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A FINITE HORIZON OPTIMAL MULTIPLE SWITCHING PROBLEM*

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Abstract. We consider the problem of optimal multiple switching in a finite horizon when the state of the system, including the switching costs, is a general adapted stochastic process. The problem is formulated as an extended impulse control problem and solved using probabilistic tools such as the Snell envelope of processes and reflected backward stochastic differential equations. Finally, when the state of the system is a Markov process, we show that the associated vector of value functions provides a viscosity solution to a system of variational inequalities with interconnected obstacles.

Key words. real options, backward stochastic differential equations, Snell envelope, stopping time, optimal switching, impulse control, variational inequalities

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1. Introduction.

Optimal control of multiple switching models arises naturally in many applied disciplines. The pioneering work by Brennan and Schwartz [4], proposing a two-mode switching model for the life cycle of an investment in the natural resource industry, is probably the first to apply this special case of stochastic impulse control to questions related to the structural profitability of an investment project or an industry whose production depends on the fluctuating market price of a number of underlying commodities or assets. Within this discipline, Carmona and Ludkovski [5] and Deng and Xia [8] suggest a multiple switching model to price energy tolling agreements, where the commodity prices are modeled as continuous time processes and the holder of the agreement exercises her managerial options by controlling the production modes of the assets. Target tracking in aerospace and electronic systems (cf. Doucet and Ristic [12]) is another class of problems where these models are very useful. These are often formulated as a hybrid state estimation problem characterized by continuous time target state and discrete time regime (mode) variables. All these applications seem agree that reformulating these problems in a multiple switching dynamic setting is a promising (if not the only) approach to fully capture the interplay between profitability, flexibility, and uncertainty.

The optimal two-mode switching problem is probably the most extensively studied in the literature starting with the above-mentioned work by Brennan and Schwartz [4] and Dixit [10] who considered a similar model but without resource extraction; see Dixit and Pindyck [11] and Trigeorgis [33] for an overview, extensions of these models, and extensive reference lists. Brekke and Øksendal [2] and [3], Shirakawa [30], Knudsen, Meister, and Zervos [26], Duckworth and Zervos [13] and [14], and Zervos [34] use the framework of generalized impulse control to solve several versions and extensions of this model in the case where the decision to start and stop the production process

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is done over an infinite time horizon and the market price process of the underlying commodity is a diffusion process, while Trigeorgis [32] models the market price process of the commodity as a binomial tree. Hamadène and Jeanblanc [23] consider a finite horizon optimal two-mode switching problem in the case of Brownian filtration setting, while Hamadène and Hdhiri [24] extend the set up of the latter paper to the case where the processes of the underlying commodities are adapted to a filtration generated by a Brownian motion and an independent Poisson process. Porchet, Touzi, and Warin [29] also study the same problem, where they assume the payoff function to be given by an exponential utility function and allow the manager to trade on the commodities market. Finally, let us mention the work by Djehiche and Hamadène [9] where it is shown that including the possibility of abandonment or bankruptcy in the two-mode switching model over a finite time horizon makes the search for an optimal strategy highly nonlinear and is not at all a trivial extension of previous results.

An example of the class of multiple switching models discussed in Carmona and Ludkovski [5] is related to the management strategies needed to run a power plant that converts natural gas into electricity (through a series of gas turbines) and sells it in the market. The payoff rate from running the plant is roughly given by the difference between the market price of electricity and the market price of gas needed to produce it.

Suppose that, besides running the plant at full capacity or keeping it completely off (the two-mode switching model), there also exists a total of $q-2$ ($q \geq 3$) intermediate operating modes, corresponding to different subsets of turbines running.

Let ℓ_{ij} and Ψ_i denote, respectively, the switching costs from state i to state j to cover the required extra fuel and various overhead costs and the payoff rate in mode i. A management strategy for the power plant is a combination of two sequences:

(i) a nondecreasing sequence of stopping times $(\tau_n)_{n\geq 1}$, where, at time τ_n, the manager decides to switch the production from its current mode to another one;

(ii) a sequence of indicators $(\xi_n)_{n\geq 1}$ taking values in $\{1,\dots,q\}$ of the state the production is switched to. At τ_n for $n \geq 1$, the station is switched from its current mode ξ_{n-1} to ξ_n. The value ξ_0 is deterministic and is the state of the station at time 0.

When the power plant is run under a strategy $\mathcal{S} = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$, over a finite horizon $[0,T]$, the total expected profit up to T for such a strategy is

$$J(\mathcal{S}, i) = \mathbb{E}\left[\int_0^T \sum_{n\geq 0} \left(\Psi_{\xi_n}(s)\mathbb{1}_{(\tau_n,\tau_{n+1}]}(s)\right) ds - \sum_{n\geq 1} \ell_{\xi_{n-1},\xi_n}(\tau_n)\mathbb{1}_{[\tau_n<T]}\right],$$

where we set $\tau_0 = 0$ and $\xi_0 = i$. The optimal switching problem we will investigate is to find a management strategy $\mathcal{S}^*$ such that $J(\mathcal{S}^*, i) = \sup_{\mathcal{S}} J(\mathcal{S}, i)$.

Using purely probabilistic tools such as the Snell envelope of processes and backward stochastic differential equations (BSDEs for short), inspired by the work of Hamadène and Jeanblanc [23], Carmona and Ludkovski [5] suggest a powerful robust numerical scheme based on Monte Carlo regressions to solve this optimal switching problem when the payoff rates are given as deterministic functions of a diffusion process. They also list a number of technical challenges, such as the continuity of the associated value function, that prevent a rigorous proof of the existence and a characterization of an optimal solution of this problem.

The objective of this work is to fill in this gap by providing a solution to the optimal multiple switching problem using the same framework. We are able to prove existence and provide a characterization of an optimal strategy of this problem when

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the payoff rates Ψ_i and the switching costs $\ell_{i,j}$ are adapted only to the filtration generated by a Brownian motion. The generalization, e.g., to a filtration generated by a Brownian motion and an independent Poisson measure is more or less straightforward and can be done as in Hamadène and Hdhiri [24].

We first provide a verification theorem that shapes the problem via the Snell envelope of processes. We show that if the verification theorem is satisfied by a vector of continuous processes $(Y^1, \dots, Y^q)$ such that, for each $i \in \{1, \dots, q\}$,

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau \Psi_i(s)ds + \max_{j \neq i}(-\ell_{ij}(\tau) + Y_\tau^j)1_{[\tau<T]}\big|\mathcal{F}_t\right], \tag{1}$$

then each Y_t^i is the value function of the optimal problem when the system is in mode i at time t:

$$Y_t^i = \operatorname*{ess\,sup}_{\mathcal{S}\in\mathcal{A}_t^i} \mathbb{E}\left[\int_t^T \sum_{n\geq 0}\left(\Psi_{\xi_n}(s)1\!\!1_{(\tau_n,\tau_{n+1}]}(s)\right)ds - \sum_{n\geq 1}\ell_{\xi_{n-1},\xi_n}(\tau_n)1\!\!1_{[\tau_n<T]}\big|\mathcal{F}_t\right],$$

where $\mathcal{A}_t^i$ is the set of admissible strategies such that $\tau_1 \geq t$ a.s. and $\xi_0 = i$. An optimal strategy $\mathcal{S}^*$ is then constructed using the relation (1). Moreover it holds that $Y_0^i = \sup_{\mathcal{S}} J(\mathcal{S}, i)$, provided that the system is in mode i at time $t = 0$.

The unique solution for the verification theorem is obtained as the limit of sequences of processes $(Y^{i,n})_{n\geq 0}$, where for any $t \leq T$, $Y_t^{i,n}$ is the value function (or the optimal yield) from t to T when the system is in mode i at time t and only at most n switchings after t are allowed. This sequence of value functions will be defined recursively (see (15) and (16)).

Finally, if the processes Ψ_i are given by $\psi_i(t, X_t)$, where X is an Itô diffusion and ψ_i are deterministic functions, and if each $\ell_{ij}(t)$ is a deterministic function of t, we prove existence of q deterministic continuous functions $v^1(t,x), \dots, v^q(t,x)$ such that for any $i \in \{1, \dots, q\}$, $Y_t^i = v^i(t, X_t)$. Moreover the vector $(v^1, \dots, v^q)$ is a viscosity solution of a system of q variational inequalities with interconnected obstacles (see (20)). This result improves the one by Tang and Yong [31] proved under rather restrictive assumptions.

The organization of the paper is as follows. In section 2, we give a formulation of the problem and provide some preliminary results. Sections 3 and 4 are devoted to establish the verification theorem and provide an optimal strategy to our problem. In section 5, we show that in the case when Ψ_i are deterministic functions of an Itô process and ℓ_{ij} are deterministic functions of t, the vector of value functions of the switching problem provides a viscosity solution of a system of variational inequalities with interconnected obstacles. This system is the deterministic version of the verification theorem. Finally, in section 6 we address the issue of numerical simulations of the solution of this switching problem.

2. Formulation of the problem, assumptions, and preliminary results. Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ will be a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0\leq t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0\leq t\leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{0\leq t\leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$. Hence $\mathbf{F}$ satisfies the usual conditions; i.e., it is right continuous and complete.

Furthermore, the following hold:

- Let $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of $\mathbf{F}$-progressively measurable sets.

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- Let $\mathcal{M}^p$ be the set of $\mathcal{P}$-measurable and $\mathbb{R}$-valued processes $w = (w_t)_{t\le T}$ such that

$$\mathbb{E}\left[\int_0^T |w_s|^p ds\right] < \infty$$

and $\mathcal{S}^p$ be the set of $\mathcal{P}$-measurable, continuous, $\mathbb{R}$-valued processes $w = (w_t)_{t\le T}$ such that $\mathbb{E}\left[\sup_{0\le t\le T}|w_t|^p\right] < \infty$ ($p > 1$ is fixed).
- For any stopping time $\tau \in [0,T]$, $\mathcal{T}_\tau$ denotes the set of all stopping times θ such that $\tau \le \theta \le T$, $\mathbb{P}$-a.s. Next, for $\tau \in \mathcal{T}_0$, we denote by $\mathcal{F}_\tau$ the σ-algebra of events prior to τ, i.e., the set $\{A \in \mathcal{F},\ A \cap \tau \le t \in \mathcal{F}_t\ \forall t \le T\}$.

The finite horizon multiple switching problem can be formulated as follows. First assume without loss of generality (w.l.o.g.) that the plant is in production mode 1 at $t = 0$ and let $\mathcal{J} := \{1,\dots,q\}$ be the set of all possible activity modes of the production of the commodity. A management strategy of the project consists, on the one hand, of the choice of a sequence of nondecreasing **F**-stopping times $(\tau_n)_{n\ge1}$ (i.e., $\tau_n \le \tau_{n+1}$) where the manager decides to switch the activity from its current mode, say, i, to another one from the set $\mathcal{J}^{-i} \subseteq \{1,\dots,i-1,i+1,\dots,q\}$. On the other hand, it consists of the choice of the mode ξ_n to which the production is switched at τ_n from the current mode i. Therefore we assume that for any $n \ge 1$, ξ_n is a r.v. $\mathcal{F}_{\tau_n}$-measurable with values in $\mathcal{J}$.

Assume that a strategy of running the plant $\mathcal{S} := ((\tau_n)_{n\ge1}, (\xi_n)_{n\ge1})$ is given. We denote by $(u_t)_{t\le T}$ its associated indicator of the production activity mode at time $t \in [0,T]$. It is given by

$$u_t = \mathbb{1}_{[0,\tau_1]}(t) + \sum_{n\ge1} \xi_n \mathbb{1}_{(\tau_n,\tau_{n+1}]}(t). \tag{2}$$

Note that $\tau := (\tau_n)_{n\ge1}$ and the sequence $\xi := (\xi_n)_{n\ge1}$ determine uniquely u and conversely, the left continuous with right limits process u determines uniquely τ and ξ. Therefore a strategy for our multiple switching problem will be simply denoted by u. A strategy $u =: ((\tau_n)_{n\ge1}, (\xi_n)_{n\ge1})$ will be called admissible if it satisfies

$$\lim_{n\to\infty} \tau_n = T \quad \mathbb{P} - \text{a.s.},$$

and the set of admissible strategies is denoted by $\mathcal{A}$.

Now for $i \in \mathcal{J}$, let $\Psi_i := (\Psi_i(t))_{0\le t\le T}$ be a stochastic process which belongs to $\mathcal{M}^p$. In what follows, it stands for the payoff rate per unit time when the plant is in state i. On the other hand, for $i \in \mathcal{J}$ and $j \in \mathcal{J}^{-i}$ let $\ell_{ij} := (\ell_{ij}(t))_{0\le t\le T}$ be a continuous process of $\mathcal{S}^p$. It stands for the switching cost of the production at time t from its current mode i to another mode $j \in \mathcal{J}^{-i}$. For completeness we adopt the convention that $\ell_{ij} \equiv +\infty$ for any $i \in \mathcal{J}$ and $j \in \mathcal{J} - \mathcal{J}^{-i}$ ($j \ne i$). This convention is set in order to exclude the switching from the state i to another state j which does not belong to $\mathcal{J}^{-i}$. Moreover we suppose that there exists a real constant $\gamma > 0$ such that for any $i, j \in \mathcal{J}$ and any $t \le T$, $\ell_{ij}(t) \ge \gamma$.

When a strategy $u := ((\tau_n)_{n\ge1}, (\xi_n)_{n\ge1})$ is implemented, the optimal yield is given by

$$J(u) = J(1,u) = \mathbb{E}\left[\int_0^T \Psi_{u_s}(s)ds - \sum_{n\ge1} \ell_{u_{\tau_{n-1}},u_{\tau_n}}(\tau_n)\mathbb{1}_{[\tau_n<T]}\right].$$

We can now formulate the multiregime starting and stopping problem as follows.

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PROBLEM 1. *Find a strategy* $u^* \equiv ((\tau_n^*)_{n\geq 1}, (\xi_n^*)_{n\geq 1}) \in \mathcal{A}$ *such that*

$$J(u^*) = \sup_{u\in\mathcal{A}} J(u).$$

An admissible strategy u is called *finite* if, during the time interval $[0,T]$, it allows the manager to make only a finite number of decisions, i.e., $\mathbb{P}[\omega, \tau_n(\omega) < T \text{ for all } n \geq 0] = 0$. Hereafter the set of finite strategies will be denoted by $\mathcal{A}^f$. The next proposition tells us that the supremum of the expected total profit can only be reached over finite strategies.

PROPOSITION 1. *The suprema over admissible strategies and finite strategies coincide:*

$$\sup_{u\in\mathcal{A}} J(u) = \sup_{u\in\mathcal{A}^f} J(u). \tag{3}$$

Proof. If u is an admissible strategy which does not belong to $\mathcal{A}^f$, then $J(u) = -\infty$. Indeed, let $B = \{\omega, \tau_n(\omega) < T \text{ for all } n \geq 0\}$ and B^c be its complement. Since $u \in \mathcal{A} \setminus \mathcal{A}^f$, then $\mathbb{P}(B) > 0$. Recall that the processes Ψ_i belong to $\mathcal{M}^p \subset \mathcal{M}^1$. Therefore

$$\begin{aligned} J(u) \leq \mathbb{E}\left[\int_0^T \max_{i\in\mathcal{J}} |\Psi_i(s)|\, ds\right] \\ -\mathbb{E}\left[\left\{\sum_{n\geq 1} \ell_{u_{\tau_{n-1}},u_{\tau_n}}(\tau_n)\right\}\mathbb{1}_B + \left\{\sum_{n\geq 1} \ell_{u_{\tau_{n-1}},u_{\tau_n}}(\tau_n)\mathbb{1}_{[\tau_n<T]}\right\}\mathbb{1}_{B^c}\right] = -\infty, \end{aligned}$$

since for any $t \leq T$ and $i,j \in \mathcal{J}$, $\ell_{ij}(t) \geq \gamma > 0$. This implies that $J(u) = -\infty$ and then (3) is proved. ∎

We finish this section by introducing the key ingredient of the proof of the main result, namely, the notion of the Snell envelope of processes and its properties. We refer to El Karoui [16], Cvitanic and Karatzas [6], Appendix D in Karatzas and Shreve [25], or Hamadène [22] for further details.

2.1. The Snell envelope. In the following proposition we summarize the main results on the Snell envelope of processes used in this paper.

PROPOSITION 2. *Let* $U = (U_t)_{0\leq t\leq T}$ *be an* $\mathbf{F}$*-adapted* $\mathbb{R}$*-valued càdlàg process that belongs to the class [D]; i.e., the set of random variables* $\{U_\tau,\ \tau \in \mathcal{T}_0\}$ *is uniformly integrable. Then there exists an* $\mathbf{F}$*-adapted* $\mathbb{R}$*-valued càdlàg process* $Z := (Z_t)_{0\leq t\leq T}$ *such that* Z *is the smallest supermartingale which dominates* U*; i.e., if* $(\bar{Z}_t)_{0\leq t\leq T}$ *is another càdlàg supermartingale of class [D] such that for all* $0 \leq t \leq T$, $\bar{Z}_t \geq U_t$, *then* $\bar{Z}_t \geq Z_t$ *for any* $0 \leq t \leq T$. *The process* Z *is called the Snell envelope of* U. *Moreover it enjoys the following properties:*

(i) *For any* $\mathbf{F}$*-stopping time* θ *we have*

$$Z_\theta = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_\theta} \mathbb{E}[U_\tau|\mathcal{F}_\theta] \quad (\textit{and then } Z_T = U_T). \tag{4}$$

(ii) *The Doob–Meyer decomposition of* Z *implies the existence of a martingale* $(M_t)_{0\leq t\leq T}$ *and two nondecreasing processes* $(A_t)_{0\leq t\leq T}$ *and* $(B_t)_{0\leq t\leq T}$ *which are, respectively, continuous and purely discontinuous predictable such that for all* $0 \leq t \leq T$,

$$Z_t = M_t - A_t - B_t \quad (\textit{with } A_0 = B_0 = 0).$$

Moreover, for any $0 \leq t \leq T$, $\{\Delta_t B > 0\} \subset \{\Delta_t U < 0\} \cap \{Z_{t-} = U_{t-}\}$.

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(iii) *If U has only positive jumps, then Z is a continuous process. Furthermore, if θ is an $\mathbf{F}$-stopping time and $\tau_\theta^* = \inf\{s \geq \theta, Z_s = U_s\} \wedge T$, then τ_θ^* is optimal after θ, i.e.,*

$$Z_\theta = \mathbb{E}[Z_{\tau_\theta^*}|\mathcal{F}_\theta] = \mathbb{E}[U_{\tau_\theta^*}|\mathcal{F}_\theta] = \operatorname*{ess\,sup}_{\tau \geq \theta} \mathbb{E}[U_\tau|\mathcal{F}_\theta]. \tag{5}$$

(iv) *If $(U^n)_{n\geq 0}$ and U are càdlàg and of class [D] and such that the sequence $(U^n)_{n\geq 0}$ converges increasingly and pointwisely to U, then $(Z^{U^n})_{n\geq 0}$ converges increasingly and pointwisely to Z^U; Z^{U_n} and Z^U are the Snell envelopes of, respectively, U_n and U. Finally, if U belongs to $\mathcal{S}^p$, then Z^U belongs to $\mathcal{S}^p$.*

For the sake of completeness, we give a proof of the stability result (iv), as we could not find it in the standard references mentioned above.

Proof of (iv). Since, for any $n \geq 0$, U^n converges increasingly and pointwisely to U, it follows that for all $t \in [0,T]$, $Z_t^{U_n} \leq Z_t^U$ $\mathbb{P}$-a.s. Therefore $\mathbb{P}$-a.s., for any $t \in [0,T]$, $\lim_{n\to\infty} Z_t^{U_n} \leq Z_t^U$. Note that the process $\left(\lim_{n\to\infty} Z_t^{U^n}\right)_{0\leq t\leq T}$ is a càdlàg supermartingale of class [D], since it is a limit of a nondecreasing sequence of supermartingales (see, e.g., Dellacherie and Meyer [7, p. 86]). But $U^n \leq Z^{U_n}$ implies that $\mathbb{P}$-a.s., for all $t \in [0,T]$, $U_t \leq \lim_{n\to\infty} Z_t^{U^n}$. Thus $Z_t^U \leq \lim_{n\to\infty} Z_t^{U^n}$ since the Snell envelope of U is the lowest supermartingale that dominates U. It follows that $\mathbb{P}$-a.s., for any $t \leq T$, $\lim_{n\to\infty} Z_t^{U^n} = Z_t^U$ and hence the desired result.

Assume now that U belongs to $\mathcal{S}^p$. Since, for any $0 \leq t \leq T$,

$$-\mathbb{E}\left[\sup_{0\leq s\leq T} |U_s| \middle| \mathcal{F}_t\right] \leq U_t \leq \mathbb{E}\left[\sup_{0\leq s\leq T} |U_s| \middle| \mathcal{F}_t\right],$$

using the Doob–Meyer inequality, it follows that Z^U also belongs to $\mathcal{S}^p$. ☐

3. A verification theorem.

In terms of a verification theorem, we show that Problem 1 is expressed in terms of q continuous processes $Y^1, \ldots, Y^q$ solutions of a system of equations expressed via Snell envelopes. The process Y_t^i for $i \in \mathcal{J}$ will stand for the optimal expected profit if, at time t, the production activity is in the state i. So for τ an $\mathbf{F}$-stopping time and $(\zeta_t)_{0\leq t\leq T}$, $(\zeta'_t)_{0\leq t\leq T}$ two continuous $\mathbf{F}$-adapted and $\mathbb{R}$-valued processes let us set

$$D_\tau(\zeta = \zeta') := \inf\{s \geq \tau, \zeta_s = \zeta'_s\} \wedge T.$$

We have the following.

THEOREM 1 (verification theorem). *Assume there exist q $\mathcal{S}^p$-processes $(Y^i := (Y_t^i)_{0\leq t\leq T},\ i = 1,\ldots,q)$ that satisfy* (1). *Then $Y^1,\ldots,Y^q$ are unique. Furthermore, the following hold:*

(i)

$$Y_0^1 = \sup_{v\in\mathcal{A}} J(v). \tag{6}$$

(ii) *Define the sequence of $\mathbf{F}$-stopping times $(\tau_n)_{n\geq 1}$ by*

$$\tau_1 = D_0\left(Y^1 = \max_{j\in\mathcal{J}^{-1}} \left(-\ell_{1j} + Y^j\right)\right) \tag{7}$$

and, for $n \geq 2$,

$$\tau_n = D_{\tau_{n-1}}\left(Y^{u_{\tau_{n-1}}} = \max_{k\in\mathcal{J}^{-\tau_{n-1}}} \left(-\ell_{\tau_{n-1}k} + Y^k\right)\right), \tag{8}$$

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where,

- $u_{\tau_1} = \sum_{j\in\mathcal{J}} j 1\!\!1_{\{\max_{k\in\mathcal{J}^{-1}}(-\ell_{1k}(\tau_1)+Y^k_{\tau_1})=-\ell_{1j}(\tau_1)+Y^j_{\tau_1}\}}$;
- *for any* $n \geq 1$ *and* $t \geq \tau_n$, $Y_t^{u_{\tau_n}} = \sum_{j\in\mathcal{J}} 1\!\!1_{[u_{\tau_n}=j]} Y_t^j$;
- *for* $n \geq 2$, $u_{\tau_n} = l$ *on the set* $\left\{ \max_{k\in\mathcal{J}^{-u_{\tau_{n-1}}}} \left(-\ell_{u_{\tau_{n-1}}k}(\tau_n) + Y^k_{\tau_n}\right) = -\ell_{u_{\tau_{n-1}}l}(\tau_n) + Y^l_{\tau_n} \right\}$, *where*

$$\ell_{u_{\tau_{n-1}}k}(\tau_n) = \sum_{j\in\mathcal{J}} 1\!\!1_{[\tau_{n-1}=j]}\ell_{jk}(\tau_n) \quad \text{and} \quad \mathcal{J}^{-u_{\tau_{n-1}}} = \sum_{j\in\mathcal{J}} 1\!\!1_{[\tau_{n-1}=j]}\mathcal{J}^{-j}.$$

Then the strategy $u = ((\tau_n)_{n\geq 1}, (\xi_n)_{n\geq 1})$ *is optimal, i.e.,* $J(u) \geq J(v)$ *for any* $v \in \mathcal{A}$.

Proof. The proof consists essentially in showing that each process Y^i, as defined by (1), is nothing but the expected total profit or the value function of the optimal problem, given that the system is in mode i at time t. More precisely,

$$Y_t^i = \operatorname*{ess\,sup}_{u\in\mathcal{A}_t^{f,i}} \mathbb{E}\left[\int_t^T \Psi_{u_s}(s)ds - \sum_{j\geq 1} \ell_{u_{\tau_{j-1}}u_{\tau_j}}(\tau_j) 1\!\!1_{[\tau_j<T]}\middle|\mathcal{F}_t\right], \tag{9}$$

where $\mathcal{A}_t^{f,i}$ is the set of finite strategies such that $\tau_1 \geq t$, $\mathbb{P}$-a.s., and $u_t = i$ (at time t the system is in mode i). This characterization implies in particular that the processes $Y^1, \ldots, Y^q$ are unique. Moreover, thanks to a repeated use of the characterization of the Snell envelope (Proposition 2 (iii)), the strategy u defined recursively by (7) and (8) is shown to be optimal.

Indeed, assuming that at time $t = 0$ the system is in mode 1, it holds true that for any $0 \leq t \leq T$,

$$Y_t^1 + \int_0^t \Psi_1(s)ds = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_0^\tau \Psi_1(s)ds + \max_{j\in\mathcal{J}^{-1}}\left(-\ell_{1j}(\tau) + Y^j_\tau\right) 1\!\!1_{[\tau<T]}\middle|\mathcal{F}_t\right]. \tag{10}$$

But Y_0^1 is $\mathcal{F}_0$-measurable. Therefore it is $\mathbb{P}$-a.s. constant and then $Y_0^1 = \mathbb{E}[Y_0^1]$.

On the other hand, according to Proposition 2 (iii), τ_1 as defined by (7) is optimal, and

$$u_{\tau_1} = \sum_{j\in\mathcal{J}} j 1\!\!1_{\left\{\max_{k\in\mathcal{J}^{-1}}\left(-\ell_{1k}(\tau_1)+Y^k_{\tau_1}\right)=-\ell_{1j}(\tau_1)+Y^j_{\tau_1}\right\}}.$$

Therefore

$$\begin{aligned} Y_0^1 &= \mathbb{E}\left[\int_0^{\tau_1} \Psi_1(s)ds + \max_{j\in\mathcal{J}^{-1}}\left(-\ell_{1j}(\tau_1) + Y^j_{\tau_1}\right) 1\!\!1_{[\tau_1<T]}\right] \\ &= \mathbb{E}\left[\int_0^{\tau_1} \Psi_1(s)ds + \left(-\ell_{1u_{\tau_1}}(\tau_1) + Y^{u_{\tau_1}}_{\tau_1}\right) 1\!\!1_{[\tau_1<T]}\right]. \end{aligned} \tag{11}$$

Next we claim that $\mathbb{P}$-a.s. for every $\tau_1 \leq t \leq T$,

$$Y_t^{u_{\tau_1}} = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_t^\tau \Psi_{u_{\tau_1}}(s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}}\left(-\ell_{u_{\tau_1}j}(\tau) + Y^j_\tau\right) 1\!\!1_{[\tau<T]}\middle|\mathcal{F}_t\right]. \tag{12}$$

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To see this, note that (1) means that the process $\left\{Y_t^i + \int_0^t \Psi_i(s)ds, 0 \le t \le T\right\}$ is a supermartingale which dominates

$$\left\{\int_0^t \Psi_i(s)ds + \max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(t) + Y_t^j\right) 1\!\!1_{[t<T]},\ 0 \le t \le T\right\}.$$

Since $\mathcal{J}$ is finite, the process $\sum_{i\in\mathcal{J}} 1\!\!1_{[u_{\tau_1}=i]}\left(Y_t^i + \int_{\tau_1}^t \Psi_i(s)ds\right)$ for $t \ge \tau_1$ is a supermartingale which dominates $\sum_{i\in\mathcal{J}} 1\!\!1_{[u_{\tau_1}=i]}\left(\int_{\tau_1}^t \Psi_i(s)ds + \max_{j\in\mathcal{J}^{-i}}\left(-\ell_{ij}(t) + Y_t^j\right)1\!\!1_{[t<T]}\right)$. Thus the process $Y_t^{u_{\tau_1}} + \int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds$ for $t \ge \tau_1$ is a supermartingale which is greater than

$$\int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} \left(-\ell_{u_{\tau_1}j}(t) + Y_t^j\right) 1\!\!1_{[t<T]}. \tag{13}$$

To complete the proof it remains to show that it is the smallest one which has this property and use the characterization of the Snell envelope given in Proposition 2.

Indeed, let $(Z_t)_{t\in[0,T]}$ be a supermartingale of class [D] such that for any $t \ge \tau_1$,

$$Z_t \ge \int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} \left(-\ell_{u_{\tau_1}j}(t) + Y_t^j\right) 1\!\!1_{[t<T]}.$$

It follows that for every $t \ge \tau_1$,

$$Z_t 1\!\!1_{[u_{\tau_1}=i]} \ge 1\!\!1_{[u_{\tau_1}=i]} \left(\int_{\tau_1}^t \Psi_i(s)ds + \max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(t) + Y_t^j\right) 1\!\!1_{[t<T]}\right).$$

But the process $(Z_t 1\!\!1_{[u_{\tau_1}=i]})_{t\in[0,T]}$ is a supermartingale for $t \ge \tau_1$ since $1\!\!1_{[u_{\tau_1}=i]}$ is $\mathcal{F}_t$-measurable and nonnegative. With again (1) it follows that for every $t \ge \tau_1$,

$$1\!\!1_{[u_{\tau_1}=i]} Z_t \ge 1\!\!1_{[u_{\tau_1}=i]} \left(Y_t^i + \int_{\tau_1}^t \Psi_i(s)ds\right).$$

Summing over i, we get for every $t \ge \tau_1$,

$$Z_t \ge Y_t^{u_{\tau_1}} + \int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds.$$

Hence the process $Y_t^{u_{\tau_1}} + \int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds$, $t \ge \tau_1$, is the Snell envelope of

$$\int_{\tau_1}^t \Psi_{u_{\tau_1}}(s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} \left(-\ell_{u_{\tau_1}j}(t) + Y_t^j\right) 1\!\!1_{[t<T]}$$

and hence (12).

Now, from (12) and the definition of τ_2 in (8), we have

$$\begin{aligned} Y_{\tau_1}^{u_{\tau_1}} &= \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \Psi_{u_{\tau_1}}(s)ds + \max_{j\in\mathcal{J}^{-u_{\tau_1}}} \left(-\ell_{u_{\tau_1}j}(\tau_2) + Y_{\tau_2}^j\right) 1\!\!1_{[\tau_2<T]} \middle| \mathcal{F}_{\tau_1}\right] \\ &= \mathbb{E}\left[\int_{\tau_1}^{\tau_2} \Psi_{u_{\tau_1}}(s)ds + \left(-\ell_{u_{\tau_1}u_{\tau_2}}(\tau_2) + Y_{\tau_2}^{u_{\tau_2}}\right) 1\!\!1_{[\tau_2<T]} \middle| \mathcal{F}_{\tau_1}\right]. \end{aligned}$$

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Setting this characterization of $Y_{\tau_1}^{u_{\tau_1}}$ in (11) and noting that $1\!\!1_{[\tau_1<T]}$ is F_{τ_1}-measurable, it follows that

$$\begin{aligned} Y_0^1 &= \mathbb{E}\left[\int_0^{\tau_1} \Psi_1(s)ds - \ell_{1u_{\tau_1}}(\tau_1)1\!\!1_{[\tau_1<T]}\right] \\ &\quad +\mathbb{E}\left[\int_{\tau_1}^{\tau_2} \Psi_{u_{\tau_1}}(s)ds.1\!\!1_{[\tau_1<T]} - \ell_{u_{\tau_1}u_{\tau_2}}(\tau_2)1\!\!1_{[\tau_2<T]} + Y_{\tau_2}^{u_{\tau_2}}1\!\!1_{[\tau_2<T]}\right] \\ &= \mathbb{E}\left[\int_0^{\tau_2} \Psi_{u_s}(s)ds - \ell_{1u_{\tau_1}}(\tau_1)1\!\!1_{[\tau_1<T]} - \ell_{u_{\tau_1}u_{\tau_2}}(\tau_2)1\!\!1_{[\tau_2<T]} + Y_{\tau_2}^{u_{\tau_2}}1\!\!1_{[\tau_2<T]}\right], \end{aligned}$$

since $[\tau_2 < T] \subset [\tau_1 < T]$.

Repeating this procedure n times, we obtain

$$Y_0^1 = \mathbb{E}\left[\int_0^{\tau_n} \Psi_{u_s}(s)ds - \sum_{j=1}^{n} \ell_{u_{\tau_{j-1}}u_{\tau_j}}(\tau_j)1\!\!1_{[\tau_j<T]} + Y_{\tau_n}^{u_{\tau_n}}1\!\!1_{[\tau_n<T]}\right]. \tag{14}$$

But the strategy $(\tau_n)_{n\geq 1}$ is finite; otherwise Y_0^1 would be equal to $-\infty$ since $\ell_{ij} \geq \gamma > 0$ contradicting the assumption that the processes Y^j belong to $\mathcal{S}^p$. Therefore taking the limit as $n \to \infty$ we obtain $Y_0^1 = J(u)$.

To complete the proof it remains to show that $J(u) \geq J(v)$ for any other finite admissible strategy $v \equiv ((\theta_n)_{n\geq 1}, (\zeta_n)_{n\geq 1})$. The definition of the Snell envelope yields

$$\begin{aligned} Y_0^1 &\geq \mathbb{E}\left[\int_0^{\theta_1} \Psi_1(s)ds + \max_{j\in\mathcal{J}^{-1}} \left(-\ell_{1j}(\theta_1) + Y_{\theta_1}^j\right)1\!\!1_{[\theta_1<T]}\right] \\ &\geq \mathbb{E}\left[\int_0^{\theta_1} \Psi_1(s)ds + \left(-\ell_{1v_{\theta_1}}(\theta_1) + Y_{\theta_1}^{v_{\theta_1}}\right)1\!\!1_{[\theta_1<T]}\right]. \end{aligned}$$

But once more using a similar characterization as (12) we get

$$\begin{aligned} Y_{\theta_1}^{v_{\theta_1}} &\geq \mathbb{E}\left[\int_{\theta_1}^{\theta_2} \Psi_{v_{\theta_1}}(s)ds + \max_{j\in\mathcal{J}^{-v_{\theta_1}}} \left(-\ell_{v_{\theta_1}j}(\theta_2) + Y_{\theta_2}^j\right)1\!\!1_{[\theta_2<T]}\middle|\mathcal{F}_{\theta_1}\right] \\ &\geq \mathbb{E}\left[\int_{\theta_1}^{\theta_2} \Psi_{v_{\theta_1}}(s)ds + \left(-\ell_{v_{\theta_1}v_{\theta_2}}(\theta_2) + Y_{\theta_2}^{v_{\theta_2}}\right)1\!\!1_{[\theta_2<T]}\middle|\mathcal{F}_{\theta_1}\right]. \end{aligned}$$

Therefore

$$\begin{aligned} Y_0^1 &\geq \mathbb{E}\left[\int_0^{\theta_1} \Psi_1(s)ds - \ell_{1v_{\theta_1}}(\theta_1)1\!\!1_{[\theta_1<T]}\right] \\ &\quad +\mathbb{E}\left[1\!\!1_{[\theta_1<T]}\int_{\theta_1}^{\theta_2} \Psi_{v_{\theta_1}}(s)ds - \ell_{v_{\theta_1}v_{\theta_2}}(\theta_2)1\!\!1_{[\theta_2<T]} + Y_{\theta_2}^{v_{\theta_2}}1\!\!1_{[\theta_2<T]}\right] \\ &= \mathbb{E}\left[\int_0^{\theta_2} \Psi_{v_s}(s)ds - \ell_{1v_{\theta_1}}(\theta_1)1\!\!1_{[\theta_1<T]} - \ell_{v_{\theta_1}v_{\theta_2}}(\theta_2)1\!\!1_{[\theta_2<T]} + Y_{\theta_2}^{v_{\theta_2}}1\!\!1_{[\theta_2<T]}\right]. \end{aligned}$$

Repeat this argument n times to obtain

$$Y_0^1 \geq \mathbb{E}\left[\int_0^{\theta_n} \Psi_{v_s}(s)ds - \sum_{j=1}^{n} \ell_{v_{\theta_{n-1}}v_{\theta_n}}(\theta_n)1\!\!1_{[\theta_n<T]} + Y_{\theta_n}^{v_{\theta_n}}1\!\!1_{[\theta_n<T]}\right].$$

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Finally, thanks to the dominated convergence theorem, taking the limit as $n \to \infty$ yields

$$Y_0^1 \geq \mathbb{E}\left[\int_0^T \Psi_{v_s}(s)ds - \sum_{j\geq 1} \ell_{v_{\theta_{n-1}} v_{\theta_n}}(\theta_n) 1\!\!1_{[\theta_n < T]}\right] = J(v)$$

since the strategy v is finite. Hence the strategy u is optimal. The proof is now complete. ∎

4. Existence of the processes $(Y^1, \ldots, Y^q)$. We will now establish existence of the processes $(Y^1, \ldots, Y^q)$. They will be obtained as a limit of a sequence of processes $(Y^{1,n}, \ldots, Y^{q,n})_{n\geq 0}$ defined recursively by means of the Snell envelope notion as follows:

For $i \in \mathcal{J}$, let us set, for any $0 \leq t \leq T$,

$$Y_t^{i,0} = \mathbb{E}\left[\int_0^T \Psi_i(s)ds \middle| \mathcal{F}_t\right] - \int_0^t \Psi_i(s)ds \tag{15}$$

and for $n \geq 1$,

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_0^\tau \Psi_i(s)ds + \max_{k\in\mathcal{J}^{-i}} \left(-\ell_{ik}(\tau) + Y_\tau^{k,n-1}\right) 1\!\!1_{[\tau<T]} \middle| \mathcal{F}_t\right] - \int_0^t \Psi_i(s)ds. \tag{16}$$

In the next proposition we collect some useful properties of $Y^{1,n}, \ldots, Y^{q,n}$. In particular we show that, as $n \to \infty$, the limit processes $\tilde{Y}^i := \lim_{n\to\infty} Y^{i,n}$ exist and are only càdlàg but have the same characterization (1) as the Y^i's. Thus the existence proof of the Y^i's will consist in showing that $\tilde{Y}^i$'s are continuous and hence satisfy the verification theorem. This will be done in Theorem 2 below.

PROPOSITION 3.

(i) *For each $n \geq 0$, the processes $Y^{1,n}, \ldots, Y^{q,n}$ are continuous and belong to $\mathcal{S}^p$.*

(ii) *For any $i \in \mathcal{J}$, the sequence $(Y^{i,n})_{n\geq 0}$ converges increasingly and pointwisely P-a.s. for any $0 \leq t \leq T$ and in $\mathcal{M}^p$ to càdlàg processes $\tilde{Y}^i$. Moreover these limit processes $\tilde{Y}^i = (\tilde{Y}^i_t)_{0\leq t\leq T}$, $i = 1, \ldots, q$, satisfy the following:*

(a) $\mathbb{E}\left[\sup_{0\leq t\leq T} \left|\tilde{Y}_t^i\right|^p\right] < \infty, \quad i \in \mathcal{J}.$

(b) *For any $0 \leq t \leq T$ we have*

$$\tilde{Y}_t^i = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau \Psi_i(s)ds + \max_{k\in\mathcal{J}^{-i}} \left(-\ell_{ik}(\tau) + \tilde{Y}_\tau^k\right) 1\!\!1_{[\tau<T]} \middle| \mathcal{F}_t\right].$$

Proof. (i) Let us show by induction that for any $n \geq 0$ and every $i \in \mathcal{J}$, $Y^{i,n} \in \mathcal{S}^p$. For $n = 0$ the property holds true since we can write $Y^{i,0}$ as the sum of a continuous process and a martingale w.r.t. to the Brownian filtration. Therefore $Y^{i,0}$ is continuous, and since the process $(\Psi_i(s))_{0\leq s\leq T}$ belongs to $\mathcal{M}^p$, using Doob's inequality, we obtain that $Y^{i,0}$ belongs to $\mathcal{S}^p$. Suppose now that the property is satisfied for some n. By Proposition 2, for every $i \in \mathcal{J}$ and up to a term, $Y^{i,n+1}$ is the Snell envelope of the process $\left(\int_0^t \Psi_i(s)ds + \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t) + Y_t^{k,n}) 1\!\!1_{[t<T]}\right)_{0\leq t\leq T}$ and verifies $Y_T^{i,n+1} = 0$. Since $\max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t) + Y_t^{k,n})_{|t=T} < 0$, this process is continuous on $[0, T)$ and has a positive jump at T. Hence $Y^{i,n+1}$ is continuous (Proposition 2 (iii)) and belongs to $\mathcal{S}^p$. This shows that, for every $i \in \mathcal{J}$, $Y^{i,n} \in \mathcal{S}^p$ for any $n \geq 0$.

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(ii) Let us now set $\mathcal{A}_t^{i,n} = \{u \in \mathcal{A} \text{ such that } u_0 = i,\ \tau_1 \geq t, \text{ and } \tau_{n+1} = T\}$. Using the same arguments as the ones of the verification theorem (Theorem 1) the following characterization of the processes $Y^{i,n}$ holds true:

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{u \in \mathcal{A}_t^{i,n}} \mathbb{E}\left[\int_t^T \Psi_{u_s}(s)ds - \sum_{j=1}^{n} \ell_{u_{\tau_{j-1}} u_{\tau_j}}(\tau_j)\mathbb{1}_{[\tau_j < T]} \middle| \mathcal{F}_t\right]. \tag{17}$$

Since $\mathcal{A}_t^{i,n} \subset \mathcal{A}_t^{i,n+1}$, we have P-a.s. for all $t \in [0,T]$, $Y_t^{i,n} \leq Y_t^{i,n+1}$ thanks to the continuity of $Y^{i,n}$. Using once more (17) and since $\ell_{ij} \geq \gamma > 0$, we obtain for each $i \in \mathcal{J}$,

$$\forall\ 0 \leq t \leq T, \quad Y_t^{i,n} \leq \mathbb{E}\left[\int_t^T \max_{i=1,\dots,q} |\Psi_i(s)|\, ds \middle| \mathcal{F}_t\right]. \tag{18}$$

Therefore, for every $i \in \mathcal{J}$, the sequence $(Y^{i,n})_{n\geq 0}$ is convergent, and then let us set $\tilde{Y}_t^i := \lim_{n\to\infty} Y_t^{i,n}$ for $t \leq T$. The process $\tilde{Y}^i$ satisfies

$$Y_t^{i,0} \leq \tilde{Y}_t^i \leq \mathbb{E}\left[\int_t^T \max_{i=1,\dots,q} |\Psi_i(s)|\, ds \middle| \mathcal{F}_t\right], \quad 0 \leq t \leq T. \tag{19}$$

Let us now show that $\tilde{Y}^i$ is càdlàg. Actually, for each $i \in \mathcal{J}$ and $n \geq 1$, by (16), the process $\left(Y_t^{i,n} + \int_0^t \Psi_i(s)ds\right)_{0\leq t\leq T}$ is a continuous supermartingale. Hence its limit process $\left(\tilde{Y}_t^i + \int_0^t \Psi_i(s)ds\right)_{0\leq t\leq T}$ is càdlàg as a limit of increasing sequence of continuous supermartingales (see Dellacherie and Meyer [7, p. 86]). Therefore $\tilde{Y}^i$ is càdlàg . Next, using (19), the L^p-properties of Ψ_i and Doob's maximal inequality yield, for each $i \in \mathcal{J}$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left|\tilde{Y}_t^i\right|^p\right] < \infty.$$

By the Lebesgue dominated convergence theorem, the sequence $(Y^{i,n})_{n\geq 0}$ also converges to $\tilde{Y}^i$ in $\mathcal{M}^p$.

Finally, the càdlàg processes $\tilde{Y}^1, \dots, \tilde{Y}^q$ satisfy (1), since they are limits of the increasing sequence of processes $Y^{i,n}$, $i \in \mathcal{J}$, that satisfy (16). We use Proposition 2 (iv) to conclude. ∎

We will now prove that the processes $\tilde{Y}^1, \dots, \tilde{Y}^q$ are continuous and satisfy the verification theorem (Theorem 1).

THEOREM 2. *The limit processes $\tilde{Y}^1, \dots, \tilde{Y}^q$ satisfy the verification theorem.*

Proof. Recall from Proposition 3 that the processes $\tilde{Y}^1, \dots, \tilde{Y}^q$ are càdlàg and uniformly L^p-integrable and satisfy (1). It remains to prove that they are continuous.

Indeed, note that, for $i \in \mathcal{J}$, the process $\left(\tilde{Y}_t^i + \int_0^t \Psi_i(s)ds\right)_{0\leq t\leq T}$ is the Snell envelope of

$$\left(\int_0^t \Psi_i(s)ds + \max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(t) + \tilde{Y}_t^k)\mathbb{1}_{[t<T]}\right)_{0\leq t\leq T}.$$

Of course the processes $\left(\int_0^t \Psi_i(s)ds\right)_{0\leq t\leq T}$ are continuous. Therefore from the property of the jumps of the Snell envelope (Proposition 2 (ii)), when there is a (necessarily negative) jump of $\tilde{Y}^i$ at t, there is a jump, at the same time t, of the process

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$(\max_{k\in\mathcal{J}^{-i}}(-\ell_{ik}(s)+\tilde{Y}^k_s))_{0\le s\le T}$. Since ℓ_{ij} are continuous, there is $j\in\mathcal{J}^{-i}$ such that $\Delta_t\tilde{Y}^j<0$, and $\tilde{Y}^i_{t-}=-\ell_{ij}(t)+\tilde{Y}^j_{t-}$.

Suppose now there is an index $i_1\in\mathcal{J}$ for which there exists $t\in[0,T]$ such that $\Delta_t\tilde{Y}^{i_1}<0$. This implies that there exists another index $i_2\in\mathcal{J}^{-i_1}$ such that $\Delta_t\tilde{Y}^{i_2}<0$ and $\tilde{Y}^{i_1}_{t-}=-\ell_{i_1i_2}(t)+\tilde{Y}^{i_2}_{t-}$. But given i_2, there exists an index $i_3\in\mathcal{J}^{-i_2}$ such that $\Delta_t\tilde{Y}^{i_3}<0$ and $\tilde{Y}^{i_2}_{t-}=-\ell_{i_2i_3}(t)+\tilde{Y}^{i_3}_{t-}$. Repeating this argument many times, we get a sequence of indices $i_1,\dots,i_j,\dots\in\mathcal{J}$ that have the property that $i_k\in\mathcal{J}^{-i_{k-1}}$, $\Delta_t\tilde{Y}^{i_k}<0$, and $\tilde{Y}^{i_{k-1}}_{t-}=-\ell_{i_{k-1}i_k}(t)+\tilde{Y}^{i_k}_{t-}$.

Since $\mathcal{J}$ is finite, there exist two indices $m<r$ such that $i_m=i_r$ and $i_m,i_{m+1},\dots,i_{r-1}$ are mutually different. It follows that

$$\begin{aligned}\tilde{Y}^{i_m}_{t-}&=-\ell_{i_mi_{m+1}}(t)+\tilde{Y}^{i_{m+1}}_{t-}=-\ell_{i_mi_{m+1}}(t)-\ell_{i_{m+1}i_{m+2}}(t)+\tilde{Y}^{i_{m+2}}_{t-}\\&=\cdots=-\ell_{i_mi_{m+1}}(t)-\cdots-\ell_{i_{r-1}i_r}(t)+\tilde{Y}^{i_r}_{t-}.\end{aligned}$$

As $i_m=i_r$ we get

$$-\ell_{i_mi_{m+1}}(t)-\cdots-\ell_{i_{r-1}i_r}(t)=0$$

which is impossible since for any $i\neq j$, all $0\le t\le T$, $\ell_{ij}(t)\ge\gamma>0$. Therefore there is no $i\in\mathcal{J}$ for which there is a $t\in[0,T]$ such that $\Delta_t\tilde{Y}^i<0$. This means that the processes $\tilde{Y}^1,\dots,\tilde{Y}^q$ are continuous. Since they satisfy (1), then by uniqueness $Y^i=\tilde{Y}^i$ for any $i\in\mathcal{J}$. Thus the verification theorem (Theorem 1) is satisfied by $Y^1,\dots,Y^q$. ∎

We end this section by the following convergence result of the sequences $(Y^{i,n})_{n\ge0}$ to Y^i's.

PROPOSITION 4. *It holds true that for any* $i\in\mathcal{J}$,

$$\mathbb{E}\left[\sup_{s\le T}\left|Y^{i,n}_s-Y^i_s\right|^p\right]\to0\quad as\quad n\to+\infty.$$

Proof. By Proposition 3, we know that $\mathbb{P}$-a.s. for any $n\ge1$, the function $t\mapsto Y^{i,n}_t(\omega)$ is continuous, and for any $0\le t\le T$ the sequence $(Y^{i,n}_t(\omega))_{n\ge1}$ converges increasingly to $Y^i_t(\omega)$. As the function $t\mapsto Y^i_t(\omega)$ is continuous, then thanks to Dini's theorem it holds true that

$$\mathbb{P}\text{-a.s. }\lim_{n\to\infty}\sup_{0\le t\le T}|Y^{i,n}_t(\omega)-Y^i_t(\omega)|=0.$$

The result now follows from the Lebesgue dominated convergence theorem. ∎

5. Connection with systems of variational inequalities. In this section we will assume that the switching processes ℓ_{ij} are deterministic functions of the time variable. An example of such a family of switching costs is

$$\ell_{ij}(t)=e^{-rt}a_{ij},$$

where a_{ij} are constant costs and $r>0$ is some discounting rate. We moreover assume that the payoff rates are given by $\Psi_i(\omega,t)=\psi_i(t,X_t)$, where ψ_i are deterministic functions and $X=(X_t)_{t\ge0}$ is a vector of stochastic processes that stands for the market price of the underlying commodities and other financial assets that influence the production of energy. When the underlying market price process X is Markov, the classical methods of solving impulse problems (cf. Brekke and Øksendal [3], Guo and

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Pham [21], and Tang and Yong [31]) formulate a verification theorem suggesting that the value function of our optimal switching problem is the unique viscosity solution of the following system of quasi-variational inequalities (QVI) with interconnected obstacles:

(20)
$$\begin{cases} \min\{v^i(t,x) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t) + v^j(t,x)), -\partial_t v^i(t,x) - \mathcal{A}v^i(t,x) - v^i(t,x)\} = 0, \\ v^i(T,x) = 0, \qquad i \in \mathcal{J}, \end{cases}$$

where $\mathcal{A}$ is the infinitesimal generator of the driving process X.

By means of yet another characterization of the Snell envelope in terms of systems of reflected BSDEs, due to El Karoui et al. [17, Theorems 7.1 and 8.5], we are able to show that the vector of value processes $(Y^1, \dots, Y^q)$ of our switching problem provides a viscosity solution of the system (20). Actually we show that under mild assumptions on the coefficients $\psi_i(t,x)$ and $\ell_{ij}(t)$,

$$Y_t^i = v^i(t, X_t), \quad 0 \le t \le T, \quad i \in \mathcal{J},$$

where $v^1(t,x), \dots, v^q(t,x)$ are continuous deterministic functions and are viscosity solutions of the system of QVI with interconnected obstacles (20). Alternative approaches to this problem include the stochastic target problem discussed in Bouchard [1] and the dynamic programming principle used in Tang and Yong [31]. In [1], Bouchard provides a solution for (20) in a weak sense since he faces an issue in connection with a lack of continuity of the solution, while, in [31], Tang and Yong obtain a unique continuous viscosity solution under rather stronger assumptions than ours. In this section, using the well known link between BSDEs and variational inequalities, we obtain the existence of a continuous viscosity solution for (20) in a more general framework than, e.g., the one in Tang and Yong [31]. Uniqueness is obtained using recent results by El Asri and Hamadène [15], where existence of a unique viscosity solution of the switching problem (20) is obtained under weaker assumptions on the switching costs ℓ_{ij}. Actually they consider the case when each of the ℓ_{ij}'s depends also on x. However, their proof of continuity is quite technical, while ours, displayed below, is easier, which is our primary motivation for including it here.

For $(t,x) \in [0,T] \times \mathbb{R}^k$, let $(X_s^{t,x})_{s\le T}$ be the solution of the following Itô diffusion:

(21) $$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \quad t \le s \le T; \qquad X_s^{t,x} = x \quad \text{for } s \le t,$$

where the functions b and σ, with appropriate dimensions, satisfy the following standard conditions: there exists a constant $C \ge 0$ such that

(22)
$$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \quad \text{and} \quad |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \le C|x-x'|$$

for any $t \in [0,T]$ and $x, x' \in \mathbb{R}^k$.

These properties of σ and b imply in particular that the process $X^{t,x} := (X_s^{t,x})_{0\le s\le T}$, solution of (21), exists and is unique. Its infinitesimal generator $\mathcal{A}$ is given by

(23) $$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^*)_{ij}(t,x)\partial_{ij}^2 + \sum_{i=1}^{k} b_i(t,x)\partial_i.$$

Moreover the following estimates hold true (see, e.g., Revuz and Yor [28] for more details).

PROPOSITION 5. *The process $X^{t,x}$ satisfies the following estimates:*

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(i) *For any $\theta \geq 2$, there exists a constant C such that*

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|X_s^{t,x}\right|^\theta\right] \leq C(1+|x|^\theta). \tag{24}$$

(ii) *There exists a constant C such that for any $t,t' \in [0,T]$ and $x,x' \in \mathbb{R}^k$,*

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|X_s^{t,x}-X_s^{t',x'}\right|^2\right] \leq C(1+|x|^2)(|x-x'|^2+|t-t'|). \tag{25}$$

Let us now introduce the following assumption on the payoff rates ψ_i and the switching cost functions ℓ_{ij}.

Assumption [H].

(H1) The running costs ψ_i, $i=1,\dots,q$ (of subsection 2.1), are jointly continuous and are of polynomial growth; i.e., there exist some positive constants C and δ such that for each $i \in \mathcal{J}$,

$$|\psi_i(t,x)| \leq C(1+|x|^\delta)\ \forall (t,x) \in [0,T]\times\mathbb{R}^k.$$

(H2) For any $i,j \in \mathcal{J}$, the switching costs ℓ_{ij} are deterministic continuous functions of t, and there exists a real constant $\gamma > 0$ such that for any $0 \leq t \leq T$, $\min\{\ell_{ij}(t), i,j \in \mathcal{J},\ i \neq j\} \geq \gamma$.

Taking into account Proposition 5, the processes $(\psi_i(s,X_s^{t,x}))_{0\leq s\leq T}$ belong to $\mathcal{M}^2$, a condition we will need to establish a characterization of the value processes of our optimal problem with a class of reflected BSDEs.

Recall the notion of viscosity solution of the system (20).

DEFINITION 1. *Let $(v_1,\dots,v_q)$ be a vector of continuous functions on $[0,T]\times\mathbb{R}^k$ with values in $\mathbb{R}^q$ and such that $(v_1,\dots,v_q)(T,x)=0$ for any $x \in \mathbb{R}^k$. The vector $(v_1,\dots,v_q)$ is called*

(i) *a viscosity supersolution (resp., subsolution) of the system* (20) *if for any $(t_0,x_0) \in [0,T]\times\mathbb{R}^k$ and any q-uplet functions $(\varphi_1,\dots,\varphi_q) \in (C^{1,2}([0,T]\times\mathbb{R}^k))^q$ such that $(\varphi_1,\dots,\varphi_q)(t_0,x_0) = (v_1,\dots,v_q)(t_0,x_0)$ and for any $i \in \mathcal{J}$, (t_0,x_0) is a maximum (resp., minimum) of $\varphi_i - v_i$, then we have for any $i \in \mathcal{J}$,*

$$\begin{aligned}\min\{v_i(t_0,x_0) - \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t_0)+v_j(t_0,x_0)), -\partial_t\varphi_i(t_0,x_0)\\ -\mathcal{A}\varphi_i(t_0,x_0)-\psi_i(t_0,x_0)\} \geq 0\ (\textit{resp.}, \leq 0);\end{aligned} \tag{26}$$

(ii) *a viscosity solution of the system* (20) *if it is both a viscosity supersolution and subsolution.*

Now let $(Y_s^{1;t,x},\dots,Y_s^{q;t,x})_{0\leq s\leq T}$ be the vector of value processes which satisfies the verification theorem (Theorem 1) associated with $(\psi_i(s,X_s^{t,x}))_{s\leq T}$ and $(\ell_{ij}(s))_{s\leq T}$. The vector $(Y^{1;t,x},\dots,Y^{q;t,x})$ exists through Theorem 2 combined with the estimates of $X^{t,x}$ of Proposition 5 and Assumption [H].

The following theorem is the main result of this section.

THEOREM 3. *Under Assumption* [H]*, there exist q deterministic functions $v^1(t,x),\dots,v^q(t,x)$ defined on $[0,T]\times\mathbb{R}^k$ and $\mathbb{R}$-valued such that the following hold:*

(i) *$v^1,\dots,v^q$ are continuous in (t,x), are of polynomial growth, and satisfy, for each $t \in [0,T]$ and for every $s \in [t,T]$,*

$$Y_s^{i;t,x} = v^i(s,X_s^{t,x}) \ \ \textit{for every } i \in \mathcal{J}.$$

(ii) *The vector of functions $(v^1,\dots,v^q)$ is a viscosity solution for the system of variational inequalities* (20).

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Proof. The proof is obtained through the three following steps.

Step 1 (an approximation scheme). For $n \geq 0$, let $(Y_s^{1,n;t,x})_{0\leq s\leq T}, \ldots, (Y_s^{q,n;t,x})_{0\leq s\leq T}$ be the continuous processes defined recursively by (15)–(16). Using Assumption [H], the estimate (24) for $X^{t,x}$, and Proposition 3, the processes $Y^{1,n;t,x}, \ldots, Y^{q,n;t,x}$ belong to $\mathcal{S}^2$. Therefore using the representation theorem of solutions of standard BSDEs (Pardoux and Peng [27]) there exist deterministic functions $v^{1,0}, \ldots, v^{q,0}$ defined on $[0,T]\times\mathbb{R}^k$, continuous and with polynomial growth such that for every $(t,x) \in [0,T]\times\mathbb{R}^k$ and every $i \in \mathcal{J}$,

$$Y_s^{i,0;t,x} = v^{i,0}\left(s, X_s^{t,x}\right), \quad t \leq s \leq T.$$

Using an induction argument and applying Theorem 8.5 in El Karoui et al. [17] at each step yields the existence of deterministic functions $v^{1,n}, \ldots, v^{q,n}$ defined on $[0,T]\times\mathbb{R}^k$ that are continuous and with polynomial growth such that, for every $(t,x) \in [0,T]\times\mathbb{R}^k$ and every $i \in \mathcal{J}$,

$$Y_s^{i,n;t,x} = v^{i,n}\left(s, X_s^{t,x}\right), \quad t \leq s \leq T.$$

Since the sequences of processes $(Y^{i,n;t,x})_{n\geq 0}$ are nondecreasing in n, then, for any $i \in \mathcal{J}$, the sequences of deterministic functions $(v^{i,n})_{n\geq 0}$ are also nondecreasing.

Moreover we have

$$\begin{aligned} v^{i,n}(t,x) \leq Y_t^{i;t,x} &\leq \mathbb{E}\left[\int_t^T \max_{i=1,\ldots,q}\left|\psi_i\left(s, X_s^{t,x}\right)\right| ds \middle| \mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\int_t^T \max_{i=1,\ldots,q}\left|\psi_i\left(s, X_s^{t,x}\right)\right| ds\right] \end{aligned} \tag{27}$$

where the last inequality is obtained after taking expectations, since $v^{i,n}(t,x)$ is a deterministic function. It follows that, for any $i \in \mathcal{J}$, the sequence $(v^{i,n})_{n\geq 0}$ converges pointwisely to a deterministic function v^i and the last inequality in (27) implies that v^i is of at most polynomial growth through ψ_i and the estimates (24) for $X^{t,x}$. Furthermore, for any $(t,x) \in [0,T]\times R^k$ we have

$$Y_s^{i;t,x} = v^i\left(s, X_s^{t,x}\right), \quad t \leq s \leq T. \tag{28}$$

Step 2 (L^2-continuity of the value functions $(t,x) \longrightarrow Y^{i;t,x}$). Let (t,x) and (t',x') be elements of $[0,T]\times\mathbb{R}^k$. Using the representation (9) we will show that

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|Y_s^{i;t',x'} - Y_s^{i;t,x}\right|^2\right] \to 0 \text{ as } (t',x') \to (t,x) \text{ for any } i \in \mathcal{J}.$$

Indeed, recall that, by (9), we have, for any $i \in \mathcal{J}$ and $s \in [0,T]$,

$$Y_s^{i;t',x'} = \operatorname*{ess\,sup}_{u\in\mathcal{A}_s^i} \mathbb{E}\left[\int_s^T \psi_{u_r}\left(r, X_r^{t',x'}\right) dr - \sum_{j\geq 1} \ell_{u_{\tau_{j-1}} u_{\tau_j}}(\tau_j) \mathbb{1}_{[\tau_j<T]} \middle| \mathcal{F}_s\right],$$

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where $\mathcal{A}_s^i$ is the set of finite strategies such that $\tau_1 \geq s$, $\mathbb{P}$-a.s., and $u_0 = i$. Therefore, taking into account the fact that ℓ_{ij} does not depend on x, we have

$$\begin{aligned} Y_s^{i;t,x} - Y_s^{i;t',x'} &\leq \operatorname*{ess\,sup}_{u \in \mathcal{A}_s^i} \mathbb{E}\left[\int_s^T \psi_{u_r}\left(r, X_r^{t,x}\right) - \psi_{u_r}\left(r, X_r^{t',x'}\right) dr \middle| \mathcal{F}_s\right] \\ &\leq \operatorname*{ess\,sup}_{u \in \mathcal{A}_s^i} \mathbb{E}\left[\int_s^T \left|\psi_{u_r}\left(r, X_r^{t,x}\right) - \psi_{u_r}\left(r, X_r^{t',x'}\right)\right| dr \middle| \mathcal{F}_s\right] \\ &\leq \mathbb{E}\left[\int_0^T \left(\sum_{i=1}^q \left|\psi_i\left(r, X_r^{t,x}\right) - \psi_i\left(r, X_r^{t',x'}\right)\right|\right) dr \middle| \mathcal{F}_s\right]. \end{aligned}$$

Note that if ℓ_{ij} depend on x, the deduction of the previous inequality would not have been correct. Hence we have

$$\left|Y_s^{i;t,x} - Y_s^{i;t',x'}\right| \leq \mathbb{E}\left[\int_0^T \left(\sum_{i=1}^q \left|\psi_i\left(r, X_r^{t,x}\right) - \psi_i\left(r, X_r^{t',x'}\right)\right|\right) dr \middle| \mathcal{F}_s\right].$$

Now, using Doob's maximal inequality (see, e.g., [28]) and taking expectation, there exists a constant $C \geq 0$ such that

(29)

$$\mathbb{E}\left[\sup_{0 \leq s \leq T}\left|Y_s^{i;t,x} - Y_s^{i;t',x'}\right|^2\right] \leq C\mathbb{E}\left[\int_0^T \left(\sum_{i=1}^q \left|\psi_i\left(r, X_r^{t,x}\right) - \psi_i\left(r, X_r^{t',x'}\right)\right|\right)^2 dr\right].$$

But using continuity and polynomial growth of ψ_i and estimates (24)–(25), one can show that the right-hand side of (29) converges to 0 as (t', x') tends to (t, x). Therefore the left-hand side of the same inequality converges also to 0 as $(t', x') \to (t, x)$. Thus we get the desired result.

Step 3. We show that the functions $v^1, \ldots, v^q$ are continuous in (t, x) and the vector of functions $(v^1, \ldots, v^q)$ is a viscosity solution of the system of variational inequalities (20).

Thanks to the result obtained in Step 2, for any $i \in \mathcal{J}$, the function $(s, t, x) \mapsto Y_s^{i;t,x}$ is continuous from $[0, T]^2 \times \mathbb{R}^k$ into $L^2(\Omega)$. Indeed, this follows from the fact that

$$\begin{aligned} \left|Y_{s'}^{i;t',x'} - Y_s^{i;t,x}\right| &\leq \left|Y_{s'}^{i;t',x'} - Y_{s'}^{i;t,x}\right| + \left|Y_{s'}^{i;t,x} - Y_s^{i;t,x}\right| \\ &\leq \sup_{0 \leq r \leq T}\left(\left|Y_r^{i;t',x'} - Y_r^{i;t,x}\right|\right) + \left|Y_{s'}^{i;t,x} - Y_s^{i;t,x}\right|. \end{aligned}$$

Therefore the function $(t, t, x) \mapsto Y_t^{i;t,x}$ is also continuous. From the result obtained in Step 1 the function v^i is continuous in (t, x). The deterministic functions v^i, $i \in \mathcal{J}$, being continuous and of polynomial growth, by Theorem 8.5 in El Karoui et al. [17], are viscosity solutions for the system (20). ∎

Remark 1. The viscosity solution $(v^1, \ldots, v^q)$ is unique in the class of continuous functions with polynomial growth (cf. [15, Theorem 4]).

6. A numerical scheme. In this section we assume $p = 2$. Using the result by El Karoui et al. ([17, Proposition 5.1]) which characterizes a Snell envelope as a solution of a one barrier reflected BSDE we deduce that the q-uplet of processes

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$(Y^1,\dots,Y^q)$ solution of the verification theorem (Theorem 1) also satisfies the following:

For any $i\in\mathcal{J}$, there exists a pair of $\mathcal{F}_t$-adapted processes (Z^i,K^i) with value in $\mathbb{R}^d\times\mathbb{R}^+$ such that

$$
(30)\qquad \begin{cases} Y^i,\, K^i\in\mathcal{S}^2 \text{ and } Z^i\in\mathcal{M}^{2,d};\ K^i \text{ is nondecreasing and } K_0^i=0,\\ Y_s^i=\displaystyle\int_s^T\Psi_i(u)du-\int_s^T Z_u^i dB_u+K_T^i-K_s^i \text{ for all } 0\le s\le T,\\ Y_s^i\ge \max_{j\in\mathcal{J}^{-i}}\left\{-\ell_{ij}(s)+Y_s^j\right\} \quad\text{for all } 0\le s\le T,\\ \displaystyle\int_0^T\left(Y_u^i-\max_{j\in\mathcal{J}^{-i}}\left\{-\ell_{ij}(u)+Y_u^j\right\}\right)dK_u^i=0. \end{cases}
$$

Now we know that the solution of a reflected BSDE can be obtained as a limit of a sequence of solutions of standard BSDE by approximation via penalization. Therefore, for any $n\in\mathbb{N}$, let us define the following system:

$$
(31)\ \ \forall i\in\mathcal{J}\ \forall t\in[0,T],\ Y_t^{i,n}=\int_t^T\Psi_i(s)ds+n\int_t^T\left(L_s^{i,n}-Y_s^{i,n}\right)^+ds-\int_t^T Z_s^{i,n}dB_s,
$$

where for every $i\in\mathcal{J}$,

$$
\forall t\in[0,T],\ L_t^{i,n}=\max_{k\in\mathcal{J}^{-i}}\left(-\ell_{ik}(t)+Y_t^{k,n}\right).
$$

Note that if we define the generator $f:=(f^1,\dots,f^q):[0,T]\times\mathbb{R}^q\to\mathbb{R}^q$ by

$$
\forall i\in\mathcal{J},\ f^i(s,y)=\Psi_i(s)+n\left(\max_{k\in\mathcal{J}^{-i}}\left(-\ell_{ik}(s)+y^k\right)-y^i\right)^+,
$$

then f is a Lipschitz function w.r.t. y uniformly w.r.t. t and the $\mathbb{R}^q$-valued process $Y^n=(Y^{1,n},\dots,Y^{q,n})$ satisfies the following BSDE:

$$
(32)\qquad \forall t\in[0,T],\ Y_t^n=\int_t^T f\left(s,Y_s^n\right)ds-\int_t^T Z_s^n dB_s.
$$

Now, from Gobet, Lemor, and Warin [19] and [20], we know that the multidimensional BSDE (32) can be solved numerically. Therefore if the sequence $(Y^{i,n})_n$ converges to Y^i, for any $i\in\mathcal{J}$, this provides a way to simulate the value function Y^i. Therefore in what follows of this section we focus on this convergence.

Proposition 6. *For every $i\in\mathcal{J}$ and all $t\in[0,T]$, the sequence $(Y_t^{i,n})_{n\in\mathbb{N}}$ is non decreasing and a.s. $Y_t^{i,n}\le Y_t^i$.*

Proof. For $n\in\mathbb{N}$ and $k\in\mathbb{N}^*$, let us introduce the following scheme: for every $i\in\mathcal{J}$,

$$
\begin{aligned}
\forall t\in[0,T],\ Y_t^{i,n,k}=\int_t^T\Psi_i(s)ds+n\int_t^T&\left(\max_{j\in\mathcal{J}^{-i}}\left(-\ell_{ij}(t)+Y_t^{j,n,k-1}\right)\right.\\
(33)\qquad &\left.-\,Y_s^{i,n,k}\right)^+ds-\int_t^T Z_s^{i,n,k}dB_s
\end{aligned}
$$

and $Y_t^{i,n,0}=E[\int_t^T\Psi_i(u)du|\mathcal{F}_t],\ t\le T$. From El Karoui, Peng, and Quenez [18], we know that $Y^{i,n,k}$ converges to $Y^{i,n}$ when k goes to infinity. Next let us prove by induction on k that for any $k\ge 0$ and any $n\ge 0$ we have

$$
\forall i\in\mathcal{J}\ \forall t\in[0,T],\ Y_t^{i,n,k}\le Y_t^{i,n+1,k}.
$$

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For $k=0$, the property holds obviously. Suppose now that for some $k \geq 1$ we have for any $n \geq 0$, $i \in \mathcal{J}$, $t \in [0,T]$, $Y_t^{i,n,k-1} \leq Y_t^{i,n+1,k-1}$. But for any k, n, i and $t \in [0,T]$ we have

$$Y_t^{i,n+1,k} = \int_t^T \Psi_i(s)ds + (n+1)\int_t^T \left(\max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(s) + Y_s^{j,n+1,k-1}\right) - Y_s^{i,n+1,k}\right)^+ ds - \int_t^T Z_s^{i,n+1,k} dB_s$$

and

$$Y_t^{i,n,k} = \int_t^T \Psi_i(s)ds + n\int_t^T \left(\max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(s) + Y_s^{j,n,k-1}\right) - Y_s^{i,n,k}\right)^+ ds - \int_t^T Z_s^{i,n,k} dB_s.$$

So thanks to the induction hypothesis we have

$$\left(\max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(s) + Y_s^{j,n+1,k-1}\right) - y\right)^+ \geq \left(\max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(s) + Y_s^{j,n,k-1}\right) - y\right)^+, \; s \in [0,T], \; y \in \mathbb{R},$$

and therefore the result follows from the comparison theorem for standard BSDEs (see [18]). Thus we have P-a.s. for any $t \in [0,T]$, and $Y_t^{i,n+1,k} \geq Y_t^{i,n,k}$ for any $k, n \geq 0$ and $i \in \mathcal{J}$. Taking now the limit w.r.t. k we obtain that for any $n \geq 0$ and $i \in \mathcal{J}$, $Y^{i,n} \leq Y^{i,n+1}$ which is the desired result.

We now focus on the second inequality. It is enough to show that for any $k \geq 0$, $n \geq 0$, and $i \in \mathcal{J}$, $Y^{i,n,k} \leq Y^i$. Once more we use induction on k. For $k = 0$ the property holds since the process K^i of (30) is nondecreasing. Suppose now that the property is valid for some k; i.e., for any $n \geq 0$ and $i \in \mathcal{J}$ we have $Y^{i,n,k} \leq Y^i$. Therefore for any $n \geq 0$, $i \in \mathcal{J}$, and $t \leq T$ we have $Y_t^i \geq \max_{j\in\mathcal{J}^{-i}}(-\ell_{ij}(t) + Y_t^{j,n,k})$ thanks to the inequality in (30). It follows that for any $i \in \mathcal{J}$, Y^i satisfies

$$Y_t^i = \int_t^T \Psi_i(s)ds + n\int_t^T \left(\max_{j\in\mathcal{J}^{-i}} \left(-\ell_{ij}(t) + Y_t^{j,n,k}\right) - Y_s^i\right)^+ ds + K_T^i - K_t^i - \int_t^T Z_s^i dB_s.$$

As K^i is a nondecreasing process, then with the comparison theorem for standard BSDEs we obtain that $Y^{i,n,k+1} \leq Y^i$ for any $n \geq 0$. Finally, taking the limit as $k \to \infty$ we complete the proof. ▯

THEOREM 4. *For every $i \in \mathcal{J}$,*

$$E\left[\sup_{t\leq T}\left|Y_t^{i,n} - Y_t^i\right|^2\right] \to 0 \text{ as } n \to \infty.$$

Proof. It is enough to show that for any $t \leq T$, $Y_t^{i,n} \nearrow Y_t^i$ and to use the same argument as in Proposition 4. For every $i \in \mathcal{J}$ and all $t \in [0,T]$, $Y_t^{i,n} \leq Y_t^{i,n+1}$. Therefore there exists a process $\bar{Y}^i$ such that

$$\forall t \in [0,T], \; \lim_{n\to+\infty} Y^{i,n} = \bar{Y}_t^i \leq Y_t^i.$$

Moreover from (31) we deduce that

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{\tau\geq t} \mathbb{E}\left[\int_t^\tau \Psi_i(s)ds + \left(L_\tau^{i,n} \wedge Y_\tau^{i,n}\right) 1_{[\tau<T]} \middle| \mathcal{F}_t\right].$$

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This is due to the link between reflected BSDEs and the Snell envelope of processes since the process $K_t^{i,n} = \int_0^t n(L_t^{i,n} - Y_t^{i,n})^+ dt$, $t \le T$, is nondecreasing and

$$\int_0^T (Y_t^{i,n} - L_t^{i,n} \wedge Y_t^{i,n}) n (L_t^{i,n} - Y_t^{i,n})^+ dt = 0.$$

Hence, up to a continuous term which does not depend on n, $Y^{i,n}$ is a continuous supermartingale which converges to a process $\bar{Y}^i$ which is also càdlàg. Then from Proposition 2 (iv) we deduce that

$$\bar{Y}_t^i = \operatorname*{ess\,sup}_{\tau \ge t} \mathbb{E}\left[\int_t^\tau \Psi_i(s) ds + \left(\bar{L}_\tau^i \wedge \bar{Y}_\tau^i \right) 1_{[\tau < T]} \middle| \mathcal{F}_t \right], \tag{34}$$

where $\bar{L}_t^i = \max_{k \in \mathcal{J}^{-i}} (-\ell_{ik}(t) + \bar{Y}_t^k)$ is the nondecreasing limit of $L^{i,n}$. Going back now to (31), dividing both sides by n, and taking expectation and finally the limit w.r.t. n, we obtain

$$\mathbb{E}\left[\int_0^T \left(\bar{L}_t^i - \bar{Y}_t^i \right)^+ dt \right] = 0.$$

It implies that $\bar{L}^i \le \bar{Y}^i$ since those processes are càdlàg and $\bar{L}_T^i \le 0 = \bar{Y}_T^i$. Going back now to (34) we have

$$\bar{Y}_t^i = \operatorname*{ess\,sup}_{\tau \ge t} \mathbb{E}\left[\int_t^\tau \Psi_i(s) ds + \bar{L}_\tau^i 1_{[\tau < T]} \middle| \mathcal{F}_t \right].$$

Now we can argue as in the proof of Theorem 2 to deduce that the $\bar{Y}^i$'s are continuous. Finally, uniqueness of the solution of the verification theorem implies that $\bar{Y}^i = Y^i$ which ends the proof. ∎

Remark 2. (i) It seems to be difficult to obtain a convergence rate in Theorem 4. Hamadène and Jeanblanc ([23, Proposition 4.2]) give such a convergence rate because the lower barrier is constant and negative.

(ii) Numerical results can be obtained when $q = 2$ (see Hamadène and Jeanblanc [23] or Porchet, Touzi, and Warin [29]). For $q \ge 3$, Carmona and Ludkovski [5] suggest a numerical scheme when the switching costs are constant. The case of nonconstant switching costs seems out of reach.

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