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On the fundamental solution of heat and stochastic heat equations

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Abstract

We consider the generic divergence form second order parabolic equation with coefficients that are regular in the spatial variables and just measurable in time. We show that the spatial derivatives of its fundamental solution admit upper bounds that agree with the Aronson type estimate and only depend on the ellipticity constants of the equation and the L^∞ norm of the spatial derivatives of its coefficients.

We also study the corresponding stochastic partial differential equations and prove that under natural assumptions on the noise the equation admits a mild solution, given by anticipating stochastic integration.

1   Introduction

In the first part of the paper we study the fundamental solution $\Gamma = \Gamma(x, t, y, s)$ of the parabolic equation

$$\frac{\partial u}{\partial t}(x,t) = \mathrm{div}\Big[\mathrm{a}\Big(x,t\Big)\nabla u(x,t)\Big], \quad (x,t) \in \mathbb{R}^d \times (0,T]. \tag{1}$$

The existence of the fundamental solution Γ and the description of its properties is an old story that has given rise to a vast literature (see among others [7, 11, 16, 6, 9] and the references therein). One of the most famous result in this field is the Aronson

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estimate (see Inequality (4) and [2, Theorem 7]), which holds under the uniform ellipticity condition on the diffusion matrix a (see condition **(H1)**). No regularity assumption on the coefficients of a is required. To obtain similar estimates on the spatial derivatives of Γ, it is usually assumed in the existing literature that the matrix a is Hölder continuous w.r.t. both x and t (see [11], Chapter IV, sections 11 to 13 or [7], Chapter I): for some $\hbar \in (0,1)$

$$\left| a\left(x,t\right) - a\left(x',t'\right) \right| \leq K_a \left(\left|x - x'\right|^\hbar + \left|t - t'\right|^{\hbar/2} \right).$$

Notice that this setting is not well adapted to the stochastic framework, for example if $a(x,t) = a(x,\xi_t)$ where ξ is a diffusion process. Indeed, in this case the constant K_a depends on the continuity properties of ξ and is random (see for example [3] for details). Hence the constants in the estimate of $\nabla_x \Gamma$ need not be uniformly bounded if we follow directly this construction.

Our first goal in the paper is to obtain Aronson type estimates for the spatial derivatives of Γ, without any regularity assumption on the dependence $t \mapsto a(x,t)$. We impose only a uniform Lipschitz continuity condition on the dependence $x \mapsto a(x,t)$. Then the upper bounds only depend on the ellipticity constants and L^∞ norm of the gradient of the coefficients (see Theorem 1). To our best knowledge, this result is new, we did not succeed to find it in the existing literature. We believe that it is of essential interest for theory of parabolic operators.

In the second part of this paper we deal with the following stochastic heat equation:

$$dv(x,t) - \mathrm{div}\left[a\left(x,t\right) \nabla v(x,t) \right] dt = G\left(x,t\right) dB_t \tag{2}$$

with the initial condition $v(x,0) = 0$ (see Remark 1 for more general initial value). B is a standard Brownian motion, generating the filtration $\mathbb{F} = (\mathcal{F}_t, \ t \geq 0)$. The matrix a is supposed to be a measurable function from $\mathbb{R}^d \times [0, +\infty[\times \Omega$ into $\mathbb{R}^{d \times d}$ and for each $(x,t) \in \mathbb{R}^d \times [0, +\infty[$, $a(x,t)$ is $\mathcal{F}_t$-measurable. This stochastic partial differential equation (SPDE in short) in divergence form is somehow classical and among many other we refer to the books [7, 11] on PDE in divergence form, [4, 5, 10, 14, 17] on SPDE (and the references therein).

Our aim is to prove that the SPDE in (2) admits a mild solution v given by:

$$v(x,t) \quad = \quad \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s) G\left(y,s\right) dy dB_s, \tag{3}$$

where Γ is the fundamental solution of the equation in (1).

If the matrix a is deterministic, Γ is also deterministic and the existence of a mild solution v given by (3) is well known (see [17, Chapter 5]). However, when a is random, the stochastic integral in (3) has to be defined properly since $\Gamma(x,t,y,s)$ is measurable w.r.t. the σ-field $\mathcal{F}_t$ generated by the random variables B_u with $u \leq t$. In other words Equation (3) involves an anticipating integral. To our best knowledge, there is only one work on this topic by Alos et al. [1]. Compared to our setting, the authors in [1] consider

a space-time Wiener process, but the matrix a is Hölder continuous in time[1] (condition (A3) in [1]).

From the first part of this paper, we know that Γ and its spatial derivative admit Aronson's type upper bounds and we extend these bounds to the Malliavin derivatives of Γ, again without regularity assumption on a w.r.t. t (see Theorem 2 and, in the diffusion case, Corollary 1).

Finally, since our noise is a one parameter Brownian motion, we also want to obtain a regular mild solution v on $\mathbb{R}^d \times (0, T)$ in the sense of Definition 1 of Equation (2). Compared to [1], since we have no space noise, we do not impose any condition on the dimension d and our solution is derivable w.r.t. x (see Theorem 3 and Corollary 2).

In a recent paper paper [15] a similar subject is handled with a parametrix construction. However, since the studied operator is not in the divergence form, the authors have to impose more regularity assumptions on the diffusion matrix a. Also, the SPDEs investigated in this paper are rearranged in such a way that the anticipating stochastic calculus can be avoided.

The paper is organized as follows. In Section 2 we consider the generic heat equation (1) and its fundamental solution Γ. We prove that the spatial derivatives of Γ admit an Aronson's type upper bound, without any time regularity condition on a. Our result is presented in Theorem 1.

In the next section 3, we assume that the matrix a is random. Using the arguments developed in the previous section, we a number of estimates for the Malliavin derivative of Γ and of $\nabla\Gamma$, see Theorem 2. We also study the particular case where the randomness is given by the solution of a SDE (diffusion case, section 3.2).

In Section 4 we construct a mild solution v of the SPDE in (2). Here we use anticipating calculus and the properties of the fundamental solution Γ of a parabolic equation with random coefficients. In the first part of this section we provide our assumptions and formulate the main result concerning a mild solution (Theorem 3 and Corollary 2 in the diffusion case). Section 4.2 is devoted to the proof of those results.

2 Estimate for the spatial derivative of the fundamental solution

Our goal here is to obtain an upper bound for the derivative of the fundamental solution Γ for the PDE (1). On the matrix a : $\mathbb{R}^d \times [0, +\infty) \to \mathbb{R}^{d\times d}$ we impose the following conditions.

(H1) Uniform ellipticity. For any $(t, x, \zeta) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$

$$\lambda^{-1}|\zeta|^2 \leq \mathrm{a}(x,t)\zeta \cdot \zeta \leq \lambda|\zeta|^2.$$

---

[1]At the end of [1, section 5], the authors make a remark and give an example on this time regularity assumption.

(H2) The matrix a is measurable on $\mathbb{R}^d \times \mathbb{R}_+$, and for any $t \geq 0$ the function $a(\cdot, t)$ is of class C^1 w.r.t. $x \in \mathbb{R}^d$. Moreover, there is a constant K_a such that for all t and x

$$|\nabla a(x,t)| \leq K_a.$$

We denote by $\mathcal{L}$ the operator: $\mathcal{L} = \mathrm{div}\left[a\left(x, t\right)\nabla\right]$, then (1) can be written:

$$\frac{\partial u}{\partial t}(x,t) = \mathcal{L}u(x,t).$$

It is well known (see among other [2] or [6]) that under condition **(H1)** there exist two constants $\varsigma > 0$ and $\varpi > 0$ depending only on the constant λ in Assumption **(H1)** and the dimension d, such that

$$0 \leq \Gamma(x,t,y,s) \leq g_{\varsigma,\varpi}(x-y, t-s); \tag{4}$$

here and in what follows, for two positive constants c and C, the function $g_{c,C}(x,t)$ is defined by

$$g_{c,C}(x,t) = c\, t^{-\frac{d}{2}} \exp\left(-\frac{C|x|^2}{t}\right), \qquad t > 0, \quad x \in \mathbb{R}^d.$$

Inequality (4) is called the **Aronson estimate**[2]. Our first result reads.

Theorem 1 *If the matrix $a = a(x,t)$ satisfies the uniform ellipticity condition* **(H1)** *and the regularity condition* **(H2)**, *then the (weak) fundamental solution Γ of equation (1) admits the following estimate: there exist two constants $\varrho > 0$ and $\varpi > 0$ such that*

$$|\nabla_x \Gamma(x,t,y,s)| \leq \frac{1}{\sqrt{t-s}}\, g_{\varrho,\varpi}(x-y, t-s); \tag{5}$$

here ϖ depends only on the uniform ellipticity constant λ and the dimension d, while ϱ might also depend on K_a and on T.

Weak fundamental solution is defined in [6, Definition VI.6]. Let us emphasize that these estimates are coherent with [6, Theorem VI.4]. The novelty is that the regularity of a w.r.t. t is not required. The rest of this section is devoted to the proof of this theorem.

2.1 When a does not depend on x.

First assume that a just depends on t. In this case the fundamental solution Γ is denoted by Z and is given by the formula: for any $s < t$ and $(x,y) \in (\mathbb{R}^d)^2$

$$Z(x-y, t, s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V(t,s,\zeta) d\zeta, \tag{6}$$

where V is the following function:

$$V(t,s,\zeta) = \exp\left(-\left\langle \int_s^t a(u)du\, \zeta, \zeta \right\rangle\right).$$

---

[2]The function Γ has a lower bound similar to the upper bound (see [2, Theorem 7])

Due to Condition **(H1)** the matrix a verifies the estimates

$$\lambda^{-1}(t-s)|\zeta|^2 \leq \left\langle \int_s^t a(u)du\ \zeta, \zeta \right\rangle \leq \lambda(t-s)|\zeta|^2.$$

From the above expression for Z, we deduce that for any $k \geq 1$ and $1 \leq j_\ell \leq d$ with $1 \leq \ell \leq k$

$$\partial_{x_{j_1}\ldots x_{j_k}}^k Z(x-y,t,s) \;\;=\;\; \frac{(i)^k}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V(t,s,\zeta)(\zeta_{j_1}\ldots\zeta_{j_k})d\zeta.$$

As in [7], Chapter 9, Theorem 1, we obtain that:

$$|\partial_{x_{j_1}\ldots x_{j_k}}^k Z(x-y,t,s)| \leq \frac{1}{(t-s)^{k/2}}\ g_{\varsigma,\varpi}(x-y,t-s). \tag{7}$$

In particular the Aronson estimates (4) and (5) can be derived.

Now we define the parametrix, also denoted by Z, as the fundamental solution of (1) for $a(\mathfrak{z},t)$ where $\mathfrak{z} \in \mathbb{R}^d$ is a fixed parameter:

$$\frac{\partial u}{\partial t}(x,t) = \text{div}\Big[a(\mathfrak{z},t)\nabla u(x,t)\Big].$$

We have again the representation

$$\forall s \leq t, \quad Z(x-y,t,s,\mathfrak{z}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V(t,s,\zeta,\mathfrak{z})d\zeta, \tag{8}$$

with

$$V(t,s,\zeta,\mathfrak{z}) = \exp\left(-\left\langle \int_s^t a(\mathfrak{z},u)du\ \zeta, \zeta \right\rangle\right).$$

The above arguments give Estimates (4) and (5). The following statement is equivalent to Lemma 5 in [7], Chapter 9, Section 3 (see also [7, Theorem I.3.2]).

In the next section, we use the parametrix method to construct Γ when a depends on both x and t. The following technical result is used several times.

Lemma 1 *Suppose that f is a measurable function on $\mathbb{R}^d \times [0,+\infty)$ that satisfies the estimate*

$$|f(x,t)| \leq k\exp(\mathfrak{k}|x|^2)$$

for some constants k and $\mathfrak{k} < \varpi/T$. Then the integral

$$F(x,t) = \int_0^t \left(\int_{\mathbb{R}^d} Z(x-\zeta,t,s,\zeta)f(\zeta,s)d\zeta\right)ds$$

is well defined for $0 \leq t \leq T$, continuous on $\mathbb{R}^d \times [0,T]$, and the derivative $\nabla_x F$ exists for $0 < t \leq T$ and

$$\nabla_x F(x,t) = \int_0^t \left(\int_{\mathbb{R}^d} \nabla_x Z(x-\zeta,t,s,\zeta)f(\zeta,s)d\zeta\right)ds.$$

Proof. We skip the proof of this Lemma because it is the same as the proof of Lemma IX.5 in [7] (see also [7], Chapter 1, Section 3 for more details). $\qquad\square$

2.2 Parametrix method and the estimate on the gradient

The parametrix method suggests to construct Γ in the form

$$
\begin{aligned}
\Gamma(x,t,y,s) &= Z(x-y,t,s,y) \\
&+ \int_s^t \int_{\mathbb{R}^d} Z(x-\zeta,t,r,\zeta)\Phi(\zeta,r,y,s)d\zeta dr.
\end{aligned}
\tag{9}
$$

If the function Φ is measurable and satisfies a suitable growth condition, we can apply Lemma 1. Then Γ is the fundamental solution if and only if

$$
\Phi(x,t,y,s) = \mathcal{K}(x,t,y,s) + \int_s^t \int_{\mathbb{R}^d} \mathcal{K}(x,t,\zeta,r)\Phi(\zeta,r,y,s)d\zeta dr,
$$

where

$$
\mathcal{K}(x,t,y,s) = \operatorname{div}\left[\left(\mathrm{a}(x,t) - \mathrm{a}(y,t)\right)\nabla_x Z(x-y,t,s,y)\right].
$$

Notice that in the expression $\mathrm{a}(x,t) - \mathrm{a}(y,t)$, the matrix is evaluated two times at the same time t. Hence formally the function Φ is the sum of iterated kernels

$$
\Phi(x,t,y,s) = \sum_{m=1}^{\infty} \mathcal{K}_m(x,t,y,s)
\tag{10}
$$

with $\mathcal{K}_m$ defined by

$$
\mathcal{K}_m(x,t,y,s) = \int_s^t \int_{\mathbb{R}^d} \mathcal{K}(x,t,\zeta,r)\mathcal{K}_{m-1}(\zeta,r,y,s)d\zeta dr.
$$

Let us follow the scheme of [7] to obtain (5). Remark that continuity of a w.r.t. t is not assumed. We will use the following notations: a_i is the i-th column of a, γ is the vector-function such that

$$
\gamma_i(x,t) = \operatorname{div}(\mathrm{a}_i(x,t)) = \sum_{j=1}^n \frac{\partial \mathrm{a}_{ji}}{\partial x_j}(x,t).
$$

Note that under **(H2)**, γ is bounded. The kernel $\mathcal{K}$ satisfies:

$$
\begin{aligned}
\mathcal{K}(x,t,y,s) &= \sum_{i,j=1}^n (\mathrm{a}_{ij}(x,t) - \mathrm{a}_{ij}(y,t))\frac{\partial^2 Z}{\partial x_i \partial x_j}(x-y,t,s,y) \\
&+ \sum_{i=1}^n \gamma_i(x,t)\frac{\partial Z}{\partial x_i}(x-y,t,s,y).
\end{aligned}
\tag{11}
$$

6

Lemma 2 *Under* **(H1)** *and* **(H2)**, *the series in* (10) *converge. The sum* Φ *is measurable and satisfies the estimate*

$$|\Phi(x,t,y,s)| \leq \frac{1}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s). \tag{12}$$

The constants ϱ *and* ϖ *depend on* λ *and* d, *whereas* ϱ *also depends on the Lispchitz constant* K_a *and on* T.

Proof. From estimate (7) considering Lipschitz continuity of a, we obtain

$$
\begin{aligned}
|\mathcal{K}(x,t,y,s)| \quad &\leq \quad K_a|x-y|\frac{1}{t-s} g_{\varsigma,\varpi}(x-y,t-s) \\
&+ \quad K_a\frac{1}{\sqrt{t-s}} g_{\varsigma,\varpi}(x-y,t-s) \\
&\leq \quad \frac{1}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s).
\end{aligned}
$$

Again ς, ϖ or ϱ may differ from line to line. Thus $\mathcal{K}$ satisfies inequality (4.6) of [7], Chapter 9, Section 4. Then the convergence of the series in (10) can be proved by the same arguments. Indeed, by Lemma IX.7 in [7] for any η, $0 < \eta < 1$, there is a constant $M(\eta,\varpi) > 0$ depending on η, ϖ and d such that

$$
\begin{aligned}
|\mathcal{K}_2(x,t,y,s)| &\leq \int_s^t \int_{\mathbb{R}^d} |\mathcal{K}(x,t,\zeta,r)|\,|\mathcal{K}(\zeta,r,y,s)|\,d\zeta dr \\
&\leq \int_s^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{(t-r)(r-s)}}\, g_{\varrho,\varpi}(x-\zeta,t-r) g_{\varrho,\varpi}(\zeta-y,r-s) d\zeta dr \\
&\leq \int_s^t \frac{M(\eta,\varpi)\varrho^2}{\sqrt{(t-r)(r-s)}} \frac{1}{(t-s)^{\frac{d}{2}}} \exp\left(-\varpi(1-\eta)\frac{|x-y|^2}{t-s}\right) dr
\end{aligned}
$$

By direct computation (see also Lemma I.2 in [7])

$$\int_s^t \frac{1}{\sqrt{(t-r)(r-s)}} dr = \pi.$$

Thereby there exist two constants $\varrho > 0$ and $\varpi > 0$ such that

$$|\mathcal{K}_2(x,t,y,s)| \leq g_{\varrho,\varpi}(x-y,t-s).$$

Iterating this computation we obtain by induction for $m \geq 2$:

$$|\mathcal{K}_m(x,t,y,s)| \leq \frac{M^m}{(1+m/2)!}(t-s)^{m/2-1} g_{\varrho,\varpi}(x-y,t-s)$$

where M is a constant depending on ϱ and ϖ, and the symbol $(\cdot)!$ stands for the gamma function (see the proof of Theorem IX.2 in [7] for the details). The convergence of the

series and estimate (12) can be then deduced. Namely,

$$
\begin{aligned}
|\Phi(x,t,y,s)| &\leq \frac{1}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s) \Big[\sum_{m \geq 1} \frac{M^m}{(1+m/2)!}(t-s)^{(m-1)/2} \Big] \\
&= \frac{1}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s)\Theta(t-s).
\end{aligned}
$$

For t and s in $[0,T]$, we get: $\Theta(t-s) \leq \Theta(T)$. $\qquad\square$

Using Lemma 1, we deduce that Γ is well-defined, and inequality (5) follows from the formula

$$
\begin{aligned}
\partial_{x_j}\Gamma(x,t,y,s) &= \partial_{x_j}Z(x-y,t,s,y) \\
&+ \int_s^t \int_{\mathbb{R}^d} \partial_{x_j}Z(x-\zeta,t,r,\zeta)\Phi(\zeta,r,y,s)d\zeta dr, \qquad (13)
\end{aligned}
$$

together with estimate (7) on Z and (12) on Φ. We underline that only the properties **(H1)** and **(H2)** of a are required to obtain (5). This completes the proof of Theorem 1.

3 Malliavin derivative of the fundamental solution

From now on we suppose that $a = a(x,t)$ are random fields defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ that carries a d-dimensional Brownian motion B and that the filtration $\mathbb{F} = (\mathcal{F}_t,\ t \geq 0)$ is generated by B, augmented with the $\mathbf{P}$-null sets. The matrix $a :$ $\mathbb{R}^d \times [0,+\infty) \times \Omega \to \mathbb{R}^{d \times d}$ depends[3] also on ω and we assume that conditions **(H1)** and **(H2)** are fulfilled uniformly w.r.t. ω. In particular the ellipticity constant λ and the bound K_a do not depend on ω. Since **(H1)** and **(H2)** hold, by Theorem 1 the fundamental solution Γ of (1) and its spatial derivatives satisfy estimates (4) and (5).

In order to define properly the stochastic integral in (3), we will use the approach developed in [13] for anticipating integrals and thus Malliavin's derivatives. In what follows we borrow some notations from Nualart [12]. Recall that B is a d-dimensional Brownian motion. Let f be an element of $C_{\mathrm{p}}^\infty(\mathbb{R}^{dn})$ (the set of all infinitely many times continuously differentiable functions such that these functions and all their partial derivatives have at most polynomial growth at infinity) with

$$
f(x) = f(x_1^1, \ldots, x_1^d; \ldots; x_n^1, \ldots, x_n^d).
$$

We define a smooth random variable F by:

$$
F = f(B(t_1), \ldots, B(t_n))
$$

for $0 \leq t_1 < t_2 < \ldots < t_n \leq T$. The class of smooth random variables is denoted by $\mathcal{S}$. Then the Malliavin derivative $D_t F$ is given by

$$
D_t^j(F) = \sum_{i=1}^d \frac{\partial f}{\partial x_i^j}(B(t_1), \ldots, B(t_n))\mathbf{1}_{[0,t_i]}(t)
$$

---

[3]Note that here and in the sequel we follow the usual convention and omit the function argument ω.

(see Definition 1.2.1 in [12]). $D_t(F)$ is the d-dimensional vector $D_t(F) = (D_t^j(F),\ j = 1, \ldots, d)$. Moreover, this derivative $D_t(F)$ is a random variable with values in the Hilbert space $L^2([0,T];\mathbb{R}^d)$. The space $\mathbb{D}^{1,p}$, $p \geq 1$, is the closure of the class of smooth random variables with respect to the norm

$$\|F\|_{1,p} = \left[\mathbf{E}(|F|^p) + \mathbf{E}\left(\|DF\|^p_{L^2([0,T];\mathbb{R}^d)} \right) \right]^{1/p}.$$

For $p = 2$, $\mathbb{D}^{1,2}$ is a Hilbert space. Then by induction we can define $\mathbb{D}^{k,p}$ the space of k-times differentiable random variables where the k derivatives are in $L^p(\Omega)$. Finally

$$\mathbb{D}^{k,\infty} = \bigcap_{p \geq 1} \mathbb{D}^{k,p}, \quad \mathbb{D}^{\infty} = \bigcap_{k \in \mathbb{N}} \mathbb{D}^{k,\infty}.$$

For the Malliavin differentiability property of Γ, we use the approach developed in Alòs et al. [1]. We assume that, in addition to **(H1)** and **(H2)**, the matrix a possesses the following properties:

(H3) For each $(x,t) \in \mathbb{R}^d \times [0,+\infty)$, $a(x,t)$ is a $\mathcal{F}_t$-measurable random variable.

(H4) For each $(x,t) \in \mathbb{R}^d \times [0,+\infty)$ the random variable $a(x,t)$ belongs to $\mathbb{D}^{1,2}$.

(H5) There exists a non negative process ψ such that for any $t \in [0,T]$ and any $x \in \mathbb{R}^d$,

$$|D_r a(x,t)| + |D_r \nabla a(x,t)| \leq \psi(r).$$

Moreover, ψ satisfies the integrability condition: for some $p > 1$

$$\mathbf{E}\left(\int_0^T \psi(r)^{2p} dr \right) < +\infty.$$

Note that if **(H5)** holds, then for all $(x, x', t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$|D_r a(x,t) - D_r a(x',t)| \leq \psi(r)|x - x'|.$$

Indeed

$$a_{ij}(x,t) - a_{ij}(x',t) = \int_0^1 \nabla a_{ij}(x' + \theta(x - x'), t) d\theta (x - x').$$

We differentiate both sides in the Malliavin sense and we use the estimate on $D_r \nabla a$. Our second main result is

Theorem 2 *Under conditions* **(H1)**–**(H5)**, *the fundamental solution* Γ *of* (1) *and its spatial derivatives belong to* $\mathbb{D}^{1,2}$ *for every* $(t,s) \in [0,T]^2$, $s < t$ *and* $(x,y) \in (\mathbb{R}^n)^2$. *Moreover, there exist two constants* ϱ *and* ϖ *that depend only on the uniform ellipticity constant* λ, *the dimension* d, *on* K_a *and on* T, *such that*

$$|D_r \Gamma(x,t,y,s)| \leq \psi(r) g_{\varrho,\varpi}(x - y, t - s), \tag{14}$$

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and

$$|D_r \nabla_x \Gamma(x,t,y,s)| \leq \frac{\psi(r)}{\sqrt{t-s}} g_{\varrho,\varpi}(x-y,t-s). \tag{15}$$

The quantity ψ is defined by (23). Finally Γ and $D_r\Gamma$ are continuous w.r.t. $(x,y) \in \mathbb{R}^{2d}$ and $0 \leq s < t \leq T$.

Let us emphasize that the constant ϖ depends only on the uniform ellipticity constant λ and the dimension d, whereas the constant ϱ also depends on K_a and T.

3.1 Proof of Theorem 2

Let us remark that the construction of Γ in Section 2 applies pathwise, ω by ω. We want to prove now that in the framework of this section Γ is also Malliavin differentiable. As a straightforward consequence of **(H3)** one obtains that for any $s < t$, the random variables $Z(x-y,t,s)$, $\Phi(x,t,y,s)$ and Γ are $\mathcal{F}_t$-measurable.

Let us first assume that a does not depend on x and consider the Malliavin derivative of Z. From the representation (6), this derivative can be computed explicitly: for $j = 1, \ldots, d$

$$D_r^j Z(x-y,t,s) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} D_r^j V(t,s,\zeta) d\zeta$$

$$= -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V(t,s,\zeta) \left\langle \int_s^t D_r^j a(u) du \, \zeta, \zeta \right\rangle d\zeta.$$

Thus

$$D_r^j Z(x-y,t,s) = \text{Trace}\left[\left(\int_s^t D_r^j a(u) du \right) \partial_x^2 Z(x-y,t,s) \right].$$

Therefore,

$$|D_r^j Z(x-y,t,s)| \leq \left| \int_s^t D_r^j a(u) du \right| \frac{1}{t-s} \, g_{\varsigma,\varpi}(x-y,t-s).$$

Since the Malliavin derivative of a is bounded by $\psi(r)$, we obtain:

$$|D_r Z(x-y,t,s)| \leq \psi(r) g_{\varsigma,\varpi}(x-y,t-s).$$

This yields (14). Similar computations give:

$$D_r \partial_{x_j} Z(x-y,t,s) = -i \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\zeta(x-y)} V(t,s,\zeta) \left\langle \int_s^t D_r^j a(u) du \, \zeta, \zeta \right\rangle \zeta_j d\zeta.$$

Using the estimate on the third derivative of Z w.r.t. x, we obtain (15):

$$|D_r \partial_{x_i} Z(x-y,t,s)| \leq \psi(r) \frac{1}{(t-s)^{1/2}} \, g_{\varsigma,\varpi}(x-y,t-s). \tag{16}$$

In other words if a does not depend on x, estimates (4), (5), (14) and (15) hold for Z. In the case $a(t) = a(\xi_t)$, the constants appearing in inequalities (14) and (15) depend on the Lipschitz constant of the matrix $a(y)$. Similar computations also show that

$$|D_r \partial_{x_i x_j}^2 Z(x-y,t,s)| \leq \psi(r) \frac{1}{(t-s)} \, g_{\varsigma,\varpi}(x-y,t-s).$$

We turn to the case of a that depends on both x and t.

Lemma 3 (Malliavin differentiability of Φ) *The function Φ belongs to $\mathbb{D}^{1,2}$ for every $(t,s) \in [0,T]^2$, $s < t$ and $(x,y) \in (\mathbb{R}^d)^2$. Moreover, there exists two constants $\varrho > 0$ and $\varpi > 0$ such that*

$$|D_r\Phi(x,t,y,s)| \leq \psi(r)\frac{1}{\sqrt{t-s}}\ g_{\varrho,\varpi}(x-y,t-s). \tag{17}$$

Proof. Recall that

$$\gamma_i(x,t) = \text{div}(\mathrm{a}_i(x,t)) = \sum_{j=1}^{d} \frac{\partial \mathrm{a}_{ji}}{\partial x_j}(x,t).$$

Note that due to Condition **(H5)** the process γ belongs also to $\mathbb{D}^{1,2}$. According to (11) the Malliavin derivative of $\mathcal{K}$ is given by:

$$
\begin{aligned}
D_r\mathcal{K}(x,t,y,s) &= \sum_{i,j=1}^{n} [D_r\mathrm{a}_{ij}(x,t) - D_r\mathrm{a}_{ij}(y,t)]\frac{\partial^2 Z}{\partial x_i \partial x_j}(x-y,t,s,y) \\
&+ \sum_{i=1}^{n} D_r\gamma_i(x,t)\frac{\partial Z^{\varepsilon}}{\partial x_i}(x-y,t,s,y) \\
&+ \sum_{i,j=1}^{n} (\mathrm{a}_{ij}(x,t) - \mathrm{a}_{ij}(y,t))D_r\frac{\partial^2 Z}{\partial x_i \partial x_j}(x-y,t,s,y) \\
&+ \sum_{i=1}^{n} \gamma_i(x,t)D_r\frac{\partial Z}{\partial x_i}(x-y,t,s,y).
\end{aligned}
$$

From our previous assumptions and properties we deduce that

$$
\begin{aligned}
|D_r\mathcal{K}(x,t,y,s)| &\leq \psi(r)\frac{|x-y|}{(t-s)}\ g_{\varrho,\varpi}(x-y,t-s) \\
&+ \psi(r)\frac{1}{\sqrt{t-s}}\ g_{\varrho,\varpi}(x-y,t-s) \\
&+ \frac{|x-y|}{(t-s)}\ \psi(r)\ g_{\varrho,\varpi}(x-y,t-s) \\
&+ \psi(r)\frac{1}{\sqrt{t-s}}\ g_{\varrho,\varpi}(x-y,t-s) \\
&\leq \psi(r)\frac{1}{\sqrt{t-s}}\ g_{\varrho,\varpi}(x-y,t-s).
\end{aligned}
$$

By induction, using the same techniques as in the proof of Lemma 2), we obtain for $m \geq 2$

$$|D_r\mathcal{K}_m(x,t,y,s)| \leq \frac{M^m}{(1+m/2)!}\psi(r)(t-s)^{m/2-1}g_{\varrho,\varpi}(x-y,t-s)$$

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with some constant $M > 0$ depending on ϱ and ϖ. Indeed, for $m = 2$

$$|D_r\mathcal{K}_2(x,t,y,s)| \leq \int_s^t \int_{\mathbb{R}^d} |D_r\mathcal{K}(x,t,\zeta,\tau)|\, |\mathcal{K}(\zeta,\tau,y,s)|\, d\zeta d\tau$$

$$+ \int_s^t \int_{\mathbb{R}^d} |\mathcal{K}(x,t,\zeta,\tau)|\, |D_r\mathcal{K}(\zeta,\tau,y,s)|\, d\zeta d\tau$$

$$\leq 2\psi(r) \int_s^t \int_{\mathbb{R}^d} \frac{1}{\sqrt{(t-\tau)(\tau-s)}}\, g_{\varrho,\varpi}(x-\zeta,t-\tau)g_{\varrho,\varpi}(\zeta-y,\tau-s)d\zeta d\tau$$

and the required estimate on the integral can be deduced by the classical arguments. By the closability of the operator D we conclude that

$$D_r\Phi(x,t,y,s) = \sum_{m=1}^{\infty} D_r\mathcal{K}_m(x,t,y,s) \tag{18}$$

and that estimate (17) holds. Since $\psi(r)$ belongs to $L^2(\Omega)$, $\Phi \in \mathbb{D}^{1,2}$. This completes the proof of the Lemma. $\qquad\square$

We turn to the proof of Theorem 2. Let us show that $\Gamma \in \mathbb{D}^{1,2}$ and that the Gaussian estimates hold for the Malliavin derivative. From the definition of Γ in (9), the two previous lemmata and the properties of the Malliavin derivative D we obtain that

$$
\begin{aligned}
D_\tau\Gamma(x,t,y,s) &= D_\tau Z(x-y,t,s,y) \\
&+ \int_s^t \int_{\mathbb{R}^d} D_\tau Z(x-\zeta,t,\tau,\zeta)\Phi(\zeta,\tau,y,s)d\zeta d\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} Z(x-\zeta,t,\tau,\zeta)D_\tau\Phi(\zeta,\tau,y,s)d\zeta d\tau.
\end{aligned}
\tag{19}
$$

Inequalities (16) and (17) imply that

$$|D_r\Gamma(x,t,y,s)| \leq \psi(r)g_{\varrho,\varpi}(x-y,t-s);$$

for the details see Lemma I.4.3 in [7]. From equation (13) one can obtain an expression for the Malliavin derivative of $\dfrac{\partial}{\partial x_i}\Gamma(x,t,y,s)$:

$$
\begin{aligned}
D_r\frac{\partial}{\partial x_i}\Gamma(x,t,y,s) &= D_r\frac{\partial Z}{\partial x_i}(x-y,t,s,y) \\
&+ \int_s^t \int_{\mathbb{R}^d} D_r\frac{\partial Z}{\partial x_i}(x-\zeta,t,\tau,\zeta)\Phi(\zeta,\tau,y,s)d\zeta d\tau \\
&+ \int_s^t \int_{\mathbb{R}^d} \frac{\partial Z}{\partial x_i}(x-\zeta,t,\tau,\zeta)D_r\Phi(\zeta,r,y,s)d\zeta d\tau.
\end{aligned}
$$

Again with the help of Lemma I.4.3 in [7], estimates (16) and (17) imply (15). This achieves the proof.

3.2 Diffusion example

Here we consider the special case $\mathrm{a}(x,t) = a(x,\xi_t)$, with a matrix-valued function a defined on $\mathbb{R}^d \times \mathbb{R}^d$ such that

a1. a is uniformly elliptic: for any $(x, y, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$

$$\lambda^{-1}|\zeta|^2 \le a(x,y)\zeta \cdot \zeta \le \lambda|\zeta|^2.$$

a2. a is continuous on $\mathbb{R}^d \times \mathbb{R}^d$ and of class C^1 w.r.t. x with a bounded derivative: for any (x, y)

$$|\nabla_x a(x,y)| \le K_a.$$

The process ξ is given as the solution of the following SDE:

$$d\xi_t = \beta(t, \xi_t)dt + \sigma(t, \xi_t)dB_t, \tag{20}$$

or, in the coordinate form, $d\xi_t^i = \beta_i(t, \xi_t)dt + \sum_{j=1}^d \sigma_{i,j}(t, \xi_t)dB_t^j$. We assume that the matrix-function σ and vector-function β possess the following properties.

c1. σ and b are globally Lipschitz continuous: there exists $K_{\beta,\sigma} > 0$ such that

$$\|\sigma(t, y') - \sigma(t, y'')\| + |\beta(t, y') - \beta(t, y'')| \le K_{\beta,\sigma}|y' - y''|.$$

c2. $t \mapsto \sigma(t, 0)$ and $t \mapsto \beta(t, 0)$ are bounded on $\mathbb{R}_+$.

c3. σ and β are at least two times differentiable w.r.t. x with uniformly bounded derivatives. The absolute value of these derivatives does not exceed a constant that is also denoted by $K_{\beta,\sigma}$.

It is well known that under the assumptions **c1** and **c2**, ξ is the unique strong solution of the SDE (20) and for any $T \ge 0$ and any $p \ge 2$

$$\mathbb{E}\left(\sup_{t \in [0,T]} |\xi_t|^p\right) \le C,$$

where C is a positive constant depending on p, T, $K_{\beta,\sigma}$ and ξ_0. The next result can be found in [12], Theorems 2.2.1 and 2.2.2.

Lemma 4 *Under conditions **c1**– **c3**, the coordinate ξ_t^i belongs to $\mathbb{D}^{1,\infty}$ for any $t \in [0,T]$ and $i = 1, \ldots, d$. Moreover for any $j = 1, \ldots, d$ and any $p \ge 1$*

$$\sup_{0 \le r \le T} \mathbf{E}\left(\sup_{r \le t \le T} |D_r^j \xi_t^i|^p\right) < +\infty. \tag{21}$$

The derivative $D_r^j \xi_t^i$ satisfies the following linear equation:

$$D_r^j \xi_t^i = \sigma_{i,j}(\xi_r) + \sum_{1 \le k,l \le d} \int_r^t \widetilde{\sigma}_{i,k}^l(s) D_r^j(\xi_s^k)dB_s^l + \sum_{k=1}^d \int_r^t \widetilde{b}_{i,k}(s) D_r^j(\xi_s^k)ds$$

for $r \leq t$ a.e. and $D_r^j \xi_t = 0$ for $r > t$ a.e., where σ^j is the column number j of the matrix σ and where for $1 \leq i, j \leq d$ and $1 \leq l \leq d$, $\widetilde{b}_{i,j}(s)$ and $\widetilde{\sigma}_{i,j}^l(s)$ are given by:

$$\widetilde{b}_{i,j}(s) = (\partial_{x_j} b_i)(\xi_s), \qquad \widetilde{\sigma}_{i,j}^l(s) = (\partial_{x_j} \sigma_{i,l})(\xi_s). \tag{22}$$

The process ξ belongs to $\mathbb{D}^{2,\infty}$ and the second derivatives $D_r^i D_s^j \xi_t^k$ satisfy also a linear stochastic differential equation with bounded coefficients.

For any $r \in [0, T]$ we define

$$\psi(r) = \sup_{t \in [r,T]} \|D_r \xi_t\|. \tag{23}$$

From Lemma 4 we have for any $p \geq 2$

$$\sup_{r \in [0,T]} \mathbf{E}\left(\psi(r)^p\right) < +\infty. \tag{24}$$

We define for any $(x, t) \in \mathbb{R}^d \times [0, +\infty)$

$$\mathrm{a}(x, t) = a(x, \xi_t).$$

Assumptions **a1** and **a2** imply that Conditions **(H1)**, **(H2)** and **(H3)** hold. Moreover let us assume that the matrix a is smooth w.r.t. y and satisfies the following regularity conditions.

a3. For any $1 \leq j, k \leq d$

$$|\nabla_y a(x, y)| + \left|\frac{\partial^2}{\partial x_j \partial y_k} a(x, y)\right| \leq K_a.$$

Using conditions **a2** and **a3**, the previous lemma and the classical chain rule (see Proposition 1.2.3 in [12]), we obtain that

$$D_r^j \mathrm{a}_{i,\ell}(x, t) = \sum_k \frac{\partial a_{i,\ell}}{\partial y_k}(x, \xi_t) D_r^j \xi_t^k.$$

Thus $D_r \mathrm{a}(x, t) = 0$ if $r > t$, while for $r \leq t$ we have

$$|D_r^k \mathrm{a}_{ij}(x, t)| \leq \left|\frac{\partial a_{ij}}{\partial y_\ell}\right| |D_r(\xi_t)| \leq K_a \psi(r). \tag{25}$$

The same computation shows that

$$D_r^k \frac{\partial \mathrm{a}_{ij}}{\partial x_\ell}(x, t) = \sum_k \frac{\partial^2 a_{i,\ell}}{\partial x_\ell \partial y_k}(x, \xi_t) D_r^j \xi_t^k.$$

Hence

$$\left|D_r^k \frac{\partial \mathrm{a}_{ij}}{\partial x_\ell}(x, t)\right| \leq K_a \psi(r).$$

We deduce that $\mathrm{a}(x, t)$ belongs to $\mathbb{D}^{1,\infty}$ (condition **(H4)**), the previous computations yield **(H5)**, and ψ satisfies the integrability condition (24). From Theorems 1 and 2 we deduce immediately the following result.

Corollary 1 *Under assumptions* **a1** *–* **a3** *on the matrix a and conditions* **c1** *–* **c3** *on the coefficients of the SDE* (20)*, if* $a(x,t) = a(x, \xi_t)$*, then the fundamental solution* Γ *of equation* (1) *and its spatial derivatives belong to* $\mathbb{D}^{1,2}$ *and satisfy Estimates* (4)*,* (5)*,* (14) *and* (15)*.*

4   Mild solution of the heat SPDE

In this last section we construct a mild solution v to the heat SPDE (2) with the initial condition $v(x, 0) = 0$, that is we construct a solution v of equation (3).

Remark 1 *If the initial condition for* v *is given by a function* $\imath$*, then by linearity of the SPDE, we should add in* (3) *one term:*

$$v(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s) G(y,s) \, dy dB_s + \int_{\mathbb{R}^d} \Gamma(x,t,y,0) \imath(y) dy$$

Under the setting of Theorem 1, this additional term is well defined provided that the function $\imath$ *increases no faster than a function* $\exp(cx^2)$ *(see [7, Theorem I.7.12]).*

Let us specify our setting. We still assume that all hypotheses **(H1)** to **(H5)** hold and we add several conditions on G.

(D1) The function $G : \mathbb{R}^d \times [0, +\infty) \times \Omega \to \mathbb{R}^d$ is a progressively measurable function that satisfies the estimate $(1 + |x|)^N |G(x,t)| \leq \mathfrak{G}(t)$ for some $N > d/2$ and some adapted process $\mathfrak{G}$ such that

$$\mathbf{E}\left(\int_0^T \mathfrak{G}(t)^{2q} dt\right) < +\infty$$

with some $q > 1$.

(D2) For each $(x,t) \in \mathbb{R}^d \times [0, +\infty)$, the random variable $G(x,t)$ belongs to $\mathbb{D}^{1,2}$, and for any $t \in [0,T]$ and any $x \in \mathbb{R}^d$,

$$|D_r G(x,t)| \leq \widetilde{G}(x,t)\psi(r).$$

The process ψ is the same as in Condition **(H5)** and $\widetilde{G}$ verifies the growth assumption: $(1 + |x|)^N |\widetilde{G}(x,t)| \leq \mathfrak{G}(t)$.

(D3) The constants p of **(H5)** and q verify: $p > q > 2d + 4$.

(D4) The process $\mathfrak{G}$ verifies $\mathbf{P}\left(\sup_{t \in [0,T]} \mathfrak{G}(t) < +\infty\right) = 1$.

Remark 2 *Under* **(D3)**, *we have the weaker condition* $\frac{1}{p} + \frac{1}{q} \leq 1$. *From the proofs, we are aware that this condition* **(D3)** *is a little bit too strong. But a relation between* p, q *and* d *is needed with our arguments. In [1], this relation is implicitly given: for example*

in Theorem 3.5, the authors impose $p > 8$ (for $d = 1$). **(D1)** *and* **(D4)** *is a little bit more general than in [1] where G is bounded with respect to (x, t).*

Moreover the following relations hold:

$$2 \leq \kappa \leq \frac{2pq}{p+q} \Rightarrow \frac{q}{q-1} \leq \frac{\kappa q}{2q - \kappa} \leq p, \quad \frac{1}{p} + \frac{1}{q} \leq 1 \Leftrightarrow \frac{2pq}{p+q} \geq 2,$$

and

$$\frac{1}{2} + \frac{p+q}{2pq} \leq \frac{q-1}{q} \leq 1 - \frac{p+q}{2pq} \leq \frac{2p-1}{2p}.$$

Let us give our third main result.

Theorem 3 *Let assumptions* **(H1)** *–* **(H5)** *be fulfilled, and assume that conditions* **(D1)** *–* **(D4)** *hold. Then on $\mathbb{R}^d \times (0, +\infty)$, the random field v given by* (3) *is well defined, is continuous w.r.t. (x, t) and has first derivatives w.r.t. x such that*

$$\mathbf{E}\left[\sup_{x,t}\left(|v(x,t)|^{\frac{2pq}{p+q}} + |\nabla v(x,t)|^{\frac{2pq}{p+q}}\right)\right] < +\infty.$$

Moreover v is a weak solution of the SPDE in (2).

The notion of a weak solution is explained in Definition 1.

4.1 The diffusion case

Again we assume that $\mathrm{a}(x, t) = a(x, \xi_t)$ where ξ is the solution of the SDE in (20). Let us fix a measurable function $g : \mathbb{R}^d \times [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ such that g is of class C^1 w.r.t. the last component and

$$G(x, t) = g(x, t, \xi_t)$$

Then the Malliavin derivative of G can be computed by a chain rule argument: $D_r G(x, t) = \nabla_y g(x, t, \xi_t) D_r \xi_t$. Hence

$$|D_r G(x, t)| \leq |\nabla_y g(x, t, \xi_t)| \psi(r).$$

Let us assume that for some $N > d/2$:

$$|g(x, t, y)| + |\nabla_y g(x, t, y)| \leq C \frac{|y|}{(1 + |x|)^N}.$$

Then $\mathfrak{G}(t) = |\xi_t|$ is continuous w.r.t. t, thus **(D4)** holds. And, for any $q > 1$,

$$\mathbb{E}\left(\sup_{t \in [0,T]} |\xi_t|^{2q}\right) \leq C.$$

Therefore, **(D1)** and **(D3)** are also satisfied. From Theorem 3 we get

Corollary 2 *Under conditions* **a1** *–* **a3** *on the matrix a and* **c1** *–* **c3** *on the coefficients of the SDE, if the previous assumptions are satisfied, then the conclusion of Theorem 3 holds in the diffusion case.*

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4.2 Construction of the mild solution

The rest of the paper is devoted to the proof of Theorem 3. Let us first specify the meaning of a weak solution of equation (2).

Definition 1 *Let $v = \{v(x,t),\ (x,t) \in \mathbb{R}^d \times [0,+\infty)\}$ be a random field. We say that v is a weak solution of equation (2) if*

- *v is continuous on $\mathbb{R}^d \times (0,+\infty)$. Moreover, a.s. for any $x \in \mathbb{R}^d$,*

$$\lim_{t \downarrow 0} v(x,t) = 0 \ ;$$

- *v has all first order partial derivatives in x on $\mathbb{R}^d \times (0,+\infty)$;*

- *for any test function $\phi \in C_0^\infty(\mathbb{R}^d)$ and for all $t \in [0,T]$ we have*

$$\int_{\mathbb{R}^d} v(x,t)\phi(x)dx + \int_0^t\!\!\int_{\mathbb{R}^d} a(x,s)\nabla\phi(x)\nabla v(x,s)dx = \int_0^t\!\!\int_{\mathbb{R}^d} G(x,s)\phi(x)dxdB_s.$$

Our aim is to prove that the random function v given by (3) is a weak solution of the SPDE (2). The stochastic integral in (3) has to be defined properly since $\Gamma(x,t,y,s)$ is measurable w.r.t. the σ-field $\mathcal{F}_t$ generated by the random variables B_u with $u \leq t$. The correct definition can be found in [13] and is based on Malliavin's calculus. To define a mild solution of (2), let us recall [13, Definition 3.1].

Definition 2 *Let $\mathbb{L}^{1,2}$ denote class of scalar processes $u \in \mathbb{L}^2([0,T] \times \Omega)$ such that $u_t \in \mathbb{D}^{1,2}$ for a.a. t and there exists a measurable version of $D_r u_t$ verifying*

$$\mathbf{E}\int_0^T\int_0^T |D_r u_t|^2 drdt < +\infty.$$

$\mathbb{L}_d^{1,2}$ is the set of d-dimensional processes whose components are in $\mathbb{L}^{1,2}$.

Proposition 1 *For any $(t,x) \in [0,T] \times \mathbb{R}^d$, the stochastic integral*

$$v(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s)G(y,s)\, dydB_s$$

is well defined and

$$\mathbf{E}\left[\int_0^T \int_{\mathbb{R}^d} (v(x,t))^2 dxdt\right] < +\infty.$$

Proof. From Theorem 2 and condition **(D1)** on G, we deduce that for each $(x,t) \in \mathbb{R}^d \times [0,T]$, the process

$$u(x,t,s) = \int_{\mathbb{R}^d} \Gamma(x,t,y,s)G(y,s)dy \tag{26}$$

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is well defined. Aronson's estimate (4), Hölder's inequality and condition **(D1)** lead to:

$$|u(x,t,s)|^2 \le C \int_{\mathbb{R}^d} g_{\varsigma,\varpi}(x-y,t-s)|G(y,s)|^2 dy \le \frac{C}{(1+|x|)^{2N}} \mathfrak{G}(s)^2. \qquad (27)$$

Therefore,

$$\mathbf{E} \int_0^t |u(x,t,s)|^2 ds \le \frac{C^2}{(1+|x|)^{2N}} \mathbf{E} \int_0^t \mathfrak{G}(s)^2 ds < +\infty. \qquad (28)$$

Moreover,

$$D_r u(x,t,s) = \int_{\mathbb{R}^d} [D_r \Gamma(x,t,y,s)G(y,s) + \Gamma(x,t,y,s)D_r G(y,s)] \, dy.$$

Therefore, from estimate (14) on $D_r\Gamma$, Hölder's inequality and conditions **(D1)** and **(D2)** for G and $\widetilde{G}$, we obtain

$$
\begin{aligned}
|D_r u(x,t,s)|^2 &\le \left(\int_{\mathbb{R}^d} |D_r\Gamma(x,t,y,s)G(y,s) + \Gamma(x,t,y,s)D_r G(y,s)| \, dy \right)^2 \\
&\le \psi(r)^2 C \int_{\mathbb{R}^d} g_{\varrho,\varpi}(x-y,t-s) \left[|G(y,s)|^2 + |\widetilde{G}(y,s)|^2 \right] dy \\
&\le \frac{C}{(1+|x|)^{2N}} \psi(r)^2 \mathfrak{G}(s)^2. \qquad (29)
\end{aligned}
$$

Applying again the Hölder inequality yields

$$\mathbf{E} \int_0^t \int_0^t |D_r u(x,t,s)|^2 ds dr$$

$$\le \frac{C^2}{(1+|x|)^{2N}} \mathbf{E}\left[\left(\int_0^t \psi(r)^2 dr \right)^{\frac{q}{q-1}} \right]^{\frac{q-1}{q}} \mathbf{E}\left[\left(\int_0^t \mathfrak{G}(s)^2 ds \right)^q \right]^{\frac{1}{q}}.$$

Since $p \ge q/(q-1)$, using Jensen's inequality, we obtain

$$\mathbf{E} \int_0^t \int_0^t |D_r u(x,t,s)|^2 ds dr < +\infty. \qquad (30)$$

Conditions (28) and (30) are exactly the ones required in Definition 2. Hence $u(x,t,s)$ belongs to the space $\mathbb{L}_d^{1,2}$ and the stochastic integral $v(x,t)$ is well-defined for any (x,t). Moreover, the isometric property of the anticipating Itô integral holds (see Eq. (3.5) in [13]):

$$\mathbf{E}((v(x,t))^2) = \mathbf{E} \int_0^t |u(x,t,s)|^2 ds + \mathbf{E} \int_0^t \int_0^t |D_r u(x,t,s)|^2 ds dr.$$

From our previous estimates (28) and (30), we obtain that

$$\mathbf{E} \int_0^T \int_{\mathbb{R}^d} (v(x,t))^2 dx dt$$

$$\le C\mathbf{E}\left[\left(\int_0^T \psi(r)^2 dr \right)^{\frac{q}{q-1}} \right]^{\frac{q-1}{q}} \mathbf{E}\left[\left(\int_0^T \mathfrak{G}(s)^2 ds \right)^q \right]^{\frac{1}{q}}.$$

$\square$

We are going to prove that $(x,t) \mapsto v(x,t)$ is continuous and $x \mapsto v(x,t)$ is differentiable. Note that we cannot directly use [13, Theorem 5.2] since Γ also depends on t. Even if Γ is continuous on $\{0 \le s < t \le T\}$, the singularity at time t should be handled carefully. We follow some ideas contained in [1, Section 3] and the regularity results concerning the volume potential (see Lemmata A.1 and A.2 in the Appendix). The main trick is to transform the anticipating stochastic integral v into a Lebesgue integral.

4.2.1 Another representation of v

Given $\alpha \in (0,1)$ define for any $(t,x) \in [0,T] \times \mathbb{R}^d$:

$$X(x,t) = \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} D_s\Gamma(x,t,y,s) G(y,s) dy ds, \tag{31}$$

$$Y(x,t) = \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} \Gamma(x,t,y,s) G(y,s) dy dB_s. \tag{32}$$

Due to the Aronson estimate (14) on $D_s\Gamma$ and hypothesis **(D1)** on G the field X is well defined for any $\alpha \in [0,1)$.

Lemma 5 *Assume that $0 \le \alpha < \frac{2p-1}{2p}$. Then a.s. $(x,t) \mapsto X(x,t)$ is continuous. Moreover, for any $\alpha < 1 - \frac{p+q}{2pq}$ and any $1 < \delta \le \frac{2pq}{p+q}$*

$$\mathbf{E}\left(\sup_{x,t} |X(x,t)|^\delta\right) + \mathbf{E}\int_0^T \int_{\mathbb{R}^d} |X(x,t)|^\delta dx dt$$

$$\le C\mathbf{E}\left[\left(\int_0^T \psi(s)^{2p} ds\right)^{\frac{q}{p+q}} \left[\mathbf{E}\left(\int_0^T \mathfrak{G}(s)^{2q} ds\right)\right]^{\frac{p}{p+q}}.$$

Assume furthermore that $0 \le \alpha < \frac{p-1}{2p}$. Then a.s. $x \mapsto X(x,t)$ is differentiable:

$$\nabla X(x,t) = \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} D_s\nabla\Gamma(x,t,y,s) G(y,s) dy ds$$

and if $0 \le 2\alpha < 1 - \frac{p+q}{pq}$, then $\mathbf{E}\left[\sup_{x,t} |\nabla X(x,t)|^{\frac{2pq}{p+q}}\right] < +\infty$.

Proof. We already know that $D_s\Gamma(x,t,y,s)$ is continuous w.r.t. (x,y) and $s < t$. Thanks to (14), we have a.s.

$$\int_{\mathbb{R}^d} (t-s)^{-\alpha} |D_s\Gamma(x,t,y,s)| dy$$

$$\le \int_{\mathbb{R}^d} (t-s)^{-\alpha} \psi(s) g_{\varrho,\varpi}(x-y, t-s) dy \le C\psi(s)(t-s)^{-\alpha}.$$

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From our assumption on α and ψ we have

$$\int_0^t \psi(s)(t-s)^{-\alpha}ds \leq \left(\int_0^t \psi(s)^{2p}ds\right)^{\frac{1}{2p}}\left(\int_0^t (t-s)^{-\frac{2p\alpha}{2p-1}}ds\right)^{\frac{2p-1}{2p}} < +\infty.$$

Moreover, a.s.

$$\sup_{y,s}|G(y,s)| \leq \sup_{y,s}\frac{\mathfrak{G}(s)}{(1+|y|)^N} < +\infty.$$

Arguing as in the proof of Lemma A.1, we get the a.s. continuity of X w.r.t. (x,t). From estimate (14) on $D_s\Gamma$ we deduce

$$\int_{\mathbb{R}^d}|D_s\Gamma(x,t,y,s)G(y,s)|\,dy \leq C\psi(s)\mathfrak{G}(s)\frac{1}{(1+|z|)^N}.$$

Let us choose $r > 1$ such that $1/r + 1/(2p) + 1/(2q) = 1$ and $\alpha r < 1$. Then

$$|X(x,t)| \leq \frac{C}{(1+|x|)^N}\int_0^t (t-s)^{-\alpha}\psi(s)\mathfrak{G}(s)ds$$

$$\leq \frac{C}{(1+|x|)^N}\left(\int_0^t (t-s)^{-r\alpha}ds\right)^{\frac{1}{r}}\left(\int_0^t \mathfrak{G}(s)^{2q}ds\right)^{\frac{1}{2q}}\left(\int_0^t \psi(s)^{2p}ds\right)^{\frac{1}{2p}}$$

$$\leq \frac{C}{(1+|x|)^N}\left(\int_0^T \mathfrak{G}(s)^{2q}ds\right)^{\frac{1}{2q}}\left(\int_0^T \psi(s)^{2p}ds\right)^{\frac{1}{2p}}.$$

Finally, the Hölder and Jensen inequalities lead to the desired result.

To obtain the differentiability observe that estimate (15) leads to:

$$(t-s)^{-\alpha}|D_s\nabla_x\Gamma(x,t,y,s)| \leq \psi(s)(t-s)^{-\alpha-1/2}g_{\varrho,\varpi}(x-y,t-s).$$

It then remains to apply the same arguments as above with $\alpha + 1/2$ instead of α. $\qquad\square$

In the next lemma we prove that Y is well defined and integrable.

Lemma 6 *For any $(t,s,x) \in [0,T]^2 \times \mathbb{R}^d$ and any $0 \leq \alpha < \frac{q-1}{q}$, the process*

$$u_\alpha(x,t,s) = (t-s)^{-\alpha}\int_{\mathbb{R}^d}\Gamma(x,t,y,s)G(y,s)dy\mathbf{1}_{[0,t)}(s)$$

belongs to $\mathbb{L}_d^{1,2}$. Moreover, for any $2 \leq \kappa \leq \frac{2pq}{p+q}$ it holds

$$\mathbf{E}\left[|Y(x,t)|^\kappa\right] \leq \frac{C}{(1+|x|)^{\kappa N}}.$$

Proof. As was shown in the proof of Proposition 1, we have the upper bound (27) on u and (29) on $D_r u$. Thus by the Hölder inequality

$$\mathbf{E}\int_0^t (t-s)^{-2\alpha}\left|\int_{\mathbb{R}^d}\Gamma(x,t,y,s)G(y,s)dy\right|^2 ds$$

$$\leq \frac{C}{(1+|x|)^{2N}}\left(\int_0^t (t-s)^{-\frac{\alpha q}{q-1}}ds\right)^{\frac{q-1}{q}}\left(\mathbf{E}\int_0^t \mathfrak{G}(s)^{2q}ds\right)^{\frac{1}{q}} < +\infty;$$

here we have also used the inequality $\frac{\alpha q}{q-1} < 1$. Similarly,

$$\mathbf{E} \int_0^t \int_0^t (t-s)^{-2\alpha} |D_r u(x,t,s)|^2 \, dr ds$$

$$\leq \frac{C}{(1+|x|)^{2N}} \left[\mathbf{E} \left(\int_0^t \mathfrak{G}(s)^{2q} ds \right) \right]^{\frac{1}{q}} \left[\mathbf{E} \left(\int_0^t \psi(r)^2 dr \right)^{\frac{q}{q-1}} \right]^{\frac{q-1}{q}}.$$

Since $p \geq q/(q-1)$, by the Jensen inequality we derive that the process u_α is in $\mathbb{L}_d^{1,2}$. Now, using [13, Proposition 3.5], we have for any $\kappa \geq 2$

$$\mathbf{E} \left(\left| \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} \Gamma(x,t,y,s) G(y,s) dy dB_s \right|^\kappa \right)$$

$$\leq c_\kappa \left(\int_0^t (t-s)^{-2\alpha} |\mathbf{E}(u(x,t,s))|^2 \, ds \right)^{\kappa/2}$$

$$+ c_\kappa \, \mathbf{E} \left[\left(\int_0^t \int_0^t (t-s)^{-2\alpha} |D_r u(x,t,s)|^2 \, dr ds \right)^{\kappa/2} \right].$$

Combining this with the previous inequalities we get

$$\mathbf{E} \left(\left| \int_0^t \int_{\mathbb{R}^d} (t-s)^{-\alpha} \Gamma(x,t,y,s) G(y,s) dy dB_s \right|^\kappa \right)$$

$$\leq \frac{C}{(1+|x|)^{\kappa N}} \left(\mathbf{E} \int_0^t \mathfrak{G}(s)^{2q} ds \right)^{\frac{\kappa}{2q}}$$

$$+ \frac{C}{(1+|x|)^{\kappa N}} \mathbf{E} \left[\left(\int_0^t \mathfrak{G}(s)^{2q} ds \right)^{\frac{\kappa}{2q}} \left(\int_0^t \psi(r)^2 dr \right)^{\frac{\kappa}{2}} \right]$$

$$\leq \frac{C}{(1+|x|)^{\kappa N}} \left[\mathbf{E} \int_0^T \mathfrak{G}(s)^{2q} ds \right]^{\frac{\kappa}{2q}} \left\{ 1 + \left[\mathbf{E} \left(\int_0^T \psi(r)^2 dr \right)^{\frac{\kappa q}{2q-\kappa}} \right]^{\frac{2q-\kappa}{2q}} \right\}.$$

This gives the conclusion of the lemma. $\qquad\square$

In particular if $N > d/2$, the process Y belongs to $L^\kappa([0,T] \times \mathbb{R}^d \times \Omega)$. We use the semigroup property of the fundamental solution to derive the desired representation of v.

Lemma 7 *For any* $0 < \alpha < \frac{q-1}{q}$, $v(x,t)$ *admits the following representation:*

$$v(x,t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \Gamma(x,t,z,r)(Y(z,r) + X(z,r)) dz dr$$

$$- \int_0^t \int_{\mathbb{R}^d} D_s \Gamma(x,t,y,s) G(y,s) dy ds, \tag{33}$$

where X *and* Y *are given by* (31) *and* (32).

Proof. Recall that for any $\alpha \in (0,1)$,

$$\Gamma(x,t,y,s) = \frac{\sin(\pi\alpha)}{\pi} \int_s^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1}(r-s)^{-\alpha}\Gamma(x,t,z,r)\Gamma(z,r,y,s)dzdr.$$

Applying Fubini's theorem for the Skorohod integral we obtain

$$v(x,t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x,t,y,s)G(y,s)dydB_s$$

$$= \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}^d} \left[\int_0^r \int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,z,r) \right.$$

$$\left. (r-s)^{-\alpha}\Gamma(z,r,y,s)G(y,s)dydB_s \right]dzdr.$$

By Lemma 6 with $0 < \alpha < \frac{q-1}{q}$ and $u_\alpha(r,x,s) \in \mathbb{L}_d^{1,2}$, and by [13, Theorem 3.2] we have

$$\int_0^r \int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,z,r)(r-s)^{-\alpha}\Gamma(z,r,y,s)G(y,s)dydB_s$$

$$= (t-r)^{\alpha-1}\Gamma(x,t,z,r)Y(z,r)$$

$$- \int_0^r \int_{\mathbb{R}^d} (t-r)^{\alpha-1}D_s\Gamma(x,t,z,r)(r-s)^{-\alpha}\Gamma(z,r,y,s)G(y,s)dyds.$$

Hence

$$v(x,t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,z,r)Y(z,r)dzdr$$

$$- \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1}\left[\int_0^r \int_{\mathbb{R}^d} D_s\Gamma(x,t,z,r) \right.$$

$$\left. (r-s)^{-\alpha}\Gamma(z,r,y,s)G(y,s)dyds \right]dzdr.$$

Since for $0 \le s < r < t \le T$ we have

$$D_s\Gamma(x,t,y,s) = \int_{\mathbb{R}^d} [D_s\Gamma(x,t,z,r)\Gamma(z,r,y,s) + \Gamma(x,t,z,r)D_s\Gamma(z,r,y,s)]\,dz,$$

then

$$v(x,t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,z,r)(Y(r,z) + X(r,z))dzdr$$

$$- \frac{\sin(\pi\alpha)}{\pi} \int_0^t (t-r)^{\alpha-1}\left[\int_0^r \int_{\mathbb{R}^d} (r-s)^{-\alpha}D_s\Gamma(x,t,y,s)G(y,s)dyds \right]dr$$

By the Fubini theorem we deduce the representation (33). $\qquad\square$

4.2.2 Regularity of the process v

Now we assume that $0 < \alpha < \frac{q-1}{q}$ and study separately the three terms in the decomposition (33) of v. Let us begin with the last one, namely

$$I_3(x,t) = \int_0^t \int_{\mathbb{R}^d} D_s \Gamma(x,t,y,s) G(y,s) \, dy \, ds.$$

Remark that I_3 is equal to X with $\alpha = 0$. By Lemma 5 with $\alpha = 0$, a.s. the mapping $(x,t) \mapsto I_3(x,t)$ is continuous, $x \mapsto I_3(x,t)$ is differentiable, and

$$\mathbf{E}\left[\sup_{x,t} \left(|I_3(x,t)|^{\frac{2pq}{p+q}} + |\nabla I_3(x,t)|^{\frac{2pq}{p+q}}\right)\right] < +\infty.$$

We proceed with the term I_2 given by

$$I_2(x,t) = \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \Gamma(x,t,z,r) X(z,r) \, dz \, dr.$$

Notice that for all $p > q$ we have $1 - \frac{p+q}{2pq} > (q-1)/q$. Therefore, for $\alpha < \frac{q-1}{q}$ by Lemma 5 we obtain

$$\mathbf{E}\left(\sup_{x,t} |X(x,t)|^{\frac{2pq}{p+q}}\right) < +\infty. \tag{34}$$

Thus a.s. X is bounded w.r.t. (x,t). Arguing as in the proof of Lemma 5, we show that for all α such that $1/2 + \frac{p+q}{2pq} < \alpha < \frac{q-1}{q}$ the term I_2 has the same regularity as I_3 with

$$\nabla I_2(x,t) = \int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \nabla\Gamma(x,t,z,r) X(z,r) \, dz \, dr.$$

Up to now the dimension d plays no role in our estimate, and we only used **(D1)**, **(D2)** and the relation $p > q > 4$. To control I_2, we used the fact that $\sup_{x,t} |X(x,t)|$ is a.s. finite. The estimate in the next statement does depend on d. Remark that if $p > q > 2d + 4$, then

$$\frac{1}{2} + \frac{(d+2)(p+q)}{4pq} < \frac{q-1}{q}.$$

Lemma 8 *Assume that $\frac{1}{2} + \frac{(d+2)(p+q)}{4pq} < \alpha < \frac{q-1}{q}$. Then*

$$\mathbf{E}\left[\sup_{x,t} |I_2(x,t)|^{\frac{2pq}{(p+q)}} + \sup_{x,t} |\nabla I_2(x,t)|^{\frac{2pq}{(p+q)}}\right] < +\infty.$$

Proof. We only detail the arguments for the gradient of I_2 ; for I_2 itself they are similar. Note that $\frac{1}{2} + \frac{(d+2)(p+q)}{4pq} < \alpha$ is equivalent to $\left(\alpha - \frac{3}{2} - \frac{d}{2\delta}\right)\frac{\delta}{\delta-1} > -1$ with $\delta = \frac{2pq}{(p+q)}$. Thus by the Hölder inequality:

$$\left|\int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \nabla\Gamma(x,t,z,r) X(z,r) \, dz \, dr\right|$$

$$\leq \int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}^d} |\nabla\Gamma(x,t,z,r)|^{\frac{\delta}{\delta-1}} dz\right)^{\frac{\delta-1}{\delta}} \left(\int_{\mathbb{R}^d} |X(z,r)|^{\delta} dz\right)^{\frac{1}{\delta}} dr.$$

From Estimate (5), we obtain

$$|\nabla_x\Gamma(x,t,y,r)|^{\frac{\delta}{\delta-1}} \leq (t-r)^{-\delta/2(\delta-1)} \, g_{\varrho,\varpi}(x-y,t-r)^{\frac{\delta}{\delta-1}}$$
$$\leq (t-r)^{-\frac{\delta}{2(\delta-1)}-\frac{d}{2}\frac{1}{\delta-1}} \, g_{\varrho',\varpi'}(x-y,t-r)$$

with $(\varrho',\varpi') = (\varrho^{\frac{\delta}{\delta-1}}, \varpi\frac{\delta}{\delta-1})$. This yields

$$(t-r)^{\alpha-1}\left(\int_{\mathbb{R}^d} |\nabla\Gamma(x,t,z,r)|^{\frac{\delta}{\delta-1}}dz\right)^{\frac{\delta-1}{\delta}} \leq C(t-r)^{\alpha-1}(t-r)^{-\frac{1}{2}-\frac{d}{2\delta}}.$$

Using again the Hölder inequality we arrive at the estimate

$$\int_0^t (t-r)^{\alpha-1}\left(\int_{\mathbb{R}^d} |\nabla\Gamma(x,t,z,r)|^{\frac{\delta}{\delta-1}}dz\right)^{\frac{\delta-1}{\delta}}\left(\int_{\mathbb{R}^d} |X(z,r)|^{\delta}dz\right)^{\frac{1}{\delta}} dr$$
$$\leq C\left(\int_0^t (t-r)^{(\alpha-\frac{3}{2}-\frac{d}{2\delta})\frac{\delta}{\delta-1}}dr\right)^{\frac{\delta-1}{\delta}}\left(\int_0^t\int_{\mathbb{R}^d} |X(z,r)|^{\delta}dzdr\right)^{\frac{1}{\delta}}$$
$$\leq C\left(\int_0^T\int_{\mathbb{R}^d} |X(z,r)|^{\delta}dzdr\right)^{\frac{1}{\delta}}.$$

Thereby

$$\sup_{x,t} |\nabla I_2(x,t)|^{\frac{2pq}{(p+q)}} \leq C\left(\int_0^T\int_{\mathbb{R}^d} |X(z,r)|^{\frac{2pq}{(p+q)}}dzdr\right).$$

Taking the expectation and considering (34) we obtain the desired statement. $\qquad\square$

It remains to estimate the term I_1 in decomposition (33). It reads

$$I_1(x,t) = \int_0^t\int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,z,r)Y(z,r)dzdr$$

with Y given by (32). Note that we are not able to obtain boundedness of Y; to do so we would have to exchange the expectation and the supremum for an anticipating stochastic integral. Recall that according to **(D3)** we have $2pq/(p+q) > 2d+4$. Hence the constant κ in Lemma 6 can be chosen in such a way that $2 < \kappa < 2d+4$. Since Y is not bounded, we will apply Lemma A.2. Denote

$$Z(x,t) = \int_0^t\int_{\mathbb{R}^d} (t-r)^{\alpha-1}\Gamma(x,t,u,r)|Y(u,r)|^{\delta}dudr,$$
$$\widehat{Z}(x,t) = \int_0^t\int_{\mathbb{R}^d} (t-r)^{\alpha-1}\nabla\Gamma(x,t,u,r)|Y(u,r)|^{\delta}dudr.$$

Lemma 9 *For any $\frac{1}{2} + \frac{(d+2)(p+q)}{4pq} < \alpha < \frac{q-1}{q}$, there exists $1 < \delta < \frac{2pq}{(p+q)}$ such that*

$$\mathbf{E}\left[\sup_{x,t}(Z(x,t))^{\frac{2pq}{(p+q)\delta}} + \sup_{x,t}\left(\widehat{Z}(x,t)\right)^{\frac{2pq}{(p+q)\delta}}\right] \leq C\mathbf{E}\int_0^T\int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}}dzdr.$$

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Proof. Choose $1 < \delta < \frac{2pq}{(p+q)}$ and $\theta = \frac{2pq}{(p+q)\delta} > 1$. Then

$$\int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \Gamma(x,t,z,r) |Y(z,r)|^\delta dz dr$$

$$\leq \int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}^d} \Gamma(x,t,z,r)^{\frac{\theta}{\theta-1}} dz \right)^{\frac{\theta-1}{\theta}} \left(\int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz \right)^{\frac{1}{\theta}} dr$$

Due to the Aronson estimate the right-hand side here admits the following upper bound:

$$\int_0^t (t-r)^{\alpha-1} \left(\int_{\mathbb{R}^d} \Gamma(x,t,z,r)^{\frac{\theta}{\theta-1}} dz \right)^{\frac{\theta-1}{\theta}} \left(\int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz \right)^{\frac{1}{\theta}} dr$$

$$\leq C \int_0^t (t-r)^{\alpha-1-\frac{d}{2\theta}} \left(\int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz \right)^{\frac{1}{\theta}} dr$$

$$\leq C \left(\int_0^t (t-r)^{(\alpha-1-\frac{d}{2\theta})\frac{\theta}{\theta-1}} dr \right)^{\frac{\theta-1}{\theta}} \left(\int_0^t \int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz dr \right)^{\frac{1}{\theta}}$$

$$\leq C \left(\int_0^T \int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz dr \right)^{\frac{1}{\theta}};$$

here the latter inequality holds if $\left(\alpha - 1 - \frac{d}{2\theta} \right) \frac{\theta}{\theta-1} > -1$, or equivalently $\alpha > \frac{d+2}{2\theta} = \frac{(d+2)(p+q)}{4pq} \delta$.

The computations similar to those in the proof of the previous lemma yield

$$\int_0^t \int_{\mathbb{R}^d} (t-r)^{\alpha-1} \nabla \Gamma(x,t,z,r) |Y(z,r)|^\delta dz dr \leq C \left(\int_0^T \int_{\mathbb{R}^d} |Y(z,r)|^{\frac{2pq}{(p+q)}} dz dr \right)^{\frac{1}{\theta}},$$

if $\left(\alpha - 1 - \frac{1}{2} - \frac{d}{2\theta} \right) \frac{\theta}{\theta-1} > -1$, or equivalently $\alpha > \frac{1}{2} + \frac{d+2}{2\theta} = \frac{1}{2} + \frac{(d+2)(p+q)}{4pq} \delta$. □

From Lemmata 9 and A.2 it follows that I_1 is a.s. continuous w.r.t. (x,t) and differentiable w.r.t. x. Arguing as in the proof of the above lemma, we obtain

$$\mathbf{E} \left[\sup_{x,t} \left(|I_1(x,t)|^{\frac{2pq}{p+q}} + |\nabla I_1(x,t)|^{\frac{2pq}{p+q}} \right) \right] < +\infty.$$

Furthermore, a careful examination of our proofs shows that there exists $\eta > 0$ such that for any $h > 0$

$$\mathbf{E} \left[\sup_{x, 0 \leq t \leq h} |v(x,t)|^{\frac{2pq}{p+q}} \right] \leq C h^\eta.$$

This implies that a.s. for any $x \in \mathbb{R}^d$, $v(x,t)$ tends to zero as t goes to zero.

To complete the proof of Theorem 3 consider a function $\phi \in C_0^\infty(\mathbb{R}^d)$ and

$$J(t) = \int_{\mathbb{R}^d} v(x,t)\phi(x)dx + \int_0^t \int_{\mathbb{R}^d} \mathrm{a}(x,u)\nabla v(x,u)\nabla\phi(x)dx du.$$

By the previous Lemmata, $J(t)$ is well defined on $[0, T]$ with

$$
\begin{aligned}
J(t) &= \int_{\mathbb{R}^d} \left(\int_0^t \int_{\mathbb{R}^d} \Gamma(x, t, y, s) G(y, s) dy dB_s \right) \phi(x) dx \\
&+ \int_0^t \int_{\mathbb{R}^d} \mathrm{a}(x, u) \left(\int_0^u \int_{\mathbb{R}^d} \nabla \Gamma(x, u, y, s) G(y, s) dy dB_s \right) \nabla \phi(x) dx du.
\end{aligned}
$$

By the Fubini theorem

$$
\begin{aligned}
J(t) &= \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \Gamma(x, t, y, s) \phi(x) dx \right) G(y, s) dy dB_s \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(\int_s^t \int_{\mathbb{R}^d} \nabla \Gamma(x, u, y, s) \mathrm{a}(x, u) \nabla \phi(x) dx du \right) G(y, s) dy dB_s \\
&= \int_0^t \int_{\mathbb{R}^d} \phi(y) G(y, s) dy dB_s,
\end{aligned}
$$

since Γ is the fundamental solution of (1).

Appendix

Recall that

$$
V(x, t) = \int_0^t \int_{\mathbb{R}^d} \Gamma(x, t, y, s) f(y, s) dy ds
$$

is the volume potential of f (see [7, Section I.3]). Here we give some results concerning the regularity of V. The first lemma is closely related to Lemma I.3.1 and Theorem I.3.3 of [7] and Theorem 1 of [8].

Lemma A.1 *Assume that f is a bounded measurable function. Then V is continuous w.r.t. $(x, t) \in \mathbb{R}^d \times (0, +\infty)$ and has first continuous derivatives w.r.t. x. Moreover, for any $t > 0$ and $x \in \mathbb{R}^d$,*

$$
\frac{\partial V}{\partial x_i}(x, t) = \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \Gamma(x, t, y, s) f(y, s) dy ds.
$$

Proof. Fix some $x \in \mathbb{R}^d$ and $t > 0$ and consider

$$
J(x, t, s) = \int_{\mathbb{R}^d} \Gamma(x, t, y, s) f(y, s) dy.
$$

This function is continuous with respect to all its arguments $x \in \mathbb{R}^d$ and $0 \le s < t$. Moreover, by (4)

$$
|J(x, t, s)| \le \int_{\mathbb{R}^d} g_{\varrho, \varpi}(x - y, t - s) |f(y, s)| dy \le C.
$$

Since the function $\int_0^{t-\varepsilon} J(x,t,s)ds$ is continuous for any sufficiently small $\varepsilon > 0$, this implies the required continuity of V. For the derivatives, let us consider

$$J(x,t,s) = \int_{\mathbb{R}^d} \Gamma(x,t,y,s)g(y,s)dy.$$

For any $s < t$, it holds

$$\frac{\partial J}{\partial x_i}(x,t,s) = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i}\Gamma(x,t,y,s)g(y,s)dy.$$

Now using (5), we have

$$\left|\frac{\partial J}{\partial x_i}(x,t,s)\right| \leq (t-s)^{-1/2}\int_{\mathbb{R}^d} g_{\varrho,\varpi}(x-y,t-s)|g(y,s)|dy \leq C(t-s)^{-1/2}.$$

Therefore the integral

$$\int_0^t \frac{\partial J}{\partial x_i}(x,t,s)ds$$

converges uniformly with respect to x and $t > 0$. It follows that for $t > 0$ and any x, the derivatives

$$\frac{\partial V}{\partial x_i}(x,t) = \int_0^t \frac{\partial J}{\partial x_i}(x,t,s)ds$$

exist and are continuous. $\qquad\qquad\Box$

Let us give another version of these results.

Lemma A.2 *Let g be a measurable function such that for some $q > 1$ there exists a constant $K \geq 0$ such that for any $(x,t) \in \mathbb{R}^d \times (0,+\infty)$*

$$\int_0^t \int_{\mathbb{R}^d} [\Gamma(x,t,y,s) + |\nabla\Gamma(x,t,y,s)|]\, |g(y,s)|^q dyds \leq K.$$

Then V is continuous w.r.t. $(x,t) \in \mathbb{R}^d \times (0,+\infty)$ and has first continuous derivatives w.r.t. x. Moreover, for any $t > 0$ and $x \in \mathbb{R}^d$,

$$\frac{\partial V}{\partial x_i}(x,t) = \int_0^t \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i}\Gamma(x,t,y,s)g(y,s)dyds.$$

Proof. By the Hölder inequality

$$|J(x,t,s)| \leq \left(\int_{\mathbb{R}^d} \Gamma(x,t,y,s)dy\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^d} \Gamma(x,t,y,s)|g(y,s)|^q dy\right)^{\frac{1}{q}}.$$

This implies the uniform convergence of the integral $\int_0^t J(x,t,s)ds$ w.r.t. x and $t > 0$. Therefore, V is continuous for $t > 0$. For the derivative, the same arguments give:

$$\left|\frac{\partial J}{\partial x_i}(x,t,s)\right|$$
$$\leq \left(\int_{\mathbb{R}^d} \left|\frac{\partial}{\partial x_i}\Gamma(x,t,y,s)\right| dy\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^d} \left|\frac{\partial}{\partial x_i}\Gamma(x,t,y,s)\right| |g(y,s)|^q dy\right)^{\frac{1}{q}}$$
$$\leq C(t-s)^{-(q-1)/(2q)}.$$

The rest of the proof is exactly the same as in the previous lemma. □

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