, $x \mapsto -\log(\mathbb{P}(\mathfrak{q}_T > x | \mathfrak{q}_T > 0))$ and $x \mapsto \frac{x q_0}{\mathbb{E}[Q_T^* | Q_T^* > 0]}$ (all estimated via simulating 10^4 sample paths).
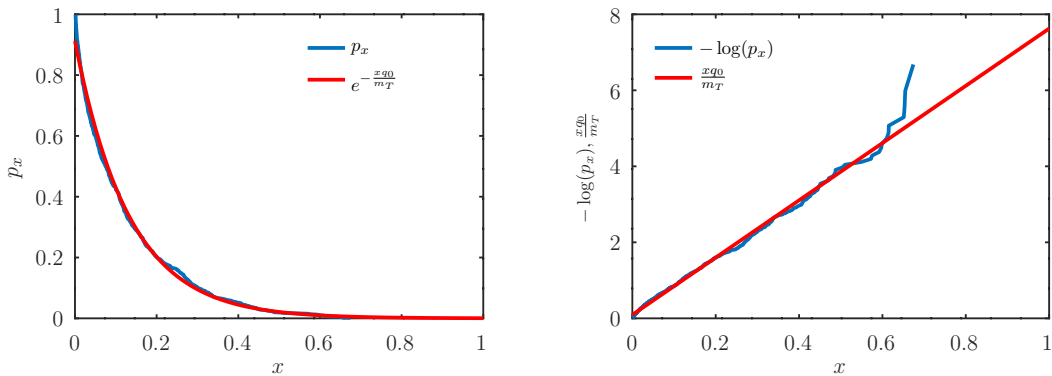

Figure 4: On the left: graph of $p_x = \mathbb{P}(Q_T^*/q_0 > x | Q_T^* > 0)$, on the right: graphs of $-\log(p_x)$ and $\frac{x q_0}{m_T}$, $m_T = \mathbb{E}[Q_T^* | Q_T^* > 0]$

These graphs, (78) and (79) suggest that the exponential distribution provides a rough approximation for the conditional distribution of Q_T^*/q_0 given the event $\{Q_T^* > 0\}$. An exponentially distributed random variable satisfies $\mathbb{E}[E] = \frac{1}{\lambda}$ and $\text{var}(E) = \frac{1}{\lambda^2}$. In the case of Q_T^*/q_0 conditioned over $\{Q_T^* > 0\}$ we have the Monte Carlo estimates $\mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0] = 0.1218$ and $\text{var}(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2} = 0.1387$ for the parameter values specified above.

Q^* depends on ℓ via the domain of the PDE (75) and on $kV/2\eta$ via the terminal condition (76). Figure 5 shows how $\mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0]$ and $\text{var}(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2}$ vary with these parameters.
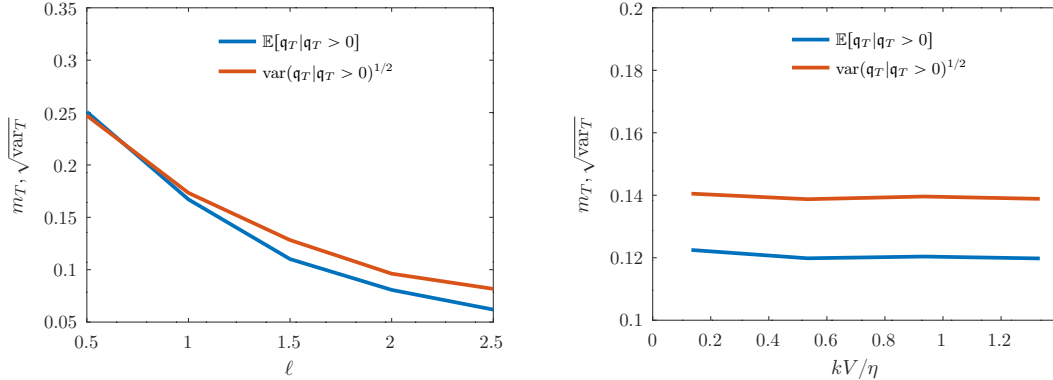
Figure 5: Graph of $m_T = \mathbb{E}[\mathfrak{q}_T | \mathfrak{q}_T > 0]$ and $\sqrt{\text{var}_T} = \text{var}(\mathfrak{q}_T | \mathfrak{q}_T > 0)^{1/2}$ as a function of ℓ

We have already noted that the permanent impact factor term A_1 of (72) is fully determined by $\mathfrak{q}_T$. We now consider the joint distribution of $(A_2, \mathfrak{q}_T)$. This distribution consists of two parts: the distribution of A_2 conditioned on $\mathfrak{q}_T = 0$ (i.e., the cases where the algorithm closes the initial position q_0 fully) and the conditional distribution of A_2 given $\mathfrak{q}_T$ for $\mathfrak{q}_T > 0$ (the cases where the algorithm closes the initial position q_0 partially). If Q^* were a deterministic function (as in the case of the standard IS order), A_2 would be normally distributed by the normal and independent increments of W (see (74)). The q-q plot of the conditional distribution of A_2 given $Q_T^* = 0$ and $Q_T^* = x$ for several values of x is shown Figure 6; (for $x > 0$ we approximate $P(A_2 \in A | Q_T^*/q_0 = x)$ with $P(A_2 \in A | Q_T^*/q_0 \in (x, x + \delta))$ where $\delta > 0$ is small and we estimate the latter by simulating 2×10^5 sample paths of W and Q^*). These plots suggest that the conditional distribution A_2 given Q_T^*/q_0 is approximately normal even though Q^* is random and a function of W.
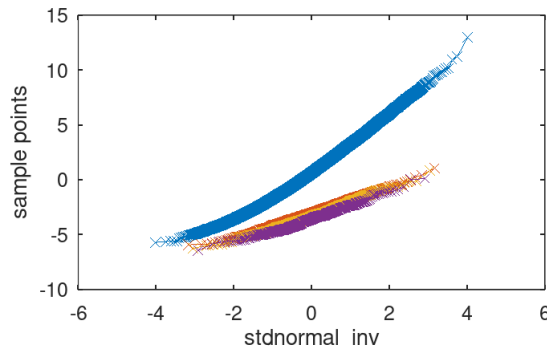

Figure 6: q-q plots of the conditional distribution of A_2 given $\mathfrak{q}_T = x$ for $x = 0$, $x = 0.06 + j0.07$, $j \in \{1, 2, 3, 4\}$

Figure 7 shows the graphs of $\mathbb{E}[A_2|\mathfrak{q}_T = x]$ and $\sqrt{\mathrm{var}(A_2|\mathfrak{q}_T = x)}$ (using the same approximation as above and then simulation).
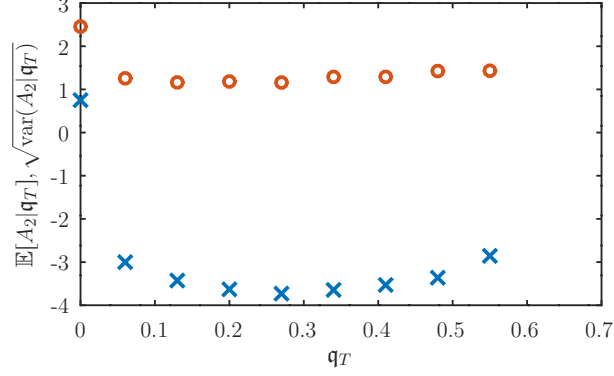

Figure 7: Graphs of $\mathbb{E}[A_2|\mathfrak{q}_T = x]$ and $\sqrt{\mathrm{var}(A_2|\mathfrak{q}_T = x)}$ for $x = 0$ and $x = 0.06 + j0.07$, $j \in \{1, \cdots, 8\}$

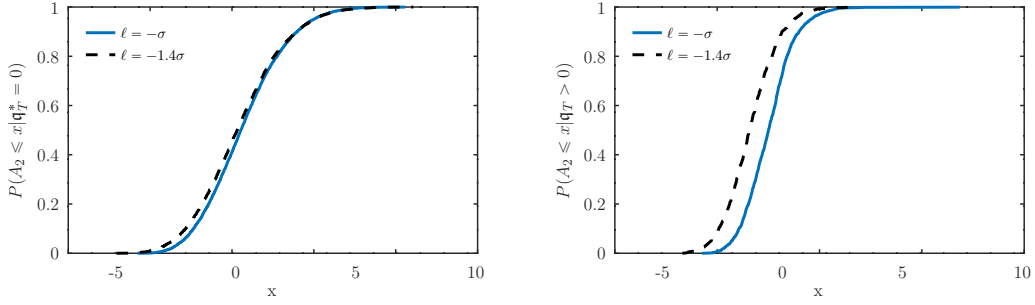Figures 8 and 9 show how the distribution of A_2 change with ℓ and kV/η.

Figure 8: Graphs of $P(A_2 \leqslant x)|\mathfrak{q}_T = 0)$ and $P(A_2 \leqslant x)|\mathfrak{q}_T > 0)$ for $\ell = -\sigma$ and $\ell = -1.4\sigma$
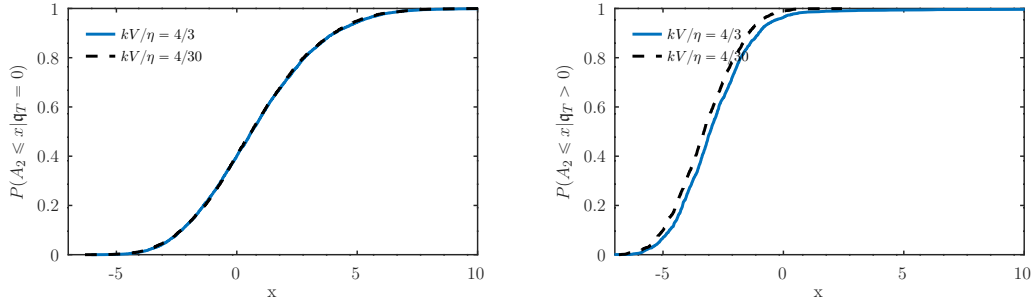

Figure 9: Graphs of $P(A_2 \leqslant x)|\mathfrak{q}_T = 0)$ and $P(A_2 \leqslant x)|\mathfrak{q}_T > 0)$ for $kV/\eta = 4/3$ and $kV/\eta = 4/30$

Lastly, we consider the distribution of A_3. Like A_2 this distribution consists of two parts: over the event $\mathfrak{q}_T = 0$ and over the event $\mathfrak{q}_T > 0$.
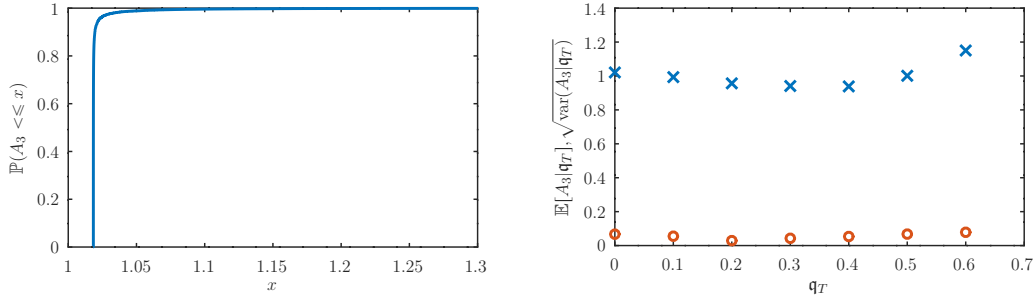

Figure 10: On the left: distribution of A_3 conditioned on $\mathfrak{q}_T = 0$, on the right: graphs of $\mathbb{E}[A_3|\mathfrak{q}_T = x]$ ($\times$) and $\mathrm{var}(A_3|\mathfrak{q}_T = x)^{1/2}$ (o)

The first part of Figure 10 shows the distribution of A_2 over $\mathfrak{q}_t = 0$; this graphs suggests that A_3 behaves approximately like $A_3^S = 1$ of the standard IS algorithm: most of the mass is concentrated around a constant near 1. The second part of Figure 10 shows how the conditional mean and variance of A_3 change with $\mathfrak{q}_T$ for $\mathfrak{q}_T > 0$. This graph suggests that the near constant behavior of A_3 persists for $\mathfrak{q}_T > 0$. Finally Figures 11 and 12 show how the distribution of A_3 changes with model parameters ℓ and kV/η.
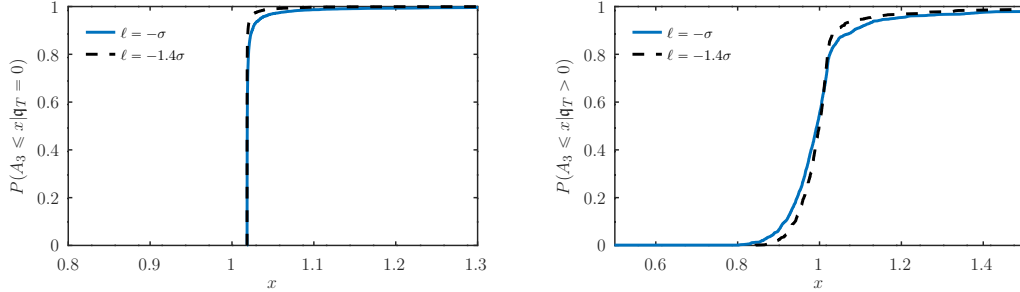

Figure 11: Graphs of $P(A_3 \leqslant x)|\mathfrak{q}_T = 0)$ and $P(A_3 \leqslant x)|\mathfrak{q}_T > 0)$ for $\ell = -\sigma$ and $\ell = -1.4\sigma$
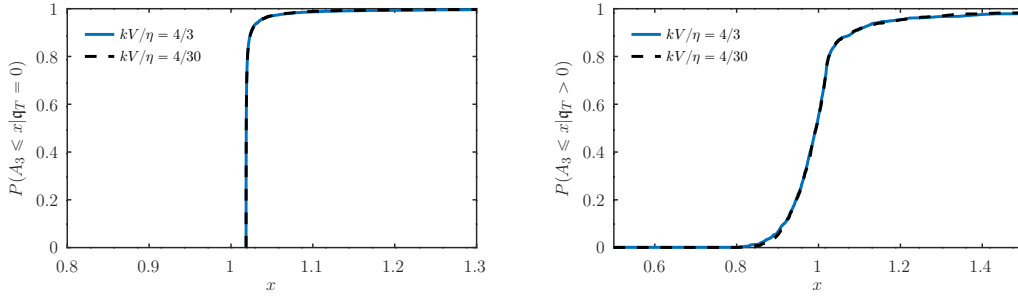
Figure 12: Graphs of $P(A_3 \leqslant x)|\mathfrak{q}_T = 0)$ and $P(A_3 \leqslant x)|\mathfrak{q}_T > 0)$ for $kV/\eta = 4/3$ and $kV/\eta = 4/30$

6 Comments on the continuity problem

Remember that the only condition on the terminal behavior of the minimal supersolution $Y^{\min}$ is (18):

$$\liminf_{t \to T} Y_t^{\min} \geqslant \xi.$$

For the class of stochastic optimal control problems studied in this paper, minimal super-solutions suffice to characterize the value function; this is the reason for the focus of this paper on minimal supersolutions. An interesting further question is whether $Y^{\min}$ satisfies the terminal condition with equality, i.e., whether

$$\lim_{t \to T} Y_t^{\min} = \xi \tag{80}$$

holds. We call this question "the continuity problem." Recall the constraint $\{Q_T = 0\}$ over the set $\boldsymbol{S}$ for the stochastic optimal control problem. Let Q^* be the optimal control defined by $Y^{\min}$. The relation (80) corresponds to $\{Q_T^* = 0\} \supset \boldsymbol{S}$, i.e., the constraint is satisfied over $\boldsymbol{S}$, but it may happen that it is satisfied beyond the event $\boldsymbol{S}$. The relation (80) on the other hand implies that the optimal control satisfies the constraint tightly: $\{Q_T^* = 0\} = \boldsymbol{S}$. This further information about the optimal control is the main significance of the continuity problem from the control perspective.

In previous works [21, 26, 1, 23] we studied the continuity problem for a number of BSDE with singular terminal conditions. The results in these works do not apply in the current setting because they all assume monotone generators. We think that the continuity problem can be resolved positively for the Markovian cases covered in Section 4 using the ideas and methods presented in these prior works. The continuity problem for the general BSDE studied in Section 3 is more challenging: results in all of the previously cited works concern $\boldsymbol{S}$ of special forms such as $\boldsymbol{S} = \{\tau \leqslant T\}$ where τ is a stopping time. We think that results and methods in [1, 23] can be extended to cover the BSDE in Section 3 for $\boldsymbol{S}$ of the form studied in these works. To the best of our knowledge, the continuity problem with terminal conditions of the form $\lim_{t \to T} Y_t^{\min} = \infty \cdot \mathbf{1}_{\boldsymbol{S}}$ for general $\boldsymbol{S} \in \mathcal{F}_T$ is an open problem even when the filtration is generated only by a Brownian motion.

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7 Conclusion

The present work studies the optimal control problem (13) of optimal liquidation where the key parameters are the measurable set $\boldsymbol{S}$ specifying conditions for full liquidation and the process I specifying when trading is allowed; we note that choosing $\boldsymbol{S} \neq \Omega$ introduces a negative term into the terminal condition of the associated BSDE determined by the permanent price impact parameter. We find the optimal control to be of the form

$$Q_t^* = q_0 \exp\left(-\int_0^t I_s \mathrm{Vol}_s |Y_s^{\min}|^{p-1}\mathrm{sgn}\ (Y_s^{\min})ds\right)$$

where $Y^{\min}$ is the first component of the minimal supersolution of the BSDE (25) with terminal condition $\xi = -\dfrac{k}{\widehat{p}}\mathbf{1}_{\boldsymbol{S}^c} + \infty\mathbf{1}_{\boldsymbol{S}}$. In Section 3 this is proved directly using the BSDE for the case $p = 2$; Section 4 uses a PDE approach and focuses on the case $p \neq 2$ where the price dynamics are assumed Markovian and I and $\boldsymbol{S}$ are chosen so that the resulting problem is either Markovian or can be broken into Markovian pieces.

In the remaining paragraphs we comment on some of our assumptions and on future research. In the present work we use the identity function as our utility function (see (7)); this corresponds to setting the risk aversion parameter γ in the exponential utility function to 0. Let us comment on the case $\gamma > 0$. Recall that we proceed in two steps: we start with the control problem (7) and then derive from it the secondary problem (13) and then work with this problem. For $\gamma > 0$ the reduction from (7) to (13) is possible only in special cases (for example, when the optimal control is known to be deterministic, as in the case of the standard IS order (see [14, Chapter 3]). To circumvent this problem, many prior works directly start from the secondary problem and add a risk measure on the size of the portfolio directly to this problem (see, e.g., [2, 4]). When the risk measure is added directly to the second problem, it can even be allowed to be a random process, an example is given in (19) (the term $\gamma_t|Q_t|^{\widehat{p}}$). As is the case in these prior works, it is straight-forward to introduce a penalization term on the size of the portfolio involving a positive process in the stochastic optimal control problem (13) and almost all of the analysis presented in sections 3 and 4 will have straightforward modifications. On the other hand, an analysis of the exponential utility case for $\gamma > 0$ (i.e., introducing $\gamma > 0$ and starting the analysis from the problem $\sup_{Q \in \mathcal{A}_{I,\boldsymbol{S}}} \mathbb{E}[-e^{-\gamma \tilde{X}_T}]$) would require significant changes from the analysis presented in the current work. In our view, one benefit of starting the analysis from (7) as we did in the present work, is that it provided a relatively simple framework to understand the role permanent price impact plays in the stochastic optimal control problem in the presence of the parameter $\boldsymbol{S}$.

An important assumption in the present work is the monetary representation of the terminal position Q_T, for which we used $m(q) = qS_T$. A simple modification that would keep the problem within the framework of the current work is to set $m(q) = qS_T + \xi'|q|^{\widehat{p}}$ where ξ' is an $\mathcal{F}_T$-measurable random variable. This would introduce an additional ξ' term into the terminal condition of the BSDE which would not impact the analysis as long as ξ'^- is bounded from above. Other choices are obviously possible depending on how the events taking place after time T are modeled. These choices will give rise to different stochastic optimal control problems whose analysis will probably require new/other tools and ideas. This is a natural direction for future research.

Recall the random variable A of (1), the percentage deviation from the target price S_0 of the average price at which the portfolio is (partially) closed at terminal time. We provided a

numerical study of the distribution of $\mathfrak{q}_T = Q_T^*/q_0$ (remaining portion of the position) and A in Section 5 for the case $I = 1$ and $\boldsymbol{S} = \{\bar{S}_T > \ell\}$. An analytical study of these distributions for this choice of $(I, \boldsymbol{S})$ and for others seems interesting and challenging. One idea in the study of the distribution of Q_T/q_0 is the use of Malliavin calculus (see [18, Theorem 2.1.3]). This would require that the product $Y^{\min}$vol has a Malliavin derivative with some additional conditions. The Malliavin derivative of the solution of a BSDE has already been studied; see among others [9] for integrable terminal condition and [20] for singular terminal condition. In the Markovian framework, it requires some Malliavin regularity on the forward processes S and ν. A study of these ideas and problems remain for future work. Another natural direction is to try to compute the performance of the algorithms of the present work (especially the joint distribution of $(\mathfrak{q}_T, A)$) on real trading data.

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