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Design for estimation of the drift parameter in fractional diffusion systems

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Abstract We consider a controlled linear differential equation which is partially observed with an additive fractional noise. In this setting, we study the asymptotic (for large observation time) design problem of the input and give an efficient estimator of the unknown signal drift parameter. The optimal estimation input is deduced. The consistency, asymptotic normality and convergence of the moments of the MLE are established.

Keywords Optimal design · Parameter estimation · Maximum likelihood · Fractional diffusion

1 Introduction

1.1 Historical survey

The problem of the choice of inputs so as to obtain maximal information from the experiments have been given a great deal of interest over the last 60 years. Background to this problem can be found in the early statistics literature (see e.g. Kiefer 1974; Wald 1943; Whittele 1973) as well as in the engineering literature (see e.g. Gevers 2005; Goodwin and Payne 1977; Goodwin et al. 1973, 2007). The focus has been predominately on experiments design for identification of directly observed dynamic system parameters.

The classical approach for experiment design consists on a two-step procedure: maximize the Fisher information and find an adaptive estimation procedure. In this area, there are several approaches like sequential design, Bayesian design, mini-max and robust design (see e.g. Brodeau and Le Breton 1979; Lopez-Toledo and Athans 1975; Goodwin et al. 2007; Levadi 1966; Mehra 1974a and the references therein). However, there has been little work on partially observed systems, even linear because of the implicit form of estimators. We can

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mention Aoki and Staley (1970), Levin (1960), Mehra (1974a,b), Ovseevich et al. (2000), where linear signal-observation model perturbed by the white noise has been considered.

In this paper, we consider a controlled linear differential equation which is partially observed with an additive fractional noise. In this setting, we study the asymptotic (for large observation time) design problem of the input and give an efficient estimator of the unknown signal drift parameter.

As in a classical approach, we can separate the initial problem in two subproblems.

In the one hand, we deduce the optimal input by maximizing the Fisher information under certain constraints on the energy of the input. In a contrast with the previous works (see e.g. Levadi 1966; Mehra 1974a; Ovseevich et al. 2000), the problem can not be reduced to the optimization on the real line and the method proposed in Ovseevich et al. (2000) does not work. We propose to use Laplace transform computations, in particular, the Cameron–Martin formula and the link between the Laplace transform of the integral of the square of a Gaussian process and the eigenvalues of its self-adjoint covariance operator.

Surprisingly, the asymptotical optimal input does not depend on the unknown parameter (see Proposition 1.1) and so the MLE is a good candidate to reach efficient estimation.

In the other hand, we study the asymptotic properties of the MLE. We establish consistency, asymptotic normality and convergence of the moments of the MLE. This is done by the method previously developed by the authors Brouste (2010), Brouste and Kleptsyna (2010).

1.2 The setting and the main result

We consider real-valued functions $x = (x_t,\ t \geq 0)$, $u = (u(t),\ t \geq 0)$ and a process $Y = (Y_t,\ t \geq 0)$, representing the signal, the control and the observation respectively, governed by the following homogeneous linear system of ordinary and stochastic differential equations interpreted as integral equations:

$$\begin{cases} dx_t = -\vartheta x_t dt + u(t)dt\,, & x_0 = 0\,, \\ dY_t = \mu x_t dt + dV_t^H\,, & Y_0 = 0. \end{cases} \tag{1}$$

Here, $V^H = (V_t^H,\ t \geq 0)$ is normalized fBm with Hurst parameter $H \in [\frac{1}{2}, 1)$ and the coefficients ϑ and $\mu \neq 0$ are real constants. System (1) has a uniquely defined solution process (x, Y) where Y is Gaussian but neither Markovian nor a semimartingale for $H \neq \frac{1}{2}$.

Suppose that parameter $\vartheta > 0$ is unknown[1] and is to be estimated given the observed trajectory $Y^T = (Y_t,\ 0 \leq t \leq T)$ for a control u in the proper class.

For a fixed value of the parameter ϑ, let $\mathbf{P}_\vartheta^T$ denote the probability measure, induced by Y^T on the function space $\mathcal{C}_{[0,T]}$ and let $\mathcal{F}_t^Y$ be the natural filtration of Y, $\mathcal{F}_t^Y = \sigma\,(Y_s, 0 \leq s \leq t)$.

Let $\mathcal{L}(\vartheta, Y^T)$ be the likelihood, *i.e.* the Radon–Nikodym derivative of $\mathbf{P}_\vartheta^T$, restricted to $\mathcal{F}_T^Y$ with respect to some reference measure on $\mathcal{C}_{[0,T]}$. The explicit representation of the likelihood function can be written thanks to the transformation of observation model (1) proposed in Kleptsyna and Le Breton (2002a). Therefore, Fisher information stands for:

$$\mathcal{I}_T\,(\vartheta, u) = -\mathbf{E}_\vartheta \frac{\partial^2}{\partial \vartheta^2} \ln \mathcal{L}_T(\vartheta, Y^T).$$

[1] In the continuous-time observation setting, there is no statistical error made for the Hurst parameter H estimation with classical methods, see for instance quadratic generalized variations method in Istas and Lang (1997).

Let us denoted $\mathcal{U}_T$ some functional space of controls, that is defined by (8) and (9) page 136. Let us also note

$$\mathcal{J}_T(\vartheta) = \sup_{u\in\mathcal{U}_T} \mathcal{I}_T(\vartheta, u).$$

Our main goal is to find estimator $\overline{\vartheta}_T$ of the parameter ϑ which is asymptotically efficient in the sense that, for any compact $\mathbb{K} \subset \mathbb{R}_*^+ = \{\vartheta \in \mathbb{R}, \vartheta > 0\}$,

$$\sup_{\vartheta\in\mathbb{K}} \mathcal{J}_T(\vartheta)\mathbf{E}_\vartheta \left(\overline{\vartheta}_T - \vartheta\right)^2 = 1 + o(1), \tag{2}$$

as $T \to \infty$.

We claim that:

Proposition 1.1 *The asymptotical optimal input in the class of controls $\mathcal{U}_T$ is $u_{opt}(t) = \frac{\kappa}{\sqrt{2\lambda}} t^{H-\frac{1}{2}}$, where*

$$\kappa = 2H\Gamma\left(\frac{3}{2} - H\right)\Gamma\left(\frac{1}{2} + H\right) \quad \textit{and} \quad \lambda = \frac{H\Gamma(3-2H)\Gamma(H+\frac{1}{2})}{2(1-H)\Gamma(\frac{3}{2}-H)} \tag{3}$$

and Γ stands for the Gamma function. Moreover,

$$\lim_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} = \frac{\mu^2}{\vartheta^4}$$

As the optimal input does not depend on ϑ (see Proposition 1.1), a possible candidate is the maximum likelihood estimator (MLE) $\hat{\vartheta}_T$, defined as the maximizer of the likelihood:

$$\hat{\vartheta}_T = \arg\max_{\vartheta>0} \mathcal{L}(\vartheta, Y^T). \tag{4}$$

Moreover, MLE reaches efficiency and we deduce its large samples asymptotic properties:

Proposition 1.2 *The MLE is uniformly consistent on compacts $\mathbb{K} \subset \mathbb{R}_*^+$, i.e. for any $\nu > 0$,*

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \mathbf{P}_\vartheta^T \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \nu \right\} = 0, \tag{5}$$

uniformly on compacts asymptotically normal: as T tends to $+\infty$,

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \left| \mathbf{E}_\vartheta f\left(\sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right)\right) - \mathbf{E} f(\xi) \right| = 0 \quad \forall f \in \mathcal{C}_b \tag{6}$$

and ξ is a zero mean Gaussian random variable of variance $\mathcal{J}(\theta)^{-1} = \frac{\vartheta^4}{\mu^2}$ **which does not depend on** *H and we have the uniform on $\vartheta \in \mathbb{K}$ convergence of the moments: for any $p > 0$,*

$$\lim_{T\to\infty} \sup_{\vartheta\in\mathbb{K}} \left| \mathbf{E}_\vartheta \left| \sqrt{T}\left(\hat{\vartheta}_T - \vartheta\right) \right|^p - \mathbf{E} \left|\xi\right|^p \right| = 0. \tag{7}$$

Finally, the MLE is efficient in the sense of (2).

Remark 1 It is worth emphasizing that the classical case $H = \frac{1}{2}$, treated in Ovseevich et al. (2000), presents the same result.

2 The proofs

2.1 Preliminaries

In what follows, all random variables and processes are defined on a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ satisfying the usual conditions and processes are $(\mathcal{F}_t)$-adapted. Moreover the *natural filtration* of a process is understood as the **P**-completion of the filtration generated by this process.

Even if fBm are not martingales, there are simple integral transformations which change the fBm to martingales (see Norros et al. 1999; Nuzman and Poor 2000). In particular, with (3), defining for $0 < s < t$,

$$k_H(t,s) = \kappa^{-1} s^{\frac{1}{2}-H} (t-s)^{\frac{1}{2}-H}, \quad N_t = \int_0^t k_H(t,s) dV_s^H,$$

then the process $N = (N_t, t \geq 0)$ is a Gaussian martingale, called in Norros et al. (1999) the *fundamental martingale* whose variance function is the function

$$w_H(t) = \frac{1}{2\lambda(2-2H)} t^{2-2H}.$$

Moreover the natural filtration of the martingale N coincides with the natural filtration of the fBm V^H.

For a given control u we define the function v by the following equation

$$v(t) = \frac{d}{dw_H(t)} \int_0^t k_H(t,s) u(s) ds; \tag{8}$$

provided that the fractional derivative exists. Let us define the space of controls $\mathcal{U}_T$ as the space of functions u s.t. v given by (8) exists and $v \in \mathcal{V}_T$ where

$$\mathcal{V}_T = \left\{ v \;\middle|\; \frac{1}{T} \int_0^T |v(t)|^2 \, dw_H(t) \leq 1 \right\}. \tag{9}$$

Note that these sets are non empty. Remark that with (8) the following relation between control u and its transformation v holds:

$$u(t) = \frac{d}{dt} \int_0^t K_H(t,s) v(s) dw_H(s) \tag{10}$$

where

$$K_H(t,s) = H(2H-1) \int_s^t r^{H-\frac{1}{2}} (r-s)^{H-\frac{3}{2}} dr \quad \text{for } 0 \leq s \leq t.$$

Following Kleptsyna and Le Breton (2002a), let us introduce $Z = (Z_t, t \geq 0)$ the *fundamental semimartingale* associated to Y, namely

$$Z_t = \int_0^t k_H(t,s) dY_s. \tag{11}$$

Note that Y can be represented as $Y_t = \int_0^t K_H(t,s)dZ_s$ and therefore the natural filtrations of Y and Z coincide. It can be proved that the following equation holds (see e.g. Kleptsyna and Le Breton 2002a):

$$dZ_t = \mu\lambda\ell(t)^*\zeta_t d\langle N\rangle_t + dN_t\,, \quad Z_0 = 0, \tag{12}$$

where v is defined by (8) and $\zeta = (\zeta_t, t \geq 0)$ is the solution of the ordinary differential equation

$$\frac{d\zeta_t}{d\langle N\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\zeta_t + b(t)v(t), \quad \zeta_0 = 0, \tag{13}$$

with

$$\ell(t) = \begin{pmatrix} t^{2H-1} \\ 1 \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} t^{2H-1} & 1 \\ t^{4H-2} & t^{2H-1} \end{pmatrix} \quad \text{and} \quad b(t) = \begin{pmatrix} 1 \\ t^{2H-1} \end{pmatrix}.$$

2.2 Likelihood function and the Fisher information

Recall that $Z^T = (Z_t,\ 0 \leq t \leq T)$ is the observed transformed trajectory. In this section, we are interested in the explicit representation of the likelihood function $\mathcal{L}_T\left(\vartheta, Z^T\right)$. The classical Girsanov theorem gives the following equality

$$\mathcal{L}_T(\vartheta, Z^T) = \exp\left\{\mu\lambda\int_0^T \ell(t)^*\zeta_t dZ_t - \frac{\mu^2\lambda^2}{2}\int_0^T \zeta_t^*\ell(t)\ell(t)^*\zeta_t d\langle N\rangle_t\right\} \tag{14}$$

where $\zeta = (\zeta_t, t \geq 0)$ is the solution of the ordinary differential Eq. (13).

The Fisher information stands for

$$\begin{aligned}
\mathcal{I}_T(\vartheta, v) &= -\mathbf{E}_\vartheta \frac{\partial^2}{\partial\vartheta^2}\ln\mathcal{L}_T(\vartheta, Z^T) \\
&= \mathbf{E}_\vartheta \int_0^T \mu^2\lambda^2\left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \\
&= \int_0^T \mu^2\lambda^2\left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \quad (\zeta \text{ is deterministic}) \\
&= \int_0^T \left(\frac{\partial\zeta_t}{\partial\vartheta}\right)^* \mu^2\lambda^2\ell(t)\ell(t)^*\frac{\partial\zeta_t}{\partial\vartheta}d\langle N\rangle_t.
\end{aligned} \tag{15}$$

From (13), we get

$$\zeta_t = \varphi(t)\int_0^t \varphi^{-1}(s)b(s)v(s)d\langle N\rangle_s \tag{16}$$

where $\varphi(t)$ is the fundamental matrix, i.e.

$$\frac{d\varphi(t)}{d\langle N\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\varphi(t)\,, \quad \varphi(0) = \mathbf{Id}, \tag{17}$$

and **Id** is the 2×2 identity matrix. Therefore

$$\begin{aligned}\mathcal{I}_T(\vartheta, v) &= \mu^2\lambda^2 \int_0^T \left(\frac{\partial \zeta_t}{\partial \vartheta}\right)^* \ell(t)\ell(t)^* \frac{\partial \zeta_t}{\partial \vartheta} d\langle N\rangle_t \\ &= \int_0^T\int_0^T \mathcal{K}_T(s,\sigma)\frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}}v(s)\frac{\sigma^{\frac{1}{2}-H}}{\sqrt{2\lambda}}v(\sigma)ds d\sigma,\end{aligned}$$

where

$$\mathcal{K}_T(s,\sigma) = \int_{\max(s,\sigma)}^T G(t,s)G(t,\sigma)dt\,, \tag{18}$$

and

$$G(t,\sigma) = \frac{\partial}{\partial \vartheta}\left(\frac{\mu}{2}t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(\sigma)b(\sigma)\sigma^{\frac{1}{2}-H}\right).$$

2.3 Optimal input and efficiency

Let us denote $||.||_{2,T}$ the usual norm in $L_2([0,T])$ and $||.||_2$ the usual norm in $L^2(\mathbb{R})$. From the previous formulas and (9), we obtain

$$\begin{aligned}\mathcal{J}_T(\vartheta) &= \sup_{v\in\mathcal{V}_T} \mathcal{I}_T(\vartheta, v) \\ &= T \sup_{\tilde{v}\in L^2([0,T]),\, ||\tilde{v}||_{2,T}\le 1} \int_0^T\int_0^T \mathcal{K}_T(s,\sigma)\tilde{v}(s)\tilde{v}(\sigma)ds d\sigma \\ &= T \sup_{\tilde{v}\in L^2([0,T]),\, ||\tilde{v}||_{2,T}\le 1} (\mathcal{K}_T\tilde{v}, \tilde{v})\,,\end{aligned}$$

with $\tilde{v}(s) = \frac{s^{\frac{1}{2}-H}}{\sqrt{2\lambda}}\frac{v(s)}{\sqrt{T}}$.

We have, due to the explicit form of the kernel (see Remark 2 page 148), that

$$\lim_{T\to+\infty} \mathcal{K}_T(s,\sigma) = \mathcal{K}_\infty(s,\sigma)$$

uniformly in any finite interval of s and σ. Contrary to the classical case $H = \frac{1}{2}$ (see Ovseevich et al. 2000), the limit kernel $\mathcal{K}_\infty(s,\sigma)$ is no more of the form

$$C(s-\sigma) = \frac{\mu^2}{4\vartheta^3}e^{-\vartheta|\sigma-s|}\left(\vartheta|\sigma-s|+1\right).$$

But we have the following result:

Lemma 2.1 *In this setting,*

$$\liminf_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} \ge \sup_{w\in L^2(\mathbb{R}),||w||_2=1} (Cw, w) = \frac{\mu^2}{\vartheta^4}.$$

Proof The proof is postponed in Sect. 2.5. □

In the classical case, the reminder $\mathcal{K}_\infty(s,\sigma)-\mathcal{K}_T(s,\sigma)$ corresponds to a positive quadratic form and

$$\sup_{\tilde{v}\in L^2([0,T]),\,||\tilde{v}||_{2,T}\le 1}(\mathcal{K}_T\tilde{v},\tilde{v})\le \sup_{w\in L^2(\mathbb{R}),||w||_2=1}(\mathcal{K}_\infty w,w)=\sup_{w\in L^2(\mathbb{R}),||w||_2=1}(Cw,w).$$

In the fractional case, we only have that:

$$\sup_{\tilde{v}\in L^2([0,T]),\,||\tilde{v}||_{2,T}\le 1}(\mathcal{K}_T\tilde{v},\tilde{v})\le \sup_{w\in L^2(\mathbb{R}),||w||_2=1}(\mathcal{K}_\infty w,w).$$

Nevertheless, we claim the following result:

Proposition 2.1

$$\lim_{T\to+\infty}\sup_{\tilde{v}\in L^2([0,T]),\,||\tilde{v}||_{2,T}\le 1}(\mathcal{K}_T\tilde{v},\tilde{v})=\sup_{w\in L^2(\mathbb{R}),||w||_2=1}(Cw,w)=\frac{\mu^2}{\vartheta^4}.$$

Proof Lemma 2.1 gives the lower bound. It remains to show the upper bound:

$$\lim_{T\to+\infty}\sup_{\tilde{v}\in L^2([0,T]),\,||\tilde{v}||_{2,T}\le 1}(\mathcal{K}_T\tilde{v},\tilde{v})\le\sup_{w\in L^2(\mathbb{R}),||w||_2=1}(Cw,w)=\frac{\mu^2}{\vartheta^4}.$$

Let us introduce the pair process $\xi=\left(\left(\xi_t^1,\xi_t^2\right),0\le t\le T\right)$ with

$$\xi_t^1=\left(\int_t^T\sigma^{\frac{1}{2}-H}\ell(\sigma)^*\varphi(\sigma)*dW_\sigma\right)\varphi^{-1}(t)\quad\text{and}\quad \xi_t^2=\frac{\partial}{\partial\vartheta}\xi_t^1, \tag{19}$$

where W is a Wiener process and $*dW_\sigma$ denotes the Itô backward integral (see e.g Rozovskii 1990). It is worth emphasizing that

$$\mathcal{K}_T(s,\sigma)=\frac{\mu^2}{4}\mathbf{E}\left(\xi_s^2 b(s)s^{\frac{1}{2}-H}\xi_\sigma^2 b(\sigma)\sigma^{\frac{1}{2}-H}\right)=\mathbf{E}\left(\mathcal{X}_\sigma\mathcal{X}_s\right),$$

where $\mathcal{X}$ is the centered Gaussian process defined by:

$$\mathcal{X}_t=\frac{\mu}{2}\xi_t^2 b(t)t^{\frac{1}{2}-H}.$$

The process ξ also satisfies the following dynamic

$$-d\xi_t=\xi_t\mathcal{A}(t)d\langle N\rangle_t+\mathcal{L}(t)\frac{t^{\frac{1}{2}-H}}{\sqrt{2\lambda}}*dW_t,\xi_T=0,$$

or

$$-d\xi_t=\xi_t\mathcal{A}(t)d\langle M\rangle_t+\mathcal{L}(t)*dM_t,\ \xi_T=0,$$

with $M=(M_t,\ t\ge 0)$ a martingale of the same variance function as $N=(N_t,\ t\ge 0)$,

$$\mathcal{A}(t)=\begin{pmatrix}-\vartheta & -1\\ 0 & -\vartheta\end{pmatrix}\otimes\lambda\mathbf{A}(t)\quad\text{and}\quad\mathcal{L}(t)=\sqrt{2\lambda}\left(\ell(t)^*\ \ 0\right).$$

Obviously, we should estimate the spectral gap (the first eigenvalue $\nu_1(T)$) of the operator associated to the kernel $\mathcal{K}_T$. The estimation of the spectral gap is based on the Laplace transform computation.

Let us compute, for sufficiently small negative $a < 0$ the Laplace transform of $\int_0^T \mathcal{X}_t^2 dt$:

$$L_T(a) = \mathbf{E}_\vartheta \exp\left\{-a\int_0^T \mathcal{X}_t^2 dt\right\}$$
$$= \mathbf{E}_\vartheta \exp\left\{-a\int_0^T \left[\frac{\mu}{2}\left(\frac{\partial}{\partial\vartheta}\xi_t^1\right) b(t) t^{\frac{1}{2}-H}\right]^2 dt\right\}$$

On the one hand, $\mathcal{X}$ is a centered Gaussian process with covariance operator $\mathcal{K}_T$. For $a > -\frac{1}{\nu_1(T)}$, using Mercer's theorem and Parseval's inequality, $L_T(a)$ can be represented as:

$$L_T(a) = \prod_{i\geq 1}(1 + 2a\nu_i(T))^{-\frac{1}{2}}, \tag{20}$$

where $\nu_i(T),\ i \geq 1$ is the sequence of positive eigenvalues of the covariance operator. On the other hand,

$$L_T(a) = \mathbf{E}_\vartheta \exp\left\{-a\frac{\mu^2\lambda}{2}\int_0^T \xi_t \mathcal{M}(t)\xi_t^* d\langle N\rangle_t\right\}$$
$$= \exp\left\{\frac{1}{2}\int_0^T \operatorname{trace}(\mathcal{H}(t)\mathcal{L}(t)^*\mathcal{L}(t)) d\langle N\rangle_t\right\},$$

where

$$\mathcal{M}(t) = \begin{pmatrix} 0 & 0 \\ 0 & b(t)b(t)^* \end{pmatrix}$$

and $\mathcal{H} = (\mathcal{H}(t), t \geq 0)$ is the solution of Ricatti differential equation:

$$\frac{d\mathcal{H}(t)}{d\langle N\rangle_t} = \mathcal{H}(t)\mathcal{A}(t)^* + \mathcal{A}(t)\mathcal{H}(t) + \mathcal{H}(t)\mathcal{L}(t)^*\mathcal{L}(t)\mathcal{H}(t) - a\mu^2\lambda\mathcal{M}(t), \tag{21}$$

with initial condition $\mathcal{H}(0)=0$, provided that the solution of Eq. (21) exists for any $0\leq t\leq T$.

It is well known that if $\det\Psi_1(t) > 0$, for any $t \in [0, T]$, then the solution $\mathcal{H}$ of Eq. (21) can be written as $\mathcal{H}(t) = \Psi_1^{-1}(t)\Psi_2(t)$, where the pair of 4×4 matrices (Ψ_1, Ψ_2) satisfies the system of linear differential equations:

$$\begin{aligned} \frac{d\Psi_1(t)}{d\langle N\rangle_t} &= -\Psi_1(t)\mathcal{A}(t) - \Psi_2(t)\mathcal{L}(t)^*\mathcal{L}(t), \quad \Psi_1(0) = \mathcal{I}d, \\ \frac{d\Psi_2(t)}{d\langle N\rangle_t} &= -a\mu^2\lambda\Psi_1(t)\mathcal{M}(t) + \Psi_2(t)\mathcal{A}(t)^*, \quad \Psi_2(0) = 0, \end{aligned} \tag{22}$$

and $\mathcal{I}d$ is the 4×4 identity matrix.

Moreover, under the condition $\det\Psi_1(t) > 0$, for any $t \in [0, T]$, the following equality holds:

$$L_T(a) = \exp\left\{-\frac{1}{2}\int_0^T \operatorname{trace} \mathcal{A}(t) d\langle N\rangle_t\right\} (\det\Psi_1(T))^{-\frac{1}{2}}$$
$$= \exp\{\vartheta T\} (\det\Psi_1(T))^{-\frac{1}{2}}, \tag{23}$$

or, equivalently using (20),

$$\prod_{i\geq 1}(1 + 2a\nu_i(T)) = \exp\{-2\vartheta T\} (\det\Psi_1(T)) . \tag{24}$$

Let us note here that the solution of linear system (22) exists for any $t > 0$ and for any $a \in \mathbb{C}$. For $a = 0$, $\det\Psi_1(t) = \exp\{2\vartheta t\} > 0$. Due to the continuity property of the solutions of linear differential equations with respect to a parameter, for all $T > 0$, there exists $a(T) < 0$ such that

$$\inf_{t\in[0,T]} \det\Psi_1(t) > 0.$$

Therefore, equality (24) holds in an open set in $\mathbb{C}$, containing 0. Compactness of the covariance operator, namely, $\int_0^T \mathcal{K}_T(s, s)ds < \infty$, implies, due to the Weierstrass theorem, the analytic property of $\prod_{i\geq 1}(1 + 2a\nu_i(T))$ with respect to a. Hence, equality (24) holds for any $a \in \mathbb{C}$.

Now let us show that for T sufficiently large and for $-\frac{\vartheta^4}{2\mu^2} < a < 0$ $\det\Psi_1(T) > 0$. Indeed, linear system (22) can be rewritten as

$$\frac{d\left(\Psi_1(t), \Psi_2(t)\mathbf{J}\right)}{d\langle N\rangle_t} = (\Psi_1(t), \Psi_2(t)\mathbf{J}) \cdot (\daleth \otimes \lambda\mathbf{A}(t)) \tag{25}$$

where

$$\daleth = \begin{pmatrix} \vartheta & 1 & 0 & 0 \\ 0 & \vartheta & 0 & -a\mu^2 \\ -2 & 0 & -\vartheta & 0 \\ 0 & 0 & -1 & -\vartheta \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{J} = \begin{pmatrix} J & J \\ J & J \end{pmatrix}.$$

Clearly, system (25) has an explicit solution:

$$(\Psi_1(t), \Psi_2(t)\mathbf{J}) = (\mathcal{I}d, 0) \cdot (\mathcal{P} \otimes \mathbf{Id})\, \mathcal{G}\left(\mathcal{P}^{-1} \otimes \mathbf{Id}\right) \tag{26}$$

where $\mathcal{G} = \operatorname{diag}(\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3, \mathbf{G}_4)$ and

$$\frac{d\mathbf{G}_i(t)}{d\langle N\rangle_t} = \lambda x_i \mathbf{G}_i \mathbf{A} \quad \mathbf{G}_i(0) = \mathbf{Id}, \quad i = 1 \ldots 4, \tag{27}$$

with $(x_i)_{i=1\ldots 4}$ the eigenvalues of matrix $\daleth$ and $\mathcal{P}$ the matrix of its eigenvectors. For

$$-\frac{\vartheta^4}{2\mu^2} < a < 0,$$

eigenvalues of matrix $\daleth$ are of the form $x_i = \pm\sqrt{\vartheta^2 \pm \mu\sqrt{-2a}}$.

It can be checked that there exists a constant $C > 0$ such that

$$\det\Psi_1(T) = \exp\left((x_1 + x_3)T\right)\left(C + O\left(\frac{1}{T}\right)\right),$$

where $x_1 = \sqrt{\vartheta^2 + \mu\sqrt{-2a}} > x_3 = \sqrt{\vartheta^2 - \mu\sqrt{-2a}}$.

Therefore, due to equality (24), we have that $\prod_{i\geq 1}(1 + 2a\nu_i(T)) > 0$ for any $a > -\dfrac{\vartheta^4}{2\mu^2}$.

It means that $-\dfrac{1}{\nu_1(T)} \leq -\dfrac{\vartheta^4}{\mu^2}$, or equivalently, that

$$\nu_1(T) \leq \frac{\mu^2}{\vartheta^4} \tag{28}$$

which achieves the proof. □

Combining the proof of Lemma 2.1 and the upper bound (28), we obtain that

$$v_{opt}(t) = \sqrt{2\lambda}t^{H-\frac{1}{2}},\ 0 \leq t \leq T$$

is optimal in the class $\mathcal{V}_T$. As in Ovseevich et al. (2000),

$$\frac{1}{T}\int\limits_0^T |v_{opt}(t)|^2 d\langle N\rangle_t = 1$$

Finally, using (10), we have

$$u_{opt}(t) = \frac{d}{dt}\int\limits_0^t K_H(t,s)v_{opt}(s)d\langle N\rangle_s = \frac{\kappa}{\sqrt{2\lambda}}t^{H-\frac{1}{2}}.$$

2.4 MLE large sample asymptotic properties

As the optimal input does not depend on ϑ, it is possible to compute directly the MLE on the following system:

$$\begin{cases} dx_t &= -\vartheta x_t dt + u_{opt}(t)dt\,, & x_0 = 0\,, \\ dY_t &= \mu x_t dt + dV_t^H\,, & Y_0 = 0. \end{cases} \tag{29}$$

where $u_{opt}(t)$ is the optimal input found in Sect. 2.3. After transformation,

$$dZ_t = \mu\lambda\ell(t)^*\zeta_t d\langle N\rangle_t + dN_t\,, \quad Z_0 = 0, \tag{30}$$

where

$$\frac{d\zeta_t}{d\langle N\rangle_t} = -\vartheta\lambda\mathbf{A}(t)\zeta_t + b(t)v_{opt}(t).$$

Actually, to compute large sample asymptotic properties of the implicit MLE, we need the explicit representation of the likelihood ratio

$$\mathcal{Z}_T(\vartheta_1,\vartheta_2,Z^T) = \frac{\mathcal{L}_T(\vartheta_2,Z^T)}{\mathcal{L}_T(\vartheta_1,Z^T)}, \tag{31}$$

which is also the Radon–Nikodym derivative of $\mathbf{P}_{\vartheta_2}^T$ with respect to $\mathbf{P}_{\vartheta_1}^T$, restricted to $\mathcal{F}_T^Y$, i.e.

$$\mathcal{Z}_T(\vartheta_1,\vartheta_2,Z^T) = \frac{\mathcal{L}_T(\vartheta_2,\zeta^T)}{\mathcal{L}_T(\vartheta_1,\zeta^T)} = \left.\frac{d\mathbf{P}_{\vartheta_2}^T}{d\mathbf{P}_{\vartheta_1}^T}\right|_{\mathcal{F}_T^Y}.$$

From Eq. (13), this ratio can be written in the following form:

$$\begin{aligned}&\mathcal{Z}_T(\vartheta_1, \vartheta_2, Z^T)\\&= \exp\left\{\mu\lambda \int_0^T \ell^*(t)\delta_{\vartheta_1,\vartheta_2}(t)d\nu_t^{\vartheta_1} - \frac{\mu^2\lambda^2}{2}\int_0^T \delta^*_{\vartheta_1,\vartheta_2}(t)\ell(t)\ell^*(t)\delta_{\vartheta_1,\vartheta_2}(t)d\langle N\rangle_t\right\}\end{aligned}$$

where $\delta_{\vartheta_1,\vartheta_2}(t)$ is the difference $\zeta_t^{\vartheta_2} - \zeta_t^{\vartheta_1}$ and $\left(\nu_t^{\vartheta_1}, t \geq 0\right)$ is defined by:

$$d\nu_t^{\vartheta_1} = dZ_t - \mu\lambda\ell(t)^*\zeta_t^{\vartheta_1} d\langle N\rangle_t\,, \quad \nu_0^{\vartheta_1} = 0.$$

We will denote by $\mathcal{Z}_T(x, Z^T)$ the perturbation of $\mathcal{Z}_T(\vartheta, \vartheta_2, Z^T)$, when $\vartheta_2 = \vartheta + \frac{x}{\sqrt{T}}$. Namely, $\mathcal{Z}_T(x, Z^T) = \mathcal{Z}_T(\vartheta, \vartheta + \frac{x}{\sqrt{T}}, Z^T)$. For this case, we will denote $\delta_{\vartheta,x,T}(t) = \delta_{\vartheta,\vartheta+\frac{x}{\sqrt{T}}}(t)$.

2.4.1 Ibragimov–Khasminskii program

It follows from Ibragimov–Khasminskii (1981, Theorem I.10.1) that in order to prove Proposition 1.2, it is sufficient to check the three following conditions:

(A.1)

$$\mathcal{Z}_T(x, Z^T) \stackrel{law}{\Longrightarrow} \exp\left\{x.\eta - \frac{x^2}{2}\mathcal{I}^{opt}(\vartheta)\right\} \text{ with } \eta \sim \mathcal{N}\left(0, \mathcal{I}^{opt}(\vartheta)\right),$$

(A.2) for some $\chi > 0$:

$$\mathbf{E}_\vartheta\sqrt{\mathcal{Z}_T(x, Z^T)} \leq \exp\left(-\chi x^2\right)$$

(A.3) there exists $C > 0$ such that

$$\mathbf{E}_\vartheta\left(\sqrt{\mathcal{Z}_T(x_1, Z^T)} - \sqrt{\mathcal{Z}_T(x_2, Z^T)}\right)^2 \leq C|x_1 - x_2|^2.$$

We present here the proof of Proposition 1.2 by checking the three conditions.

Proof Since (see Lemma 2.1 and Proposition 2.1)

$$\lim_{T\to\infty}\frac{\mu^2\lambda^2}{2}\int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t)d\langle N\rangle_t = \frac{x^2}{2}\lim_{T\to\infty}\frac{\mathcal{I}(\vartheta, \upsilon_{opt})}{T} = \frac{x^2}{2}\mathcal{I}^{opt}(\vartheta)\,,$$

with $\mathcal{I}^{opt}(\vartheta) = \frac{\mu^2}{\vartheta^4}$, we have that

$$\mu\lambda\int_0^T \ell^*(t)\delta_{\vartheta_1,\vartheta_2}(t)d\nu_t^{\vartheta_1} \stackrel{law}{\Longrightarrow} \mathcal{N}(0, \mathcal{I}^{opt}(\vartheta))$$

and the condition (A.1) is checked. The condition (A.2) holds thanks to the following chain of inequalities:

$$\begin{aligned}
&\mathbf{E}_\vartheta \sqrt{\mathcal{Z}_T(x)} \\
&= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2} \int_0^T \ell^*(t)\delta_{\vartheta,x,T}(t) d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{4} \int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t) d\langle N\rangle_t \right) \\
&= \mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2} \int_0^T \ell^*(t)\delta_{\vartheta,x,T}(t) d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{8} \int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t) d\langle N\rangle_t \right) \\
&\quad \times \exp\left(-\frac{\mu^2\lambda^2}{8} \int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t) d\langle N\rangle_t \right) \\
&\overset{(a)}{\leq} \exp\left(-\frac{\mu^2\lambda^2}{8} \int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t) d\langle N\rangle_t \right) \\
&\overset{(b)}{\leq} \exp\left(-\chi x^2\right),
\end{aligned}$$

where (a) is Girsanov theorem since

$$\mathbf{E}_\vartheta \exp\left(\frac{\mu\lambda}{2} \int_0^T \ell^*(t)\delta_{\vartheta,x,T}(t) d\nu_t^\vartheta - \frac{\mu^2\lambda^2}{8} \int_0^T \delta^*_{\vartheta,x,T}(t)\ell(t)\ell^*(t)\delta_{\vartheta,x,T}(t) d\langle N\rangle_t \right) \leq 1,$$

and (b) comes from the proof of (A.1). To prove (A.3), let us note that

$$\begin{aligned}
\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(x_1)} - \sqrt{\mathcal{Z}_T(x_2)}\right)^2 &= 2\left(1 - \mathbf{E}_\vartheta\, \mathcal{Z}_T(x_1) \sqrt{\frac{\mathcal{Z}_T(x_2)}{\mathcal{Z}_T(x_1)}}\right) \\
&= 2\left(1 - \mathbf{E}_{\vartheta_1} \sqrt{\mathcal{Z}_T(\vartheta_1, \vartheta_2)}\right).
\end{aligned}$$

The same chain of inequalities (with reverse Hölder inequality and Girsanov theorem) gives:

$$\begin{aligned}
\mathbf{E}_\vartheta \left(\sqrt{\mathcal{Z}_T(x_1)} - \sqrt{\mathcal{Z}_T(x_2)}\right)^2 &\leq 2\left(1 - \exp\left(-\chi_1 (x_2 - x_1)^2\right)\right) \\
&\leq C|x_1 - x_2|^2,
\end{aligned}$$

which ends the proof of Proposition 1.2. $\square$

2.5 Proof of Lemma 2.1

From (16), we have

$$\begin{aligned}\ell(t)^*\zeta_t &= \ell(t)^*\varphi(t)\int_0^t \varphi^{-1}(s)b(s)v(s)d\langle N\rangle_s \\ &= t^{H-\frac{1}{2}}\int_0^t \left(t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H}\right)\frac{s^{\frac{1}{2}-H}}{2\lambda}v_s ds \\ &= t^{H-\frac{1}{2}}\int_0^t g(t,s)\frac{s^{\frac{1}{2}-H}}{2\lambda}v_s ds,\end{aligned}$$

where

$$g(t,s) = t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H}$$

and $(\varphi(t), t \geq 0)$ satisfies Eq. (17). From Kleptsyna and Le Breton (2002b), explicit expression of $\varphi(t)$, $t \geq 0$, can be deduced, namely

$$\varphi(t) = e^{-\frac{\vartheta t}{2}}\begin{pmatrix} f_{2,2}(t) & -f_{1,2}(t) \\ -f_{2,1}(t) & f_{1,1}(t)\end{pmatrix}, \quad \det\varphi(t) = e^{-\vartheta t}$$

where

$$\begin{aligned} f_{1,1}(t) &= \left(\frac{\vartheta}{4}\right)^H \Gamma(1-H)\, t^H I_{-H}\left(\frac{\vartheta t}{2}\right) \\ f_{1,2}(t) &= \left(\frac{\vartheta}{4}\right)^H \Gamma(1-H)\, t^{1-H} I_{1-H}\left(\frac{\vartheta t}{2}\right) \\ f_{2,1}(t) &= \left(\frac{\vartheta}{4}\right)^{1-H} \Gamma(H)\, t^H I_{H}\left(\frac{\vartheta t}{2}\right) \\ f_{2,2}(t) &= \left(\frac{\vartheta}{4}\right)^{1-H} \Gamma(H)\, t^{1-H} I_{H-1}\left(\frac{\vartheta t}{2}\right), \end{aligned}$$

and I_ν is the modified Bessel function of the first kind and order ν. Direct computation leads to

$$t^{\frac{1}{2}-H}\ell(t)^*\varphi(t) = e^{-\frac{\vartheta t}{2}}\sqrt{t}\left(C_H\left(I_{H-1}-I_H\right)\left(\frac{\vartheta t}{2}\right), C_{1-H}\left(I_{-H}-I_{1-H}\right)\left(\frac{\vartheta t}{2}\right)\right)$$

where $C_H = \left(\frac{\vartheta}{4}\right)^{1-H}\Gamma(H)$.
Let us remark that,

$$\mathbf{Id} = \frac{1}{C_H C_{1-H}}\begin{pmatrix} C_{1-H} & 0 \\ 0 & C_H\end{pmatrix}\begin{pmatrix} C_H & 0 \\ 0 & C_{1-H}\end{pmatrix}$$

and

$$\mathbf{Id} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1\end{pmatrix}.$$

Therefore, we can compute

$$\begin{aligned}(a) &= t^{\frac{1}{2}-H}\ell(t)^*\varphi(t)\begin{pmatrix} C_{1-H} & 0 \\ 0 & C_H\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\ &= e^{-\frac{\vartheta t}{2}}\left(g_1(t), g_2(t)\right)\end{aligned}$$

with

$$\begin{aligned}g_1(t) &= C_H C_{1-H}\sqrt{t}\left(I_{H-1} - I_H + I_{-H} - I_{1-H}\right)\left(\frac{\vartheta t}{2}\right)\\ g_2(t) &= C_H C_{1-H}\sqrt{t}\left(I_{H-1} - I_H - I_{-H} + I_{1-H}\right)\left(\frac{\vartheta t}{2}\right).\end{aligned}$$

With the same computation we have

$$\begin{aligned}(b) &= \frac{1}{2C_H C_{1-H}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix} C_H & 0 \\ 0 & C_{1-H}\end{pmatrix}\varphi^{-1}(s)b(s)s^{\frac{1}{2}-H}\\ &= e^{\frac{\vartheta s}{2}}\begin{pmatrix}\tilde{g}_1(s)\\ \tilde{g}_2(s)\end{pmatrix}\end{aligned}$$

with

$$\begin{aligned}\tilde{g}_1(s) &= \frac{\sqrt{s}}{2}\left(I_{H-1} + I_H + I_{-H} + I_{1-H}\right)\left(\frac{\vartheta s}{2}\right)\\ \tilde{g}_2(s) &= \frac{\sqrt{s}}{2}\left(-I_{H-1} - I_H + I_{-H} + I_{1-H}\right)\left(\frac{\vartheta s}{2}\right).\end{aligned}$$

Hence

$$\begin{aligned}g(t,s) = (a)(b) &= e^{-\frac{\vartheta t}{2}}e^{\frac{\vartheta s}{2}}\left(g_1(t)\tilde{g}_1(s) + g_2(t)\tilde{g}_2(s)\right)\\ &= e^{-\vartheta t}e^{\vartheta s}\left(e^{\frac{\vartheta t}{2}}g_1(t)e^{-\frac{\vartheta s}{2}}\tilde{g}_1(s)\right) + e^{-\frac{\vartheta t}{2}}g_2(t)e^{\frac{\vartheta s}{2}}\tilde{g}_2(s).\end{aligned} \tag{32}$$

Since $C_H C_{1-H} = \frac{\vartheta}{4}\frac{\pi}{\sin\pi H}$, $(I_H - I_{-H})\left(\frac{\vartheta t}{2}\right) \underset{t\to\infty}{\sim} \frac{2\sin\pi H}{\sqrt{\vartheta\pi t}}e^{-\frac{\vartheta t}{2}}$, we get

$$e^{\frac{\vartheta t}{2}}g_1(t) \underset{t\to\infty}{\longrightarrow} \sqrt{\pi\vartheta}.$$

With property (see Olver 1997)

$$I_\nu\left(\frac{\vartheta t}{2}\right) = \frac{e^{\frac{\vartheta t}{2}}}{\sqrt{\pi\vartheta t}}\left(1 - \frac{4\nu^2-1}{4\vartheta t} + \underset{t\to\infty}{O}\left(\frac{1}{t^2}\right)\right),$$

we have that

$$\begin{aligned}e^{-\frac{\vartheta s}{2}}\tilde{g}_1(s) &\underset{s\to\infty}{\longrightarrow} \frac{2}{\sqrt{\pi\vartheta}},\\ e^{-\frac{\vartheta t}{2}}g_2(t) &\underset{t\to\infty}{\sim} \frac{2H-1}{2t\sin(\pi H)}\sqrt{\frac{\pi}{\vartheta}},\\ e^{\frac{\vartheta s}{2}}\tilde{g}_2(s) &\underset{s\to\infty}{\sim} \frac{(2H-1)\sin(\pi H)}{s\vartheta\sqrt{\pi\vartheta}}.\end{aligned}$$

It follows from (15) that

$$\begin{aligned}
\mathcal{I}_T(\vartheta, v) &= \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta}\ell(t)^*\zeta_t\right)^2 d\langle N\rangle_t \\
&= \int_0^T \mu^2\lambda^2 \left(\frac{\partial}{\partial\vartheta} t^{H-\frac{1}{2}} \int_0^t g(t,s)\frac{s^{\frac{1}{2}-H}}{2\lambda} v_s ds\right)^2 \frac{t^{1-2H}}{2\lambda} dt \\
&= \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T \left(\int_0^t \frac{\partial}{\partial\vartheta} g(t,s)\frac{s^{\frac{1}{2}-H} v_s}{\sqrt{2\lambda}} ds\right)^2 dt\right].
\end{aligned}$$

Now if we take $v_{opt}(s) = \sqrt{2\lambda} s^{H-\frac{1}{2}}$ then

$$\frac{1}{T}\int_0^T (v_{opt}(s))^2 d\langle N\rangle_s = \frac{1}{T}\int_0^T (v_{opt}(s))^2 \frac{s^{1-2H}}{2\lambda} ds = 1$$

and

$$\mathcal{I}_T\left(\vartheta, v_{opt}\right) = \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T \left(\int_0^t \frac{\partial}{\partial\vartheta} g(t,s) ds\right)^2 dt\right] = \frac{T\mu^2}{4}\left[\frac{1}{T}\int_0^T (\Psi(t))^2 dt\right].$$

It follows from the previous asymptotic estimates that for $s \geq 0$ we have

$$\lim_{t\to+\infty} \frac{\partial}{\partial\vartheta} g(t,s) = 0$$

and by Lebesgue's theorem for any $M \geq 0$

$$\lim_{t\to+\infty} \int_0^M \frac{\partial}{\partial\vartheta} g(t,s)\mathbf{1}_{(0,t)}(s) ds = 0.$$

Moreover for s and t large enough we obtain

$$g(t,s) \sim 2e^{-\vartheta(t-s)} + \frac{(2H-1)^4}{2\vartheta^2 ts}.$$

The recurrence relations for the derivatives of Bessel functions

$$I'_\nu = I_{\nu+1} + \frac{\nu}{x} I_\nu \quad \text{and} \quad I'_\nu = \frac{1}{2}\left(I_{\nu+1} + I_{\nu-1}\right) \tag{33}$$

imply that

$$\frac{\partial}{\partial\vartheta} g(t,s) \sim -2(t-s)e^{-\vartheta(t-s)} - \frac{2(2H-1)^4}{2\vartheta^3 ts}.$$

Therefore

$$\int_M^t \frac{\partial}{\partial \vartheta} g(t,s)ds \sim -2 \int_M^t (t-s)e^{-\vartheta(t-s)}ds - \frac{2(2H-1)^4}{2\vartheta^3} \frac{\ln t - \ln M}{t}$$
$$= \frac{-2}{\vartheta^2}\left[1-(1+\vartheta(t-M))e^{-\vartheta(t-M)}\right] - \frac{2(2H-1)^4}{2\vartheta^3}\frac{\ln t - \ln M}{t}.$$

Finally we obtain

$$\Psi(t) \underset{t\to\infty}{\longrightarrow} \frac{-2}{\vartheta^2}$$

which implies the the mean converges to the same value and therefore

$$\mathcal{I}_T\left(\vartheta, v_{opt}\right) \underset{T\to\infty}{\sim} \frac{T\mu^2}{\vartheta^4}.$$

To conclude recall that

$$\mathcal{J}_T(\vartheta) = \sup_{v\in\mathcal{V}_T} \mathcal{I}_T(\vartheta, v) \geq \mathcal{I}_T\left(\vartheta, v_{opt}\right).$$

which implies

$$\liminf_{T\to+\infty} \frac{\mathcal{J}_T(\vartheta)}{T} \geq \frac{\mu^2}{\vartheta^4} = \sup_{w\in L^2(\mathbb{R}), ||w||_2=1} (Cw, w).$$

Remark 2 It is possible to obtain the behavior, as $t \to \infty$, of

$$G(t,s) = \frac{\mu}{2}\frac{\partial}{\partial\vartheta} g(t,s)$$

for any s in a finite interval using the explicit form of $g(t,s)$ in (32) associated to the recurrence relation for the derivatives of Bessel functions (33).

References

Aoki H, Staley R (1970) On input signal synthesis in parameter identification. Atomatica 6:431–440

Brodeau F, Le Breton A (1979) Identification de paramètres pour un système excité par des bruits gaussiens et poissoniens. Annales de l'Institut Henri Poincaré 15:1–23

Brouste A (2010) Asymptotic properties of MLE for partially observed fractional diffusion system with dependent noises. J Stat Plan Inference 140(2):551–558

Brouste A, Kleptsyna M (2010) Asymptotic properties of MLE for partially observed fractional diffusion system. Stat Inference Stoch Process 13(1):1–13

Gevers M (2005) From the early achievement to the revival of experiment design. Eur J Control 11:1–18

Goodwin G, Murdoch J, Payne R (1973) Optimal test signal design for linear single input—single output system identification. Int J Control 17:45–55

Goodwin G, Payne R (1977) Dynamic system identification: experiment design and data analysis. Academic Press, New York

Goodwin G, Rojas C, Welsh J, Feuer A (2007) Robust optimal experiment design for system identification. Automatica 43:993–1008

Ibragimov IA, Khasminskii RZ (1981) Statistics of random processes. Springer, New York

Istas J, Lang G (1997) Quadratic variations and estimation of the local Hölder index of a gaussian process. Annales de l'Institut Henri Poincaré 33(4):407–436

Kiefer J (1974) On the efficient design of statistical investigations. Ann Stat 2:849–879

Kleptsyna ML, Le Breton A (2002a) Extension of the Kalman–Bucy filter to elementary linear systems with fractional Brownian noises. Stat Inference Stoch Process 5:249–271
Kleptsyna ML, Le Breton A (2002b) Statistical analysis of the fractional Ornstein–Uhlenbeck type process. Stat Inference Stoch Process 5:229–241
Levadi VS (1966) Design of input signals for parameter estimation. IEEE Trans Automat Control 11:205–211
Levin M (1960) Estimation of impulse response in the presence of noise. IRE Trans Circuit Theory 7:50–56
Lopez-Toledo A, Athans M (1975) Optimal policies for identification of stochastic linear systems. IEEE Trans Automat Control 20:754–765
Mehra RK (1974a) Optimal inputs for linear system identification. IEEE Trans Automat Control 19:192–200
Mehra RK (1974b) Optimal inputs signals for parameter estimation in dynamic systems—survey and new results. IEEE Trans Automat Control 19:753–768
Norros I, Valkeila E, Virtamo J (1999) An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5:571–587
Nuzman CJ, Poor HV (2000) Linear estimation of self-similar processes via Lamperti's transformation. J Appl Probab 37:429–452
Olver FWJ (1997) Asymptotics and special functions. AKP classics, vol xviii. A K Peters, Wellesley, p 572 (Reprint)
Ovseevich I, Khasminskii RZ, Chow PL (2000) Adaptative design for estimation of unknown parameters in linear systems. Probl Inf Transm 36(2):125–153
Rozovskii BL (1990) Stochastic evolution systems. Kluwer, Dordrecht
Wald A (1943) On the efficient design of statistical investigations. Annals Math Stat 14:134–140
Whittele P (1973) Some general points in the theory of optimal experimental design. J R Stat Soc 1:123–140