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Asymptotic approach for backward stochastic differential equation with singular terminal condition*

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Abstract

In this paper, we provide a one-to-one correspondence between the solution Y of a BSDE with singular terminal condition and the solution H of a BSDE with singular generator. This result provides the precise asymptotic behavior of Y close to the final time and enlarges the uniqueness result to a wider class of generators.

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Keywords. Backward stochastic differential equation, singular terminal condition, asymptotic approach, singular generator.

1 Introduction

This paper is devoted to the study of the *asymptotic behavior* of the solution of backward stochastic differential equations (BSDEs) with *singular* terminal condition. We adopt from [26] and [21] the notion of a weak (super) solution (Y, Z) to a BSDE of the following form

$$-dY_t = \frac{1}{\eta_t} f(Y_t)dt + \lambda_t dt - Z_t dW_t \tag{1}$$

where W is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The filtration $\mathbb{F}$ is the natural filtration generated by W and is supposed to be complete and right continuous. The function $f : \mathbb{R} \to \mathbb{R}$ is called the *driver* (or *generator*) of the BSDE. The particularity here is that we allow the *terminal condition* ξ to be *singular*, in the sense that $\xi = +\infty$ a.s.

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Since the seminal paper by Pardoux and Peng [23] BSDEs have proved to be a powerful tool to solve stochastic optimal control problems (see e.g. the survey article [8] or the book [25]). BSDEs with singular terminal condition provide a purely probabilistic solution of a stochastic control problem with a terminal constraint on the controlled process. The analysis of optimal control problems with state constraints on the terminal value is motivated by models of optimal portfolio liquidation under stochastic price impact. The traditional assumption that all trades can be settled without impact on market dynamics is not always appropriate when investors need to close large positions over short time periods. In recent years models of optimal portfolio liquidation have been widely developed, see, e.g. [1], [2], [10], [11], [15], or [19], among many others. In [4], the following problem is considered: minimizing the cost functional

$$J(X) = \mathbb{E}\left[\int_0^T (\mu_s|\alpha_s|^p + \lambda_s|X_s|^p)\, ds\right] \tag{2}$$

over all progressively measurable processes X that satisfy the dynamics

$$X_s = x + \int_0^s \alpha_u du$$

with the terminal constraint that $X_T = 0$ a.s. Here $p > 1$ and the processes μ and λ are non-negative and progressively measurable. In this framework the state process X denotes the agent's position in the financial market. At each point in time t she can trade in the primary venue at a rate α_t which generates costs $\mu_t|\alpha_t|^p$ incurred by the stochastic price impact parameter η_t. The term $\gamma_t|X_t|^p$ can be understood as a measure of risk associated to the open position. $J(X)$ thus represents the overall expected costs for closing an initial position x over the time period $[0, T]$ using strategy X. In [4], optimal strategies and the value function of this control problem (2) are characterized with the BSDE

$$-\, dY_t = -(p-1)\frac{Y_t^q}{\mu_t^{q-1}}dt + \lambda_t dt - Z_t dW_t \tag{3}$$

with $\lim\limits_{t\to T} Y_t = +\infty$. Here $q > 1$ is the Hölder conjugate of p. The generator f is here a polynomial function. Variants of the position targeting problem (2) have been studied in [5], [12], [13] or [29]. Note that these problems are particular cases of the stochastic calculus of variations (see [3]).

Let us explain the methodology to obtain a solution for the BSDE (3). The most common approach in the literature is the so-called *penalization approach*, see, e.g., [26], [27], [4], [12], [21], and the references therein. The idea of the penalization approach is to relax the binding liquidation constraint by penalizing open position in the underlying liquidation problem. In [4], as in [21] for more general driver, the authors use the penalization approach, replacing the singular terminal by a constant n and letting n go to $+\infty$. The convergence is obtained by a comparison principle for solution of BSDEs (see [20] or [24]). In [13], the approach consists in the study of the precise asymptotic behavior at time T of the solution Y of (3). Roughly speaking, the major singular term of Y is then removed to obtain a non-singular problem. The key of this *asymptotic approach* is to establish sharp a priori estimates of the singular solution at the terminal time. In [13], the authors consider a time-homogeneous Markov setting and obtain the a priori estimates and uniqueness by establishing a general comparison principle for singular viscosity solution to (3). This results are based on time-shifting arguments, applied similar before in [26], which in general do not apply in a non-time-homogeneous setting. However, it is outlined in [13] how the shifting argument may be applied in non-Markov settings to obtain sharp a priori estimates of the singular solution to (3). One major result of [13] is the uniqueness of the solution of (3) (under boundedness assumptions on the coefficients μ and λ).

Let us outline in which directions our findings generalize some results from these papers. In [21] the generator may depend also on Z in a non trivial way; here our generator has a special form. However in the previously mentioned papers f is assumed to be a polynomial function or of polynomial growth w.r.t. y, that is $f(y) \leq -y|y|^q$. Here we essentially assume that $1/f$ is integrable on the neighborhood of $+\infty$. If η and λ are deterministic, the BSDE becomes an ODE and this condition is necessary and sufficient to ensure that the solution can be equal to $+\infty$ at time T, but finite at any time $t < T$. Under this condition (called **(C1)**), we prove *existence of a minimal solution* (Y, Z^Y) of the BSDE (1). The function $f(y) = -(y+1)|\log(y+1)|^q$ is an example satisfying **(C1)** but not covered by the preceding papers.

Our second main result concerns the *decomposition of this minimal solution.* We prove that Y is equal to:

$$Y_t = \phi\left(\mathbb{E}\left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t\right]\right) + \psi\left(\frac{T-t}{\eta^\star}\right) H_t, \quad \forall t \in [0,T], \tag{4}$$

where

- ϕ solves the ODE: $\phi' = f \circ \phi$ with initial condition $\phi(0) = +\infty$ and $\psi = -\phi'$,
- $\eta^\star$ is the deterministic upper bound on the process η.

The process H is the minimal non-negative solution of a BSDE with terminal condition 0 and with a *singular generator* F in the sense of [17] (Theorem 1). As a consequence, we provide a *one-to-one correspondence* between the BSDE (1) with singular terminal condition and a BSDE with singular generator. We give a *self-contained construction of the solution* (H, Z^H) (without any reference to Y) which extends the existence result of [17]. The *asymptotic behavior* follows from the boundedness of the process $\psi H/\phi$ on some neighborhood of T:

$$\phi(A_t) = \phi\left(\mathbb{E}\left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t\right]\right) \leq Y_t \leq (1+\kappa)\phi\left(\mathbb{E}\left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t\right]\right),$$

where the constant κ depends on the coefficients η, λ and f. At this stage it is important to note that there is some asymmetry in (4) since the first term with ϕ is random, whereas the second with ψ is deterministic. However this method avoids assuming extra assumptions on f.

To deal with a symmetric expansion, we suppose that f is concave and we decompose Y as follows: for any $t \in [0, T]$

$$Y_t = \phi \left(\mathbb{E} \left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t \right] \right) + \psi \left(\mathbb{E} \left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t \right] \right) H_t = \phi(A_t) + \psi(A_t) H_t. \tag{5}$$

Again H solves a BSDE with a singular generator. As in the case (4), one singularity comes from the explosion at time T and creates trouble only close to time T. But since H is multiplied now by a random process, there are extra linear terms including the martingale part of the process A. These terms have to be controlled on the whole interval $[0, T]$, and not only on a neighborhood of T. Nonetheless we prove that under a technical condition, called **(H)**, (H, Z^H) is the unique solution of the BSDE with singular generator and as a by-product, we obtain *uniqueness of the solution of the BSDE* (1). Let us emphasize that there was only one result about uniqueness, namely [13, Theorem 6.3] for the power case. Uniqueness was proved by showing that any solution (Y, Z^Y) is the value function of the control problem (2). Here the proof is only based on the comparison principle for BSDEs.

The condition **(H)** is a stronger Novikov condition. For a general process η, this assumption may be false; some regularity on its Malliavin derivative is required in **(H)**. Thereby, assuming that η is an Itô process, we provide sufficient conditions under which **(H)** holds. Under this Itô setting with bounded coefficients, we provide another decomposition of Y, where again H is the unique solution of a BSDE with a singular generator, but without the troubling linear part.

Up to now, the construction of H is based on the comparison principle for BSDEs and H is the monotone limit of a sequence of solutions of "standard" monotone BSDEs. In the power case $f(y) = -y|y|^q$, we follow the arguments of the paper [13] for a PDE and show that the process H can be *obtained by Picard iterations* in the suitable space $\mathcal{H}$. This construction has two main advantages: first we have a more accurate behavior of H at time T, secondly this construction is more tractable for numerical approximation.

In addition to the precise behavior of the solution Y, that is the behavior of the value function of the control problem (2) in the power case, or the uniqueness result for (1), our result establishes a link between Y and H. The main drawback for BSDE with singular terminal condition is the lack of approximation scheme with some rate of convergence. Moreover most of numerical schemes for BSDE are based on backward induction starting at the terminal value. The correspondence between Y and H could be a promising solution for numerical scheme, since the terminal value of H is zero. The singularity of the generator of H is a serious obstacle. But if H is obtained by a fixed point argument in a weighted space, we strongly believe that it could be a way to compute H, and thus Y. This point is left for further research.

The paper is decomposed as follows. In the next section we explain our assumptions on the coefficients η, λ and f of the BSDE (1). The reader finds here several examples of functions f, for which the asymptotic behaviour of Y holds. Let us emphasize these assumptions only imply the behaviour of f on an interval $[R, +\infty)$ for R sufficiently large. In Section 3, we recall and extend several results concerning the *existence of the solution*

(Y, Z^Y) *of the BSDE* (1) *with singular terminal condition* $+\infty$ and provide some a priori estimates on this solution Y and on Z^Y. Section 4 is dedicated to the decomposition (4), by proving the *existence of the minimal non-negative solution of the BSDE (Equation* (21)*) with a singular generator* and with terminal condition 0 (Theorem 1) and the *one-to-one correspondence* between the minimal solutions Y and H. In Section 5, we study the symmetric decomposition (5) of Y and prove *uniqueness of the solutions* Y for BSDE (1) and H for the BSDE (42). In the power case we prove that H can be constructed by a Picard approximation scheme. In the last section, we briefly explain the relations between the different expansions of Y. Let us emphasize that all results from Section 4 are ordered from the more general to the less general drivers.

In the continuation, unimportant constants will be denoted by C and they could vary from line to line.

2 Assumptions on the generator

In the BSDE (1), the generator is of the form:

$$(\omega, t, y) \mapsto \frac{1}{\eta_t(\omega)} f(y) + \lambda_t(\omega).$$

In the rest of the paper, the following conditions hold:

(A1) *There exist three constants* $0 < \eta_\star < \eta^\star$ *and* $\|\lambda\| \geq 0$ *such that a.s. for any* t

$$\eta_\star \leq \eta_t \leq \eta^\star, \qquad 0 \leq \lambda_t \leq \|\lambda\|.$$

(A2) *The function* f *is continuous and non increasing, with* $f(0) = 0$ *and with continuous derivative.*

Supposing that f is continuous and non increasing[1] is coherent with the existence and uniqueness results concerning monotone BSDEs (see [24, Chapter 5.3.4]). Note that if $f(0) \neq 0$, then

$$\frac{f(y)}{\eta_t} + \lambda_t = \frac{f(y) - f(0)}{\eta_t} + \lambda_t + \frac{f(0)}{\eta_t} = \frac{\widetilde{f}(y)}{\eta_t} + \widetilde{\lambda}_t,$$

provided that $\widetilde{\lambda}_t \geq 0$. The non-negativity of λ is natural for the control problem (2) and leads to a more accurate expansion of Y. However it is not necessary (see Section 6.1 for a short discussion on this point). Somehow and to summarize, the only stronger condition on our type of generator is the regularity: $f \in C^1(\mathbb{R})$.

Now let us consider the ordinary differential equation (ODE): $y' = -f(y)$ with the terminal condition $y(T) = +\infty$. There exists a solution if and only if the function G given by:

$$G(x) := \int_x^\infty \frac{1}{-f(t)}\, dt$$

[1] For monotone BSDEs, the classical assumption is: for some $\mu \in \mathbb{R}$ and any $(y, y') \in \mathbb{R}^2$, $(f(y) - f(y'))(y - y') \leq \mu(y - y')^2$. By a very standard transformation (see [24, Proof of Corollary 5.26]) we may assume w.l.o.g. that $\mu = 0$, thus f is non increasing.

is well-defined at least on some interval $(\kappa, +\infty)$, with $\kappa = \sup\{y \geq 0,\ f(y) = 0\}$, meaning that $f \equiv 0$ on $[0, \kappa]$. Note that the function G is positive, strictly decreasing and convex, such that $G(\infty) = 0$ and the smoothness of f implies that $G(\kappa) = +\infty$. Then the solution y is given by: $y(t) = G^{-1}(T-t)$ on $[0, T]$. Defining $f^\kappa(x) = f(x+\kappa)$, $G^\kappa(x) = G(x+\kappa)$ yields that G^κ is defined on $(0, +\infty)$. Moreover the solution y is given by: $y(t) = G^{-1}(T-t) = (G^\kappa)^{-1}(T-t) - \kappa$ and solves $y' = -f^\kappa(y)$ together with $y(T) = +\infty$. Hence w.l.o.g. we assume from on now that:

(A3) *For any $x > 0$, the function*

$$G(x) := \int_x^\infty \frac{1}{-f(t)}\, dt$$

is well-defined on $(0, \infty)$.

We define the two functions:

$$\phi(x) := G^{-1}(x) > 0, \qquad \psi(x) := -\phi'(x) > 0. \tag{6}$$

The function ϕ being decreasing and C^2 on $(0, \infty)$ solves $\phi' = f \circ \phi$.

Under the previous conditions, there exists a minimal non-negative solution (Y, Z^Y) for the BSDE (1) (Proposition 1), and Y verifies the a priori estimate (19). Note that we can extend the result when the generator also depends in a particular way on Z (see Remark 1).

For the asymptotic behavior of Y, we also consider the next condition:

(C1) *There exists a constant $\delta > 0$ and $R > 0$ such that $x \mapsto \left(G(x)^{-\delta} = \int_x^\infty \frac{1}{-f(y)} dy \right)^{-\delta}$ is convex on $[R, +\infty)$.*

Let us emphasize that this condition only involves the function f on some interval $[R, +\infty)$ and the value of R may be large. From Lemma 5, Condition **(C1)** is equivalent to the boundedness of $x \mapsto -x\dfrac{\phi''(x)}{\phi'(x)}$ on a neighborhood of zero.

Example 1 *If $f(y) = -(y+1)|\log(y+1)|^q$ for some $q > 1$ and $y \geq 0$, all conditions* **(A1)**, **(A2)** *and* **(A3)** *are verified and if p is the Hölder conjugate of q then*

$$\forall x > 0, \quad G(x) = \frac{1}{q-1} \log(x+1)^{1-q}.$$

Thereby

$$\phi(x) = \exp\left(((q-1)x)^{1-p} \right) - 1.$$

Direct computations show that

$$-\frac{\phi''(x)}{\phi'(x)} x = \left[(p-1)^p x^{-p} + p \right]$$

is not a bounded function near zero and for any $\delta > 0$, $G^{-\delta}$ is not convex. Somehow this function f is "not enough non linear".

Example 2 *Here we study several functions f, ordered by their "non linearity". All of them verify* **(C1)**.

- *If $f(y) = -y|y|^q$ for some $q > 0$, then $G(x) = \frac{1}{qx^q}$ and $\phi(x) = \left(\frac{1}{qx}\right)^{\frac{1}{q}}$. The assumption* **(C1)** *holds for any $\delta > 0$ and*
$$-\frac{\phi''(x)}{\phi'(x)}x = \frac{q+1}{q}.$$

- *If $f(y) = -(\exp(ay) - 1)$ for some $a > 0$, then*
$$G(x) = -\frac{1}{a}\log\left(1 - e^{-ax}\right) = G^{-1}(x) = \phi(x).$$
And
$$-\frac{\phi''(x)}{\phi'(x)}x = \frac{axe^{ax}}{e^{ax}-1} \underset{x\to 0}{\sim} 1,$$
is bounded near zero.

- *If $f(y) = -\exp(ay^2)$ for some $a > 0$ and $y \geq 0$ (note that $f(0) = -1$ to simplify the computations). Then*
$$G(x) = \sqrt{\frac{\pi}{a}}\left[1 - \mathcal{N}(x\sqrt{2a})\right], \quad \phi(x) = \frac{1}{\sqrt{2a}}\mathcal{N}^{-1}\left(1 - x\sqrt{\frac{a}{\pi}}\right),$$
where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the normal law. Thereby
$$\phi'(x) = -\exp(a\phi(x)^2), \quad \phi''(x) = 2a\phi(x)(\phi'(x))^2,$$
and with $\sqrt{a}\phi(x) = z/\sqrt{2}$
$$\begin{aligned} -\frac{\phi''(x)}{\phi'(x)}x &= 2ax\phi(x)\exp(a\phi(x)^2) = z\sqrt{2a}G(z/\sqrt{2a})\exp(z^2/2) \\ &= z\sqrt{2\pi}\exp(z^2/2)\left[1 - \mathcal{N}(z)\right] \underset{z\to+\infty}{\sim} 1. \end{aligned}$$
using the classical tail estimate of the normal law. Hence $x\phi''(x)/\phi'(x)$ is bounded near zero and **(C1)** *holds (again by Lemma 5).*

Let us define
$$A_t = \mathbb{E}\left[\int_t^T \frac{1}{\eta_s}ds \middle| \mathcal{F}_t\right] \tag{7}$$
together with[2]
$$\phi_t = \phi(A_t), \quad \psi_t = \psi(A_t). \tag{8}$$
Let us emphasize that $(t \mapsto \phi_t,\ t \geq 0)$ and $(t \mapsto \psi_t,\ t \geq 0)$ are processes. Remark that from the boundedness of η
$$\frac{1}{\eta^\star}(T-t) \leq A_t = \mathbb{E}\left[\int_t^T \frac{1}{\eta_s}ds \middle| \mathcal{F}_t\right] \leq \frac{1}{\eta_\star}(T-t). \tag{9}$$

[2] In the following, X_t denotes a random process, whereas $X(t)$ is a deterministic function.

From the monotonicity of ϕ and ψ, we get

$$\phi_\star(t) = \phi\left(\frac{T-t}{\eta_\star}\right) \leq \phi_t \leq \phi\left(\frac{T-t}{\eta^\star}\right) = \phi^\star(t). \tag{10}$$

and

$$\psi_\star(t) = \psi\left(\frac{T-t}{\eta_\star}\right) \leq \psi_t \leq \psi\left(\frac{T-t}{\eta^\star}\right) = \psi^\star(t). \tag{11}$$

We introduce our next condition on f:

(C2) *For some $p \geq 1$, some $\tau < T$ and for any $r \geq 0$*

$$\mathbb{E}\left[\left(\int_\tau^T \frac{f(\phi_t + r)}{\psi^\star(t)} dt\right)^p\right] < +\infty.$$

Since the process ϕ is bounded from above by $\phi^\star$, **(C2)** holds if

$$\int_\tau^T \frac{-f\left(\phi^\star(t) + r\right)}{\psi^\star(t)} dt < +\infty$$

since this integral w.r.t. t is now deterministic. Hence **(C2)** depends only on the behavior of f on a neighborhood of $+\infty$. In particular if the function $-f$ is submultiplicative: $-f(x+y) \leq C(-f(x))(-f(y))$ for some fixed constant C, then

$$-\frac{f(\phi^\star(t) + r)}{\psi^\star(t)} \leq C(-f(r))\frac{-f(\phi^\star(t))}{\psi^\star(t)} \leq C(-f(r)).$$

In Remark 3, we show that all functions of Example 2 verifying **(C1)**, also satisfy **(C2)**, even if they are not submultiplicative.

In Section 5, we add several conditions on f:

(C3) *f is concave and of class C^2 on $(0, +\infty)$.*

(C4) *If $\mathfrak{F}$ is the increasing and concave function $\mathfrak{F} : x \mapsto G^{-1}(x^{-1/\delta})$ for $x > 0$, then $(-f) \circ \mathfrak{F}$ is also increasing and concave on a neighborhood of $+\infty$.*

From **(C1)**, we know that $\mathfrak{F}$ is increasing. Since f is concave, $-f'$ is a non-decreasing function and there exists a rank such that for any x greater than this rank, $-f' > 0$. In other words $(-f) \circ \mathfrak{F}$ is an increasing function, at least on a neighborhood of ∞. Hence the main assumption in **(C4)** is the concavity of $-f \circ \mathfrak{F}$. We prove that all functions of Example 2 satisfying **(C1)** also verify **(C3)** and **(C4)** (see computations after Lemma 11).

Finally our last condition on f is the following. Let us define for some $\rho \in (0,1)$, the non-negative function $h(y) = -f(y)G(y)^{1-\rho}$.

(C5) *There exists $\rho \in (0,1)$ such that the function $y \mapsto \dfrac{y}{h(y)}$ remains bounded on a neighborhood of $+\infty$.*

Again this property only depends on the behavior of f near $+\infty$. Note that

$$\int_y^{+\infty} \frac{1}{h(t)} dt = \frac{1}{\rho} G(y)^\rho,$$

thus $1/h$ is integrable on $[1,+\infty)$. It is known (see [14, Section 178]) that if h is non-decreasing, then we have

$$\lim_{y\to+\infty} \frac{y}{h(y)} = 0.$$

Thus **(C5)** holds. But since

$$h'(y) = G(y)^{-\rho}\left[-f'(y)G(y) - (1-\rho)\right] = G(y)^{-\rho}\left[\frac{G''(y)G(y) + (\rho-1)(G'(y))^2}{(G'(y))^2}\right],$$

the non-negativity of h' is equivalent to the non-negativity of the second derivative of G^ρ. In other words h is non decreasing if and only if G^ρ is convex. For all functions of Example 2, direct computations show that $\lim_{y\to+\infty} \dfrac{y}{h(y)} = 0$, for any $\rho \in (0,1)$. Hence **(C5)** holds.

3 BSDEs with singular terminal condition

In this section the assumptions **(A1)**, **(A2)** and **(A3)**, except for the last result (Lemma 4) where we suppose besides that f is concave and that **(C1)** is verified. Let us introduce the following spaces for $p \geq 1$.

- $\mathbb{D}^p(0,T)$ is the space of all adapted càdlàg[3] processes X such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < +\infty.$$

- $\mathbb{H}^p(0,T)$ is the subspace of all predictable processes X such that

$$\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

- $\mathbb{S}^p(0,T) = \mathbb{D}^p(0,T) \times \mathbb{H}^p(0,T)$ and $\mathbb{S}^\infty(0,T) = \bigcap_{p\geq 1} \mathbb{S}^p(0,T)$.

From [21, Theorem 1], if $f(y) \leq -y|y|^q$ for some $q > 0$, we know that the singular BSDE (1) has a minimal solution (Y, Z^Y), in the sense of the next definition:

Definition 1 (BSDE with singular terminal condition) *The process* (Y, Z^Y) *is a solution of the BSDE* (1) *with terminal condition* $+\infty$ *if:*

- *For any* $\varepsilon > 0$, $(Y, Z^Y) \in \mathbb{S}^\infty(0, T-\varepsilon)$;
- *for any* $0 \leq s \leq t < T$,

$$Y_s = Y_t + \int_s^t \left[\frac{1}{\eta_u} f(Y_u) + \lambda_u\right] du - \int_s^t Z_u^Y dW_u;$$

[3] French acronym for right-continuous with left-limit

- $Y_t \geq 0$ *a.s. for any* $t \in [0, T]$ *;*

- *a.s.* $\lim_{t \to T} Y_t = +\infty$.

Minimality means that for any other process $(\widetilde{Y}, \widetilde{Z})$ satisfying the previous four items, a.s. $\widetilde{Y}_t \geq Y_t$ for any t. Moreover from [13, Theorem 6.3], if $f(y) = -y|y|^q$, the solution is unique.

To obtain the existence of a solution, we need some a priori estimates on Y (see [4], [13] or [21]). Using the arguments of [13, Proposition 6.1], we obtain that if $f(u) \leq -u|u|^q$, any solution of the BSDE (1) satisfies:

$$Y_t \leq \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T \left(\left(\frac{\eta_s}{q} \right)^{\frac{1}{q}} + (T-s)^{q^\dagger} \lambda_s \right) ds \Big| \mathcal{F}_t \right] \tag{12}$$

where $q^\dagger$ is the Hölder conjugate of $q+1$. Under our setting and from this estimate we have:

$$Y_t \leq \left(\frac{\eta^\star}{q(T-t)} \right)^{\frac{1}{q}} + \frac{(T-t)}{(q^\dagger + 1)} \|\lambda\| = \phi\left(\frac{T-t}{\eta^\star} \right) + \frac{(T-t)}{(q^\dagger+1)} \|\lambda\|.$$

The goal of this section is to extend these results to our class of drivers. Let us first begin with a lower bound, similar to [4, Estimate 3.7], but for a more general driver f.

Lemma 1 *The minimal solution Y satisfies a.s. for any $t \in [0, T]$,*

$$Y_t \geq \phi_t = \phi(A_t). \tag{13}$$

Proof. Indeed the process A satisfies

$$-dA_t = \frac{1}{\eta_t} dt + Z_t^A dW_t \tag{14}$$

for some $Z^A \in \mathbb{H}^2(0, T)$. For some $L > 0$, define $A_t^L = \frac{1}{L} + A_t$. Since ϕ is a smooth function, if $U^L = \phi(A^L)$, Itô's formula leads to

$$\begin{aligned} -dU_t^L &= \phi'(A_t^L) \left[\frac{1}{\eta_t} dt + Z_t^A dW_t \right] - \frac{1}{2} \phi''(A_t^L)(Z_t^A)^2 dt \\ &= \frac{1}{\eta_t} f(U_t^L) dt + \Theta_t dt + Z^{U^L} dW_t. \end{aligned}$$

Note that $\phi''(x) = f'(\phi(x)) f(\phi(x)) \geq 0$, thus $\Theta_t \leq 0$. Since $U_T^L = \phi(1/L)$, from the comparison principle for monotone BSDE (see [24, Proposition 5.34]) and the construction of Y by approximation, we obtain that $Y_t \geq U_t^L$. Passing through the limit on L leads to the conclusion. □

Let us now give an upper bound on Y, similar to [13, Proposition 6.1] but again for a general driver f. Let us consider the function

$$\mathfrak{G}(x) = \int_x^\infty \frac{-1}{\|\lambda\| + \frac{f(y)}{\eta^\star}} dy = \eta^\star \int_x^\infty \frac{1}{-\mathfrak{C} - f(y)} dy$$

defined on the interval $(\Upsilon = f^{-1}(-\mathfrak{C}), +\infty)$ with $\mathfrak{C} = \|\lambda\|\eta^\star$. Since f is a function with continuous derivative, $\mathfrak{G}(\Upsilon) = +\infty$. If we define $\vartheta = \mathfrak{G}^{-1}$, this function is well-defined on $(0, +\infty)$, with $\vartheta(0) = +\infty$ and satisfies:

$$\vartheta' = \|\lambda\| + \frac{f(\vartheta)}{\eta^\star}. \tag{15}$$

Note that the function ϑ strongly depends on $\eta^\star$, $\|\lambda\|$ and f.

Lemma 2 *Assume that the process (U, Z^U) satisfies the dynamics: for any $\varepsilon > 0$ and $0 \leq t \leq T - \varepsilon$*

$$U_t + \zeta_t = U_{T-\varepsilon} + \int_t^{T-\varepsilon} \left[\lambda_s + \frac{1}{\eta_s} f(U_s) \right] ds - \int_t^{T-\varepsilon} \Theta_s ds - \int_t^{T-\varepsilon} Z_s^U dW_s, \tag{16}$$

where ζ and Θ are two non-negative processes. Then a.s. for all $t \in [0, T)$,

$$0 \leq U_t \leq \vartheta(T - t). \tag{17}$$

Proof. We proceed as in the proof of [13, Proposition 6.1], namely we shift the singularity. Take any $0 < \varepsilon$ such that $0 \leq T - \varepsilon < T$. The function $(\vartheta\,(T - \varepsilon - t)\,,\ t \in [\tau, T - \varepsilon])$ solves the ODE: $y(T - \theta) = +\infty$ and

$$y' = -\|\lambda\| - \frac{f(y)}{\eta^\star}.$$

By the comparison principle again we have that $U_t \leq \vartheta\,(T - \varepsilon - t)$ on $[0, T - \varepsilon]$. Since U does not depend on ε, we obtain that a.s.

$$\forall t \in [0, T), \quad U_t \leq \vartheta\,(T - t)\,.$$

This achieves the proof of the lemma. □

As a by-product, our proof implies that for any non-negative solution (Y, Z^Y) of the BSDE (1), we have a.s. on $[0, T]$:

$$\phi_\star(t) = \phi\left(\frac{T - t}{\eta_\star}\right) \leq \phi_t \leq Y_t \leq \vartheta_t = \vartheta(T - t). \tag{18}$$

The first inequality comes from (10). Compared to (12), in the power case $f(y) = -y|y|^q$, this estimate is less accurate. However it holds for functions without polynomial growth. For the upper bound, note that if $\lambda = 0$, then $\vartheta = \psi^\star$. In general we have

Lemma 3 *For any $\varepsilon > 0$, there exists a deterministic time $T^\varepsilon \in [0, T)$ such that a.s. for any $t \in [T^\varepsilon, T]$*

$$Y_t \leq \phi\left(\frac{T - t}{(1 + \varepsilon)\eta^\star}\right) = \phi^\star_\varepsilon(t). \tag{19}$$

Proof. We write

$$\mathfrak{G}(x) \quad = \quad \eta^\star \int_x^\infty \frac{1}{-\mathfrak{C} - f(y)} dy = \eta^\star G(x) + \eta^\star \mathfrak{C} \int_x^\infty \frac{1}{(-\mathfrak{C} - f(y))(-f(y))} dy.$$

Therefore $\eta^\star G(x) \leq \mathfrak{G}(x)$ and

$$\mathfrak{G}(x) \quad \leq \quad \eta^\star G(x) + \eta^\star \mathfrak{C} \frac{1}{(-\mathfrak{C} - f(x))} G(x) = \eta^\star G(x) \left(\frac{-f(x)}{-\mathfrak{C} - f(x)} \right).$$

We deduce that on the interval $(f^{-1}(-(1+\varepsilon)\mathfrak{C}), +\infty)$, $\eta^\star G(x) \leq \mathfrak{G}(x) \leq (1+\varepsilon)\eta^\star G(x)$, and thereby on the neighborhood of zero $(0, G^{-1}(f^{-1}(-(1+\varepsilon)\mathfrak{C})/\eta^\star)) = (0, \widetilde{\Upsilon})$,

$$G^{-1}(x/\eta^\star) \leq \vartheta(x) \leq G^{-1}(x/((1+\varepsilon)\eta^\star)) = \phi(x/((1+\varepsilon)\eta^\star)).$$

Thus provided that $T^\varepsilon = T - \widetilde{\Upsilon} \leq t \leq T$

$$Y_t \leq \vartheta(T-t) \leq \phi^\star_\varepsilon(t).$$

□

Hence we deduce that there exists a constant $\tau \in [0, T)$ such that a.s. for any $t \in [\tau, T]$

$$\phi_\star(t) \leq Y_t \leq \phi^\star_1(t)$$

with two deterministic functions $\phi_\star$ and $\phi^\star_1$ on a deterministic neighborhood[4] of T.

Let us state the following result. Note that if $f(y) \leq -y|y|^q$, there is nothing new here. But since we strength the integrability conditions on η and λ, we can remove this growth condition on f. A typical example is $f(y) = -(y+1)|\log(y+1)|^q$ for some $q > 1$.

Proposition 1 *Under our setting, the BSDE* (1) *has a minimal non-negative solution* (Y, Z^Y).

Proof. The existence of a non-negative solution can be obtained by the same penalization arguments as in [4] or [21]. We use the a priori estimate (19) in order to obtain the convergence of the penalization scheme on any interval $[0, T-\varepsilon]$. Minimality can be proved as in [21, Proposition 4]. Thus we skip the details here. □

Remark 1 (Generator depending on Z) *Assume that the generator has the form:*

$$(t, \omega, y, z) = \frac{f(y)}{\eta_t(\omega)} + \lambda_t(\omega) + \zeta(t, \omega, z),$$

where there exists a constant C such that for any (t, ω, z, z')

$$0 \leq \zeta(t, \omega, 0) \leq C, \qquad |\zeta(t, \omega, z) - \zeta(t, \omega, z')| \leq C|z - z'|.$$

Using the Girsanov theorem, existence of a solution can be derived directly from Proposition 1. Moreover all results in this paper remain valid under some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$.

[4]Note that we can consider the solution $\widehat{\vartheta}$ of the ODE (15) starting at the point $\phi^\star_1(\tau)$. Then defining for $t \in [0, \tau]$, $\phi^\star_1(t) = \widehat{\vartheta}(T-t)$, we can extend the estimate on the whole interval $[0, T]$.

To finish this section, let us give an estimate of Z^Y. Let us also emphasize that this upper bound is valid for any solution of the BSDE (1), since the proof only uses the dynamics on $[0,T)$ and the a priori estimate (18) on Y, but not the construction by penalization of Y. In the power case ($f(y) = -y|y|^q$), it is known (see [6, 26]) that

$$\mathbb{E}\left[\left(\int_0^T (T-s)^{2/q}(Z_s^Y)^2 ds\right)\right] < +\infty.$$

Lemma 4 *Assume that f is concave and that* **(C1)** *holds. Any solution (Y, Z^Y) of (1) satisfies for all $p \in [1,+\infty)$*

$$\mathbb{E}\left[\left(\int_0^T \frac{1}{(T-s)(\psi^\star(s))^2}(Z_s^Y)^2 ds\right)^p\right] < +\infty.$$

Let us immediately remark that this estimate is not optimal in the power case since we only have

$$\mathbb{E}\left[\left(\int_0^T (T-s)^{2/q+1}(Z_s^Y)^2 ds\right)\right] < +\infty.$$

Nevertheless it is sufficient for our purpose in Section 5.

Proof. The following argument will be used several times through the paper. From the definition of a solution, we have for any $\varepsilon > 0$

$$\mathbb{E}\left[\left(\int_0^{T-\varepsilon} (Z_s^Y)^2 ds\right)^p\right] < +\infty.$$

Since $s \mapsto \frac{1}{(T-s)(\psi^\star(s))^2}$ is bounded on $[0, T-\varepsilon]$, we have to prove only that

$$\mathbb{E}\left[\left(\int_\tau^T \frac{1}{(T-s)(\psi^\star(s))^2}(Z_s^Y)^2 ds\right)^p\right] < +\infty$$

for some deterministic $\tau \in [0,T)$.

From (18), Y remains bounded away from zero on $[0,T]$. Thus let us apply the function G to Y:

$$\begin{aligned} G(Y_t) - G(Y_0) &= \int_0^t \frac{1}{f(Y_s)}\left(-\frac{1}{\eta_s}f(Y_s) - \lambda_s\right) ds + \int_0^t \frac{1}{f(Y_s)} Z_s^Y dW_s \\ &+ \frac{1}{2}\int_0^t \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds. \end{aligned}$$

Hence

$$0 \le \frac{1}{2}\int_0^t \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds \le G(Y_t) + \int_0^t \left(\frac{1}{\eta_s}\right) ds - \int_0^t \frac{1}{f(Y_s)} Z_s^Y dW_s.$$

Now for $p \ge 1$, there exists C_p such that

$$0 \le \left(\int_0^t \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds\right)^p \le C_p\left((G(Y_t))^p + \frac{t^p}{\eta_\star^p} + \sup_{u\in[0,t]}\left|\int_0^u \frac{1}{f(Y_s)} Z_s^Y dW_s\right|^p\right).$$

Recall that $\phi_\star(t) \leq Y_t \leq \phi^\star(t)$ (Equation (19)). Since G is non-increasing

$$0 \leq G(Y_t) \leq G(\phi_\star(t))$$

and since $-f'$ and $\phi_\star$ are non-decreasing (that is f is concave), for any $s \in [0,T]$

$$0 \leq \frac{1}{-f'(Y_s)} \leq \frac{1}{-f'(\phi_\star(T))} < +\infty$$

Taking the expectation and using BDG's inequality we obtain

$$\begin{aligned}
\mathbb{E}\left(\int_0^t \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds\right)^p &\leq C_p\left((G(\phi_\star(t)))^p + \frac{t^p}{\eta_\star^p}\right) \\
&+ C_p\mathbb{E}\left(\sup_{u\in[0,t]}\left|\int_0^u \frac{1}{f(Y_s)} Z_s^Y dW_s\right|^p\right) \\
&\leq C_p\left(G(\phi_\star(t))^p + \frac{T^p}{\eta_\star^p}\right) \\
&+ \widehat{C}_p\mathbb{E}\left[\left(\int_0^t \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds\right)^{p/2}\right].
\end{aligned}$$

Therefore

$$\mathbb{E}\left[\left(\int_0^T \frac{-f'(Y_s)}{(f(Y_s))^2}(Z_s^Y)^2 ds\right)^p\right] < +\infty.$$

Using the monotonicity of f and f', using (19) we get

$$\mathbb{E}\left[\left(\int_0^T \frac{-\psi_\star'(s)}{\psi_\star(s)(\psi^\star(s))^2}(Z_s^Y)^2 ds\right)^p\right] = \frac{1}{\eta^\star}\mathbb{E}\left[\left(\int_0^T \frac{-f'(\phi_\star(s))}{(f(\phi^\star(s)))^2}(Z_s^Y)^2 ds\right)^p\right] < +\infty.$$

Recall that under **(C1)**, from Lemma 5, the function $s \mapsto \frac{-\psi_\star'(s)}{\psi_\star(s)}(T-s)$ is bounded. This leads to the conclusion. $\square$

4 Asymptotic behavior for a general driver f

In this section, we assume that the hypotheses **(A1)** to **(A3)**, **(C1)** and **(C2)** hold. Recall that

$$\psi(x) = -\phi'(x) = -f(\phi(x)) \geq 0$$

and if $\phi_t = \phi(A_t)$, assume that

$$Y_t = \phi_t + \psi\left(\frac{T-t}{\eta^\star}\right) H_t = \phi_t + \psi^\star(t) H_t. \tag{20}$$

Let us derive formally the dynamics of H. From the proof of Lemma 1

$$\begin{aligned}
-dY_t &= \phi'(A_t)\left[\frac{1}{\eta_t}dt + Z_t^A dW_t\right] - \frac{1}{2}\phi''(A_t)(Z_t^A)^2 dt \\
&+ \frac{1}{\eta^\star}\psi'\left(\frac{T-t}{\eta^\star}\right) H_t dt - \psi^\star(t) dH_t
\end{aligned}$$

But we also know that

$$-dY_t = \frac{1}{\eta_t} f(Y_t)dt + \lambda_t dt - Z_t^Y dW_t.$$

Then

$$\begin{aligned}
-\psi^\star(t)dH_t &= \left[\frac{1}{\eta_t} f(Y_t) - \frac{1}{\eta_t}\phi'(A_t)\right] dt + \left[\lambda_t + \frac{1}{2}\phi''(A_t)(Z_t^A)^2\right] dt \\
&- \frac{1}{\eta^\star}\psi'\left(\frac{T-t}{\eta^\star}\right) H_t dt - \left[\phi'(A_t) Z_t^A + Z_t^Y\right] dW_t.
\end{aligned}$$

And we deduce

$$\begin{aligned}
-dH_t &= \frac{1}{\psi^\star(t)\eta_t}\left[f(\phi_t + \psi^\star(t)H_t) - f(\phi_t)\right] dt - \frac{1}{\eta^\star\psi^\star(t)}\psi'\left(\frac{T-t}{\eta^\star}\right) H_t dt \\
&+ \frac{\lambda_t}{\psi^\star(t)} dt + \frac{1}{2}\frac{\psi(A_t)}{\psi^\star(t)}\left[\frac{\phi''(A_t)}{\psi(A_t)} A_t\right] \frac{(Z_t^A)^2}{A_t} dt \\
&- \frac{1}{\psi^\star(t)}\left[\phi'(A_t) Z_t^A + Z_t^Y\right] dW_t.
\end{aligned}$$

From Lemma 1, we know that $Y_t \geq \phi_t = \phi(A_t)$, thus $H_t \geq 0$ a.s. In other words H should solve the BSDE:

$$H_t = \int_t^T F(s, H_s)ds - \int_t^T Z_s^H dW_s, \tag{21}$$

with generator

$$F(t,h) = \alpha_t \frac{(Z_t^A)^2}{A_t} + \gamma_t + \left[\frac{\beta_t}{T-t} h + \frac{1}{\psi^\star(t)\eta_t}\left[f(\phi_t + \psi^\star(t)h) - f(\phi_t)\right]\right] \mathbf{1}_{h\geq 0} \tag{22}$$

with

$$\begin{aligned}
\alpha_t &= \frac{1}{2}\frac{\psi(A_t)}{\psi^\star(t)}\left[\frac{\phi''(A_t)}{\psi(A_t)} A_t\right] = \frac{1}{2}\frac{\psi(A_t)}{\psi^\star(t)}\left[-\frac{\psi'(A_t)}{\psi(A_t)} A_t\right], \\
\beta_t &= -\frac{T-t}{\eta^\star}\frac{\psi'}{\psi}\left(\frac{T-t}{\eta^\star}\right), \\
\gamma_t &= \frac{\lambda_t}{\psi^\star(t)}.
\end{aligned}$$

Let us emphasize that the generator is *singular* in the sense of [17], since

$$\int_0^T \frac{\beta_t}{T-t} dt = +\infty.$$

Hence we will adopt their definition ([17, Definition 2.1]) of a solution.

Definition 2 (BSDE with singular generator) *We say that* (H, Z^H) *solves the BSDE* (21) *if the relation* (21) *holds a.s. for any* $t \in [0,T]$ *and if*

$$\mathbb{E}\left[\int_0^T |F(s,H_s)|ds + \left(\int_0^T (Z_s^H)^2 ds\right)^{\frac{1}{2}}\right] < +\infty.$$

The aim of this section is to prove existence of a minimal non-negative solution (H, Z^H) of this BSDE (21), without using the existence of Y, such that the relation (20) holds a.s. on $[0,T]$.

4.1 Properties of the generator F

In order to construct the process H, let us describe the properties of the generator F given by (22).

4.1.1 On the coefficients α, β and γ

Since $\phi'' = (f' \circ \phi)\phi' \geq 0$ and $\psi' = -\phi'' \leq 0$, the three processes α, β and γ are non-negative. Moreover the functions ϕ and $\psi = -\phi'$ are continuous and bounded on $[\eta, +\infty)$ for any $\eta > 0$ and ψ never reaches zero on compact subset on $(0, \infty)$. Thereby the coefficients α, β and γ are bounded on any time interval $[0, T - \theta]$ for $0 < \theta < T$. The next result shows that they are also bounded on the whole interval $[0, T]$.

Lemma 5 *The next two assertions are equivalent.*

1. *There exists a constant $\delta > 0$ and $R > 0$ such that $x \mapsto G(x)^{-\delta}$ is convex on $[R, +\infty)$ (condition* **(C1)***).*
2. *The functions ϕ and ψ verify the next property: there exists $K > 1$ and $\varrho > 0$ such that for all $x \in (0, \varrho]$*
$$\left| x \frac{\phi''(x)}{\phi'(x)} \right| + \left| x \frac{\psi'(x)}{\psi(x)} \right| \leq K.$$

The constants are related by: $\varrho = 1/R$ and $\delta = K - 1$.

Proof. Remark that
$$\psi'(x) = -\phi''(x) \Rightarrow -\frac{\psi'(x)}{\psi(x)} x = \frac{\phi''(x)}{\psi(x)} x.$$

Moreover $x \dfrac{\phi''(x)}{\phi'(x)} \leq 0$. Hence it is enough to show that there exists $K > 0$ such that $-x \dfrac{\phi''(x)}{\phi'(x)} \leq K$. W.l.o.g. we can assume that $K > 1$. Now let us define φ by $\varphi(x) = \phi(1/x)$ for any $x > 0$. Then

$$\phi'(x) = -\frac{1}{x^2} \varphi'(1/x), \qquad \phi''(x) = \frac{2}{x^3} \varphi'(1/x) + \frac{1}{x^4} \varphi''(1/x).$$

Thus
$$-x \frac{\phi''(x)}{\phi'(x)} = 2 + \frac{1}{x} \frac{\varphi''(1/x)}{\varphi'(1/x)}.$$

Hence to establish Lemma 5 it is sufficient to prove that there exists $K > 1$ and $\varrho > 0$ such that for all $t \geq 1/\varrho = R$,
$$-t \frac{\varphi''(t)}{\varphi'(t)} \geq 2 - K = -(K - 2).$$

Let us rewrite this condition in terms of the so-called Arrow-Pratt coefficient of absolute risk aversion by interpreting φ as utility function,

$$\alpha_\varphi(t) := -\frac{\varphi''(t)}{\varphi'(t)} \geq -\frac{(K - 2)}{t} =: \alpha_{K-2}(t), \tag{23}$$

where the utility function to α_K is given (up to positive affine transformations) by $u_K(t) = t^{K-1}$. By a classical theorem due to Pratt ([28], see also [9, Proposition 2.44]), Condition (23) holds if and only if

$$\varphi = \mathfrak{F} \circ u_K \tag{24}$$

for a strictly increasing concave function $\mathfrak{F}$. As $\varphi = G^{-1}(1/\cdot)$, Pratt's condition is equivalent to

$$\mathfrak{F}(t) := G^{-1}\left(\frac{1}{u_K^{-1}(t)}\right) = G^{-1}\left(t^{-\frac{1}{K-1}}\right).$$

defines a strictly increasing concave function. In other words $x \mapsto G(x)^{1-K}$ is strictly increasing and convex. This achieves the proof of the Lemma. □

Under the condition **(C1)**, using the second assertion of the previous lemma, the process β_t is non negative and bounded provided that $T-t \leq \eta^\star \varrho = \eta^\star/R$, that is $T-\eta^\star/R \leq t \leq T$. The process λ is bounded and since ψ tends to ∞ when x goes to zero, γ is bounded on $[0,T]$.

Concerning the process α, using (9), $A_t \leq 1/R$, if $T - \frac{\eta_\star}{R} \leq t \leq T$. Thus the process $\left[-\frac{\psi'(A)}{\psi(A)}A\right]$ is bounded on this interval. Since ψ is non increasing, $\psi(A_t) \leq \psi^\star(t)$. Therefore we deduce that $\frac{\psi(A)}{\psi^\star(t)}$ is also bounded. Finally under condition **(C1)** and with our assumption **(A1)** on η and on λ, α, β and γ are bounded processes at least on some interval $[\tau, T]$ with

$$\tau = \max\left(T - \frac{\eta^\star}{R}, T - \frac{\eta_\star}{R}, 0\right). \tag{25}$$

For $t \in [\tau, T]$

$$|\alpha_t| \leq \left(1 + \frac{K}{2}\right)\max(1, \|\eta\|^K), \quad |\beta_t| \leq K, \quad |\gamma_t| \leq \|\lambda\|(\psi(T))^{-1}.$$

On the rest of the interval $[0,\tau]$ these coefficients are also bounded due to the regularity of f (Conditions **(A1)** and **(A2)**).

Remark 2 *Using the previous lemma, integration leads to: for any $y \in (0, 1/R)$ and $a \leq 1$,*

$$1 \leq \frac{\psi(ay)}{\psi(y)} \leq \frac{1}{a^K}.$$

Hence for any $\delta > 1$: $1 \leq \frac{\psi^\star_\delta(t)}{\psi^\star(t)} \leq \delta^K$. If we assume that for some $\delta > 1$

$$Y_t = \phi_t + \psi\left(\frac{T-t}{\delta\eta^*}\right)\widehat{H}_t,$$

we have: $\widehat{H}_t \leq H_t \leq \delta^K \widehat{H}_t$. Hence, up to some constant, this new development of Y is equivalent to (20).

4.1.2 Properties of Z^A

In the generator F given by (22), we also have to control the process Z^A. First note that the martingale $\left(\int_0^t Z_s^A dW_s,\ t\in[0,T]\right)$ is a BMO martingale (see [18]) and $Z^A \in \mathbb{H}^q((0,T))$, $q>1$, due to the assumption that η is bounded above and away from zero.

Lemma 6 *For any $\rho\in(0,1)$ and $p>1$, we have*

$$\mathbb{E}\left[\left(\int_0^T \frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds\right)^p\right] < +\infty. \tag{26}$$

Proof. Let us apply Itô's formula to $A^{1-\rho}$ on $[0,T-\varepsilon]$:

$$\begin{aligned} A_t^{1-\rho} &= (A_{T-\varepsilon})^{1-\rho} + \int_t^{T-\varepsilon}(1-\rho)\frac{(A_s)^{-\rho}}{\eta_s}ds - \frac{1}{2}\int_t^{T-\varepsilon}(1-\rho)(-\rho)\frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds \\ &+ \int_t^{T-\varepsilon}(1-\rho)(A_s)^{-\rho}Z_s^A dW_s. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{(1-\rho)\rho}{2}\int_0^{T-\varepsilon}\frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds &= A_0^{1-\rho} - (A_{T-\varepsilon})^{1-\rho} - (1-\rho)\int_0^{T-\varepsilon}\frac{1}{\eta_s(A_s)^\rho}ds \\ &- (1-\rho)\int_0^{T-\varepsilon}(A_s)^{-\rho}Z_s^A dW_s. \end{aligned} \tag{27}$$

Taking the expectation and using (9) and the fact that $t\mapsto(T-t)^{-\rho}$ is integrable at time T, we can apply Lebesgue monotone convergence theorem to get

$$\mathbb{E}\int_0^T \frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds = \frac{2}{\rho}\mathbb{E}\left((A_0)^{1-\rho} - \int_0^T \frac{1}{\eta_s(A_s)^\rho}ds\right) < +\infty.$$

Using (27) for any $p>1$ we obtain for some constant $C_p>0$,

$$\begin{aligned} \frac{1}{C_p}\left|\int_0^{T-\varepsilon}\frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds\right|^p &\leq |(A_{T-\varepsilon})^{1-\rho}|^p + |(A_0)^{1-\rho}|^p + \left|\int_0^{T-\varepsilon}\frac{1}{\eta_s(A_s)^\rho}ds\right|^p \\ &+ \left|\int_0^{T-\varepsilon}(A_s)^{-\rho}Z_s^A dW_s\right|^p \\ &\leq |(A_0)^{1-\rho}|^p + |(T/\eta_\star)^{1-\rho}|^p + \frac{1}{\eta_\star^p}\left(\int_0^T \frac{1}{(A_s)^\rho}ds\right)^p \\ &+ \left|\int_0^{T-\varepsilon}(A_s)^{-\rho}Z_s^A dW_s\right|^p. \end{aligned}$$

From the BDG and Hölder inequalities, taking the expectation leads to

$$\begin{aligned} \frac{1}{C_p}\mathbb{E}\left|\int_0^{T-\varepsilon}\frac{(Z_s^A)^2}{(A_s)^{1+\rho}}ds\right|^p &\leq \mathbb{E}\left[|(A_0)^{1-\rho}|^p + |(T/\eta_\star)^{1-\rho}|^p + \frac{1}{\eta_\star^p}\left(\int_0^T \frac{1}{(A_s)^\rho}ds\right)^p\right] \\ &+ \left\{\mathbb{E}\left[\left|\int_0^{T-\varepsilon}((A_s)^{-\rho}Z_s^A)^2 ds\right|^p\right]\right\}^{1/2}. \end{aligned}$$

The function $x \mapsto x^{1-\rho}$ is bounded on $[0, T/\eta_\star]$ by some constant C. Thus

$$\begin{aligned}\frac{1}{C_p}\mathbb{E}\left|\int_0^{T-\varepsilon} \frac{(Z^A_s)^2}{(A_s)^{1+\rho}}\, ds\right|^p &\leq \mathbb{E}\left[|(A_0)^{1-\rho}|^p + |(T/\eta_\star)^{1-\rho}|^p + \frac{1}{\eta_\star^p}\left(\int_0^T \frac{1}{(A_s)^\rho}\, ds\right)^p\right] \\ &+ C^{p/2}\left\{\mathbb{E}\left[\left|\int_0^{T-\varepsilon} \frac{(Z^A_s)^2}{(A_s)^{1+\rho}} ds\right|^p\right]\right\}^{1/2}.\end{aligned}$$

In other words if $\gamma_\varepsilon = \mathbb{E}\left|\int_0^{T-\varepsilon} \frac{(Z^A_s)^2}{(A_s)^{1+\rho}}\, ds\right|^p$, then there exists C independent of ε such that

$$0 \leq \gamma_\varepsilon \leq C(1 + (\gamma_\varepsilon)^{1/2}),$$

which leads to the existence of some constant C such that $\gamma_\varepsilon \leq C$. Using the monotone convergence theorem, we obtain the desired estimate. $\square$

From this estimate on Z^A, using (9), we have for any $p > 1$

$$\mathbb{E}\left[\left(\int_0^T \frac{(Z^A_s)^2}{A_s} ds\right)^p\right] \leq \left(\frac{T}{\eta_\star}\right)^{\rho p} \mathbb{E}\left[\left(\int_0^T \frac{(Z^A_s)^2}{(A_s)^{1+\rho}} ds\right)^p\right] < +\infty. \tag{28}$$

4.2 Construction of the process H

Recall that the generator F is given by:

$$F(t,h) = \alpha_t \frac{(Z^A_t)^2}{A_t} + \gamma_t + \left[\frac{\beta_t}{T-t} h + \frac{1}{\psi^\star(t)\eta_t}\left[f(\phi_t + \psi^\star(t)h) - f(\phi_t)\right]\right] \mathbf{1}_{h\geq 0}.$$

Let us summarize its properties.

- $F(t,h) = F(t,0) \geq 0$ for any $h \leq 0$.
- F is continuous and monotone w.r.t. h: for any h and h',

  $$(h-h')(F(t,h) - F(t,h')) \leq \frac{\beta_t}{T-t}(h-h')^2,$$

  since f is itself monotone.
- For any $|h| \leq r$,

  $$|F(t,h) - F(t,0)| \leq \frac{\beta_t}{T-t} r - \frac{f(\phi_t + \psi^\star(t) r)}{\psi^\star(t)\eta_t}.$$

- The process $F(\cdot, 0)$ equal to

  $$F(t,0) = \alpha_t \frac{(Z^A_t)^2}{A_t} + \gamma_t$$

  belongs to $\mathbb{L}^p([0,T]\times\Omega)$ for any $p > 1$ (Inequality (28) and boundedness of the coeffcients α, β and γ due to **(C1)**).

Our aim is to prove that the BSDE (21) has a solution (H, Z^H). However we cannot apply directly the results of [24], since the previous functions $t \mapsto \frac{\beta_t}{T-t}$ and $t \mapsto -\frac{f(\phi_t+\psi^\star(t)r)}{\psi^\star(t)\eta_t}$ are not necessarily integrable on $[0,T]$. The cases of BSDEs with singular generator studied in [16, 17] are also not adapted to our problem. In order to solve the problem, we modify the generator F. Let us consider $\delta > 0$ and $\varepsilon > 0$ and define

$$F^{\delta,\varepsilon}(t,h) = \alpha_t \frac{(Z^A_t)^2}{A_t} + \gamma_t + \left[\frac{\beta_t}{T+\varepsilon-t} h + \frac{1}{\psi^\star(t)\eta_t} \Big[f(\phi_t + \psi^\delta(t)h) - f(\phi_t) \Big] \right] \mathbf{1}_{h\geq 0} \quad (29)$$

with

$$\psi^\delta(t) = \psi\left(\frac{T+\delta-t}{\eta^\star} \right) = \psi^\star(t-\delta).$$

We consider the following BSDE

$$H_t = \int_t^T F^{\delta,\varepsilon}(s,H_s)ds - \int_t^T Z^H_s dW_s \quad (30)$$

on the interval $[0,T]$.

Lemma 7 *Assume that* **(C1)** *and* **(C2)** *hold. Define*

$$\mu^\varepsilon_t = \int_\tau^t \frac{\beta_s}{T+\varepsilon-s} ds.$$

Then there exists a unique solution $(H^{\delta,\varepsilon}, Z^{H,\delta,\varepsilon})$ to the BSDE (30) such that a.s. for all $t \in [0,T]$

$$|H^{\delta,\varepsilon}_t| \leq \mathbb{E}\left[\int_t^T e^{\mu^\varepsilon_s - \mu^\varepsilon_t} |F(s,0)| ds \middle| \mathcal{F}_t \right],$$

and

$$\mathbb{E}\left[\sup_{s\in[t,T]} \left| e^{\mu^\varepsilon_s} H^{\delta,\varepsilon}_s \right|^p + \left(\int_t^T e^{2\mu^\varepsilon_s} |Z^{H,\delta,\varepsilon}_s|^2 ds \right)^{p/2} \middle| \mathcal{F}_t \right] \leq C_q \mathbb{E}\left[\left(\int_t^T e^{\mu^\varepsilon_s} |F(s,0)| ds \right)^p \middle| \mathcal{F}_t \right].$$

Finally a.s. for any $t \in [0,T]$, $H^{\delta,\varepsilon}_t \geq 0$.

Proof. Let us check that all conditions of [24, Proposition 5.24] hold (we keep also the same notations). First we have for all (h, h')

$$(h-h')(F^{\delta,\varepsilon}(t,h) - F^{\delta,\varepsilon}(t,h')) \leq \frac{\beta_t}{T+\varepsilon-t}(h-h')^2$$

and the process β is bounded. Moreover if $|h| \leq r$

$$\begin{aligned} |F^{\delta,\varepsilon}(t,h)| &\leq |F(t,0)| + \frac{\beta_t}{T+\varepsilon-t} r - \frac{f(\phi_t + \psi^\delta(t)r)}{\psi^\star(t)\eta_t} \\ &= |F(t,0)| + \frac{\beta_t}{T+\varepsilon-t} r - \frac{f(\phi_t + \psi(\delta)r)}{\psi^\star(t)\eta_t} = \Phi^\sharp_r(t). \end{aligned}$$

From our assumptions, in particular here Condition **(C2)**, the definition of ψ and the properties of η, using Inequality (28), we deduce that

$$\mathbb{E}\left[\left(\int_0^T e^{\mu_s^\varepsilon}\Phi_r^\sharp(t)ds\right)^p\right] < +\infty.$$

Using [24, Proposition 5.24], we deduce that there exists a unique solution $(H^{\delta,\varepsilon}, Z^{H,\delta,\varepsilon})$ satisfying the desired estimate. □

Remark 3 (Comments on (C2)) *In Example 2, all functions are submultiplicative (and thus* **(C2)** *holds), except* $f(y) = -\exp(ay^2)$*. Nevertheless for this case*

$$\frac{-f(\phi_t + r)}{\psi^\star(t)} \leq C\exp(ar^2)\exp(2ar\phi_t) = \exp(ar^2)\exp(2arG^{-1}(A_t)).$$

And using (9)

$$\begin{aligned}
\int_\tau^T \frac{-f(\phi_t + r)}{\psi^\star(t)}dt &= \exp(ar^2)\int_\tau^T \exp(2arG^{-1}(A_t))dt \\
&\leq \exp(ar^2)\int_\tau^T \exp\left(2arG^{-1}\left(\frac{T-t}{\eta^\star}\right)\right)dt \\
&\leq \eta^\star\exp(ar^2)\int_\zeta^\infty \exp(2arz)\exp(-az^2)dz < +\infty.
\end{aligned}$$

Thereby **(C2)** *holds also in this case.*

Let us begin with an a priori estimate on $H^{\delta,\varepsilon}$. Recall that the function ϑ is defined just before Lemma 2.

Lemma 8 *For all* $t \in [0, T)$,

$$0 \leq H_t^{\delta,\varepsilon} \leq \frac{\vartheta(T-t)}{\psi^\star(t)}. \tag{31}$$

In particular $H^{\delta,\varepsilon}$ *is bounded on any interval* $[0, T-\theta]$, $0 < \theta < T$.

Proof. For fixed δ and ε, the dynamics of $\phi_t + \psi^\star(t)H_t^{\delta,\varepsilon}$ is given by:

$$\begin{aligned}
-d(\phi_t + \psi^\star(t)H_t^{\delta,\varepsilon}) &= \phi'(A_t)\frac{1}{\eta_t}dt - \frac{1}{2}\phi''(A_t)(Z_t^A)^2dt \\
&+ \frac{1}{\eta^\star}\psi'\left(\frac{T-t}{\eta^\star}\right)H_t^{\delta,\varepsilon}dt + \psi^\star(t)F^{\delta,\varepsilon}(t, H_t^{\delta,\varepsilon})dt \\
&+ \phi'(A_t)Z_t^AdW_t - \psi^\star(t)Z_t^{H,\delta,\varepsilon}dW_t.
\end{aligned}$$

Recall that $F^{\delta,\varepsilon}$ is given by (29). Therefore we obtain

$$\begin{aligned}
-d(\phi_t + \psi^\star(t)H_t^{\delta,\varepsilon}) &= \left[\lambda_t + \frac{1}{\eta_t}f(\phi_t + \psi_t^\delta H_t^{\delta,\varepsilon})\right]dt \\
&+ \frac{1}{\eta^\star}\psi'\left(\frac{T-t}{\eta^\star}\right)\frac{1}{(T-t)^2}\frac{\varepsilon}{T+\varepsilon-t}H_t^{\delta,\varepsilon}dt \\
&+ \left[\phi'(A_t)Z_t^A - \psi^\star(t)Z_t^{H,\delta,\varepsilon}\right]dW_t.
\end{aligned}$$

In other words the process $U_t = \phi_t + \psi^\delta_t H^{\delta,\varepsilon}_t$ satisfies the BSDE:

$$\begin{aligned} U_t + (\psi^\star(t) - \psi^\delta_t) H^{\delta,\varepsilon}_t &= U_{T-\theta} + \int_t^{T-\theta} \left[\lambda_s + \frac{1}{\eta_s} f(U_s) \right] ds \\ &- \int_t^{T-\theta} \Theta_s ds - \int_t^{T-\theta} Z^U_s dW_s, \end{aligned}$$

with a non negative Θ, $(\psi^\star - \psi^\delta) H^{\delta,\varepsilon} \geq 0$. Using Lemma 2, we deduce that

$$\forall t \in [\tau, T), \quad U_t \leq \vartheta (T-t).$$

This leads to the conclusion of the Lemma. □

Again as a by-product, our proof implies that for any solution (H, Z^H) of the BSDE (21)

$$0 \leq H_t \leq \frac{\vartheta(T-t)}{\psi^\star(t)}. \tag{32}$$

Now by the comparison principle, for a fixed $\delta > 0$, since $\beta_t \geq 0$, $(H^{\delta,\varepsilon}_t,\ \varepsilon > 0)$ is a increasing sequence when ε decreases to zero, and for a fixed $\varepsilon > 0$, $(H^{\delta,\varepsilon}_t,\ \delta > 0)$ is a decreasing sequence when δ decreases to zero. Thereby for any $\varepsilon_1 < \varepsilon_2$ and $\delta_1 < \delta_2 \leq \delta$ for some $\delta > 0$, we have the following inequalities: a.s.

$$0 \leq H^{\delta_1,\varepsilon_1}_t \leq H^{\delta_2,\varepsilon_1}_t \leq H^\delta_t,$$

and

$$0 \leq H^{\delta_1,\varepsilon_2}_t \leq H^{\delta_1,\varepsilon_1}_t \leq H^\delta_t,$$

where

$$H^\delta_t = \lim_{\varepsilon \downarrow 0} H^{\delta,\varepsilon}_t. \tag{33}$$

Note that H^δ also satisfies (31). Now for a fixed $\varepsilon > 0$, we define

$$H^\varepsilon_t = \lim_{\delta \downarrow 0} H^{\delta,\varepsilon}_t, \tag{34}$$

and

$$H_t = \lim_{\varepsilon \downarrow 0} H^\varepsilon_t. \tag{35}$$

Since $H^{\delta,\varepsilon}_T = 0$ a.s., we have immediately that a.s. $H_T = 0$ and for all $t \in [0,T]$, $H_t \geq 0$.

Proposition 2 *There exists $Z^H \in \mathbb{H}^p(0, T-\theta)$ for any $\theta > 0$, such that the couple (H, Z^H) solves the BSDE (21) with generator F on the interval $[0, T-\theta]$ for any $0 < \theta < T$.*

Proof. Let us define $c_p = p(1 \wedge (p-1))$.
Step 1. Given $\varepsilon_1 < \varepsilon_2$ and $\delta_1 < \delta_2$, applying Itô's formula to $\Delta H = H^{\delta_1,\varepsilon_1} - H^{\delta_2,\varepsilon_2}$ on the interval $[t, T-\theta]$, $\theta > 0$, leads to:

$$\begin{aligned} & e^{p\widehat{\mu}_t} |\Delta H_t|^p + \frac{c_p}{2} \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z^H_s|^2 ds \\ & \quad \leq e^{p\widehat{\mu}_T} |\Delta H_{T-\theta}|^p \\ & \qquad + p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \Delta H_s (F^{\delta_1,\varepsilon_1}(s, H^{\delta_1,\varepsilon_1}_s) - F^{\delta_2,\varepsilon_2}(s, H^{\delta_2,\varepsilon_2}_s)) ds \\ & \qquad - p \int_t^{T-\theta} \left(\frac{\beta_s}{\theta} + 2\frac{p-1}{p} \right) e^{p\widehat{\mu}_s} |\Delta H_s|^p ds - p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \Delta H_s \Delta Z^H_s dW_s \end{aligned}$$

with $\Delta Z^H = Z^{H,\delta_1,\varepsilon_1} - Z^{H,\delta_2,\varepsilon_2}$ and

$$\widehat{\mu}_t = \int_0^t \left(\frac{\beta_s}{\theta} + 2\frac{p-1}{p} \right) ds.$$

Remark that from the monotonicity of f:

$$\begin{aligned}
\Delta H_s(F^{\delta_1,\varepsilon_1}(s,H_s^{\delta_1,\varepsilon_1}) - F^{\delta_2,\varepsilon_2}(s,H_s^{\delta_2,\varepsilon_2})) &= \Delta H_s \left[\frac{\beta_s}{T+\varepsilon_1-s} H_s^{\delta_1,\varepsilon_1} - \frac{\beta_s}{T+\varepsilon_2-s} H_s^{\delta_2,\varepsilon_2} \right] \\
&\quad + \frac{1}{\psi_s \eta_s} \Delta H_s \left[f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_1,\varepsilon_1}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right] \\
&\leq \Delta H_s \left[\frac{\beta_s}{T+\varepsilon_1-s} H_s^{\delta_1,\varepsilon_1} - \frac{\beta_s}{T+\varepsilon_2-s} H_s^{\delta_2,\varepsilon_2} \right] \\
&\quad + \frac{1}{\psi_s \eta_s} \Delta H_s \left[f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right] \\
&\leq \frac{\beta_s}{T+\varepsilon_1-s} (\Delta H_s)^2 + \Delta H_s H_s^{\delta_2,\varepsilon_2} \beta_s \frac{\varepsilon_2-\varepsilon_1}{(T+\varepsilon_1-s)(T+\varepsilon_2-s)} \\
&\quad + \frac{1}{\psi_s \eta_s} \Delta H_s \left[f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right].
\end{aligned}$$

We deduce that

$$\begin{aligned}
& e^{p\widehat{\mu}_t} |\Delta H_t|^p + \frac{c_p}{2} \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z_s^H|^2 ds \\
& \leq e^{p\widehat{\mu}_{T-\theta}} |\Delta H_{T-\theta}|^p + p \int_t^{T-\theta} e^{p\widehat{\mu}_s} \left[\frac{\beta_s}{T+\varepsilon_1-s} - \frac{\beta_s}{\theta} - 2\frac{p-1}{p} \right] |\Delta H_s|^p ds \\
& \quad + p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \Delta H_s H_s^{\delta_2,\varepsilon_2} \beta_s \frac{\varepsilon_2-\varepsilon_1}{(T+\varepsilon_1-s)(T+\varepsilon_2-s)} ds \\
& \quad + p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \frac{\Delta H_s}{\psi_s \eta_s} \left[f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right] ds \\
& \quad - p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \Delta H_s \Delta Z_s^H dW_s.
\end{aligned}$$

By Young's inequality

$$\begin{aligned}
& p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \Delta H_s H_s^{\delta_2,\varepsilon_2} \beta_s \frac{\varepsilon_2-\varepsilon_1}{(T+\varepsilon_1-s)(T+\varepsilon_2-s)} ds \\
& \leq (p-1) \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^p ds + \int_t^{T-\theta} e^{p\widehat{\mu}_s} \left[H_s^{\delta_2,\varepsilon_2} \beta_s \frac{\varepsilon_2-\varepsilon_1}{(T+\varepsilon_1-s)(T+\varepsilon_2-s)} \right]^p ds
\end{aligned}$$

and

$$\begin{aligned}
& p \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^{p-2} \mathbf{1}_{\Delta H_s \neq 0} \frac{\Delta H_s}{\psi_s \eta_s} \left[f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right] ds \\
& \leq (p-1) \int_t^{T-\theta} e^{p\widehat{\mu}_s} |\Delta H_s|^p ds \\
& \quad + \int_t^{T-\theta} e^{p\widehat{\mu}_s} \left[\frac{1}{\psi_s \eta_s} \left| f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right| \right]^p ds.
\end{aligned}$$

Recall that β, $1/\eta$ are bounded on $[0,T]$ whereas $1/\psi$ is bounded on $[0,T-\theta]$. From Estimate (31), on the interval $[0,T-\theta]$, $H^{\varepsilon_2,\delta_2}$ is also bounded and we have:

$$
\begin{aligned}
& e^{p\widehat{\mu}_t}|\Delta H_t|^p + \frac{c_p}{2}\int_t^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}|\Delta Z^H_s|^2 ds \\
& \quad \leq e^{p\widehat{\mu}_{T-\theta}}|\Delta H_{T-\theta}|^p + C(\varepsilon_2-\varepsilon_1)^p \\
& \qquad + C\int_t^{T-\theta} \left| f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon_2}) - f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon_2}) \right|^p ds \\
& \qquad - p\int_t^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}\Delta H_s \Delta Z^H_s dW_s. \qquad (36)
\end{aligned}
$$

The constant C depends on all bounds of our coefficients and on θ. This constant explodes when θ goes to zero.

Step 2. Let us fix $\varepsilon = \varepsilon_1 = \varepsilon_2 > 0$. Since for $\delta_2 \leq \delta$, $H^{\delta_2,\varepsilon} \leq H^\delta$ and H^δ satisfies the estimate (31), using the dominated convergence theorem

$$
\mathbb{E}\int_0^{T-\theta} e^{p\widehat{\mu}_s} \left| f(\phi_s + \psi_s^{\delta_2} H_s^{\delta_2,\varepsilon}) - f(\phi_s + \psi_s^{\delta_1} H_s^{\delta_2,\varepsilon}) \right|^p ds \to 0,
$$

as δ_1 and δ_2 tend to zero. Therefore using (36) and taking the expectation we deduce that

$$
\mathbb{E}\int_0^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}|\Delta Z^H_s|^2 ds
$$

tends to zero when δ_1 and δ_2 go to zero. Moreover remark that if

$$
\Lambda_t = \int_0^t e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}\Delta H_s \Delta Z^H_s dW_s,
$$

then the bracket $[\Lambda]^{1/2}_{T-\theta}$ can be handled as in [7]: for any $C > 0$

$$
\begin{aligned}
\mathbb{E}\left([\Lambda]^{1/2}_{T-\theta}\right) & \leq \frac{C}{2}\mathbb{E}\left(\sup_{t\in[0,T-\theta]} e^{p\widehat{\mu}_t}|\Delta H_t|^p\right) \\
& + \frac{1}{2C}\mathbb{E}\left(\int_0^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}|\Delta Z^H_s|^2 ds\right).
\end{aligned}
$$

Thereby $(H^{\delta,\varepsilon},\ \delta > 0)$ is a Cauchy sequence:

$$
\mathbb{E}\left(\sup_{t\in[0,T-\theta]} e^{p\widehat{\mu}_t}|\Delta H_t|^p\right) \to 0
$$

as δ_1 and δ_2 tend to zero. Finally

$$\begin{aligned}
\mathbb{E}\left(\int_0^{T-\theta} e^{2\widehat{\mu}_s}|\Delta Z^H_s|^2 ds\right)^{p/2} &= \mathbb{E}\left(\int_0^{T-\theta} e^{2\widehat{\mu}_s} (\Delta H_s)^{2-p} (\Delta H_s)^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z^H_s|^2 ds\right)^{p/2} \\
\leq \mathbb{E}\left[\left(\sup_{t\in[0,T-\theta]} e^{\widehat{\mu}_t}|\Delta H_t|\right)^{p(2-p)/2} \left(\int_0^{T-\theta} e^{p\widehat{\mu}_s} (\Delta H_s)^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z^H_s|^2 ds\right)^{p/2}\right] \\
\leq \left\{\mathbb{E}\left(\sup_{t\in[0,T-\theta]} e^{p\widehat{\mu}_t}|\Delta H_t|^p\right)\right\}^{(2-p)/2} \left\{\mathbb{E}\int_0^{T-\theta} e^{p\widehat{\mu}_s} (\Delta H_s)^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z^H_s|^2 ds\right\}^{p/2} \\
\leq \frac{2-p}{2}\mathbb{E}\left[\sup_{t\in[0,T-\theta]} e^{p\widehat{\mu}_t}|\Delta H_t|^p\right] + \frac{p}{2}\mathbb{E}\int_0^{T-\theta} e^{p\widehat{\mu}_s} (\Delta H_s)^{p-2} \mathbf{1}_{\Delta H_s \neq 0} |\Delta Z^H_s|^2 ds
\end{aligned}$$

where we have used Hölder's and Young's inequality with $\frac{2-p}{2} + \frac{p}{2} = 1$.

Hence we obtain that $(H^{\delta,\varepsilon}, Z^{H,\delta,\varepsilon})$ converges in $\mathbb{S}^p(0,T-\theta)$ to some process $(H^\varepsilon, Z^{H,\varepsilon})$. The process H^ε is non negative and also satisfies the a priori estimate (31) with $H^\varepsilon_T = 0$, and we have for any $0 \leq t \leq T-\theta < T$

$$\begin{aligned}
H^\varepsilon_t &= H^\varepsilon_{T-\theta} + \int_t^{T-\theta} \left[\alpha_s \frac{(Z^A_s)^2}{A_s} + \gamma_s\right] ds + \int_t^{T-\theta} \frac{\beta_s}{T+\varepsilon-s} H^\varepsilon_s ds \\
&+ \int_t^{T-\theta} \frac{1}{\psi^\star(s)\eta_s} \left[f(\phi_s + \psi^\star(s)H^\varepsilon_s) - f(\phi_s)\right] ds - \int_t^{T-\theta} Z^{H,\varepsilon}_s dW_s.
\end{aligned}$$

Step 3. Let us prove the convergence of $(H^\varepsilon, Z^{H,\varepsilon})$ when ε tends to zero. The arguments are almost the same as in the second step. Indeed the formula (36) becomes:

$$\begin{aligned}
& e^{p\widehat{\mu}_t}|\Delta H_t|^p + \frac{c_p}{2}\int_t^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}|\Delta Z^H_s|^2 ds \\
&\quad \leq e^{p\widehat{\mu}_{T-\theta}}|\Delta H_{T-\theta}|^p + C(\varepsilon_2-\varepsilon_1)^p \\
&\qquad -p\int_t^{T-\theta} e^{p\widehat{\mu}_s}|\Delta H_s|^{p-2}\mathbf{1}_{\Delta H_s\neq 0}\Delta H_s \Delta Z^H_s dW_s.
\end{aligned}$$

with $\Delta H = H^{\varepsilon_1} - H^{\varepsilon_2}$ and $\Delta Z^H = Z^{H,\varepsilon_1} - Z^{H,\varepsilon_2}$. The conclusion follows from the same arguments as in step 2. □

Note that from (31), the arguments to prove (19) and the remark 2, we obtain that a.s. on $[0,T)$

$$0 \leq H_t \leq C\frac{1}{\psi^\star(t)}\phi^\star(t) = C\frac{\phi^\star(t)}{-f(\phi^\star(t))}, \tag{37}$$

and

$$0 \leq \phi_t + \psi^\star(t)H_t \leq \phi^\star_\varepsilon(t). \tag{38}$$

Since f is a non positive and non increasing function, from **(A3)**, the function $y \mapsto -1/f(y)$ is an integrable, non-negative and non increasing function. Thereby we know that (see [14, Section 178])

$$\lim_{y\to+\infty} \frac{y}{-f(y)} = 0.$$

Thereby using the a priori estimate (37), the process H satisfies a.s.

$$\lim_{t\to T} H_t = 0.$$

Let us emphasize that H solves the BSDE (21), in the sense that for any $0 \leq t \leq u < T$

$$\begin{aligned}
H_t &= H_u + \int_t^u F(s,H_s)ds - \int_t^u Z_s^H dW_s \\
&= \int_t^u \left(\alpha_s \frac{(Z_s^A)^2}{A_s} + \gamma_s\right) ds - \int_t^u Z_s^H dW_s \\
&+ \int_t^u \frac{\beta_s}{T-s} H_s ds + \int_t^u \frac{1}{\psi^\star(s)\eta_s} \left[f(\phi_s + \psi^\star(s)H_s) - f(\phi_s)\right] ds. \qquad (39)
\end{aligned}$$

It is important to note that if we define H by the relation (20) on the basis of the minimal solution of the BSDE (1), then this process H satisfies all properties described previously. Proposition 2 shows that H can be constructed "from scratch" if **(C1)** and **(C2)** hold.

Now we prove that this solution (H, Z^H) is a solution of the BSDE (21) (in the sense of Definition 2) and that this solution is minimal. The a priori estimate (37) is crucial here. The processes α and γ are bounded and non-negative on $[0,T]$. Hence using Inequality (28), the integral

$$\int_t^T \left(\alpha_s \frac{(Z_s^A)^2}{A_s} + \gamma_s\right) ds$$

is well defined and is the increasing limit of the same integral on the interval $[t,u]$ for $u < T$. Since H is non-negative we also have

$$\int_t^u \left|\frac{\beta_s}{T-s} H_s\right| ds = \int_t^u \frac{\beta_s}{T-s} H_s ds,$$

and

$$\int_t^u \frac{1}{\psi^\star(s)\eta_s} \left|f(\phi_s + \psi^\star(s)H_s) - f(\phi_s)\right| ds = -\int_t^u \frac{1}{\psi^\star(s)\eta_s} \left[f(\phi_s + \psi^\star(s)H_s) - f(\phi_s)\right] ds.$$

Lemma 9 *The process* $\left(\dfrac{\beta_s}{T-s} H_s,\ s \in [0,T)\right)$ *is integrable on* $[0,T]$.

Proof. Indeed using the very definition of β_s we have

$$\begin{aligned}
0 \leq \frac{\beta_s}{T-s} H_s &= -\frac{1}{\eta^\star}\frac{\psi'}{\psi}\left(\frac{T-s}{\eta^\star}\right) H_s \leq -\frac{1}{\eta^\star}\frac{\psi'}{\psi}\left(\frac{T-s}{\eta^\star}\right) \frac{\phi\left(\frac{T-t}{\eta^\star}\right)}{\psi\left(\frac{T-t}{\eta^\star}\right)} \\
&= \frac{1}{\eta^\star}\left(\frac{\phi''\phi}{(\phi')^2}\right)\left(\frac{T-s}{\eta^\star}\right).
\end{aligned}$$

Now

$$\int_t^u \frac{1}{\eta^\star}\left(\frac{\phi''\phi}{(\phi')^2}\right)\left(\frac{T-s}{\eta^\star}\right) ds = \int_{(T-u)/(\eta^\star)}^{(T-t)/(\eta^\star)} \left(\frac{\phi''\phi}{(\phi')^2}\right)(x)\, dx$$

and our result follows if $(\phi''\phi)/(\phi')^2$ is integrable at zero. Remark that

$$\left(\frac{\phi}{\phi'}\right)' = 1 - \frac{\phi''\phi}{(\phi')^2}.$$

Hence the integrability is equivalent to the existence of the limit at zero of ϕ/ϕ', that is the limit at infinity of $y \mapsto y/(-f(y))$, which is zero. This achieves the proof of the lemma. $\square$

Coming back to (39) and taking the conditional expectation, we get:

$$\begin{aligned} H_t &= \mathbb{E}\left[\int_t^u \left(\alpha_s \frac{(Z_s^A)^2}{A_s} + \gamma_s\right) ds \middle| \mathcal{F}_t\right] \\ &+ \mathbb{E}\left[\int_t^u \frac{\beta_s}{T-s} H_s ds \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^u \frac{1}{\psi^\star(s)\eta_s}\left[f(\phi_s + \psi^\star(s)H_s) - f(\phi_s)\right] ds \middle| \mathcal{F}_t\right]. \end{aligned}$$

From the previous lemma, we deduce that

$$\mathbb{E}\left[\int_t^T \frac{1}{\psi^\star(s)\eta_s}\left|f(\phi_s + \psi^\star(s)H_s) - f(\phi_s)\right| ds\right] < +\infty.$$

In other words taking $t = 0$

$$\mathbb{E}\left[\int_0^T |F(s,H_s)| ds\right] < +\infty.$$

Then using again (39), we easily deduce that

$$\mathbb{E}\left[\sup_{0\le u\le T}\left|\int_0^u Z_s^H dW_s\right|\right] < +\infty.$$

By Burkholder-Davis-Gundy's inequality, we deduce that Z^H is an element of $\mathbb{H}^1(0,T)$. Let us summarize our results.

Theorem 1 *Assume that* **(C1)** *and* **(C2)** *hold. There exists a process (H, Z^H), which the minimal non-negative solution of the BSDE* (21)*, that is:*

- *H is non negative and essentially bounded: for any $0 \le t < T$, $0 \le \sup_{s\in[0,t]} H_s < +\infty$ a.s. and*

$$\mathbb{E}\int_0^T |F(s,H_s)| ds < +\infty.$$

- *The process Z^H belongs to $\mathbb{H}^1(0,T) \cap \mathbb{H}^p(0,T-\theta)$ for any $\theta > 0$ and $p > 1$.*

- *For any $0 \le t \le T$*

$$H_t = \int_t^T F(s,H_s)ds - \int_t^T Z_s^H dW_s.$$

  In particular

$$\lim_{t\to T} H_t = 0 = H_T.$$

- *For any $(\widehat{H}, \widehat{Z})$ solution of the BSDE* (21)*, a.s. for any $t \in [0,T]$, $\widehat{H}_t \ge H_t$.*

Proof. The only thing to prove is the minimality. Let $(\widehat{H},\widehat{Z})$ be another solution of the BSDE (21). Let us first show that $\widehat{H}$ is a non-negative process. Applying Itô's formula for the non-positive part of $\widehat{H}$ and the very definition (22) of F leads to:

$$\begin{aligned}\left(\widehat{H}_t\right)^- &\leq -\int_t^T F(s,\widehat{H}_s)\mathbf{1}_{\widehat{H}_s\leq 0}ds + \int_t^T (\widehat{Z}_s)\mathbf{1}_{\widehat{H}_s\leq 0}dW_s \\ &= -\int_t^T \left(\alpha_s\frac{(Z^A_s)^2}{A_s}+\gamma_s\right)\mathbf{1}_{\widehat{H}_s\leq 0}ds + \int_t^T (\widehat{Z}_s)\mathbf{1}_{\widehat{H}_s\leq 0}dW_s.\end{aligned}$$

Since α and γ are non-negative, taking the conditional expectation knowing $\mathcal{F}_t$ yields to the non-negativity of $\widehat{H}$.

Now for $\varepsilon>0$, the process $\Delta H=\widehat{H}-H^\varepsilon$ satisfies for any $\theta>0$:

$$\begin{aligned}\Delta H_t &= \Delta H_{T-\theta} + \int_t^{T-\theta}\left[F(s,\widehat{H}_s)-F^\varepsilon(s,H^\varepsilon_s)\right]ds - \int_t^{T-\theta}(\widehat{Z}_s-Z^\varepsilon_s)dW_s \\ &= \Delta H_{T-\theta} + \int_t^{T-\theta}\left[\varepsilon\frac{\beta_s}{T-s}\widehat{H}_s + \left(\frac{\beta_s}{T+\varepsilon-s}+\kappa_s\right)\Delta H_s\right]ds \\ &- \int_t^{T-\theta}(\widehat{Z}_s-Z^\varepsilon_s)dW_s\end{aligned}$$

where

$$\kappa_s = \frac{1}{\psi^\star(s)\eta_s}\left[f(\phi_s+\psi^\star(s)\widehat{H}_s)-f(\phi_s+\psi^\star(s)H^\varepsilon_s)\right]\frac{1}{\Delta H_s}\mathbf{1}_{\Delta H_s\neq 0}.$$

This process κ is bounded from above by zero since f is monotone. Thus if

$$\Gamma_{t,s} = \exp\left(\int_t^s\left(\frac{\beta_u}{T+\varepsilon-u}+\kappa_u\right)du\right),$$

by standard arguments concerning linear BSDE (see [20, Lemma 10] or [24, Proposition 5.31]), we have:

$$\Delta H_t = \mathbb{E}\left[\Delta H_{T-\theta}\Gamma_{t,T-\theta} + \int_t^{T-\theta}\varepsilon\frac{\beta_s}{T-s}\widehat{H}_s\Gamma_{t,s}ds\middle|\mathcal{F}_t\right] \geq \mathbb{E}\left[\Delta H_{T-\theta}\Gamma_{t,T-\theta}\middle|\mathcal{F}_t\right].$$

By Fatou's lemma, letting θ going to zero, we obtain that for any $\varepsilon>0$, $\widehat{H}_t\geq H^\varepsilon_t$. Hence the minimality of H is proved. □

The process (H,Z^H) solves a BSDE with singular driver in the sense of [17]. As mentioned in [17, Proposition 3.1], uniqueness is not an obvious property for such kind of BSDEs. In our case assume that $(\widehat{H},\widehat{Z})$ be another non negative solution of the BSDE (21). Then

$$\begin{aligned}\Delta H_t &= \int_t^T\left[F(s,\widehat{H}_s)-F(s,H_s)\right]ds - \int_t^T(\widehat{Z}_s-Z^H_s)dW_s \\ &= \int_t^T\lambda_s\Delta H_s ds - \int_t^T\Delta Z_s dW_s.\end{aligned}$$

Hence we have a linear BSDE with singular generator with

$$\lambda_s = \frac{\beta_s}{T-s} + \frac{1}{\psi^\star(s)\eta_s}\left[f(\phi_s+\psi^\star(s)\widehat{H}_s)-f(\phi_s+\psi^\star(s)H_s)\right]\frac{1}{\Delta H_s}\mathbf{1}_{\Delta H_s\neq 0}.$$

Nevertheless we cannot apply the result in [17, Propositions 3.1 and 3.5], since we don't know the sign of the drift λ.

Remark 4 *If $\widehat{H}_t = \lim_{\delta \downarrow 0} H_t^\delta$, then $H_t \leq \widehat{H}_t$. The proof of the previous proposition shows that $\widehat{H}$ also satisfies the BSDE* (21) *(in the sense of the previous theorem). It seems difficult to prove that these two processes are equal. In other words, as remarked above, we don't have any comparison or uniqueness result concerning the BSDE* (21).

4.3 Asymptotics of the minimal solution

Let us consider the process $\widehat{Y}_t = \phi_t + \psi^\star(t) H_t$ on $[0,T)$. Then from our heuristic study, for any $0 \leq t \leq s < T$, we have:

$$\widehat{Y}_t = \widehat{Y}_s + \int_t^s \left[\frac{f(\widehat{Y}_u)}{\eta_u} + \gamma_u \right] du - \int_t^s Z_u^{\widehat{Y}} dW_u$$

and a.s. $\lim_{t \uparrow T} \widehat{Y}_t = +\infty$. Note that this process $(\widehat{Y}, Z^{\widehat{Y}})$ belongs to any $\mathbb{S}^\infty(0, T-\theta)$ for any $\theta > 0$.

If $f(y) = -y|y|^q$, by uniqueness proved in [13], $\widehat{Y} = Y$ and thus

$$Y_t = \left(\frac{1}{qA_t} \right)^{\frac{1}{q}} + \left(\frac{\eta^\star}{q(T-t)} \right)^{\frac{1}{q}+1} H_t$$

with $0 \leq H_t \leq C(T-t)$ (Inequality (37)). In other words, the non-negative process

$$Y_t \left(A_t \right)^{\frac{1}{q}} = Y_t \left(\mathbb{E} \left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t \right] \right)^{\frac{1}{q}}$$

is bounded.

In general since (Y, Z) is the minimal non-negative solution of (1) (see Proposition 1), we have a.s.

$$\forall t \in [0,T], \quad \phi_t \leq Y_t \leq \widehat{Y}_t = \phi_t + \psi^\star(t) H_t.$$

But from our heuristic computations, we have $Y_t = \phi_t + \psi^\star(t) \widetilde{H}_t$. Thus $0 \leq \widetilde{H}_t \leq H_t$ and $\widetilde{H}$ satisfies the same BSDE (21), at least on any interval $[0, T-\varepsilon]$. Since $\widetilde{H} \leq H$, the preceding arguments show that $\widetilde{H}$ is also a solution of the same BSDE on the whole interval $[0,T]$. Since H is the minimal solution, we have proved that $\widetilde{H} = H$ and thus $Y = \widehat{Y}$. In other words:

Theorem 2 *The minimal solution (Y, Z^Y) of the BSDE* (1) *is given by* (20)*:*

$$Y_t = \phi_t + \psi \left(\frac{T-t}{\eta^\star} \right) H_t,$$

where (H, Z^H) is the minimal solution of the BSDE (21).

Recall that from Inequality (38), on the interval $[\tau, T]$ with τ given by (25),

$$\frac{\psi^\star(t) H_t}{\phi_t} \leq \frac{\phi_1^\star(t)}{\phi_\star(t)}.$$

Lemma 10 *There exists a constant κ depending on $\eta^\star$ and $\eta_\star$ such that for any $t \in [\tau, T]$,*

$$0 \leq \frac{\psi^\star(t) H_t}{\phi_t} \leq \kappa.$$

Proof. Recall that since ϕ is non increasing, $\phi(y) \leq \phi(ay)$. Moreover from Lemma 5 we know that for any $y \in (0, 1/R)$ and $0 < a \leq 1$,

$$1 \leq \frac{\psi(ay)}{\psi(y)} \leq \frac{1}{a^K}.$$

From the very definition of ψ, this inequality can be written as: $-\phi'(ay) \leq -a^{-K}\phi'(y)$. Integrating this inequality (between y and η) leads to:

$$\phi(ay) \leq a^{1-K}\phi(y) + C$$

for some constant $C \geq 0$. Hence

$$1 \leq \frac{\phi(ay)}{\phi(y)} \leq a^{1-K} + \frac{C}{\phi(y)}.$$

Since $\phi(0) = \infty$, the conclusions follows from this inequality. $\square$

Hence we have proved that the minimal solution Y of the BSDE (1) satisfies on the interval $[\tau, T]$:

$$\phi_t \leq Y_t \leq \phi_t(1 + \kappa).$$

5 The concave case

In the expansion (20) of Y, there is an asymmetry between ψ_t which is random, and the deterministic $\psi^\star(t)$. This asymmetry has the advantages to avoid the presence of Z^H in the generator of H and of an extra term with the second derivative of ψ. However it leads to the fact that

$$f'\left(\phi^\star(t)\right)\eta_t \neq f'(\phi_t) = f'\left(\phi\left(A_t\right)\right). \tag{40}$$

Thereby we cannot interpret the bracket

$$\begin{aligned}&\frac{\beta_t}{T-t}h + \frac{1}{\psi^\star(t)\eta_t}[f(\phi_t + \psi^\star(t)h) - f(\phi_t)] \\ &= \frac{1}{\psi^\star(t)\eta_t}\left[f(\phi_t + \psi^\star(t)h) - f(\phi_t) - f'\left(\phi^\star(t)\right)\eta_t\psi^\star(t)h\right]\end{aligned}$$

as the reminder to the first Taylor polynomial of f at ϕ_t.

Here we study a possible workaround: define ϕ_t and ψ_t symmetrically, i.e. $\phi_t = \psi(A_t)$ and $\psi_t = \psi(A_t)$. Then (40) is satisfied. This, however, leads to an additional linear term in the driver of H. This linear term creates some main difficulties. To overcome them, we add several assumptions on f and on η.

5.1 Symmetric development for general process η

Recall that A satisfies (14), ϕ verifies: $\phi' = f \circ \phi$ and $\psi = -\phi'$. Setting $\phi_t := \phi(A_t)$ and $\psi_t := \psi(A_t)$ yields

$$d\phi_t = -f(\phi_t)\left[\frac{1}{\eta_t}dt + Z_t^A dW_t\right] + \frac{1}{2}\phi''(A_t)(Z_t^A)^2 dt,$$
$$d\psi_t = -f'(\phi_t)\psi_t\left[\frac{1}{\eta_t}dt + Z_t^A dW_t\right] + \frac{1}{2}\psi''(A_t)(Z_t^A)^2 dt.$$

Recall that (9) leads to (10):

$$\phi_\star(t) = \phi\left(\frac{T-t}{\eta_\star}\right) \le \phi_t \le \phi\left(\frac{T-t}{2\eta^\star}\right) = \phi^\star(t).$$

And since ϕ' is non-decreasing, we also have: $\psi_\star(t) \le \psi_t \le \psi^\star(t)$. For

$$-dY_t = \frac{1}{\eta_t} f(Y_t)\, dt + \lambda_t\, dt - Z_t^Y\, dW_t$$

we make the ansatz $Y_t = \phi_t + \psi_t H_t$ and hence obtain the heuristic dynamics of H, namely:

$$\begin{aligned}
-dH_t &= \frac{1}{\eta_t\psi_t}\left[f(\phi_t+\psi_t H_t) - f(\phi_t) - f'(\phi_t)\psi_t H_t\right] dt \\
&+ \left[\frac{\lambda_t}{\psi_t} - \frac{\phi''(A_t)A_t}{2\phi'(A_t)}\frac{(Z_t^A)^2}{A_t}\right] dt \\
&+ \left[\frac{\psi''(A_t)(A_t)^2}{2\psi(A_t)}\left(\frac{Z_t^A}{A_t}\right)^2 H_t - \frac{A_t\psi'(A_t)}{\psi(A_t)}\frac{Z_t^A}{A_t} Z_t^H\right] dt - Z_t^H\, dW_t \\
&= \frac{1}{\eta_t\psi_t}\left[f(\phi_t+\psi_t H_t) - f(\phi_t) - f'(\phi_t)\psi_t H_t\right] dt \\
&+ \left[\frac{\lambda_t}{\psi_t} + \kappa_t^1\frac{(Z_t^A)^2}{A_t}\right] dt + \left[\kappa_t^2\left(\frac{Z_t^A}{A_t}\right)^2 H_t + \kappa_t^3\frac{Z_t^A}{A_t}Z_t^H\right] dt - Z_t^H\, dW_t
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
\kappa_t^1 &= -\frac{\phi''(A_t)A_t}{2\phi'(A_t)} \\
\kappa_t^2 &= \frac{\psi''(A_t)(A_t)^2}{2\psi(A_t)} = \left(-A_t\frac{\psi''(A_t)}{\psi'(A_t)}\right)\kappa_t^1 \\
\kappa_t^3 &= -\frac{A_t\psi'(A_t)}{\psi(A_t)}.
\end{aligned}$$

Note that from (13), $H_t \ge 0$ a.s. Under **(C1)**, using Lemma 5, we deduce that κ^1 and κ^3 are bounded and non-negative.

Compared to the previous section and the BSDE (21), the dynamics (41) of H has a new linear term, namely

$$(t,h,z) \mapsto \kappa_t^2\left(\frac{Z_t^A}{A_t}\right)^2 h + \kappa_t^3\frac{Z_t^A}{A_t}z.$$

In the next lemma we prove that under the additional conditions **(C3)** and **(C4)** of f, κ^2 is bounded.

Lemma 11 *Assume that* **(C1)**, **(C3)** *and* **(C4)** *hold. Then the term* $\kappa^2(x) := x^2 \dfrac{\psi''(x)}{\psi(x)}$ *is non-negative and bounded on a neighborhood of zero.*

Proof. Indeed if f is concave, we have

$$\psi''(x) = (-f'' \circ \phi)(x)(\phi'(x))^2 - (f' \circ \phi(x))\phi''(x) \geq 0.$$

Hence $x\psi''(x)/\psi'(x) \leq 0$. The conclusion of the lemma is equivalent to the boundedness from below of $-x\widehat{\psi}''(x)/\widehat{\psi}'(x)$ in the neighborhood of ∞ with $\widehat{\psi}(x) = \psi(1/x)$. From the proof of Lemma 5, we have

$$\widehat{\psi}(x) = (-f)(\mathfrak{F} \circ u_K) = \widehat{\mathfrak{F}} \circ u_K$$

where $\mathfrak{F}$ and $\widehat{\mathfrak{F}}$ are increasing and concave. Note that we can assume w.l.o.g. that the constant K is the same. Indeed if **(C1)** holds for some $\delta > 0$, the same condition holds for any $\delta' \geq \delta$. Hence boundedness is equivalent to the existence of $K > 1$ such that $\mathfrak{F} : x \mapsto G^{-1}(x^{-1/(K-1)})$ (condition **(C1)**) and $(-f) \circ \mathfrak{F}$ are increasing and concave. $\square$

Remark 5 *Note that under* **(C3)**, *the boundedness of* κ_2 *is equivalent to condition* **(C4)**.

Let us consider again the functions of Example 2.

- If $f(y) = -y|y|^q$ for some $q > 0$, then we can take

  $$\mathfrak{F}(x) = \left(\frac{1}{q}\right)^{\frac{1}{q}} x^{\frac{1}{q+1}}, \quad ((-f) \circ \mathfrak{F})(x) = \left(\frac{q+1}{q}\right)^{\frac{1}{q}} x$$

  and $K = 2 + 1/q$.

- If $f(y) = -(\exp(ay) - 1)$ for some $a > 0$, then $\phi(x) = -\frac{1}{a}\log\left(1 - e^{-ax}\right)$ and $\psi(x) = -\phi'(x) = \dfrac{1}{e^{ax} - 1}$. Hence

  $$-\frac{\phi''(x)}{\phi'(x)}x = \frac{axe^{ax}}{e^{ax} - 1} \underset{x\to 0}{\sim} 1, \quad -\frac{\psi''(x)}{\psi'(x)}x = \frac{ax(1 + e^{ax})}{e^{ax} - 1} \underset{x\to 0}{\sim} 2,$$

  are bounded near zero.

- If $f(y) = -\exp(ay^2)$ for some $a > 0$, then

  $$\phi(x) = \frac{1}{\sqrt{2a}}\mathcal{N}^{-1}\left(1 - x\sqrt{\frac{a}{\pi}}\right), \quad \psi(x) = -\phi'(x) = \exp\left(-a\phi(x)^2\right).$$

  And

  $$\psi'(x) = -\phi''(x) = -2a\phi(x)\psi(x)^2, \quad \psi''(x) = 2a(\psi(x)^3) - 4a\phi(x)\psi(x)\psi'(x).$$

  Thus
  $$-\frac{\psi''(x)}{\psi'(x)}x = x\frac{\psi(x)}{\phi(x)} - 2x\frac{\psi'(x)}{\psi(x)}.$$

  From **(C1)** and Lemma 5, the second term is bounded. Arguing as at the end of Example 2 yields to:

  $$x\frac{\psi(x)}{\phi(x)} = \frac{\sqrt{2a}}{z}G\left(\frac{z}{\sqrt{2a}}\right)e^{-z^2/2} \underset{z\to+\infty}{\longrightarrow} 0.$$

In other words all functions considered in Example 2 verify **(C3)** and **(C4)**.

However the preceding lemma is not sufficient to obtain existence of a solution. Indeed if we consider this BSDE (41) with $f \equiv 0$, we get a linear BSDE. From our best knowledge, existence of a solution is proved only under some exponential moment condition on the coefficients (see [24, Proposition 5.31]). Even if we avoid the final time T, then $1/A$ is bounded on $[0, T-\varepsilon]$ (Inequality (9)), but Z^A is only BMO. Hence the stochastic exponential of the martingale $M = \int_0^\cdot Z^A_s dW_s$ is uniformly integrable. But controlling the exponential of the bracket of M is more difficult. If $\xi = \int_0^T 1/\eta_s ds$ is Malliavin differentiable, then we require that its Malliavin derivative has exponential moments. Finally, if we do not control the quantity $(Z^A)^2/A^2$, then from [17, Proposition 3.1], we may have infinitely many solutions.

First we show that any solution (H, Z^H) of (41) on $[0,T)$ is a solution of the BSDE on $[0,T]$, that is:

Proposition 3 *Under the hypotheses* **(C1)** *to* **(C5)***, there exists a non-negative process* (H, Z^H) *solution (in the sense of Definition 2) of: for any* $t \in [0,T]$*,*

$$\begin{aligned} H_t &= \int_t^T \frac{1}{\eta_s \psi_s} \left[f(\phi_s + \psi_s H_s) - f(\phi_s) - f'(\phi_s)\psi_s H_s \right] \mathbf{1}_{H_s \geq 0}\, ds \\ &+ \int_t^T \left[\frac{\lambda_s}{\psi_s} + \kappa^1_s \frac{(Z^A_s)^2}{A_s} + \kappa^2_s \left(\frac{Z^A_s}{A_s} \right)^2 H_s + \kappa^3_s \frac{Z^A_s}{A_s} Z^H_s \right] ds - \int_t^T Z^H_s\, dW_s. \end{aligned} \tag{42}$$

Proof. Here we do not construct (H, Z^H) from scratch, but we use the existence of a minimal solution (Y, Z^Y) of (1). Indeed our previous computations show that if $H = (Y - \phi)/\psi$, then the process (H, Z^H) verifies:

- It satisfies the dynamics given by (41) on any interval $[0, T-\varepsilon]$.
- H verifies an a priori estimate similar to (32):

$$0 \leq H_t \leq \frac{\phi^\star(t)}{\psi_t} \leq \frac{\phi^\star(t)}{\psi_\star(t)} \leq \left(\frac{2\eta^\star}{\eta_\star} \right)^K \frac{\phi^\star(t)}{\psi^\star(t)}$$

  (we use again Remark 2).
- Z^H belongs to $\mathbb{H}^p(\widehat{\tau}, T-\varepsilon)$ for any $p > 1$.

Thus we only have to extend the assertions on $[0,T]$.

Compared to Section 4 and the discussion above Lemma 9, we need to control the additional term:

$$\left[\kappa^2_t \left(\frac{Z^A_t}{A_t} \right)^2 H_t + \kappa^3_t \frac{Z^A_t}{A_t} Z^H_t \right].$$

We already know that κ^2 and κ^3 are bounded and that Z^A satisfies the inequality (26). Let us precise the relation between Z^H and Z^Y. Since $Y = \phi + \psi H$, we have:

$$Z^Y_t = -f(\phi_t) Z^A_t - f'(\phi_t)\psi_t H_t Z^A_t + \psi_t Z^H_t = \psi_t Z^A_t - f'(\phi_t)\psi_t H_t Z^A_t + \psi_t Z^H_t$$

thus

$$Z_t^H = \left[-1 + f'(\phi_t) H_t\right] Z_t^A + \frac{1}{\psi_t} Z_t^Y$$

and

$$\begin{aligned}
\frac{Z_t^H}{(A_t)^{(1-\rho)/2}} &= \left[-1 + f'(\phi_t)H_t\right] \frac{Z_t^A}{(A_t)^{(1-\rho)/2}} + \frac{1}{\psi_t (A_t)^{(1-\rho)/2}} Z_t^Y \\
&= \left[-(A_t)^\rho - \kappa_t^3 \frac{H_t}{(A_t)^{1-\rho}}\right] \frac{Z_t^A}{(A_t)^{(1+\rho)/2}} + \frac{1}{\psi_t (A_t)^{(1-\rho)/2}} Z_t^Y. \qquad (43)
\end{aligned}$$

Using (9), (26) and Lemma 4, for any $p > 1$,

$$\mathbb{E}\left[\left(\int_0^T \frac{(Z_t^A)^2}{(A_t)^{1+\rho}} dt\right)^p + \left(\int_0^T \frac{(Z_t^Y)^2}{(\psi_t)^2 (A_t)^{(1-\rho)}} dt\right)^p\right] < +\infty.$$

Combining (43) together this estimate, we have

$$\mathbb{E} \int_0^T \left[\kappa_t^2 \left(\frac{Z_t^A}{A_t}\right)^2 H_t + \left|\kappa_t^3 \frac{Z_t^A}{A_t} Z_t^H\right|\right] < +\infty$$

if we can prove that for some $p > 1$

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} \frac{H_t}{(A_t)^{1-\rho}}\right)^p\right] < +\infty.$$

We know that

$$0 \le \frac{H_t}{(A_t)^{1-\rho}} \le (\eta^\star)^{1-\rho} \frac{H_t}{(T-t)^{1-\rho}} \le (\eta^\star)^{1-\rho} \left(\frac{2\eta^\star}{\eta_\star}\right)^K \frac{\phi^\star(t)}{\psi^\star(t)(T-t)^{1-\rho}}.$$

The last term is deterministic and if we prove that this term remains bounded on $[0, T]$, the result follows. Note that

$$\frac{(2\eta^\star)^{1-\rho}\phi^\star(t)}{\psi^\star(t)(T-t)^{1-\rho}} = \frac{\phi^\star(x)}{\psi^\star(x)x^{1-\rho}} = \frac{\phi^\star(x)}{-f(\phi^\star(x))x^{1-\rho}} = \frac{y}{-f(y)G(y)^{1-\rho}} = \frac{y}{h(y)}.$$

with $x = (T-t)/(2\eta^\star)$ and $y = \phi^\star(x)$. From Condition **(C5)**, we deduce that (H, Z^H) is a solution of our BSDE on $[0, T]$. □

Let us point out again that in the BSDE (42), the driver has a triple singularity:

- $(s, h) \mapsto \dfrac{1}{\eta_s \psi_s} f'(\phi_s)\psi_s h$ (as in Lemma 9);
- $(s, z) \mapsto \dfrac{Z_s^A}{A_s} z$, which can be controlled if we can apply Girsanov's theorem, that is if we control the martingale $\left(t \mapsto \displaystyle\int_0^t \frac{Z_s^A}{A_s} dW_s,\ t \in [0, T]\right)$;
- $(s, h) \mapsto \left(\dfrac{Z_s^A}{A_s}\right)^2 h$, which requires to control the quadratic variation of the previous martingale.

Hence we add the next condition:

(H) *There exist a deterministic time $\widehat{\tau} < T$ and a positive constant $\mathfrak{C} > \frac{1}{2} \vee \|\kappa^2\|_\infty \vee \|\kappa^3\|_\infty^2$ such that*

$$\mathbb{E}\left[\exp\left(\mathfrak{C}\int_{\widehat{\tau}}^T \left(\frac{Z_s^A}{A_s}\right)^2 ds\right)\right] < +\infty.$$

Assuming that **(C1)** to **(C4)** and **(H)** hold and arguing as in the proof of Proposition 2 in Section 4, we construct directly a process (H, Z^H) solving the dynamics (41) on any interval $[\widehat{\tau}, T-\varepsilon]$. To get the existence result similar to Lemma 7, we use [24, Theorem 5.30] and Condition **(H)** is designed to fulfill the assumptions of this theorem.

More important than the existence, we obtain also uniqueness under **(H)**[5].

Proposition 4 *Under the hypotheses* **(C1)** *to* **(C5)** *and* **(H)**, *there exists a unique non-negative process (H, Z^H) solution (in the sense of Definition 2) of the BSDE* (42).

Proof. Let us show first that any solution $(\widehat{H}, \widehat{Z})$ of (42) is non-negative. From the Itô formula for the non-positive part of $\widehat{H}$ we obtain for $t \in [\widehat{\tau}, T]$:

$$\begin{aligned}
\left(\widehat{H}_t\right)^- &\leq -\int_t^T F(s, \widehat{H}_s)\mathbf{1}_{\widehat{H}_s\leq 0} ds + \int_t^T (\widehat{Z}_s)\mathbf{1}_{\widehat{H}_s\leq 0} dW_s \\
&= -\int_t^T \left[\frac{\lambda_s}{\psi_s} + \kappa_s^1\frac{(Z_s^A)^2}{A_s} + \kappa_s^2\left(\frac{Z_s^A}{A_s}\right)^2 \widehat{H}_s + \kappa_s^3\frac{Z_s^A}{A_s}\widehat{Z}_s\right]\mathbf{1}_{\widehat{H}_s\leq 0} ds \\
&\quad + \int_t^T (\widehat{Z}_s)\mathbf{1}_{\widehat{H}_s\leq 0} dW_s \\
&\leq \int_t^T \kappa_s^2\left(\frac{Z_s^A}{A_s}\right)^2 \left(\widehat{H}_s\right)^- ds - \int_t^T \kappa_s^3\frac{Z_s^A}{A_s}\widehat{Z}_s\mathbf{1}_{\widehat{H}_s\leq 0} ds + \int_t^T (\widehat{Z}_s)\mathbf{1}_{\widehat{H}_s\leq 0} dW_s.
\end{aligned}$$

From the condition **(H)**, the martingale

$$\mathcal{E}(Z^A)_t = \exp\left(\int_{\widehat{\tau}}^t \kappa_s^3\frac{Z_s^A}{A_s} dW_s + \frac{1}{2}\int_{\widehat{\tau}}^t (\kappa_s^3)^2\left(\frac{Z_s^A}{A_s}\right)^2 ds\right), \qquad t \in [\widehat{\tau}, T],$$

is uniformly integrable. Using Girsanov's theorem and the expression of the solution of a linear BSDE (see [24, Proposition 5.31]), we get that a.s. for any $t \in [\widehat{\tau}, T]$, $\left(\widehat{H}_t\right)^- = 0$. The arguments used in Section 4 show that the process (H, Z^H) is the minimal non-negative solution of (42), that is if $(\widehat{H}, \widehat{Z})$ is another solution of (42), then $\widehat{H}_t \geq H_t$.

Now we prove uniqueness of the solution. Since f is concave, and if $\Delta H = \widehat{H} - H$,

[5] Without **(H)**, even the existence of a minimal solution for the BSDE (42) is unclear.

$\Delta Z = \widehat{Z} - Z^H$, then

$$\begin{aligned}
(\Delta H_t)^+ &\leq \int_t^T \frac{1}{\eta_s \psi_s} \left\{ f(\phi_s + \psi_s \widehat{H}_s) - f(\phi_s + \psi_s H_s) - f'(\phi_s + \psi_s H_s)\psi_s \Delta H_s \right\} \mathbf{1}_{\Delta H_s \geq 0} ds \\
&+ \int_t^T \frac{1}{\eta_s} \left\{ f'(\phi_s + \psi_s H_s) - f'(\phi_s) \right\} \Delta H_s \mathbf{1}_{\Delta H_s \geq 0} ds \\
&+ \int_t^T \left[\kappa_s^2 \left(\frac{Z_s^A}{A_s} \right)^2 \Delta H_s + \kappa_s^3 \frac{Z_s^A}{A_s} \Delta Z_s^H \right] \mathbf{1}_{\Delta H_s \geq 0} \, ds - \int_t^T \Delta Z_s^H \mathbf{1}_{\Delta H_s \geq 0} dW_s \\
&\leq \int_t^T \frac{1}{\eta_s} \left\{ f'(\phi_s + \psi_s H_s) - f'(\phi_s) \right\} \mathbf{1}_{H_s \leq 0} (\Delta H_s)^+ ds \\
&+ \int_t^T \left[\kappa_s^2 \left(\frac{Z_s^A}{A_s} \right)^2 \Delta H_s + \kappa_s^3 \frac{Z_s^A}{A_s} \Delta Z_s^H \right] \mathbf{1}_{\Delta H_s \geq 0} \, ds - \int_t^T \Delta Z_s^H \mathbf{1}_{\Delta H_s \geq 0} dW_s \\
&= \int_t^T \left[\kappa_s^2 \left(\frac{Z_s^A}{A_s} \right)^2 (\Delta H_s)^+ + \kappa_s^3 \frac{Z_s^A}{A_s} \Delta Z_s^H \mathbf{1}_{\Delta H_s \geq 0} \right] ds - \int_t^T \Delta Z_s^H \mathbf{1}_{\Delta H_s \geq 0} dW_s,
\end{aligned}$$

since H is non-negative and f' is non-increasing. Arguing as before yields that $(\Delta H)^+$ is equal to zero. Then uniqueness holds on $[\widehat{\tau}, T]$.

Let us extend uniqueness on the whole time interval $[0, T]$. If $(\widehat{H}, \widehat{Z})$ still denotes another solution, then the two processes solve the same BSDE (42) on $[0, \widehat{\tau}]$ with the same terminal condition $H_{\widehat{\tau}} = \widehat{H}_{\widehat{\tau}}$. Since the generator of (42) remains singular on the whole interval (due to the linear term), uniqueness on the rest of the time interval $[0, \widehat{\tau}]$ is not trivial. But if we define

$$\widehat{Y}_t = \phi_t + \psi_t \widehat{H}_t,$$

then $\widehat{Y}$ is the first part of the solution of the BSDE (1) on $[0, \widehat{\tau}]$ with the bounded terminal condition $\phi_{\widehat{\tau}} + \psi_{\widehat{\tau}} \widehat{H}_{\widehat{\tau}}$. Since uniqueness holds for the BSDE (1), we deduce that $\widehat{H} = H$ also on $[0, \widehat{\tau}]$. □

Theorem 3 *Under the hypotheses* **(C1)** *to* **(C5)** *and* **(H)***, the BSDE* (1) *has a unique solution* (Y, Z^Y)*. This solution is given by:* $Y_t = \phi_t + \psi_t H_t$ *a.s. for any* $t \in [0, T]$*, where* H *is the unique solution of the BSDE* (42).

Proof. Let us consider $(\widehat{Y}, \widehat{Z})$ solution of the BSDE (1) (in the sense of Definition 1). Then it satisfies (18) and the property of Lemma 4. Therefore if we define $\widehat{H} = (\widehat{Y} - \phi)/\psi$, then this process $\widehat{H}$ and the related $\widehat{Z}^H$ solve the BSDE (42) (in the sense of Definition 2). From the previous proposition, we obtain the desired result. □

5.2 About Condition (H)

The condition **(H)** is very strong and seems difficult to be checked in general. However in the Itô setting on the process η, this assumption may hold. Let us suppose that $\dfrac{1}{\eta} =: \gamma$ is an Itô process

$$d\gamma_t = \mathfrak{d}_t^\gamma \, dt + \sigma_t^\gamma \, dW_t. \tag{44}$$

Note that with Condition **(A1)**, it is equivalent to assume that η is an Itô process.

First of all the next result holds.

Lemma 12 *For $p > 2$, if $\mathfrak{d}^\gamma$ and σ^γ belong to $\mathbb{L}^{2p}((0,T)\times\Omega)$, the process Z^A/A belongs to $\mathbb{H}^{2p}(0,T)$, that is*

$$\mathbb{E}\left[\left(\int_0^T \left(\frac{Z_s^A}{A_s}\right)^2 ds\right)^p\right] < +\infty. \tag{45}$$

Proof. We consider the process $\bar{A}_t := A_t/(T-t)$, which satisfies the BSDE

$$-d\bar{A}_t = \frac{1/\eta_t - \bar{A}_t}{T-t}\,dt - Z^{\bar{A}}\,dW_t, \qquad \bar{A}_T = 1/\eta_T.$$

Since $Z_t^{\bar{A}} = Z_t^A/(T-t)$, to verify (45) it is sufficient to establish $Z^{\bar{A}} \in \mathbb{H}^{2p}(0,T)$. For the later again it is sufficient to establish that the driver to $\bar{A}$ is in L^{2p}. (Here we used frequently that η is bounded above and away from zero.)

To establish $(1/\eta - \bar{A})(T-\cdot) \in L^{2p}$ we first check Kolgomorov's criterion for $1/\eta$: For $0 \le t \le s \le T$, by Jensen and BDG inequality,

$$\mathbb{E}[|1/\eta_s - 1/\eta_t|^{2p}] \le C|s-t|^{p-1}\mathbb{E}\left[\int_t^s \left(|\mathfrak{d}_r^\gamma|^{2p} + |\sigma_r^\gamma|^{2p}\right) dr\right].$$

Hence, by Kolgomorov's criterion, for any $\alpha \in (0, \frac{p-2}{2p})$ there exits a random variable $\xi \in L^{2p}(\Omega)$ such that

$$|1/\eta_t - 1/\eta_s| \le \xi|t-s|^\alpha, \qquad t,s \in [0,T].$$

Therefore, using the mean value theorem,

$$\mathbb{E}\left[\left(\int_0^T \left|\frac{1/\eta_t - \bar{A}_t}{T-t}\right| dt\right)^{2p}\right] \le \mathbb{E}\left[\left(\int_0^T \frac{\xi(T-t)^\alpha}{T-t} dt\right)^{2p}\right] \le C\mathbb{E}[\xi^{2p}],$$

which completes the proof. $\square$

The coefficients in the linear part of the BSDE (41) are in $\mathbb{H}^{2p}(0,T)$. However it is not sufficient to get **(H)**. Let us remark that:

$$\begin{aligned} A_t &= \mathbb{E}\left[\int_t^T \frac{1}{\eta_s} ds \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \gamma_s ds \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\int_t^T \left(\gamma_t + \int_t^s \mathfrak{d}_u^\gamma du + \int_t^s \sigma_u^\gamma dW_u\right) ds \middle| \mathcal{F}_t\right] \\ &= \gamma_t(T-t) + \mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right]. \end{aligned} \tag{46}$$

If we denote

$$\widetilde{A}_t = \mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right],$$

then

$$Z_t^A = \sigma_t^\gamma(T-t) + Z_t^{\widetilde{A}}, \quad \text{with} \quad -d\widetilde{A}_t = (T-t)\mathfrak{d}_t^\gamma dt + Z_t^{\widetilde{A}} dW_t.$$

If the quantity $\xi = \int_0^T (T-u)\mathfrak{d}_u^\gamma du$ is in $\mathbb{D}^{1,2}$ (see [22] for the notations concerning the Malliavin calculus), then by the Clark-Ocone formula, we have

$$Z_t^A = \sigma_t^\gamma(T-t) + \int_t^T (T-u)\mathbb{E}\left[D_t\mathfrak{d}_u^\gamma \middle| \mathcal{F}_t\right] du.$$

In the next two lemmas we give sufficient conditions on the coefficients of (44) such that **(H)** holds.

Lemma 13 *If $\mathfrak{d}^\gamma$ and σ^γ are essentially bounded, then Condition* **(H)** *holds.*

Proof. In this setting, Condition **(H)** holds if and only if

$$\mathbb{E}\exp\left[\widehat{\mathfrak{C}}\int_\tau^T \left(\frac{Z_s^{\widetilde{A}}}{A_s}\right)^2 ds\right] < +\infty, \tag{47}$$

for some $\widehat{\mathfrak{C}} > \mathfrak{C}$. Since $\mathfrak{d}^\gamma$ is essentially bounded, then

$$\left|\widetilde{A}_t\right| \leq \mathbb{E}\left[\int_t^T (T-u)\,\|\mathfrak{d}^\gamma\|\,du\middle|\mathcal{F}_t\right] = \frac{\|\mathfrak{d}^\gamma\|}{2}(T-t)^2.$$

Itô's formula leads to

$$-d\left(\frac{\widetilde{A}_t}{(T-t)}\right) = -\frac{\widetilde{A}_t}{(T-t)^2}dt + \mathfrak{d}_t^\gamma dt + \frac{Z_t^{\widetilde{A}}}{(T-t)}dW_t.$$

Hence we obtain that the martingale $\left(\widetilde{M}_u = \displaystyle\int_0^u \frac{Z_t^{\widetilde{A}}}{(T-t)}dW_t,\ u\in[0,T]\right)$ is a BMO martingale:

$$\forall u\in[0,T],\ |\widetilde{M}_T - \widetilde{M}_u| \leq 2\|\mathfrak{d}^\gamma\|(T-u) \Rightarrow \sup_{u\in[t,T]}\mathbb{E}\left[|\widetilde{M}_T-\widetilde{M}_u|\middle|\mathcal{F}_u\right] \leq 2\|\mathfrak{d}^\gamma\|(T-t).$$

Therefore we can choose $\widehat{\tau}$ very close to T such that the BMO norm of $\widetilde{M}$ on $[\widehat{\tau},T]$ is as small as required. Using the Nirenberg inequality (see [18, Theorem 2.2]), there exists a constant C depending on the BMO norm of $\widetilde{M}$, such that

$$\mathbb{E}\left[\exp\left(C\int_t^T\left(\frac{Z_s^{\widetilde{A}}}{A_s}\right)^2 ds\right)\right] < +\infty.$$

Precisely C should be smaller than the inverse of the BMO norm of $\widetilde{M}$. Thereby choosing $\widehat{\tau}$ sufficiently close to T, we get Condition (47) and the conclusion of the lemma. □

Let us now assume that the process $\gamma = 1/\eta$ solves a SDE:

$$d\gamma_t = \mathfrak{d}(\gamma_t)\,dt + \sigma(\gamma_t)\,dW_t \tag{48}$$

where $\mathfrak{d}$ and σ are Lipschitz continuous functions defined on $\mathbb{R}$. From [22, Theorems 2.2.1 and 2.2.2], the coordinate γ_t belongs to $\mathbb{D}^{1,\infty}$ for any $t\in[0,T]$. Moreover for any $p\geq 1$

$$\sup_{0\leq r\leq T}\mathbb{E}\left(\sup_{r\leq t\leq T}|D_r\gamma_t|^p\right) < +\infty. \tag{49}$$

The derivative $D_r\gamma_t$ satisfies the following linear equation:

$$D_r\gamma_t = \sigma(\gamma_r) + \int_r^t \widetilde{\sigma}(s)D_r\gamma_s dW_s + \int_r^t \widetilde{\mathfrak{d}}(s)D_r\gamma_s ds$$

for $r \leq t$ a.e. and $D_r \gamma_t = 0$ for $r > t$ a.e., where $\widetilde{\mathfrak{d}}(s)$ and $\widetilde{\sigma}(s)$ are two bounded processes, such that if $\mathfrak{d}$ and σ are of class C^1, they are given by:

$$\widetilde{\mathfrak{d}}(s) = (\partial_x \mathfrak{d})(\gamma_s), \qquad \widetilde{\sigma}(s) = (\partial_x \sigma)(\gamma_s).$$

In this case $\xi = \int_0^T (T-u)\mathfrak{d}^\gamma_u du$ is in $\mathbb{D}^{1,2}$ and by the Clark-Ocone formula, we have

$$Z^{\widetilde{A}}_t = \int_t^T (T-u)\mathbb{E}\left[D_t \mathfrak{d}^\gamma_u \middle| \mathcal{F}_t \right] du = \int_t^T (T-u)\mathbb{E}\left[\widetilde{\mathfrak{d}}(u) D_t \gamma_u \middle| \mathcal{F}_t \right] du.$$

Lemma 14 *If γ is a diffusion process solution of a SDE with Lipschitz continuous coefficients and a bounded diffusion coefficient, then Condition* **(H)** *holds.*

Proof. Since $\zeta_u = D_t \gamma_u$ satisfies the linear one-dimensional SDE:

$$\zeta_u = \sigma(\gamma_t) + \int_t^u \widetilde{\sigma}(s)\zeta_s dB_s + \int_t^u \widetilde{\mathfrak{d}}(s)\zeta_s ds,$$

an explicit formula for $|\zeta_u|$ reads

$$|\zeta_u| \quad = \quad |\sigma(\gamma_t)| \exp\left[\int_t^u \widetilde{\sigma}(s) dB_s - \frac{1}{2}\int_t^u \widetilde{\sigma}(s)^2 ds \right] \exp\left[\int_t^u \widetilde{\mathfrak{d}}(s) ds \right].$$

Since the function $\mathfrak{d}$ is supposed to be Lipschitz continuous, $\widetilde{\mathfrak{d}}$ is essentially bounded. Thereby

$$\begin{aligned} \left| Z^{\widetilde{A}}_t \right| &\leq \|\widetilde{\mathfrak{d}}\| \int_t^T (T-u)\mathbb{E}\left[|D_t \gamma_u| \middle| \mathcal{F}_t \right] du \\ &= \|\widetilde{\mathfrak{d}}\| |\sigma(\gamma_t)| \int_t^T (T-u)\mathbb{E}\left[\exp\left[\int_t^u \widetilde{\mathfrak{d}}(s) ds \right] \middle| \mathcal{F}_t \right] du \\ &\leq \frac{1}{2}\|\widetilde{\mathfrak{d}}\| e^{T\|\widetilde{\mathfrak{d}}\|} |\sigma(\gamma_t)| (T-t)^2 = C(T-t)^2 |\sigma^\gamma_t|. \end{aligned}$$

Finally it implies that

$$\frac{Z^A_t}{T-t} = \sigma^\gamma_t \left(1 + (T-t)\varsigma^\gamma_t \right),$$

where ς^γ is a bounded process. □

Therefore by Lemma 13 (for (44)) or Lemma 14 (for (48)), using Proposition 4, we deduce that the BSDE (42) with singular generator has a unique solution (H, Z^H) and, using Theorem 3, that the BSDE (1) with singular terminal condition has also a unique solution (Y, Z^Y).

5.3 A different asymptotic development under the bounded Itô setting

If $1/\eta$ is given by (44) with essentially bounded coefficients, we can change our approach. Coming back to (46), we deduce also that

$$A_t \leq \gamma_t (T-t) + \frac{\|\mathfrak{d}^\gamma\|}{2}(T-t)^2.$$

Therefore since ϕ is non increasing, we have

$$\phi(A_t) \geq \phi_t := \phi\left(\gamma_t(T-t) + \frac{\|\mathfrak{d}^\gamma\|}{2}(T-t)^2\right).$$

Define

$$d\widetilde{\gamma}_t = \left(\mathfrak{d}_t^\gamma - \frac{\|\mathfrak{d}^\gamma\|}{2}\right) dt + \sigma_t^\gamma\, dW_t = \widetilde{\mathfrak{d}}_t^\gamma\, dt + \sigma_t^\gamma\, dW_t \tag{50}$$

leads to

$$\phi(A_t) \geq \phi_t = \phi\left(\widetilde{\gamma}_t(T-t)\right).$$

Setting

$$\phi_t := \phi(\widetilde{\gamma}_t(T-t)) \quad \text{and} \quad \psi_t := \psi(\widetilde{\gamma}_t(T-t))$$

yields

$$\begin{aligned}
d\phi_t &= -\widetilde{\gamma}_t\phi'(\widetilde{\gamma}_t(T-t))\, dt + (T-t)\phi'(\widetilde{\gamma}_t(T-t))\, d\widetilde{\gamma}_t + \frac{1}{2}(T-t)^2\phi''(\widetilde{\gamma}_t(T-t))(\sigma_t^\gamma)^2 dt \\
&= -\widetilde{\gamma}_t f(\phi_t)\, dt + \left\{\widetilde{\mathfrak{d}}_t^\gamma(T-t)\phi'(\widetilde{\gamma}_t(T-t)) + \tfrac{1}{2}(\sigma_t^\gamma)^2(T-t)^2\phi''(\widetilde{\gamma}_t(T-t))\right\} dt \\
&\quad + \sigma_t^\gamma(T-t)\phi'(\widetilde{\gamma}_t(T-t))\, dW_t \\
&:= -\widetilde{\gamma}_t f(\phi_t)\, dt + \left\{\widetilde{\mathfrak{d}}_t^\gamma(T-t)\phi_t' + \tfrac{1}{2}(\sigma_t^\gamma)^2(T-t)^2\phi_t''\right\} dt + \sigma_t^\gamma(T-t)\phi_t'\, dW_t
\end{aligned}$$

and

$$\begin{aligned}
d\psi_t &= -\widetilde{\gamma}_t\psi'(\widetilde{\gamma}_t(T-t))\, dt + (T-t)\psi'(\widetilde{\gamma}_t(T-t))\, d\widetilde{\gamma}_t + \frac{1}{2}(T-t)^2\psi''(\widetilde{\gamma}_t(T-t))(\sigma_t^\gamma)^2 dt \\
&:= -\widetilde{\gamma}_t\psi_t f'(\phi_t)\, dt + \left\{\widetilde{\mathfrak{d}}_t^\gamma(T-t)\psi_t' + \tfrac{1}{2}(\sigma_t^\gamma)^2(T-t)^2\psi_t''\right\} dt + \sigma_t^\gamma(T-t)\psi_t'\, dW_t.
\end{aligned}$$

For (Y, Z^Y) solution of (1), we make the ansatz $Y_t = \phi_t + \psi_t H_t$ and hence obtain that $H_t \geq 0$ and:

$$\begin{aligned}
-dH_t &= \frac{\gamma_t}{\psi_t} f(Y_t)\, dt + \frac{\lambda_t}{\psi_t}\, dt + \frac{1}{\psi_t}\, d\phi_t + \frac{H_t}{\psi_t}\, d\psi_t + -\frac{Z_t}{\psi_t}\, dW_t \\
&= \frac{\gamma_t}{\psi_t}\left\{f(\phi_t + \psi_t H_t) - f(\phi_t) - f'(\phi_t)\psi_t H_t\right\} dt + \frac{\lambda_t}{\psi_t}\, dt - \frac{\|\mathfrak{d}^\gamma\|}{2}(T-t)dt \\
&\quad + \frac{1}{\psi_t}\left\{\widetilde{\mathfrak{d}}_t^\gamma(T-t)\phi_t' + \frac{1}{2}(\sigma_t^\gamma)^2(T-t)^2\phi_t''\right\} dt + \sigma_t^\gamma(T-t)\frac{\psi_t'}{\psi_t} Z_t^H\, dt \\
&\quad + \frac{H_t}{\psi_t}\left\{\widetilde{\mathfrak{d}}_t^\gamma(T-t)\psi_t' - \frac{\|\mathfrak{d}^\gamma\|}{2}(T-t)\psi_t' + \frac{1}{2}(\sigma_t^\gamma)^2(T-t)^2\psi_t''\right\} dt - Z_t^H\, dW_t \\
&= \frac{\gamma_t}{\psi_t}\left\{f(\phi_t + \psi_t H_t) - f(\phi_t) - f'(\phi_t)\psi_t H_t\right\} dt + \frac{\lambda_t}{\psi_t}\, dt \\
&\quad + (T-t)\left\{\widehat{\mathfrak{d}}_t^\gamma + (\sigma_t^\gamma)^2\kappa_t^1\right\} dt + \sigma_t^\gamma\kappa_t^3 Z_t^H\, dt + H_t\left\{\widehat{\mathfrak{d}}_t^\gamma\kappa_t^3 + (\sigma_t^\gamma)^2\kappa_t^2\right\} dt - Z_t^H\, dW_t
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mathfrak{d}}^\gamma &= \mathfrak{d}_t^\gamma - \|\mathfrak{d}^\gamma\| \\
\kappa_t^1 &= \frac{\phi''(\gamma_t(T-t))(T-t)}{2\phi'(\gamma_t(T-t))} \\
\kappa_t^2 &= \frac{\psi''(\gamma_t(T-t))(T-t)^2}{2\psi(\gamma_t(T-t))} = \left((T-t)\frac{\psi''(\gamma_t(T-t))}{\psi'(\gamma_t(T-t))}\right)\kappa_t^1 \\
\kappa_t^3 &= \frac{(T-t)\psi'(\gamma_t(T-t))}{\psi(\gamma_t(T-t))}.
\end{aligned}$$

Using Lemma 11, we obtain that κ^1, κ^2 and κ^3 are bounded. Under **(C1)**, **(C3)** and **(C4)**, we deduce that

$$\begin{aligned} -dH_t &= \frac{\gamma_t}{\psi_t}\left[f(\phi_t+\psi_t H_t)-f(\phi_t)-f'(\phi_t)\psi_t H_t\right]\mathbf{1}_{H_t\geq 0}dt \\ &+ \left\{\frac{\lambda_t}{\psi_t}+(T-t)\widehat{\kappa}^1_t+\widehat{\kappa}^2_t H_t+\widehat{\kappa}^3_t Z^H_t\right\}dt - Z^H_t\,dW_t, \end{aligned} \tag{51}$$

where the coefficients $\widehat{\kappa}^1, \widehat{\kappa}^2, \widehat{\kappa}^3$ are essentially bounded. Compared to (42), the linear term has now bounded coefficients and thus the new BSDE can be solved without any reference to the singular BSDE (1).

As in Section 4.2, we may define the generator

$$\begin{aligned} F^{\delta,\varepsilon}(t,h,z) &= \frac{\gamma_t}{\psi_t}\left\{f(\phi_t+\psi^\delta_t h)-f(\phi_t)\right\}-f'(\phi^\varepsilon_t)\gamma_t h \\ &+ \left(\frac{\lambda_t}{\psi_t}+(T-t)\widehat{\kappa}^1_t+\widehat{\kappa}^2_t h+\widehat{\kappa}^3_t z\right). \end{aligned}$$

The terminal condition is again equal to zero. From [24, Theorem 5.30], there exists a unique solution $(H^{\delta,\varepsilon}, Z^{H,\delta,\varepsilon}) \in \mathbb{S}^p(0,T)$, $p>1$, to the BSDE:

$$H_t = \int_t^T F^{\delta,\varepsilon}(s,H_s,Z_s)ds - \int_t^T Z_s dW_s.$$

From Lemma 1, we deduce that H is non-negative. The upper bound of Lemma 8 holds since the proof is based on a control on $\phi_t+\psi_t H^{\delta,\varepsilon}_t$, which satisfies the same dynamics. Hence we can pass to the limit and define

$$H_t = \lim_{\varepsilon\downarrow 0}\left(\lim_{\delta\downarrow 0} H^{\delta,\varepsilon}_t\right).$$

The sequence $Z^{\delta,\varepsilon}$ also converges to Z^H and clearly (H,Z^H) satisfies the desired dynamics on any interval $[0,T)$: for any $0\leq t\leq u<T$

$$\begin{aligned} H_t &= H_u + \int_t^u \frac{\gamma_s}{\psi_s}\left\{f(\phi_s+\psi_s H_s)-f(\phi_s)-f'(\phi_s)\psi_s H_s\right\}\mathbf{1}_{H_s\geq 0}\,ds \\ &+ \int_t^u \left\{\frac{\lambda_s}{\psi_s}+(T-s)\widehat{\kappa}^1_s+\widehat{\kappa}^2_s H_s\right\}ds - \int_t^u Z^H_s dW^{\mathbb{Q}}_s, \end{aligned}$$

where the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ with density $\mathcal{E}(\int \widehat{\kappa}^3_s ds)$ and $W^{\mathbb{Q}} = W - \int \widehat{\kappa}^3$ is a Brownian motion under $\mathbb{Q}$. Using Lemma 9, we deduce that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T (-f'(\phi_t))\gamma_t(H_t)dt\right] < +\infty.$$

Indeed our upper bound on H is deterministic and thus does not depend on a particular choice of $\mathbb{Q}$ equivalent to $\mathbb{P}$. The monotonicity of f leads to

$$\frac{\gamma_t}{\psi_t}\left\{f(\phi_t+\psi_t H_t)-f(\phi_t)\right\}\leq 0.$$

Taking the expectation under $\mathbb{Q}$ and letting u go to T, we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{\gamma_t}{\psi_t}\left|f(\phi_t+\psi_t H_t)-f(\phi_t)-f'(\phi_t)\psi_t H_t\right| dt\right] < +\infty$$

and thus

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{0\le t\le T}\left|\int_t^T Z^H_s dW^{\mathbb{Q}}_s\right| dt\right] < +\infty.$$

We have proved that (H, Z^H) verifies for any $0 \le t \le T$:

$$\begin{aligned} H_t &= \int_t^T \frac{\gamma_s}{\psi_s}\left\{f(\phi_s+\psi_s H_s)-f(\phi_s)-f'(\phi_s)\psi_s H_s\right\}\mathbf{1}_{H_s\ge 0}\, ds \\ &+ \int_t^T \left\{\frac{\lambda_s}{\psi_s}+(T-s)\widehat{\kappa}^1_s+\widehat{\kappa}^2_s H_s+\widehat{\kappa}^3_s Z^H_s\right\} ds - \int_t^T Z^H_s dW_s. \end{aligned} \tag{52}$$

Adapting the arguments of the proof of Proposition 4, (H, Z^H) is the unique solution of (52).

5.4 The power case $f(y) = -y|y|^q$

In this case recall that $\phi(x) = \left(\frac{1}{qx}\right)^{1/q}$ and $\psi(x) = \left(\frac{1}{qx}\right)^{1+1/q}$ and we assume that η is an Itô process,

$$d\eta_t = \mathfrak{d}^\eta_t\, dt + \sigma^\eta_t\, dW_t, \tag{53}$$

such that $\mathfrak{d}^\eta \in L^\infty([0,T]\times\Omega;\mathbb{R})$ and $\sigma^\eta \in L^2([0,T]\times\Omega;\mathbb{R}^d)$. Then the process ϕ_t is equal to

$$\phi_t := \left(\frac{\eta_t}{q(T-t)}\right)^{1/q} = \frac{\zeta_t}{(q(T-t))^{1/q}} = \zeta_t\phi(t)$$

and again from condition **(A1)**, ζ is an Itô process with drift $\mathfrak{d}^\zeta \in L^\infty([0,T]\times\Omega;\mathbb{R})$ and diffusion matrix $\sigma^\zeta \in L^2([0,T]\times\Omega;\mathbb{R}^d)$.

We assume

$$Y_t = \zeta_t\phi(t) + \psi(t)H_t = (\eta_t)^{1/q}\,\phi(t) + \psi(t)H_t. \tag{54}$$

and formally obtain the dynamics for H:

$$\begin{aligned} -dH_t &= \psi(t)^{-1}\left\{\lambda_t + \phi(t)\mathfrak{d}^\zeta_t\right\} dt - Z^H_t dW_t \\ &+ \frac{1}{\psi(t)\eta_t}\left[f(\zeta_t\phi(t)+\psi(t)H_t) - f(\zeta_t\phi(t)) - f'(\zeta_t\phi(t))\psi(t)H_t\right] dt \\ &=: F(t,H_t)\,dt - Z^H_t dW_t, \end{aligned} \tag{55}$$

where F can be rewritten as

$$\begin{aligned} F(t,H) &= \frac{\lambda_t}{\psi(t)} + \frac{\phi(t)}{\psi(t)}\mathfrak{d}^\zeta_t + \frac{\psi(t)H^2}{\eta_t}\int_0^1 f''(\zeta_t\phi(t)+a\psi(t)H)(1-a)\,da \\ &= F(t,0) - (q+1)q\frac{\psi(t)H^2}{\eta_t}\int_0^1 |\zeta_t\phi(t)+a\psi(t)H|^{q-1}\operatorname{sign}(\zeta_t\phi(t)+a\psi(t)H)(1-a)\,da \\ &= F(t,0) - \frac{q+1}{\eta_t}\frac{H^2}{(T-t)^2}\int_0^1\left(\zeta_t + a\frac{H}{T-t}\right)^{q-1}\operatorname{sign}\left(\zeta_t + a\frac{H}{T-t}\right)(1-a)\,da. \end{aligned}$$

Let us remark that this generator is again *singular* and that the second derivative of f is not well-defined at zero if $0 < q < 1$.

To establish local existence for (55), we don't use monotonicity arguments (as in the preceding sections). But instead, we proceed very similar as in [13] and carry out the Picard iteration in the space

$$\mathcal{H}^\delta := \{H \in L^\infty(\Omega; C([T-\delta, T]; \mathbb{R})) : \|H\|_{\mathcal{H}^\delta} < +\infty\}$$

endowed with the weighted norm

$$\|H\|_{\mathcal{H}^\delta} = \left\| \sup_{t\in[T-\delta,T)} (T-t)^{-2}|H_t| \right\|_\infty .$$

Lemma 15 *Let $R > 0$ and $\delta \in (0, (\eta_\star)^{1/q}/R)$ then for every $H \in \overline{B}_{\mathcal{H}^\delta}(R)$ we have $(F(t, H_t))_{t\in[T-\delta,T]} \in L^\infty([T-\delta, T] \times \Omega; \mathbb{R})$.*

Proof. From our assumptions, the first part of $F(t, H_t)$

$$(q(T-t))^{1+1/q}\lambda_t + q(T-t)\mathfrak{d}_t^\zeta$$

is bounded and thus in $L^\infty([0, T] \times \Omega; \mathbb{R})$. By definition if $H \in \overline{B}_{\mathcal{H}^\delta}(R)$, then a.s. for any $t \in [T-\delta, T]$

$$\left| \frac{q+1}{\eta_t} \frac{H_t^2}{(T-t)^2} \right| \le \frac{q+1}{\eta_\star} R^2\delta^2.$$

Note that $\delta \in (0, (\eta_\star)^{1/q}/R)$ ensures that $\zeta_t + aH_t/(T-t) > 0$ for all $t \in [T-\delta, T]$, $a \in [0, 1]$. And

$$\int_0^1 \left|\zeta_t + a\frac{H_t}{T-t}\right|^{q-1} (1-a)\, da \le \Big((\eta^\star)^{1/q} + R\delta\Big)^{q-1}.$$

The lemma is now proved. □

The preceding lemma allows to define by

$$\Gamma(H) = \left(\mathbb{E}\left[\int_t^T F(s, H_s)\, ds \middle| \mathcal{F}_t\right]\right)_{t\in[T-\delta,T]}$$

the operator $\Gamma : \overline{B}_{\mathcal{H}^\delta}(R) \to L^\infty(\Omega; C([T-\delta, T]; \mathbb{R}))$.

Lemma 16 *For every $R > 0$ there exists a constant $L > 0$ independent of $\delta \in (0, \eta_\star^{1/q}/R)$ such that*

$$|F(t, H_t) - F(t, H'_t)| \le L|H_t - H'_t| \qquad \forall t \in [T-\delta, T]\ \forall H, H' \in \overline{B}_{\mathcal{H}^\delta}(R),\ a.s.$$

Proof. We have for $q \neq 1$

$$\begin{aligned}\frac{dF}{dH}(t, H) = &-\frac{2(q+1)}{\eta_t} \frac{H}{(T-t)^2} \int_0^1 \left(\zeta_t + a\frac{H}{T-t}\right)^{q-1} (1-a)\, da \\ &- \frac{(q+1)\mathbf{1}_{q\neq1}}{(q-1)\eta_t} \frac{H^2}{(T-t)^2} \int_0^1 \left(\zeta_t + a\frac{H}{T-t}\right)^{q-2} \frac{a(1-a)}{T-t}\, da.\end{aligned}$$

Hence, there exists $L > 0$ such that

$$\left\| \frac{dF}{dH}(t, (T-t)^2 R) \right\|_\infty \le L \qquad \forall t \in [T-\delta, T].$$

The assertion then follows by the mean value theorem. $\square$

We are now ready to prove that Γ maps $\overline{B}_{\mathcal{H}^\delta}(R)$ contractiv into itself (for appropriate R and $\delta \in (0, \eta_\star^{1/q}/R)$): For $R > 0$ specified below choose $L > 0$ as in Lemma 16. For $H, H' \in \overline{B}_{\mathcal{H}^\delta}(R)$ it then holds for all $t \in [T-\delta, T]$

$$\begin{aligned} |\Gamma(H)_t - \Gamma(H')_t| \le & \mathbb{E}\left[\int_t^T |F(s, H_s) - F(s, H'_s)|\, ds \middle| \mathcal{F}_t \right] \\ \le & (T-t)^3 L \|H - H'\|_{\mathcal{H}^\delta}. \end{aligned}$$

This yields, as long as $0 < \delta \le 1/(2L)$,

$$\|\Gamma(H) - \Gamma(H')\|_{\mathcal{H}^\delta} \le \frac{1}{2}\|H - H'\|_{\mathcal{H}^\delta}.$$

Hence, Γ is an 1/2-contraction on $\overline{B}_{\mathcal{H}^\delta}(R)$ if $\delta \le 1/(2L)$. Furthermore, for $H \in \overline{B}_{\mathcal{H}^\delta}(R)$,

$$\begin{aligned} |\Gamma(H)_t| &\le |\Gamma(H)_t - \Gamma(0)_t| + |\Gamma(0)_t| \\ &\le (T-t)^2 \frac{R}{2} + \mathbb{E}\left[\int_t^T \left[(q(T-s))^{1+1/q} \lambda_s + q(T-s)|\mathfrak{d}_s^\zeta| \right] ds \middle| \mathcal{F}_t \right] \\ &\le (T-t)^2 \frac{R}{2} + (T-t)^2 (\delta^{1/q} q^{1+1/q} \|\lambda\| + q\|\mathfrak{d}^\zeta\|_\infty). \end{aligned}$$

Thus, choosing $R = 2(q^{1+1/q}\|\lambda\| + q\|\mathfrak{d}^\zeta\|_\infty)$ and $\delta = \min\{1, 1/2L, \eta_\star^{1/q}/R\}$ yields $\|\Gamma(H)\|_{\mathcal{H}^\delta} \le R$.

Theorem 4 *The BSDE* (55) *has a unique solution* (H, Z^H) *on* $[0,T]$ *such that:*

$$\left\| \sup_{t \in [0,T)} (T-t)^{-2} |H_t| \right\|_\infty < +\infty.$$

Moreover $\int_0^\cdot Z^H dW$ *is a BMO-martingale.*

Proof. Using the property of the map Γ, we deduce that there exists $\delta > 0$ such that there exists a unique process $H \in \mathcal{H}^\delta$ such that a.s. for any $t \in [T-\delta, T]$:

$$H_t = \mathbb{E}\left[\int_t^T F(s, H_s)\, ds \middle| \mathcal{F}_t \right].$$

By the martingale representation, we obtain Z^H and since $H \in \mathcal{H}^\delta$, from Lemma 15, we deduce that the martingale $\int_{T-\delta}^\cdot Z^H dW$ is a BMO martingale.

In particular the random variable $H_{T-\delta}$ is bounded. If we consider the BSDE (55) starting at time $T-\delta$ from the terminal condition $H_{T-\delta}$, we can apply directly [24, Proposition 5.24] to obtain a unique solution (H, Z^H) on $[0, T-\delta]$ such that H is bounded. $\square$

6 Comparison of the asymptotics and extension

Let us summarize our results.

- Under **(C1)** and **(C2)**, Y and H are related by (20):

$$Y_t = \phi(A_t) + \psi\left(\frac{T-t}{\eta^\star}\right) H_t$$

  where H is the minimal solution of the BSDE (21):

$$H_t = \int_t^T F(s, H_s) ds - \int_t^T Z_s^H dW_s,$$

  with a singular generator F given by (22).

- Under the additional assumptions **(C3)** to **(C5)** and **(H)**, Y can be developed as follows:

$$Y_t = \phi(A_t) + \psi(A_t) \widehat{H}_t$$

  where $\widehat{H}$ is the unique solution of the BSDE with singular generator (42).

- In the Itô setting with bounded coefficients, we get

  – Uniqueness for $\widehat{H}$, since **(H)** holds.

  – Another possible decomposition of Y:

$$Y_t = \phi(\widetilde{\gamma}_t(T-t)) + \psi(\widetilde{\gamma}_t(T-t))\widetilde{H}_t,$$

  where $\widetilde{H}$ is the unique solution of the BSDE (52) and $\widetilde{\gamma}$ solves (50).

  – In the power case $f(y) = -y|y|^q$, we can use (54):

$$Y_t = \phi(\gamma_t(T-t)) + \frac{1}{(T-t)^{1+1/q}} H_t^{\#},$$

  where $H^{\#}$ solves the BSDE (55) .

First let us remark that if η or $1/\eta$ is an Itô process, then using (46):

$$\begin{aligned}
\phi(A_t) &= \phi\left(\gamma_t(T-t) + \mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right]\right) \\
&= \phi(\gamma_t(T-t)) \\
&+ \mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right] \int_0^1 \psi\left(\gamma_t(T-t) + a\mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right]\right) da.
\end{aligned}$$

From Remark 2 and Condition **(A1)** and for a bounded process $\mathfrak{d}^\gamma$, we deduce that there exists a constants C such that

$$\frac{1}{C}\psi(T-t) \leq \psi\left(\gamma_t(T-t) + a\mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}_u^\gamma du \middle| \mathcal{F}_t\right]\right) \leq C\psi(T-t).$$

Thus

$$\phi(A_t) = \phi(\gamma_t(T-t)) + \psi(T-t)\kappa_t(T-t)^2$$

with a bounded process κ. In other words, in the Itô setting, all developments coincide.

The second point we want to stress is the behavior in the power case $f(y) = -y|y|^q$ under the Itô setting. From the construction of $H^{\#}$, we know that $|H^{\#}_t| \leq C(T-t)^2$. Using our different asymptotics, the previous development of $\phi(A)$ and uniqueness of the (minimal) solution, we obtain that H, $\widehat{H}$ and $\widetilde{H}$ verify also this estimate, which is better than (37). However if we use the estimate (12), we have

$$\begin{aligned}
\phi_t \leq Y_t &\leq \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T \left(\left(\frac{\eta_s}{q}\right)^{\frac{1}{q}} + (T-s)^{q^\dagger}\lambda_s\right) ds \middle| \mathcal{F}_t\right] \\
&\leq \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T \left(\left(\frac{\eta_s}{q}\right)^{\frac{1}{q}}\right) ds \middle| \mathcal{F}_t\right] + \frac{\|\lambda\|}{q^\dagger+1}(T-t).
\end{aligned}$$

where $q^\dagger$ is the Hölder conjugate of $q+1$. Using that $\hat{\eta} = \eta^{1/q}$ is an Itô process with essentially bounded drift $\mathfrak{d}^{\eta,q}$, we have

$$\begin{aligned}
\phi_t \leq Y_t &\leq \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T \left(\left(\frac{\eta_s}{q}\right)^{\frac{1}{q}}\right) ds \middle| \mathcal{F}_t\right] + \frac{\|\lambda\|}{q^\dagger+1}(T-t) \\
&\leq \left(\frac{1}{q}\right)^{\frac{1}{q}} \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T \left(\hat{\eta}_t + \int_t^s \mathfrak{d}^{\eta,q}_u du\right) ds \middle| \mathcal{F}_t\right] + \frac{\|\lambda\|}{q^\dagger+1}(T-t) \\
&= \left(\frac{\eta_t}{q(T-t)}\right)^{\frac{1}{q}} + \left(\frac{1}{q}\right)^{\frac{1}{q}} \frac{1}{(T-t)^{q^\dagger}} \mathbb{E}\left[\int_t^T (T-u)\mathfrak{d}^{\eta,q}_u du \middle| \mathcal{F}_t\right] + \frac{\|\lambda\|}{q^\dagger+1}(T-t) \\
&\leq \left(\frac{\eta_t}{q(T-t)}\right)^{\frac{1}{q}} + \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}+1} (T-t)^2 \|\mathfrak{d}^{\eta,q}\| + \frac{\|\lambda\|}{q^\dagger+1}(T-t) \\
&= \phi(\gamma_t(T-t)) + \psi(T-t)(T-t)^2\|\mathfrak{d}^{\eta,q}\| + \frac{\|\lambda\|}{q^\dagger+1}(T-t).
\end{aligned}$$

Thus we have the desired result

$$0 \leq H_t \leq C(T-t)^2.$$

6.1 Non-negativity of λ

From the comparison principle for monotone BSDE (see [24, Proposition 5.34]), any solution of (1) with a non-negative terminal condition is bounded from below by the solution $(\bar{Y}, \bar{Z})$ of the BSDE with generator

$$f_\star(\omega, t, y) = \frac{1}{\eta_t(\omega)}(f(y) - f(0)) - (f(0) + \lambda_t(\omega))^-$$

and terminal condition 0. $\bar{Y}$ is non-positive and if λ is bounded, $\bar{Y}$ is also bounded. Thus the negative part of Y is bounded and we can consider only the positive part of the solution.

If the sign of λ is unknown, then Lemma 1 does not hold. However the minimal solution of (1) is bounded from below by the minimal solution $(Y_\star, Z^{Y_\star})$ of the BSDE with generator

$f_\star$ and terminal condition $+\infty$. And we can adapt the proof of Lemma 2 in order to prove that there exist two functions $\vartheta_\star$ and $\vartheta^\star = \vartheta$ such that:

$$\vartheta_\star(T-t) \leq (Y_\star)_t \leq Y_t \leq \vartheta^\star(T-t),$$

where $\vartheta_\star$ is the solution of the ODE:

$$y' = \lambda_\star - \frac{f(y)}{\eta_\star}$$

with $\lambda_\star \leq f(0) + \lambda_t(\omega) \leq \|\lambda\|$ and $\vartheta_\star(0) = +\infty$. Arguing as in the proof of Lemma 3 we get that for any $0 \leq \varepsilon < 1$, on some deterministic and non-empty interval $[T^\varepsilon, T]$, a.s.

$$\phi\left(\frac{T-t}{(1-\varepsilon)\eta_\star}\right) \leq Y_t \leq \phi\left(\frac{T-t}{(1+\varepsilon)\eta^\star}\right).$$

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