Available online at www.sciencedirect.com

ScienceDirect

Stochastic Processes and their Applications 125 (2015) 1926–1944

stochastic processes and their applications

www.elsevier.com/locate/spa

Homogenization of random parabolic operators. Diffusion approximation

M. Kleptsyna[a], A. Piatnitski[b,c,*], A. Popier[a]

[a] *Université du Maine, Département de Mathématiques, Laboratoire Manceau de Mathématiques, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France*

[b] *Faculty of Technology, Narvik University College, Norway*

[c] *Lebedev Physical Institute RAS, Moscow, Russia*

Received 17 October 2013; received in revised form 4 December 2014; accepted 4 December 2014

Available online 12 December 2014

Abstract

This paper deals with homogenization of divergence form second order parabolic operators whose coefficients are periodic with respect to the spatial variables and random stationary in time. Under proper mixing assumptions, we study the limit behaviour of the normalized difference between solutions of the original and the homogenized problems. The asymptotic behaviour of this difference depends crucially on the ratio between spatial and temporal scaling factors. Here we study the case of self-similar parabolic diffusion scaling.

Keywords: Homogenization; Diffusion approximation; Operator with random coefficients

1. Introduction

The goal of this paper is to characterize the rate of convergence in the homogenization problem for a second order divergence form parabolic operator with random stationary in time and periodic in spatial variables coefficients. We also aim at describing the limit behaviour of a normalized difference between solutions of the original and homogenized problems.

* Corresponding author at: Faculty of Technology, Narvik University College, Norway.

E-mail addresses: marina.kleptsyna@univ-lemans.fr (M. Kleptsyna), andrey@sci.lebedev.ru (A. Piatnitski), alexandre.popier@univ-lemans.fr (A. Popier).

http://dx.doi.org/10.1016/j.spa.2014.12.002

0304-4149/

To avoid boundary effects we study a Cauchy problem that takes the form

$$\begin{aligned} &\partial_t u^\varepsilon = \operatorname{div}\Big(a\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\big)\nabla u^\varepsilon\Big), \quad x \in \mathbb{R}^n,\ t > 0, \\ &u^\varepsilon(x,0) = g(x). \end{aligned} \tag{1}$$

with $\alpha > 0$. In this paper we consider the case $\alpha = 2$. We assume that the matrix $a(z,s) = \{a^{ij}(z,s)\}$ is uniformly elliptic, $(0,1)^n$-periodic in z variable, and random stationary ergodic in s. We denote $Y = (0,1)^n$ and in what follows identify Y-periodic function with functions define on the torus $\mathbb{T}^n$.

It is known (see [14,8]) that under these assumptions problem (1) admits homogenization. More precisely, for any $g \in L^2(\mathbb{R}^n)$, almost surely (a.s.) solutions u^ε of problem (1) converge, as $\varepsilon \to 0$, to a solution of the homogenized problem

$$\begin{aligned} &\partial_t u^0 = \operatorname{div}\big(a^{\text{eff}}\nabla u^0\big) \\ &u^0(x,0) = g(x) \end{aligned} \tag{2}$$

with a constant (non-random) positive definite matrix a^{eff}. The convergence is in $L^2(\mathbb{R}^n \times (0,T))$. More detailed description of the existing homogenization results is given in Sections 3 and 3.1.

The paper focuses on the rate of this convergence and on higher order terms of the asymptotics of u^ε. Our goal is to describe the limit behaviour of the normalized difference $\varepsilon^{-1}(u^\varepsilon - u^0)$.

Clearly, the main oscillating term of the asymptotics of this normalized difference should be expressed in terms of the corrector. We recall (see [8,3]) that the equation

$$\partial_s \chi(z,s) = \operatorname{div}_z\big(a(z,s)\big(\nabla_z\chi(z,s) + \mathbf{I}\big)\big)$$

has a unique up to an additive (random) constant periodic in z and stationary in s solution. Thus, the gradient $\nabla_z\chi$ is uniquely defined. The principal corrector takes the form $\varepsilon\chi\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\big) \cdot \nabla u^0(x,t)$. We study the limit behaviour of the expression

$$U^\varepsilon(x,t) := \frac{u^\varepsilon(x,t) - u^0(x,t)}{\varepsilon} - \chi\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\big) \cdot \nabla u^0(x,t).$$

For generic stationary ergodic coefficients $a(z,s)$ the family $\{U^\varepsilon\}$ needs not be compact or tight in $L^2(\mathbb{R}^n \times (0,T))$.

For this reason we assume that (see Section 2 for further details)

- The coefficients $a(z,s)$ have good mixing properties.
- The initial function g is sufficiently smooth.

Under these conditions we show (see Theorem 3, Section 6) that U^ε converges in law in $L^2(\mathbb{R}^n \times (0,T))$ equipped with the strong topology to a solution of a SPDE with constant coefficients and an additive noise. This SPDE reads

$$\begin{aligned} &dU^0 = \operatorname{div}\Big(a^{\text{eff}}\nabla U^0 + \mu\frac{\partial^3}{\partial x^3}u^0\Big)dt + \Lambda^{1/2}\frac{\partial^2}{\partial x^2}u^0\, dW_t, \\ &U^0(x,0) = 0; \end{aligned} \tag{3}$$

where U^0 is a scalar-valued function of x and t, a^{eff} is the homogenized coefficients matrix, u^0 is a solution of (2), $W_t = W_t^{ij}$ is a standard n^2-dimensional Wiener process, and $\mu = \mu^{ijk}$ and

$\Lambda^{1/2} = (\Lambda^{1/2})^{ijkl}$ are constant tensors with three and four indices, respectively, so that the two driving terms in (3) take the form

$$\mu \frac{\partial^3}{\partial x^3} u^0 = \mu^{ijk} \frac{\partial^3 u^0}{\partial x^i \partial x^j \partial x^k}, \qquad \Lambda^{1/2} \frac{\partial^2}{\partial x^2} u^0 \, dW_t = (\Lambda^{1/2})^{ijkl} \frac{\partial^2 u^0}{\partial x^i \partial x^j} \, dW_t^{kl};$$

here and in what follows we assume summation over repeated indices.

The tensors μ and $\Lambda^{1/2}$ are defined in Section 6. We show that problem (3) is well-posed and, thus, defines the limit law of U^ε uniquely.

Notice that under proper choice of an additive constant the mean value of $\chi(z, s)$ is equal to zero. Therefore, the function $\chi\big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\big)\nabla u^0(x, t)$ converges a.s. to zero weakly in $L^2(\mathbb{R}^n \times (0, T))$, as $\varepsilon \to 0$. Therefore, in the weak topology of $L^2(\mathbb{R}^n \times (0, T))$, the limit in law of the normalized difference $\varepsilon^{-1}(u^\varepsilon(x, t) - u^0(x, t))$ coincides with that of U^ε.

The first results on homogenization of elliptic operators with random statistically homogeneous coefficients were obtained in [9,11]. At present there is an extensive literature on this topic. However, optimal estimate for the rate of convergence is an open issue. In [13] some power estimates for the rate of convergence were obtained in dimension three and more. In the recent work [5] a further important progress has been made in this problem.

Parabolic operators with random coefficients depending both on spatial and temporal variables have been considered in [14]. In the case of a diffusive scaling, the a.s. homogenization theorem has been proved.

The case of non-diffusive scaling has been studied in [7] under the assumption that the coefficients are periodic in spatial variables and random stationary in time.

It turns out that the structure of the higher order terms of the asymptotics of u^ε depends crucially on whether the scaling is diffusive or not. Here we study the diffusive scaling. The case of non-diffusive scaling will be addressed elsewhere.

2. The setup

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space equipped with a measure preserving ergodic dynamical system $\mathcal{T}_s$, $s \in \mathbb{R}$.

Given a measurable matrix function $\tilde{a}(z, \omega) = \{\tilde{a}^{ij}(z, \omega)\}_{i,j=1}^n$ which is periodic in z variable with a period one in each coordinate direction, we define a random field $a(z, s)$ by

$$a(z, s) = \tilde{a}(z, \mathcal{T}_s\omega).$$

Then $a(z, s)$ is periodic in z and stationary ergodic in s.

We consider the following Cauchy problem in $\mathbb{R}^n \times (0, T]$, $T > 0$:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\Big)\nabla u^\varepsilon\Big) \\ u^\varepsilon(0, x) = g(x) \end{cases} \tag{4}$$

with a small positive parameter ε.

We assume that the coefficients in (4) possess the following properties.

H1 The matrix $a(z, s)$ is symmetric and satisfies a uniform ellipticity conditions that is there exists $\lambda > 0$ such that for all (z, ω) the following inequality holds:

$$\lambda|\zeta|^2 \le \tilde{a}(z, \omega)\xi \cdot \xi \le \lambda^{-1}|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n.$$

H2 The initial condition g is four times continuously differentiable, and for any $K > 0$ there is $C_K > 0$ such that

$$\sum_{|\mathbf{j}|\le 4}\Big|\frac{\partial^{\mathbf{j}}}{(\partial x^1)^{j_1}\dots(\partial x^n)^{j_n}}g(x)\Big| \le C_K(1+|x|)^{-K}$$

for all $x\in\mathbb{R}^n$, where the sum is taken over all $\mathbf{j}=(j_1,\dots,j_n)$ with $\sum_{i=1}^n j_i\le 4$.

It should be noted that under condition H2 for any $K>0$ there is $C_K(T)>0$ such that a solution of problem (2) satisfies the estimate

$$\sum_{|\mathbf{j}|\le 4}\Big|\frac{\partial^{\mathbf{j}}}{(\partial x^1)^{j_1}\dots(\partial x^n)^{j_n}}u^0(x,t)\Big| \le C_K(T)(1+|x|)^{-K} \tag{5}$$

for all $(x,t)\in\mathbb{R}^n\times[0,T]$.

In order to formulate one more condition we introduce the so-called maximum correlation coefficient. Setting $\mathcal{F}_{\le r}=\sigma\{a(z,s):s\le r\}$ and $\mathcal{F}_{\ge r}=\sigma\{a(z,s):s\ge r\}$, we define

$$\rho(r)=\sup_{\xi_1,\xi_2}\mathbf{E}(\xi_1\xi_2)$$

where the supremum is taken over all $\mathcal{F}_{\le 0}$-measurable ξ_1 and $\mathcal{F}_{\ge r}$-measurable ξ_2 such that $\mathbf{E}\xi_1 = \mathbf{E}\xi_2=0$, and $\mathbf{E}\{(\xi_1)^2\}=\mathbf{E}\{(\xi_2)^2\}=1$. We then assume that

H3 The function ρ satisfies the estimate $\int_0^\infty\rho(r)dr<+\infty$.

Remark 1. Condition H3 is somehow implicit. In applications various sufficient conditions are often used. In particular, H3 is fulfilled if $\rho(r)\le cr^{-(1+\delta)}$ for some $\delta>0$.

Remark 2. In an important particular case we set

$$a(z,s)=\approx{a}(z,\xi_s),$$

where ξ_s is a stationary process with values in $\mathbb{R}^N$, and $\tilde{\tilde{a}}(z,y)$ satisfies the uniform ellipticity conditions

$$\lambda|\zeta|^2\le\tilde{\tilde{a}}(z,y)\xi\cdot\xi\le\lambda^{-1}|\zeta|^2\quad\text{for all }\zeta\in\mathbb{R}^n,\ (z,y)\in\mathbb{Z}^n\times\mathbb{R}^N.$$

If ξ_s is Gaussian then condition H3 follows from integrability of the correlation function of ξ..

If ξ_s is a diffusion process, then condition H3 can be replaced with some conditions on the generator of ξ_s. This case is considered in Sections 3.1 and 7.

3. Homogenization results

In this section we remind of the existing homogenization results for problem (1). Although we only deal in this paper with the case $\alpha=2$, for convenience of the reader we formulate the homogenization results for all $\alpha>0$. To this end we first introduce the so-called cell problem. For $\alpha=2$ it reads

$$\partial_s\chi(z,s)=\operatorname{div}\big(a(z,s)(\mathbf{I}+\nabla\chi(z,s))\big),\quad (z,s)\in\mathbb{T}^n\times(-\infty,+\infty) \tag{6}$$

with $\mathbf{I}$ being the unit matrix; here $\chi=\{\chi^j\}_{j=1}^n$ is a vector function. In what follows for the sake of brevity we denote $\operatorname{div}a=\operatorname{div}(a\mathbf{I})=\frac{\partial}{\partial z^i}a^{ij}(z)$. Also, we assume summation over repeated indices.

According to Lemma 4.1, under assumption H1 this equation has a stationary periodic in y vector-valued solution. This solution is unique up to an additive constant. We define

$$a^{\text{eff}} = \mathbb{E}\int_{\mathbb{T}^n} a(z,s)\big(\mathbf{I} + \nabla\chi(z,s)\big)\,dz. \tag{7}$$

Notice that due to stationarity the expression on the right-hand side does not depend on s.

If $\alpha < 2$, the cell problem reads

$$\operatorname{div}\big(a(z,s)(\mathbf{I} + \nabla\chi_-(z,s))\big) = 0, \quad z \in \mathbb{T}^n; \tag{8}$$

here s is a parameter. This equation has a unique up to a multiplicative constant solution. We then set

$$a^{\text{eff}}_- = \mathbb{E}\int_{\mathbb{T}^n} a(z,s)\big(\mathbf{I} + \nabla\chi_-(z,s)\big)\,dz. \tag{9}$$

For $\alpha > 2$ we first define $\overline{a}(z) = \mathbb{E}a(z,s)$, then introduce a deterministic function $\chi_+(z)$ as a periodic solution to the problem

$$\operatorname{div}\big(\overline{a}(z)(\mathbf{I} + \nabla\chi_+(z))\big) = 0, \quad z \in \mathbb{T}^n, \tag{10}$$

and finally define

$$a^{\text{eff}}_+ = \int_{\mathbb{T}^n} \overline{a}(z)\big(\mathbf{I} + \nabla\chi_+(z)\big)\,dz. \tag{11}$$

The following statement has been obtained in [14,3].

Theorem 1. *Let $g \in L^2(\mathbb{R}^n)$, and assume that condition* H1 *holds. If $\alpha = 2$, then a solution u^ε of problem (1) converges a.s. in $L^2(\mathbb{R}^n \times (0,T))$ to a solution of the limit problem (2) with a^{eff} given by (7).*

If $\alpha < 2$, then a solution u^ε of problem (1) converges in probability in $L^2(\mathbb{R}^n \times (0,T))$ to a solution of the limit problem (2) with $a^{\text{eff}} = a^{\text{eff}}_-$ defined in (9).

If $\alpha > 2$, then a solution u^ε of problem (1) converges in probability in $L^2(\mathbb{R}^n \times (0,T))$ to a solution of the limit problem (2) with $a^{\text{eff}} = a^{\text{eff}}_+$ defined in (11).

Notice that only symmetric part of the matrix a^{eff} matters. In particular, a solution of (2) does not change if we replace a^{eff} with any constant matrix having the same symmetric part.

Remark 3. An alternative way of defining the effective matrix a^{eff} is related to the operator with reversed time. We define χ_- as a stationary solution of the problem

$$-\partial_s\chi_-(z,s) = \operatorname{div}\big(a(z,s)(\mathbf{I} + \nabla\chi_-(z,s))\big), \quad (z,s) \in (-\infty,+\infty) \times \mathbb{T}^n \tag{12}$$

and set

$$a^{\text{eff}} = \Big(\mathbb{E}\int_{\mathbb{T}^n} a(z,s)\big(\mathbf{I} + \nabla\chi_-(z,s)\big)\,dz\Big)^t, \tag{13}$$

where $(\cdot)^t$ denotes a transposed matrix. In order to show that (13) and (7) define the same effective matrix, we multiply the ith component of Eq. (12) by χ^j, and the jth component of Eq. (6) by χ^i_- and integrate the resulting relations over $\mathbb{T}^n \times (0,1)$. Subtracting the second relation from the first one and taking the expectation, we obtain the desired equality.

3.1. Diffusive dependence on time

In this section as a particular case of (4) we introduce the following problem

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \operatorname{div}\Big(\tilde{a}\Big(\dfrac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^2}}\Big)u^\varepsilon\Big) \\ u^\varepsilon(0,x) = g(x) \end{cases} \tag{14}$$

with a diffusion process ξ_s, $s \in (-\infty, +\infty)$, with values in $\mathbb{R}^N$ or on a compact manifold. This process is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the sake of definiteness we consider here the case of a diffusion in $\mathbb{R}^N$. The corresponding Itô equation reads

$$d\xi_t = b(\xi_t)dt + \sigma(\xi_t)d\mathtt{W}_t,$$

here $\mathtt{W}.$ stands for a standard N-dimensional Wiener process. The infinitesimal generator of ξ is denoted by $\mathcal{L}$:

$$\mathcal{L}f(y) = q^{ij}(y)\frac{\partial^2}{\partial y^i \partial y^j} f(y) + b(y)\cdot\nabla f(y), \quad y \in \mathbb{R}^N,$$

with a $N \times N$ matrix $q(y) = \frac{1}{2}\sigma(y)\sigma^*(y)$. We also introduce the operator

$$\mathcal{A}f(x) = \operatorname{div}_x\left(a\,(x,y)\,\nabla_x f\right);$$

here y is a parameter. Applied to a function $f(z,y)$, $\mathcal{L}$ acts on the function $y \mapsto f(z,y)$ for z fixed, and $\mathcal{A}$ acts on the function $z \mapsto f(z,y)$ for y fixed.

In the diffusive case condition H3 can be replaced with certain assumptions on the generator $\mathcal{L}$. More precisely, we suppose that the following conditions hold true.

A1. The coefficients a and q are uniformly bounded as well as their first order derivatives in all variables:

$$|a(z,y)| + |\nabla_z a(z,y)| + |\nabla_y a(z,y)| \le C_1,$$
$$|q(y)| + |\nabla q(y)| \le C_1.$$

The function b as well as its derivatives satisfy polynomial growth condition:

$$|b(y)| + |\nabla b(y)| \le C_1(1+|y|)^{N_1}.$$

A2. Both $\mathcal{A}$ and $\mathcal{L}$ are uniformly elliptic:

$$C_2\mathbf{I} \le a(z,y), \qquad C_2\mathbf{I} \le q(y), \quad \text{with } C_2 > 0,$$

where $\mathbf{I}$ stands for a unit matrix of the corresponding dimension.

A3. There exist $N_2 > -1$, $R > 0$ and $C_3 > 0$ such that

$$b(y)\frac{y}{|y|} \le -C_3|y|^{N_2}$$

for all y, $|y| > R$.

Under above assumptions the process ξ has a unique invariant probability measure (see [12]). We assume that ξ_s is stationary. Then

$$\mathbb{E}f(z,\xi_s) = \int_{\mathbb{R}^N} f(z,y)\pi(y)dy,$$

where π is a density of the invariant measure.

Remark 4. Notice that conditions A1–A3 need not imply condition H3. In general, mixing properties that follow from A1–A3 are weaker than those stated by H3. However, in the diffusive case these conditions are sufficient for the CLT type results used in the proofs below. This makes the diffusive case interesting. It should also be noted that in this case the conditions are given in terms of the process generator, which might be more comfortable in applications.

Let us recall the result of [7] (see also [2]).

Theorem 2. *Under Assumptions* A1–A3, *the solution* u^ε *of* (14) *converges almost surely in the space* $L^2((0,T)\times\mathbb{R}^n)$ *to the solution of problem* (2) *with*

$$a^{\mathrm{eff}} = \Big(\int_{\mathbb{R}^N}\int_{\mathbb{T}^n} a(\mathbf{I}+\nabla_z\chi^0)\pi(y)\,dzdy\Big)^t, \tag{15}$$

χ^0 *being the solution of the following equation*

$$(\mathcal{A}+\mathcal{L})\chi^0 = -\mathrm{div}_z a(z,y). \tag{16}$$

Remark 5. Let us show that formulae (15) and (7) are consistent. To this end we apply Ito's formula to the product $\chi^i(z,s)\chi_0^j(z,\xi_s)$. This yields

$$\begin{aligned} d(\chi^i\chi_0^j) &= \chi^i\mathcal{L}\chi_0^j\,ds + \chi^i\sigma\nabla_y\chi_0^j\,d\mathtt{W}_s + \chi_0^j\mathrm{div}_z\big(a(\nabla_z\chi^i+\mathbf{e}^i)\big)\,ds\\ &= -\chi^i\mathrm{div}_z\big(a(\nabla_z\chi_0^j+\mathbf{e}^j)\big)\,ds + \chi^i\sigma\nabla_y\chi_0^j\,d\mathtt{W}_s + \chi_0^j\mathrm{div}_z\big(a(\nabla_z\chi^i+\mathbf{e}^i)\big)\,ds; \end{aligned}$$

here we have used (16), $\mathbf{e}^j$ stands for the jth coordinate vector in $\mathbb{R}^n$. Integrating this relation in s from 0 to 1, taking the expectation and considering the stationarity of χ and ξ, we obtain

$$\begin{aligned} &\mathbb{E}\int_{\mathbb{T}^n}\chi^i(z,s)\mathrm{div}_z\big(a(z,\xi_s)(\nabla_z\chi_0^j(z,\xi_s)+\mathbf{e}^j)\big)\,dz\\ &\quad=\mathbb{E}\int_{\mathbb{T}^n}\chi_0^j(z,\xi_s)\mathrm{div}_z\big(a(z,\xi_s)(\nabla_z\chi^i(z,s)+\mathbf{e}^i)\big)\,dz. \end{aligned}$$

Since $a(z,y)$ is symmetric, this implies

$$\mathbb{E}\int_{\mathbb{T}^n}\nabla_z\chi^i(z,s)\cdot a(z,\xi_s)\mathbf{e}^j\,dz = \mathbb{E}\int_{\mathbb{T}^n}\nabla_z\chi_0^j(z,\xi_s)\cdot a(z,\xi_s)\mathbf{e}^i\,dz,$$

and (15) follows.

4. Technical statements

In this section we provide a number of technical statements required for formulating and proving the main results.

Consider an equation

$$\partial_s\psi(z,s) - \mathrm{div}\big(a(z,s)\nabla\psi(z,s)\big) = \phi(z,s) \tag{17}$$

with a stationary in s and periodic in z random function ϕ.

Lemma 4.1. *Let* $\phi\in L^2_{\mathrm{loc}}(\mathbb{R};H^{-1}(\mathbb{T}^n))$, *and assume that* $\|\phi\|^2_{L^2((0,1);H^{-1}(\mathbb{T}^n))}\le C$ *with a non-random constant* C. *Assume, moreover, that*

$$\int_{\mathbb{T}^n}\phi(z,s)dz = 0 \quad a.s. \tag{18}$$

Then Eq. (17) *has a stationary solution* $\psi \in L^\infty_{loc}((-\infty,+\infty); L^2(\mathbb{T}^n)) \cap L^2_{loc}((-\infty,+\infty); H^1(\mathbb{T}^n))$. *It is unique up to an additive (random) constant, and*

$$\|\psi\|^2_{L^\infty(\mathbb{R};L^2(\mathbb{T}^n))} \le C_1, \qquad \|\psi\|^2_{L^2((0,1);H^1(\mathbb{T}^n))} \le C_1. \tag{19}$$

Proof. Since a proof of this statement is similar to that of Lemmata 2 and 4 in [8], we provide here only a sketch of the proof. Consider the Green function of (17). It solves a Cauchy problem

$$\partial_s \mathcal{G}(z,z_0,s,s_0) - \operatorname{div}\big(a(z,s)\nabla\mathcal{G}(z,z_0,s,s_0)\big) = 0, \quad z \in \mathbb{T}^n,\ s \ge s_0,$$
$$\mathcal{G}(z,z_0,s_0,s_0) = \delta(z-z_0).$$

From the Harnack inequality and maximum principle it easily follows (see [8]) that for all $s \ge s_0 + 1$

$$\|\mathcal{G}(\cdot,z_0,\cdot,s_0) - 1\|_{L^2((s,s+1);H^1(\mathbb{T}^n))} \le Ce^{-\nu(s-s_0)} \tag{20}$$

with deterministic constants C and $\nu > 0$. Then we have

$$\psi(z,s) = \int_{-\infty}^{s}\int_{\mathbb{T}^n} \mathcal{G}(z,\hat z,s,\hat s)\phi(\hat z,\hat s)\,d\hat z d\hat s$$
$$= \int_{-\infty}^{s-1}\int_{\mathbb{T}^n} \big(\mathcal{G}(z,\hat z,s,\hat s) - 1\big)\phi(\hat z,\hat s)\,d\hat z d\hat s + \int_{s-1}^{s}\int_{\mathbb{T}^n} \big(\mathcal{G}(z,\hat z,s,\hat s)\big)\phi(\hat z,\hat s)\,d\hat z d\hat s,$$

here we have also used (18). The first term on the right-hand side can be estimated with the help of (20), the second one by means of the standard energy inequality. This yields the first bound in (19). By construction, $\psi(z,s)$ is a stationary solution of (17). The second bound in (19) readily follows from the first one. □

Corollary 1. *If the function* ϕ *in* (17) *belongs to* $L^\infty(\mathbb{R}; W^{-1,\infty}(\mathbb{T}^n))$, *then* $\psi \in L^\infty(\mathbb{R}\times\mathbb{T}^n)$ *and*

$$\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{T}^n)} \le C\|\phi\|_{L^\infty(\mathbb{R};W^{-1,\infty}(\mathbb{T}^n))}$$

with a deterministic constant C.

Proof. This statement follows from Lemma 4.1 due to the Nash type estimates for solutions of parabolic equations (see [4, Theorem VII, 3.1]). □

Denote by $\mathcal{F}^{a,\phi}_{\le T}$ the σ-algebra $\sigma\{a(z,s),\phi(x,s)\ :\ s \le T\}$. The σ-algebra $\mathcal{F}^{a,\phi}_{\ge T}$ is defined accordingly. Let $\rho_{a,\phi}(r)$ be maximum correlation coefficient of (a,ϕ). Denote also

$$l(s) = \int_{\mathbb{T}^n} \big(a(z,s)\nabla_z\psi(z,s) - \mathbb{E}(a(z,s)\nabla_z\psi(z,s))\big)\,dz.$$

Lemma 4.2. *For the vector-function* $l(\cdot)$ *the following estimate holds*

$$\|\mathbb{E}\{l(s)\mid\mathcal{F}^{a,\phi}_{\le 0}\}\|_{L^2(\Omega)} \le C\big(e^{-\nu s/2} + \rho_{a,\phi}(s/2)\big), \quad \nu > 0.$$

Proof. This inequality has been proved in [8, Proof of Lemma 3]. Here we provide an outline of the proof. We represent

$$\psi(z,s) = \psi^1(z,s) + \psi^2(z,s)$$
$$= \int_{-\infty}^{s/2}\int_{\mathbb{T}^n} \big(\mathcal{G}(z,\hat z,s,\hat s) - 1\big)\phi(\hat z,\hat s)\,d\hat z d\hat s + \int_{s/2}^{s}\int_{\mathbb{T}^n} \big(\mathcal{G}(z,\hat z,s,\hat s)\big)\phi(\hat z,\hat s)\,d\hat z d\hat s.$$

Then

$$l(s) = l^1(s) + l^2(s)$$

with

$$l^i(s) = \int_{\mathbb{T}^n} \big(a(z,s)\nabla_z \psi^i(z,s) - \mathbb{E}(a(z,s)\nabla_z \psi^i(z,s))\big)\, dz, \quad i = 1,\ 2.$$

Considering (20) we get $\|l^1(s)\|_{L^2(\Omega)} \le Ce^{-\nu s/2}$. Since $l^2(s)$ is $\mathcal{F}^{a,\phi}_{\ge s/2}$-measurable, we obtain $\|\mathbb{E}\{l^2(s) \mid \mathcal{F}^{a,\phi}_{\le 0}\}\|_{L^2(\Omega)} \le C\rho_{a,\phi}(s/2)$. This yields the desired inequality. □

5. Formal asymptotic expansion

In this section we deal with the formal asymptotic expansion of a solution of problem (1). Although, in contrast with the periodic case, this method fails to work in full generality in the case under consideration, we can use it in order to understand the structure of the leading terms of the difference $u^\varepsilon - u^0$. As usually in the multi-scale asymptotic expansion method we consider $z = x/\varepsilon$ and $s = t/\varepsilon^2$ as independent variables and use repeatedly the formulae

$$\frac{\partial}{\partial x^j} f\Big(x, \frac{x}{\varepsilon}\Big) = \Big(\frac{\partial}{\partial x^j} f(x,z) + \frac{1}{\varepsilon}\frac{\partial}{\partial z^j} f(x,z)\Big)_{z=\frac{x}{\varepsilon}},$$
$$\frac{\partial}{\partial t} f\Big(t, \frac{t}{\varepsilon^2}\Big) = \Big(\frac{\partial}{\partial t} f(t,s) + \frac{1}{\varepsilon^2}\frac{\partial}{\partial s} f(t,s)\Big)_{s=\frac{t}{\varepsilon}}.$$

We represent a solution u^ε as the following asymptotic series in integer powers of ε:

$$u^\varepsilon(x,t) = u^0(x,t) + \varepsilon u^1\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) + \varepsilon^2 u^2\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) + \cdots; \tag{21}$$

here all the functions $u^j(x,t,z,s)$ are periodic in z. The dependence in s is not always stationary.

Substituting the expression on the right-hand side of (21) for u^ε in (4) and collecting power-like terms in (4) yield

(ε^{-1}): $\partial_s u^1 - \mathrm{div}_z\big(a(z,s)\nabla_z u^1\big) = -\mathrm{div}_z\big(a(z,s)\nabla_x u^0\big)$.

(ε^0): $\partial_s u^2 - \mathrm{div}_z\big(a(z,s)\nabla_z u^2\big) = -\partial_t u^0 + \mathrm{div}_x\big(a(z,s)\nabla_x u^0\big) + \mathrm{div}_z\big(a(z,s)\nabla_x u^1\big) + \mathrm{div}_x\big(a(z,s)\nabla_z u^1\big)$.

(ε^1): $\partial_s u^3 - \mathrm{div}_z\big(a(z,s)\nabla_z u^3\big) = -\partial_t u^1 + \mathrm{div}_x\big(a(z,s)\nabla_x u^1\big) + \mathrm{div}_z\big(a(z,s)\nabla_x u^2\big) + \mathrm{div}_x\big(a(z,s)\nabla_z u^2\big)$.

We will see later on that dealing with the first three equations is sufficient.

In equation (ε^{-1}) the variables x and t are parameters. By Lemma 4.1 this equation has a unique stationary solution. The fact that the right-hand side of the equation is of the form $[\mathrm{div}_z\big(a(z,s)\big)] \cdot \nabla_x u^0$ suggests that

$$u^1(x,t,z,s) = \chi(z,s)\nabla_x u^0(x,t)$$

with a vector-function $\chi = \{\chi^j(z,s)\}_{j=1}^n$ solving Eq. (6) that reads

$$\partial_s \chi - \mathrm{div}_z\big(a(z,s)\nabla_z \chi\big) = \mathrm{div}_z\big(a(z,s)\big),$$

$\mathrm{div}_z a(z,s)$ stands for $\frac{\partial}{\partial z^i} a^{ij}(z,s)$. By Lemma 4.1 and Corollary 1 we have $\chi \in (L^\infty(\mathbb{R}\times\mathbb{T}^n))^n \cap (L^2_{\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{T}^n)))^n$, and

$$\|\chi^j\|_{L^\infty(\mathbb{R}\times\mathbb{T}^n)} \le C, \qquad \|\chi^j\|_{L^2([0,1];H^1(\mathbb{T}^n))} \le C, \quad j = 1, \dots, n \tag{22}$$

with a deterministic constant C. For the sake of definiteness we assume from now on that

$$\int_{\mathbb{T}^n} \chi(z, s) dz = 0. \tag{23}$$

One can easily check that this integral does not depend on s so that the normalization condition makes sense.

We turn to the terms of order ε^0. We do not reprove here the homogenization results (see [14]) and assume that u^0 satisfies problem (2) with $a^{\rm eff}$ given by (7). Then assuming the formulae (ε^{-1})–(ε^0), the right-hand side of equation (ε^0) takes the form

$$\begin{aligned}&-\partial_t u^0 + \mathrm{div}_x\big(a(z, s)\nabla_x u^0\big) + \mathrm{div}_z\big(a(z, s)\nabla_x u^1\big) + \mathrm{div}_x\big(a(z, s)\nabla_z u^1\big)\\ &\quad = \mathrm{div}_x\big(\{a(z, s)(\mathbf{I} + \nabla_z\chi(z, s)) - a^{\rm eff}\}\nabla_x u^0\big) + \mathrm{div}_z\big(a(z, s)\nabla_x u^1\big).\end{aligned}$$

By the definition of $a^{\rm eff}$ (see (7)) we have

$$\mathbb{E}\int_{\mathbb{T}^n} \{a(z, s)(\mathbf{I} + \nabla_z\chi(z, s)) - a^{\rm eff}\}\, dz = 0.$$

Letting

$$\Psi_{2,1}(s) = \int_{\mathbb{T}^n} \{a(z, s)(\mathbf{I} + \nabla_z\chi(z, s)) - a^{\rm eff}\}\, dz \tag{24}$$

and

$$\Psi_{2,2}(z, s) = \{a(z, s)(\mathbf{I} + \nabla_z\chi(z, s)) - a^{\rm eff}\} - \Psi_{2,1}(s) + \mathrm{div}_z\big(a(z, s) \otimes \chi(z, s)\big) \tag{25}$$

with

$$\mathrm{div}_z\big(a(z, s) \otimes \chi(z, s)\big) = \left\{\frac{\partial}{\partial z^i}\big(a^{ij}(z, s)\chi^k(z, s)\big)\right\}_{j,k=1}^{n},$$

we rewrite equation (ε^0) as follows

$$\partial_s u^2 - \mathrm{div}_z\big(a(z, s)\nabla_z u^2\big) = \big(\Psi_{2,1}^{ij}(s) + \Psi_{2,2}^{ij}(z, s)\big)\frac{\partial^2}{\partial x^i \partial x^j} u^0. \tag{26}$$

Since the process $\int_0^s \Psi_{2,1}(r)dr$ need not be stationary, we cannot follow any more the same strategy as in the periodic case. Instead, we consider the equation

$$\begin{cases} \dfrac{\partial V^{\varepsilon,1}}{\partial t} = \mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon}, \dfrac{t}{\varepsilon^2}\Big)\nabla V^{\varepsilon,1}\Big) + \Psi_{2,1}^{ij}\Big(\dfrac{t}{\varepsilon^2}\Big)\dfrac{\partial^2}{\partial x^i \partial x^j} u^0(x, t) \\ V^{\varepsilon,1}(0, x) = 0. \end{cases} \tag{27}$$

This suggests the representation

$$u^\varepsilon(x, t) = u^0(x, t) + \varepsilon\chi\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\Big)\nabla u^0(x, t) + V^{\varepsilon,1} + \varepsilon^2 v^2\Big(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\Big) + \cdots \tag{28}$$

with

$$v^2(x, t, z, s) = \chi_{2,2}^{ij}(z, s)\frac{\partial^2}{\partial x^i \partial x^j} u^0(x, t),$$

where $\chi^{ij}_{2,2}(z,s)$ is a stationary zero mean solution of the equation

$$\partial_s \chi^{ij}_{2,2}(z,s) - \mathrm{div}_z\big(a(z,s)\nabla_z \chi^{ij}_{2,2}(z,s)\big) = \Psi^{ij}_{2,2}(z,s). \tag{29}$$

It is straightforward to check that due to (22)–(25) we have

$$\|\Psi^{ij}_{2,2}\|_{L^2((0,1);H^{-1}(\mathbb{T}^n))} \le C, \quad i,j=1,\dots,n.$$

Then the conditions of Lemma 4.1 are fulfilled for Eq. (29) and, therefore, this equation has a stationary solution that satisfies the estimate

$$\|\chi^{ij}_{2,2}\|_{L^2([0,1];H^1(\mathbb{T}^n))} + \|\chi^{ij}_{2,2}\|_{L^\infty((-\infty.+\infty);L^2(\mathbb{T}^n))} \le C, \quad i,j=1,\dots,n \tag{30}$$

with a deterministic constant C.

By its definition, $\Psi_{2,1}(s)$ is a stationary zero mean process. Denote

$$\chi^{ij}_{2,1}(s) = \int_0^s \Psi^{ij}_{2,1}(r)dr.$$

Estimates (22) imply that

$$\|\Psi^{ij}_{2,1}\|_{L^2(0,1)} \le C, \quad i,j=1,\dots,n$$

with a deterministic constant C. It follows from Lemmata 4.1 and 4.2 that under condition H3 it holds

$$\int_0^\infty \|\mathbb{E}\{\Psi_{2,1}(s) \mid \mathcal{F}^{\Psi_{2,1}}_{\le 0}\}\|_{(L^2(\Omega))^{n^2}}\, ds \le C\int_0^\infty \big(e^{-\nu s/2} + \rho_{\Psi_{2,1}}(s/2)\big)dy < \infty.$$

Therefore, the invariance principle holds for this process (see [6, Theorem VIII.3.79]), that is for any $T>0$

$$\varepsilon\chi_{2,1}\Big(\frac{\cdot}{\varepsilon^2}\Big) \stackrel{\varepsilon\to 0}{\longrightarrow} \Lambda^{1/2}W. \tag{31}$$

in law in the space $(C[0,T])^{n^2}$ with

$$\Lambda^{ijkl} = \int_0^\infty \mathbb{E}\big(\Psi^{ij}_{2,1}(0)\,\Psi^{kl}_{2,1}(s) + \Psi^{kl}_{2,1}(0)\,\Psi^{ij}_{2,1}(s)\big)\,ds,$$

here W is a standard n^2-dimensional Wiener process. Since the $n^2\times n^2$ matrix Λ is symmetric and positive semi-definite, its square root is well defined.

Remark 6. One can see that the processes $\chi_{2,1}$ and $\chi_{2,2}$ show rather different behaviour. In fact, since the process $\chi_{2,2}$ is stationary, the function $\varepsilon\chi_{2,2}(x/\varepsilon, t/\varepsilon^2)$ goes to zero, as $\varepsilon\to 0$. To the contrary, by the Central Limit Theorem type arguments, the process $\varepsilon\chi_{2,1}(t/\varepsilon^2)$ need not tend to zero on $[0,T]$, and, thus, it contributes to the asymptotics in question. Under our standing conditions, this process is of order one.

Lemma 5.1. *The functions $\varepsilon^{-1}V^{\varepsilon,1}$ converge in law, as $\varepsilon\to 0$, in the space $C((0,T);L^2(\mathbb{R}^n))$ to the unique solution of the following SPDE with a finite dimensional additive noise:*

$$\begin{cases} dV^{0,1} = \mathrm{div}(a^{\mathrm{eff}}\nabla V^{0,1})dt + \big(\Lambda^{1/2}\big)^{ijkl}\dfrac{\partial^2}{\partial x^i\partial x^j}u^0(x,t)dW^{kl}_t \\ V^{0,1}(0,x)=0. \end{cases} \tag{32}$$

Proof. The proof is a consequence of (31) and the fact that $u^0(x,t)$ is a smooth deterministic function that satisfies estimate (5). To see this we introduce an auxiliary function $\check{V}^\varepsilon$ as the solution to the following Cauchy problem

$$\begin{cases} \dfrac{\partial \check{V}^\varepsilon}{\partial t} = \operatorname{div}(a^{\mathrm{eff}}\nabla \check{V}^\varepsilon) + \dfrac{1}{\varepsilon}\Psi^{ij}_{2,1}\Big(\dfrac{t}{\varepsilon^2}\Big)\dfrac{\partial^2}{\partial x^i \partial x^j}u^0(x,t) \\ \check{V}^\varepsilon(0,x) = 0. \end{cases}$$

For the sake of brevity we denote $v^0_{ij}(x,t) = \frac{\partial^2}{\partial x^i \partial x^j}u^0(x,t)$. Notice that v^0_{ij} solves the equation $\partial_t v^0_{ij} = \operatorname{div}(a^{\mathrm{eff}}\nabla v^0_{ij})$ for all $i, j = 1, \dots, n$. Then one can easily check that

$$\check{V}^\varepsilon(x,t) = \varepsilon \chi^{ij}_{2,1}\Big(\frac{t}{\varepsilon^2}\Big)v^0_{ij}(x,t). \tag{33}$$

Our first goal is to show that

$$\|\varepsilon^{-1}V^{\varepsilon,1} - \check{V}^\varepsilon\|_{L^2((0,T)\times\mathbb{R}^n)} \longrightarrow 0 \quad \text{in probability.} \tag{34}$$

To this end we represent $\varepsilon^{-1}V^{\varepsilon,1}$ as

$$\varepsilon^{-1}V^{\varepsilon,1} = \varepsilon \chi^{ij}_{2,1}\Big(\frac{t}{\varepsilon^2}\Big)v^0_{ij}(x,t) + Z^\varepsilon(x,t)$$

and substitute it in (27). This yields the following equation for Z^ε:

$$\begin{cases} \dfrac{\partial Z^\varepsilon}{\partial t} = \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla Z^\varepsilon\Big) + \varepsilon\chi^{ij}_{2,1}\Big(\dfrac{t}{\varepsilon^2}\Big)\Big\{\partial_t v^0_{ij} - \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla v^0_{ij}\Big)\Big\} \\ Z^\varepsilon(0,x) = 0. \end{cases}$$

Let $\zeta = \zeta(t)$ be a continuous function on $[0,T]$. Then

$$\Big\|\zeta(\cdot)\Big\{\partial_t v^0_{ij} - \operatorname{div}\Big(a\Big(\frac{\cdot}{\varepsilon},\frac{\cdot}{\varepsilon^2}\Big)\nabla v^0_{ij}\Big)\Big\}\Big\|_{L^2(0,T;H^{-1}(\mathbb{R}^n))} \le C\|\zeta\|_{L^\infty(0,T)}, \tag{35}$$

where the constant C does not depend on ε. Next, we consider the following Cauchy problem:

$$\begin{cases} \dfrac{\partial \mathcal{Z}^\varepsilon}{\partial t} = \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla \mathcal{Z}^\varepsilon\Big) + \zeta(t)\Big\{\partial_t v^0_{ij} - \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla v^0_{ij}\Big)\Big\} \\ \mathcal{Z}^\varepsilon(0,x) = 0. \end{cases} \tag{36}$$

With the help of energy estimates we derive from (35) that

$$\|\mathcal{Z}^\varepsilon\|_{L^2(0,T;H^1(\mathbb{R}^n))} + \|\partial_t \mathcal{Z}^\varepsilon\|_{L^2(0,T;H^{-1}(\mathbb{R}^n))} \le C\|\zeta\|_{L^\infty(0,T)}.$$

Due to (5) and the definition of v^0, for any $K > 0$ we have

$$|v^0_{ij}(x,t)| + |\nabla_x v^0_{ij}(x,t)| \le C_K(T)(1+|x|)^{-K}, \quad (x,t) \in \mathbb{R}^n \times [0,T],$$

for some $C_K(T) > 0$. From this estimate we deduce (see [10]) that almost surely for a subsequence the function $\mathcal{Z}^\varepsilon$ converges in $C([0,T]; L^2(\mathbb{R}^n))$ to some function $\mathcal{Z}^0$. In order to characterize $\mathcal{Z}^0$, assume for a while that ζ is smooth. For an arbitrary $\varphi \in C^\infty_0((0,T)\times\mathbb{R}^n)$ we use in the integral identity of problem (36) the following test function

$$\varphi^\varepsilon(x,t) = \varphi(x,t) + \varepsilon\chi_-\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla\varphi(x,t)$$

with χ_- defined in (12). Setting $a^\varepsilon(x,t)=a\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$, $\chi_-^\varepsilon(x,t)=\chi_-\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$, after integration by parts in this integral identity and straightforward rearrangements we obtain

$$
\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^n}\mathcal{Z}^\varepsilon\big(\partial_t\varphi+(a^\varepsilon)^{ij}\partial_{x^i}\partial_{x^j}\varphi+(a^\varepsilon)^{ij}\partial_{z^j}(\chi_-^\varepsilon)^k\partial_{x^i}\partial_{x^k}\varphi\\
&\quad+\partial_{z^i}[(a^\varepsilon)^{ij}(\chi_-^\varepsilon)^k]\partial_{x^j}\partial_{x^k}\varphi\big)\,dxdt\\
&\quad-\varepsilon^{-1}\int_0^T\int_{\mathbb{R}^n}\mathcal{Z}^\varepsilon\big(\partial_{z^i}(a^\varepsilon)^{ij}\partial_{x^j}\varphi+\partial_s(\chi_-^\varepsilon)^j\partial_{x^j}\varphi+\partial_{z^i}[(a^\varepsilon)^{ij}\partial_{z^j}(\chi_-^\varepsilon)^k]\partial_{x^k}\varphi\big)\,dxdt\\
&\quad-\varepsilon\int_0^T\int_{\mathbb{R}^n}\mathcal{Z}^\varepsilon\big((a^\varepsilon)^{ij}(\chi_-^\varepsilon)^k\partial_{x^i}\partial_{x^j}\partial_{x^k}\varphi+(\chi_-^\varepsilon)^j\partial_t\partial_{x^j}\varphi\big)\,dxdt\\
&=\int_0^T\int_{\mathbb{R}^n}\big(\zeta\varphi\partial_t v_{lm}^0-\zeta v_{lm}^0\{(a^\varepsilon)^{ij}\partial_{x^i}\partial_{x^j}\varphi\\
&\quad-(a^\varepsilon)^{ij}\partial_{z^i}(\chi_-^\varepsilon)^k\partial_{x^j}\partial_{x^k}\varphi-\partial_{z^j}[(a^\varepsilon)^{ij}(\chi_-^\varepsilon)^k]\partial_{x^i}\partial_{x^k}\varphi\}\big)\,dxdt\\
&\quad-\varepsilon^{-1}\int_0^T\int_{\mathbb{R}^n}\big(\zeta v_{lm}^0\{\partial_s(\chi_-^\varepsilon)^k+\partial_{z^i}((a^\varepsilon)^{ij}\partial_{z^j}(\chi_-^\varepsilon)^k)+\partial_{z^i}(a^\varepsilon)^{ik}\}\partial_{x^k}\varphi\big)\,dxdt\\
&\quad-\varepsilon\int_0^T\int_{\mathbb{R}^n}\big(v_{lm}^0(\chi_-^\varepsilon)^k\partial_t(\zeta\varphi)+\zeta v_{lm}^0(a^\varepsilon)^{ij}(\chi_-^\varepsilon)^k\partial_{x^i}\partial_{x^j}\partial_{x^k}\varphi\big)\,dxdt.
\end{aligned}
$$

Notice that due to Eq. (12) all the terms of order ε^{-1} are equal to zero. Passing to the limit, as $\varepsilon\to 0$ yields

$$\int_0^T\int_{\mathbb{R}^n}\mathcal{Z}^0(\partial_t\varphi+\operatorname{div}(a^{\text{eff}}\nabla\varphi))\,dxdt=\int_0^T\int_{\mathbb{R}^n}\big(\zeta\varphi\partial_t v_{lm}^0-\zeta v_{lm}^0\operatorname{div}(a^{\text{eff}}\nabla\varphi)\big)\,dxdt.$$

Since v_{lm}^0 solves the effective equation, the integral on the right-hand side is equal to zero. Therefore,

$$\partial_t\mathcal{Z}^0-\operatorname{div}(a^{\text{eff}}\nabla\mathcal{Z}^0)=0.$$

Since $\mathcal{Z}^0(x,0)=0$, we conclude that $\mathcal{Z}^0=0$.

By the density arguments, $\mathcal{Z}^0=0$ for any continuous ζ. Due to the tightness of the family $\big\{\varepsilon\chi_{2,1}^{ij}\big(\frac{t}{\varepsilon^2}\big)\big\}$ in $C[0,T]$ this implies that Z^ε converges to zero in probability in $L^2(\mathbb{R}^n\times(0,T))$, and (34) follows.

It remains to pass to the limit in (33) and check that the limit process satisfies (32). Due to (31) and (33), $\check{V}^\varepsilon$ converges in law in $C(0,T;L^2(\mathbb{R}^n))$ to the process $\Lambda^{1/2}W.v^0$ with n^2-dimensional Wiener process W_t. Recalling the definition of v_{ij}^0, we obtain the desired convergence. □

We proceed with equation (ε^1). Its right-hand side can be rearranged as follows:

$$
\begin{aligned}
&-\partial_t u^1+\operatorname{div}_x\big(a(z,s)\nabla_x u^1\big)+\operatorname{div}_z\big(a(z,s)\nabla_x v^2\big)+\operatorname{div}_x\big(a(z,s)\nabla_z v^2\big)\\
&=\big\{-a^{\text{eff}}\otimes\chi(z,s)+a(z,s)\otimes\chi(z,s)+\operatorname{div}_z[a(z,s)\otimes\chi_{2,2}(z,s)]\\
&\quad+a(z,s)\nabla_z\chi_{2,2}(z,s)\big\}\frac{\partial^3}{\partial x^3}u^0(x,t):=\Psi_3(z,s)\frac{\partial^3}{\partial x^3}u^0(x,t);
\end{aligned}
$$

here and in what follows the symbol $\frac{\partial^3}{\partial x^3}u^0(x,t)$ stands for the tensor of third order partial derivatives of u^0, that is $\frac{\partial^3}{\partial x^3}=\big\{\frac{\partial^3}{\partial x^i\partial x^j\partial x^k}\big\}_{i,j,k=1}^n$; we have also denoted

$$a(z,s)\otimes\chi(z,s)=\big\{a^{ij}(z,s)\chi^k(z,s)\big\}_{i,j,k=1}^n$$

and

$$\mathrm{div}_z[a(z,s)\otimes\chi_{2,2}(z,s)]=\big\{\partial_{z^i}[a^{ij}(z,s)\chi_{2,2}^{kl}(z,s)]\big\}_{j,k,l=1}^n.$$

We introduce the following constant tensor $\mu=\{\mu^{ijk}\}_{i,j,k=1}^n$:

$$\mu=\mathbb{E}\int_{\mathbb{T}^n}\big\{-a^{\mathrm{eff}}\otimes\chi(z,s)+a(z,s)\otimes\chi(z,s)+a(z,s)\nabla_z\chi_{2,2}(z,s)\big\}dz$$

with $a(z,s)\nabla_z\chi_{2,2}(z,s)=\{a^{ij}(z,s)\partial_{z^l}\chi_{2,2}^{lk}(z,s)\}_{i,j,k=1}^n$, and consider the following problems:

$$\begin{cases}\dfrac{\partial\Xi_{\varepsilon,1}}{\partial t}=\mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla\Xi_{\varepsilon,1}\Big)+\Big(\Psi_3\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)-\mu\Big)\dfrac{\partial^3}{\partial x^3}u^0(x,t)\\ \Xi_{\varepsilon,1}(x,0)=0,\end{cases}\tag{37}$$

and

$$\begin{cases}\dfrac{\partial\Xi_{\varepsilon,2}}{\partial t}=\mathrm{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla\Xi_{\varepsilon,2}\Big)+\mu\dfrac{\partial^3}{\partial x^3}u^0(x,t)\\ \Xi_{\varepsilon,2}(0,x)=0\end{cases}\tag{38}$$

with

$$\mu\frac{\partial^3}{\partial x^3}u^0=\mu^{ijk}\frac{\partial^3u^0}{\partial x^i\partial x^j\partial x^k},\qquad \Psi_3\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^3}{\partial x^3}u^0=\Psi_3^{ijk}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^3u^0}{\partial x^i\partial x^j\partial x^k}.$$

Notice that $\Xi_{\varepsilon,1}$ and $\Xi_{\varepsilon,2}$ are scalar-valued functions.

Lemma 5.2. *The solution of problem (37) tends to zero a.s., as $\varepsilon\to0$, in $L^2(\mathbb{R}^n\times[0,T])$. Moreover,*

$$\lim_{\varepsilon\to0}\mathbb{E}\big(\|\Xi_{\varepsilon,1}\|^2_{L^2(\mathbb{R}^n\times[0,T])}\big)=0.$$

Proof. Splitting further the term $(\Psi_3-\mu)$ on the right-hand side of (37) into two parts

$$\begin{aligned}\Psi_3(z,s)-\mu&=\mathrm{div}_z[a(z,s)\otimes\chi_{2,2}(z,s)]+\big\{(a(z,s)-a^{\mathrm{eff}})\otimes\chi(z,s)\\&\quad+a(z,s)\nabla_z\chi_{2,2}(z,s)-\mu\big\}\\&=\Psi_{3,1}(z,s)+(\Psi_{3,2}(z,s)-\mu),\end{aligned}\tag{39}$$

we represent the solution $\Xi_{\varepsilon,1}$ as the sum $\Xi_{\varepsilon,1}^1$ and $\Xi_{\varepsilon,1}^2$, respectively. Denote $\Psi_{3,1}^\varepsilon(x,t)=\Psi_{3,1}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$ and $\Psi_{3,2}^\varepsilon(x,t)=\Psi_{3,2}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$. Combining (5) with (30) we conclude that

$$\Big\|\Psi_{3,1}^\varepsilon\frac{\partial^3}{\partial x^3}u_0\Big\|_{L^2([0,T];H^{-1}(\mathbb{R}^n))}\le C\varepsilon.\tag{40}$$

Indeed, for any $\phi\in L^2(0,T:H^1(\mathbb{R}^n))$ we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}\Big(\Psi_{3,1}^\varepsilon\frac{\partial^3}{\partial x^3}u_0\Big)\phi\,dxdt\\&\quad=-\varepsilon\int_0^T\int_{\mathbb{R}^n}a^{ij,\varepsilon}\chi_{2,2}^{kl,\varepsilon}\Big(\frac{\partial^3u_0}{\partial x^j\partial x^k\partial x^l}\frac{\partial\phi}{\partial x^i}+\frac{\partial^4u_0}{\partial x^i\partial x^j\partial x^k\partial x^l}\phi\Big)dxdt\end{aligned}$$

with $a^{ij,\varepsilon}(x,t)=a^{ij}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$ and $\chi_{2,2}^{kl,\varepsilon}(x,t)=\chi_{2,2}^{kl}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)$. By (5) and (30),

$$\|a^{ij,\varepsilon}\chi_{2,2}^{kl,\varepsilon}\frac{\partial^3 u_0}{\partial x^j\partial x^k\partial x^l}\|_{L^2(\mathbb{R}^n\times(0,T))}\le C,$$

$$\|a^{ij,\varepsilon}\chi_{2,2}^{kl,\varepsilon}\frac{\partial^4 u_0}{\partial x^i\partial x^j\partial x^k\partial x^l}\|_{L^2(\mathbb{R}^n\times(0,T))}\le C.$$

Therefore,

$$\Big|\int_0^T\int_{\mathbb{R}^n}\big(\Psi_{3,1}^{\varepsilon}\frac{\partial^3}{\partial x^3}u_0\big)\phi\,dxdt\Big|\le C\varepsilon\|\phi\|_{L^2(0,T;H^1(\mathbb{R}^n))},$$

and (40) follows. As a consequence of (40) one obtains

$$\|\Xi_{\varepsilon,1}^1\|_{L^2([0,T];H^1(\mathbb{R}^n))}\le C\varepsilon. \tag{41}$$

Due to (30) and the properties of u_0, we have

$$\|\big(\Psi_{3,2}^{\varepsilon}-\mu\big)\frac{\partial^3}{\partial x^3}u_0\|_{L^2(\mathbb{R}^n\times(0,T))}\le C$$

with a deterministic C. Using Theorem 1.5.1 in [10] we derive from this estimate that a.s. the family $\Xi_{\varepsilon,1}^2$ is compact in $L^2((0,T);L^2_{\rm loc}(\mathbb{R}^n))$. Considering condition H2 and Aronson's estimate (see [1]), we then conclude that the family $\Xi_{\varepsilon,1}^2$ is compact in $L^2(\mathbb{R}^n\times(0,T))$.

Our next goal is to show that the function $\big(\Psi_{3,2}^{\varepsilon}-\mu\big)$ converges a.s. to zero weakly in $L^2(Q\times(0,T))$ for any cube $Q\subset\mathbb{R}^d$. To this end we represent $\Psi_{3,2}(z,s)$ as

$$\Psi_{3,2}(z,s)=\overline{\Psi}_{3,2}(s)+[\Psi_{3,2}(z,s)-\overline{\Psi}_{3,2}(s)],\qquad \overline{\Psi}_{3,2}(s)=\int_{\mathbb{T}^n}\Psi_{3,2}(z,s)\,dz.$$

By (22), (30) and the definition of $\Psi_{3,2}$ (see (39)) we have

$$\|\Psi_{3,2}-\overline{\Psi}_{3,2}\|_{L^2(\mathbb{T}^n\times(s,s+1))}\le C,\qquad \int_{\mathbb{T}^n}(\Psi_{3,2}(z,s)-\overline{\Psi}_{3,2}(s))\,dz=0\quad\text{for all } s.$$

Therefore, $\Psi_{3,2}^{\varepsilon}-\overline{\Psi}_{3,2}^{\varepsilon}$ converges weakly to zero, as $\varepsilon\to0$, in $L^2(Q\times(0,T))$. From the definition of $\overline{\Psi}_{3,2}$ and μ it follows that $\overline{\Psi}_{3,2}$ is stationary, and $\mathbb{E}(\overline{\Psi}_{3,2}(s)-\mu)=0$. By the Birkhoff ergodic theorem, the function $\big(\overline{\Psi}_{3,2}^{\varepsilon}-\mu\big)$ converges a.s. to zero weakly in $L^2(0,T)$. Then it also converges a.s. to zero weakly in $L^2(Q\times(0,T))$ for any cube $Q\subset\mathbb{R}^d$. This yields the desired convergence of $\big(\Psi_{3,2}^{\varepsilon}-\mu\big)$.

Due to (5) and periodicity of $\Psi_{3,2}$ in spatial variable this implies that $\big(\Psi_{3,2}\big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\big)-\mu\big)\frac{\partial^3}{\partial x^3}u_0$ converges a.s. to zero weakly in $L^2(\mathbb{R}^n\times(0,T))$. Combining this with the above compactness arguments, we conclude that a.s. $\Xi_{\varepsilon,1}^2$ converges to zero in $L^2(\mathbb{R}^n\times(0,T))$. Then in view of (41), $\Xi_{\varepsilon,1}$ tends to zero in $L^2(\mathbb{R}^n\times(0,T))$ a.s. This yields the first statement of the lemma. The second statement follows from the first one by the Lebesgue dominated convergence theorem. □

According to [14], problem (38) admits homogenization. In particular, $\Xi_{\varepsilon,2}$ converges a.s. in $L^2(\mathbb{R}^n\times(0,T))$ to a solution of the following problem:

$$\begin{cases}\dfrac{\partial\Xi_{0,2}}{\partial t}=\operatorname{div}\big(a^{\rm eff}\nabla\Xi_{0,2}\big)+\mu\dfrac{\partial^3}{\partial x^3}u^0(x,t)\\ \Xi_{0,2}(0,x)=0\end{cases} \tag{42}$$

This is not the end of the story with the asymptotic expansion because the initial condition is not satisfied at the level ε^1. In order to fix this problem we introduce one more term of order ε^1 so that the expansion takes the form

$$u^\varepsilon(x,t) = u^0(x,t) + \varepsilon\Big\{\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) + \mathcal{I}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\Big\}\nabla u^0(x,t) \\ + V^{\varepsilon,1} + \varepsilon^2 v^2\Big(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big) + \cdots. \tag{43}$$

The initial layer type function $\mathcal{I}$ has been added in order to compensate the discrepancy in the initial condition. This function solves the following problem:

$$\begin{cases} \dfrac{\partial \mathcal{I}}{\partial s} = \mathrm{div}\big(a(z,s)\nabla\mathcal{I}\big) \\ \mathcal{I}(0,z) = -\chi(0,z). \end{cases} \tag{44}$$

Lemma 5.3. *The solution of problem (44) decays exponentially as $s \to \infty$. We have*

$$\|\mathcal{I}(\cdot,s)\|_{L^\infty(\mathbb{T}^n)} \le Ce^{-\nu s}, \qquad \|\mathcal{I}\|_{L^\infty([s,s+1];H^1(\mathbb{T}^n))} \le Ce^{-\nu s}$$

Proof. The desired statement is an immediate consequence of the fact that $\int_{\mathbb{T}^n}\mathcal{I}(z,s)dz = \int_{\mathbb{T}^n}\mathcal{I}(z,0)dz = 0$, the maximum principle and the parabolic Harnack inequality (see [8] for further details). □

6. Main results

In this section we present the main result. Consider the expression

$$U^\varepsilon(x,t) = \frac{u^\varepsilon(x,t) - u^0(x,t)}{\varepsilon} - \chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla_x u^0. \tag{45}$$

It is easily seen that U^ε is equal to the normalized difference between u^ε and the first two terms of asymptotic expansion (43). The limit behaviour of U^ε is described by the following statement.

Theorem 3. *Under the assumptions* H1–H3 *the function U^ε converges in law, as $\varepsilon \to 0$, in the space $L^2(\mathbb{R}^n \times (0,T))$ to a solution of the following SPDE*

$$dU^0 = \mathrm{div}\Big(a^{\mathrm{eff}}\nabla U^0 + \mu\frac{\partial^3}{\partial x^3}u^0\Big)dt + \Lambda^{1/2}\frac{\partial^2}{\partial x^2}u^0\,dW_t, \\ U^0(x,0) = 0. \tag{46}$$

Proof. We set

$$\mathcal{V}^\varepsilon(x,t) = U^\varepsilon(x,t) - \varepsilon^{-1}V^{\varepsilon,1}(x,t) - \Xi_{\varepsilon,2}(x,t) \\ -\mathcal{I}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0(x,t) - \varepsilon\chi_{2,2}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\frac{\partial^2}{\partial x^2}u^0(x,t) - \Xi_{\varepsilon,1}(x,t).$$

Substituting this expression in (1) for u^ε and combining the above equations, we obtain after straightforward computations that $\mathcal{V}^\varepsilon$ satisfies the problem

$$\frac{\partial}{\partial t}\mathcal{V}^\varepsilon - \mathrm{div}\Big(a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla\mathcal{V}^\varepsilon\Big) = R^\varepsilon, \\ \mathcal{V}^\varepsilon(x,0) = R_1^\varepsilon$$

with

$$R^\varepsilon = \varepsilon^{-1}\big\{\partial_{z_i}[(a^\varepsilon)^{ij}(\chi^\varepsilon_{\rm il})^k] + ((a^\varepsilon)^{ji}\partial_{z^i}(\chi^\varepsilon_{\rm il})^k)\big\}\partial_{x^j}\partial_{x^k}u^0 - (\chi^\varepsilon)^j\partial_t\partial_{x^j}u^0$$
$$- (\mathcal{I}^\varepsilon)^j\partial_t\partial_{x^j}u^0 + \varepsilon(a^\varepsilon)^{ij}(\chi^\varepsilon_{2,2})^{lk}\partial_{x^i}\partial_{x^j}\partial_{x^l}\partial_{x^k}u^0 - \varepsilon(\chi^\varepsilon_{2,2})^{ij}\partial_t\partial_{x^i}\partial_{x^j}u^0$$

and

$$R_1^\varepsilon = \varepsilon\chi_{2,2}\Big(\frac{x}{\varepsilon},0\Big)\partial_x\partial_x u^0(x,0).$$

It follows from Lemma 5.3 that

$$\big\|\varepsilon^{-1}\partial_{z_i}[(a^\varepsilon)^{ij}(\mathcal{I}^\varepsilon)^k]\partial_{x^j}\partial_{x^k}u^0\big\|_{L^2((0,T);H^{-1}(\mathbb{R}^n))} + \big\|(\mathcal{I}^\varepsilon)^j\partial_t\partial_{x^j}u^0\big\|_{L^2((0,T)\times\mathbb{R}^n)} \le C\varepsilon.$$

By (22), (23) we obtain

$$\big\|(\chi^\varepsilon)^j\partial_t\partial_{x^j}u^0\big\|_{L^2((0,T);H^{-1}(\mathbb{R}^n))} \le C\varepsilon.$$

Then by (30) we have

$$\big\|\varepsilon(a^\varepsilon)^{ij}(\chi^\varepsilon_{2,2})^{lk}\partial_{x^i}\partial_{x^j}\partial_{x^l}\partial_{x^k}u^0\big\|_{L^2((0,T)\times\mathbb{R}^n)} + \big\|\varepsilon(\chi^\varepsilon_{2,2})^{ij}\partial_t\partial_{x^i}\partial_{x^j}u^0\big\|_{L^2((0,T)\times\mathbb{R}^n)} \le C\varepsilon$$

and

$$\Big\|\varepsilon\chi_{2,2}\Big(\frac{x}{\varepsilon},0\Big)\partial_x\partial_x u^0(x,0)\Big\|_{L^2(\mathbb{R}^n)} \le C\varepsilon.$$

It remains to estimate the contribution of the term $\varepsilon^{-1}((a^\varepsilon)^{ji}\partial_{z^i}(\chi^\varepsilon_{\rm il})^k)\partial_{x^j}\partial_{x^k}u^0$. From the estimates of Lemma 5.3 it is easy to deduce that

$$\big\|\varepsilon^{-1}\big((a^\varepsilon)^{ji}\partial_{z^i}(\mathcal{I}^\varepsilon)^k\big)\partial_{x^j}\partial_{x^k}u^0\big\|_{L^2((0,T)\times\mathbb{R}^n)} \le C$$

and that a.s. the family $\big\{\varepsilon^{-1}((a^\varepsilon)^{ji}\partial_{z^i}(\mathcal{I}^\varepsilon)^k)\partial_{x^j}\partial_{x^k}u^0\big\}$ converges to zero weakly in $L^2((0,T)\times\mathbb{R}^n)$. Then, using the same compactness arguments as those in the proof of Lemma 5.2 one can show that the solution of problem

$$\begin{cases} \dfrac{\partial\Xi_{\varepsilon,3}}{\partial t} = \operatorname{div}\Big(a\Big(\dfrac{x}{\varepsilon},\dfrac{t}{\varepsilon^2}\Big)\nabla\Xi_{\varepsilon,3}\Big) + \varepsilon^{-1}\big((a^\varepsilon)\partial_z(\mathcal{I}^\varepsilon)\big)\partial_x\partial_x u^0 \\ \Xi_{\varepsilon,3}(0,x) = 0 \end{cases}$$

converges a.s. to zero in $L^2((0,T)\times\mathbb{R}^n)$. Moreover,

$$\lim_{\varepsilon\to 0}\mathbb{E}\big(\|\Xi_{\varepsilon,3}\|^2_{L^2(\mathbb{R}^n\times[0,T])}\big) = 0.$$

Combining the above estimates we conclude that R^ε a.s. tends to zero in $L^2(\mathbb{R}^n\times(0,T))$, as $\varepsilon\to 0$, and R_1^ε a.s. tends to zero in $L^2(\mathbb{R}^n)$. Furthermore,

$$\mathbb{E}\|R^\varepsilon\|^2_{L^2(\mathbb{R}^n\times(0,T))} \to 0, \qquad \mathbb{E}\|R_1^\varepsilon\|^2_{L^2(\mathbb{R}^n)} \to 0.$$

By Lemmata 5.2 and 5.3 and estimate (30) it follows that $(U^\varepsilon - \varepsilon^{-1}V^{\varepsilon,1} - \Xi_{\varepsilon,2})$ tends a.s. to zero in $L^2(\mathbb{R}^n\times(0,T))$, and

$$\mathbb{E}\|U^\varepsilon - \varepsilon^{-1}V^{\varepsilon,1}(x,t) - \Xi_{\varepsilon,2}(x,t)\|^2_{L^2(\mathbb{R}^n\times(0,T))} \to 0.$$

By Lemma 5.1 the function $\varepsilon^{-1}V^{\varepsilon,1}$ converges in law to a solution of (32). Also, $\Xi_{\varepsilon,2}$ converges a.s. to $\Xi_{0,2}$ in $L^2(\mathbb{R}^n\times(0,T))$. This yields the convergence

$$U^\varepsilon \longrightarrow V^{0,1} + \Xi_{0,2}$$

in law in the space $L^2(\mathbb{R}^n \times (0, T))$. It remains to note that due to (32) and (42) the random function $U^0 := (V^{0,1} + \Xi_{0,2})$ satisfies the stochastic PDE (46) as required. □

7. Diffusive case

The goal of this section is to extend the statement of Theorem 3 to the diffusive case.

Theorem 4. *Let assumptions* A1–A3 *be fulfilled. Then the function U^ε defined in* (45) *converges in law, as $\varepsilon \to 0$, in the space $L^2(\mathbb{R}^n \times (0, T))$ to the solution of* (46).

Proof. The arguments used in the proof of Theorem 3 also apply in the case under consideration. We used assumption H3 only once, when justified convergence (31). Thus, this convergence should be reproved under our standing assumptions.

Lemma 7.1. *Under assumptions* A1–A3 *for any $K > 0$ there exists C_K such that the following estimate holds*

$$\|\mathbb{E}\{\Psi_{2,1}(s) \mid \mathcal{F}_{\le 0}\}\|_{L^2(\Omega)} \le C_K\big(e^{-\nu s/2} + (1+s)^{-K}\big), \quad \nu > 0$$

the function $\Psi_{2,1}$ has been defined in (24)

Proof. We follow the scheme of proof of Lemma 4.2. Denote

$$\begin{aligned}\chi(z,s) &= \widehat{\chi}^1(z,s) + \widehat{\chi}^2(z,s)\\ &= \int_{-\infty}^{s/2}\int_{\mathbb{T}^n}\big(\mathcal{G}(z,\hat z,s,\hat s) - 1\big)\mathrm{div}_z a(\hat z, \xi_{\hat s})\, d\hat z d\hat s\\ &\quad + \int_{s/2}^{s}\int_{\mathbb{T}^n}\big(\mathcal{G}(z,\hat z,s,\hat s)\big)\mathrm{div}_z a(\hat z, \xi_{\hat s})\, d\hat z d\hat s.\end{aligned}$$

Then $\Psi_{2,1}(s) = \widehat{\Psi}^1(s) + \widehat{\Psi}^2(s)$ with

$$\widehat{\Psi}^i(s) = \int_{\mathbb{T}^n}\big(a(z,\xi_s)\nabla_z\widehat{\chi}^i(z,s) - \mathbb{E}(a(z,\xi_s)\nabla_z\widehat{\chi}^i(z,s))\big)\, dz, \quad i = 1,\ 2.$$

Considering (20) we obtain the inequality $\|\widehat{\Psi}^1(s)\|_{L^2(\Omega)} \le Ce^{-\nu s/2}$. Since $\widehat{\Psi}^2(s)$ is $\mathcal{F}_{\ge s/2}$-measurable, we have

$$\begin{aligned}\|\mathbb{E}\{\widehat{\Psi}^2(s) \mid \mathcal{F}_{\le 0}\}\|_{L^2(\Omega)} &= \|\mathbb{E}\big\{\mathbb{E}\{\widehat{\Psi}^2(s) \mid \mathcal{F}_{\le s/2}\} \mid \mathcal{F}_{\le 0}\big\}\|_{L^2(\Omega)}\\ &= \|\mathbb{E}\big\{\mathbb{E}\{\widehat{\Psi}^2(s) \mid \mathcal{F}_{= s/2}\} \mid \mathcal{F}_{\le 0}\big\}\|_{L^2(\Omega)}\\ &= \|\mathbb{E}\big\{\mathcal{R}(\xi_{s/2}) \mid \mathcal{F}_{\le 0}\big\}\|_{L^2(\Omega)};\end{aligned}$$

here we have used the Markov property of $\xi_\cdot$. According to [12, Section 2] this yields the desired inequality. □

From the last Lemma it follows that the invariance principle holds for the process $\chi_{2,1}(s)$ (see [6, Theorem VIII.3.79]), that is (31) holds for any $T > 0$. The rest of proof of Theorem 4 is exactly the same as that of Theorem 3. □

Acknowledgement

The work of the first and the third authors was partially supported by ANR STOSYMAP project ANR-11-BS01-015-02.

References

[1] D. Aronson, Non-negative solutions of linear parabolic equations, Ann. Sc. Norm. Super. Pisa 22 (1968) 607–694.

[2] F. Campillo, M. Kleptsyna, A. Piatnitski, Homogenization of random parabolic operator with large potential, Stochastic Process. Appl. 93 (1) (2001) 57–85.

[3] M. Diop, B. Iftimie, E. Pardoux, A. Piatnitski, Singular homogenization with stationary in time and periodic in space coefficients, J. Funct. Anal. 231 (1) (2006) 1–46.

[4] D. Gilbarg, N. Trudinger, Continuity of solutions of elliptic and parabolic equations, Amer. J. Math. 80 (1958) 931–954.

[5] A. Gloria, F. Otto, An optimal error estimate in stochastic homogenization of discrete elliptic equations, Ann. Appl. Probab. 22 (1) (2012) 1–28.

[6] J. Jacod, A. Shiryaev, Limit Theorems for Stochastic Processes, second ed., in: Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 288, Springer-Verlag, Berlin, 2003.

[7] M. Kleptsyna, A. Piatnitski, Homogenization of random parabolic operators, in: Homogenization and Applications to Material Sciences (Nice, 1995), in: GAKUTO Internat. Ser. Math. Sci. Appl., vol. 9, Gakkotosho, Tokyo, 1995, pp. 241–255.

[8] M. Kleptsyna, A. Piatnitski, Homogenization of random nonstationary convection–diffusion problem, Russian Math. Surveys 57 (4) (2002) 729–751.

[9] S. Kozlov, The averaging of random operators, Mat. Sb. 109 (2) (1979) 188–202.

[10] J.L. Lions, Quelques Méthodes de Résolution des Problémes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris, 1969.

[11] G. Papanicolaou, S.R.S. Varadhan, Boundary value problems with rapidly oscillating random coefficients, in: Random Fields, Vol. I, II (Esztergom, 1979), in: Colloq. Math. Soc. Jonos Bolyai, vol. 27, North-Holland, Amsterdam, New York, 1981, pp. 835–873.

[12] E. Pardoux, A. Veretennikov, On the Poisson equation and diffusion approximation. I, Ann. Probab. 29 (3) (2001) 1061–1085.

[13] V. Yurinski, Averaging of an elliptic boundary value problem with random coefficients, Sibirsk. Mat. Zh. 21 (3) (1980) 209–223.

[14] V. Zhikov, S. Kozlov, O. Oleinik, Averaging of parabolic operators, Tr. Mosk. Mat. Obs. 45 (1982) 182–236.