Composition of approximations of two SDEs with jumps with non-finite Lévy measures.

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Composition of approximations of two SDEs with jumps with non-finite Lévy measures. *

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Abstract

The purpose of this paper is to extend the results of [13] and [20] concerning the approximation of the solution of some nonlinear second order stochastic PDEs like those satisfied by a consistent dynamic utilities, see [9, 18]. Indeed, in this works, authors showed that the solution of a SPDE of this class is the compound of two monotonic stochastic flows satisfying two SDE. The objective is then to take advantage of this representation to establish a numerical scheme approximating the SPDE's solution using Euler's approximations of the two stochastic flows. This allows us to avoid a complicated discretization in time and space of the SPDE for which it seems really difficult to obtain error estimates. The case where the two flows are solutions of two time-continuous SDEs has been treated in [13] and then extended to the framework with jumps but with a finite Lévy measure in [20]. In the case where the measure is infinite, additional terms specific to the truncation method will appear in our error estimates. In many cases, an optimal choice of parameters allows us to find a convergence rate equal to those established in [13] and [20]. However we provide some examples of Lévy measures with much slower convergence rate.

1 Introduction

The main idea of this work and that of [13] and [20] finds its sources in the results of El Karoui and Mrad [9] and Matoussi and Mrad [18] on the field of dynamic consistent utilities, introduced under the name of forward utilities by Musiela and Zariphopoulou [21]. This concept which generalizes the classic utility functions, models possible changes over the time of both the individual preferences of an agent and the dynamic of the universe of investment. Researchers in this area have shown that a consistent dynamic utility $\{U(t,z,\omega),\ t \geq 0, z \in \mathbb{R}, \omega \in \Omega\}$ is a solution of a nonlinear second-order stochastic PDE, which has proven to be very complicated to study and solve theoretically by standard tools. It is only in [9] that the answers to a large part of the questions concerning this stochastic PDE have been given. The originality is that authors have succeeded in linking the solutions of these SPDEs with those of two SDEs. They have shown that it is more appropriate to consider the SPDE satisfied by the marginal utility $\{U_z(t,z,\omega)\}$ and then, along the path of a well-chosen process Y, the marginal utility satisfies a

*Strong approximation, Euler scheme, stochastic flow, method of stochastic characteristics, SPDE driven by Lévy noise, Utility-SPDE, Garsia-Rodemich-Rumsey lemma

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regular SDE whose unique solution X necessarily satisfies a.s. $X_t(u_z(y)) = U_z(t, Y_t(y))$ for any t (u is the initial given utility function).

Therefore under the assumption that the process Y is such that $y \mapsto Y_t(y)$ is homeomorphic flow, one can easily characterize, by making a variable change, the marginal U_z which is written $U_z(t,z) = X_t(u_z(Y_t^{-1}(z)))$ and then U by integration. These results were later generalized to the framework with jumps in [18]. Unfortunately, apart from a few special cases, it is complicated, if not impossible, to obtain closed formulas for the characteristics X and Y (or its inverse Y^{-1}). This explains the need to study the composition of the approximations of the two SDEs instead of trying to directly discretize the SPDEs, which is not an easy task.
Let X^n be an approximation of X and ξ^n be an approximation of $\xi := Y^{-1}$ for some convergence parameter $n \to +\infty$, our aim is to show that the compound approximation $X^n(\omega, \xi^n(\omega))$ converges to $X(\omega, \xi(\omega))$. For this we need to answer the following questions: Which scheme to approximate the inverse flow ξ? Under which assumptions does the compound approximation $\omega \mapsto X^n(\omega, \xi^n(\omega))$ converge to the compound map $\omega \mapsto X(\omega, \xi(\omega))$ and in which manner? And finally, what is the convergence rate and how does it depend on those related to the approximations X^n to X and ξ^n to ξ?

Note that our study becomes simple if we make the hypothesis that the pair (X, X^n) is independent of (ξ, ξ^n), by using a conditioning argument. This is not the case in this work as we will explain later. The study is therefore more complex and the classic arguments are not necessarily effective.
In [13], authors answer some of these questions in a very general framework and establishes a strong convergence rates for compound maps in complicated situations where the error analysis was not available so far. Theorem 1 in [13] gives a strong approximation of the compound $F_t(\Theta_t(\theta))$ where the random fields F and Θ are arbitrary (not a necessarily semimartingales) and satisfy some space regularities assumptions. The scope of application of this result is potentially large: the application to utility-SPDE without jumps is developed in Section 3 of [13] and applications to stochastic processes (possibly non semimartingales) at random times (possibly non stopping times) are considered in [12], where the nice interplay with unbiased simulation scheme of Rhee-Glynn [26] is presented.
In this work, we first study the convergence of the compound of two approximations of two SDEs with jumps and then we develop an application to the resolution of the SPDE utility. To get an idea of this equation and its complexity, a dynamic utility in our general framework solves, denoting by U_z and U_{zz} the first and the second derivative of U with respect to z, the following second-order fully nonlinear Stochastic partial integro-differential equation (SPIDE) driven by Lévy noise and of HJB type [18],

$$\begin{aligned} dU(t,z) &= \Big(-zU_z(t,z)r_t + \int_{\mathbb{R}} (U(t,z) + H(t,z,e))\nu(\mathrm{d}e) - \mathcal{Q}(t,z,\kappa^*) \Big)\mathrm{d}t \\ &+ \gamma(t,z)dW_t + \int_{\mathbb{R}} H(t,z,e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \end{aligned} \tag{1}$$

where W is a d-dimensional Brownian motion and $\tilde{N}$ the compensated version of an independent Poisson random measure N on $[0, \infty[\times\mathbb{R}$ with intensity measure $\ell(t)\mathrm{d}t \times \nu(\mathrm{d}e)$. (γ, H) denotes the pair diffusion/jump coefficients of U with values in $\times\mathbb{R}^d \times \mathbb{R}$. The quantity $\mathcal{Q}(t,z,\kappa^*)$ is the quadratic form given by

$$\begin{aligned} \mathcal{Q}(t,z,\kappa^*) &= \int_{\mathbb{R}} (U+H)\big(t, z(1+\kappa^*(t,z)h^S(t,e))\big)\nu(\mathrm{d}e) - \frac{1}{2U_{zz}}\|\gamma_z^{\mathcal{R}} + U_z(t,z)(\eta_t - \alpha_t)\|^2 \\ &+ \frac{1}{2U_{zz}(t,z)}\|\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1+\kappa^*(t,z)h^S(t,e))\big)h^S(t,e)\nu(\mathrm{d}e)\|^2. \end{aligned}$$

With r and η denoting the interest rate and the risk premium of the market. h^S is a column vector $(d \times 1)$ representing the jump coefficient of the vector of risky-assets prices $S = (S^i)_{i=1}^d$ of the considered financial market. The process α_t is equal to $\int_{\mathbb{R}} h^S(t,e)\nu(\mathrm{d}e)$ and the vector $z\kappa^*(t,z)$ (required to lie at any time t in a set of constraints $\mathcal{R}_t$, assumed to be a linear space of $\mathbb{R}^d$) denotes the optimal strategy of the investor (with capital z at time t) and is given by

$$z\kappa^*(t,z) = -\frac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)(\eta_t - \alpha_t)}{U_{zz}(t,z)} - \frac{\displaystyle\int_{\mathbb{R}} (U_z + H_z)\big(t, z(1+\kappa^*(t,z)h^S(t,e))\big)h^S(t,e)\nu(\mathrm{d}e)}{U_{zz}(t,z)},$$

$\gamma_z^{\mathcal{R}}$ denotes the orthogonal projection of the derivative γ_z of the diffusion vector γ into $\mathcal{R}$. Both in continuous and discontinuous cases [9, 18], the spatial first derivative of the SPDE solution is the compound of the solutions of two SDEs with explicit coefficients and driven by the same Brownian motion W and the same Poisson random measure N; denote them by $(X_t(x) : t \geq 0)$ and $(Y_t(y) : t \geq 0)$, parameterized by their initial space conditions x and y at time 0. More precisely, the utility-SPDE (1) admits a unique concave (with respect to the space variable z) solution with marginal U_z characterized by composition of stochastic flows:

$$U_z(t,z,\omega) = X_t\big(u_z(\xi_t(z,\omega)),\omega\big), \quad U(0,z,\omega) = u(z) \tag{2}$$

where $\xi_t(z)$ denotes the inverse flow of $y \mapsto Y_t(y)$.

The case of continuous semimartingale $(H \equiv 0)$ was considered in [13, Theorem 8] where authors have shown, using Euler's schemes to approximate X and ξ, that $X_t^n\big(u_z(\xi_t^n(z))\big)$ converge with order $\frac{1}{2}$ to the first derivative $U_z(t,z)$ of the solution U of (1) with initial condition $U(0,z) = u(z)$. The same convergence rate was also established in [20] for a finite measure ν. The purpose of this paper is to to study the case where the Lévy measure ν is infinite.

The paper is organized as follows. In Section 2, we recall a general convergence result [13, Theorem 1] estimating the $\mathbf{L}_p$-error $\|F^n(\Theta^n) - F(\Theta)\|_{\mathbf{L}_p}$ by assuming locally uniform approximations of $F^n - F$, and local-Hölder continuity on F. In Section 3, we give more details on our model, the main assumptions and then study local and uniform estimates for SDE's solution and it space-differential, this results are crucial for establishing our main result (Theorem 5) for compound Euler schemes with the cutoff of the small jumps related to SDEs with jumps, through their initial conditions. Contrary to the frameworks of [13] and [20] an optimal convergence rate $n^{-1/2}$ is not always guaranteed, it depends a lot on the measure ν and on the regularity of the jump coefficients of the SDEs in the neighborhood of 0. Examples are then detailed in Section 3.5. Section 4 is dedicated to the proof of Theorem 5. The major difficulty is to show that the hypotheses of [13, Theorem 1] are satisfied in order to be able to apply this result to the framework of this paper. Several intermediate results, whose proofs are long and complex, are necessary to verify all these assumptions. In Section 5, we come back to the application to utility-SPDE.

2 $\mathbf{L}_p$-approximation of compound random maps

Consider a separable Banach space $(\mathcal{E}, |.|)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let F be a random field, i.e. an $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto F(\omega, x) \in \mathcal{E}$, continuous in x for a.e. ω and let $\Theta : \Omega \to \mathbb{R}^d$ an $\mathbb{R}^d$-valued $\mathcal{F}$-random variable. Denote by F^n and Θ^n the approximations of F and Θ, where $n \to +\infty$ is an asymptotic parameter.

Along this paper, for $p > 0$ and for a random variable $Z : \Omega \to \mathcal{E}$ or $\mathbb{R}^d$, we set $\|Z\|_{\mathbf{L}_p} = (\mathbb{E}|Z|^p)^{1/p}$ and say that $Z \in \mathbf{L}_p$ if $\|Z\|_{\mathbf{L}_p} < +\infty$. Despite $\|.\|_{\mathbf{L}_p}$ not being a norm for $p < 1$, we refer to it as $\mathbf{L}_p$-norm to simplify the discussion.
The following assumptions implies, as it has been established in [13], a general convergence result for the compound $F^n(\Theta^n)$ to $F(\Theta)$. They also ensure that all the quantities of interest to us belong to any $\mathbf{L}_p$, with some locally uniform estimates w.r.t. the space dependence.

(H1) For any $p > 0$, there exist constants $\alpha_p^{(\mathbf{H1})} \in [0, +\infty[$ and $C_p^{(\mathbf{H1})} \in [0, +\infty[$ such that

$$\left\| \sup_{|x|\le\lambda} |F(\cdot, x)| \right\|_{\mathbf{L}_p} \le C_p^{(\mathbf{H1})} \lambda^{\alpha_p^{(\mathbf{H1})}}, \quad \forall \lambda \ge 1. \tag{H1}$$

(H2) There is a $\kappa \in]0, 1]$ such that for any $p > 0$, there exist constants $\alpha_p^{(\mathbf{H2})} \in [0, +\infty[$ and $C_p^{(\mathbf{H2})} \in [0, +\infty[$ such that

$$\left\| \sup_{x\neq y, |x|\le\lambda, |y|\le\lambda} \frac{|F(\cdot, y) - F(\cdot, x)|}{|y - x|^\kappa} \right\|_{\mathbf{L}_p} \le C_p^{(\mathbf{H2})} \lambda^{\alpha_p^{(\mathbf{H2})}}, \quad \forall \lambda \ge 1. \tag{H2}$$

(H3) For any $p > 0$, there exist a constant $\alpha_p^{(\mathbf{H3})} \in [0, +\infty[$ and a sequence $(\varepsilon_p^{n,(\mathbf{H3})})_{N\ge1}$ with $\varepsilon_p^{n,(\mathbf{H3})} \in [0, +\infty[$ such that

$$\left\| \sup_{|x|\le\lambda} |F^n(\cdot, x) - F(\cdot, x)| \right\|_{\mathbf{L}_p} \le \varepsilon_p^{n,(\mathbf{H3})} \lambda^{\alpha_p^{(\mathbf{H3})}}, \quad \forall \lambda \ge 1, \forall N \ge 1. \tag{H3}$$

(H4) For any $p > 0$, there exist a constant $C_p^{(\mathbf{H4}\text{-a})} \in [0, +\infty[$ and a sequence $(\varepsilon_p^{n,(\mathbf{H4}\text{-b})})_{N\ge1}$ with $\varepsilon_p^{n,(\mathbf{H4}\text{-b})} \in [0, +\infty[$ such that

$$\|\Theta\|_{\mathbf{L}_p} \vee \|\Theta^n\|_{\mathbf{L}_p} \le C_p^{(\mathbf{H4}\text{-a})}, \quad \forall N \ge 1, \tag{H4-a}$$

$$\|\Theta^n - \Theta\|_{\mathbf{L}_p} \le \varepsilon_p^{n,(\mathbf{H4}\text{-b})}, \quad \forall N \ge 1. \tag{H4-b}$$

These assumptions being satisfied, the following Theorem states an error estimate on the approximation of $F(\Theta)$ by $F^n(\Theta^n)$, as a function of n, through the sequences $(\varepsilon_\cdot^{n,(\mathbf{H3})})_{n\ge1}$ and $(\varepsilon_\cdot^{n,(\mathbf{H4}\text{-b})})_{n\ge1}$.

Theorem 1 (Gobet-Mrad [13]). *Assume* (**H1**)-(**H2**)-(**H3**)-(**H4**-a)-(**H4**-b). *Then for any $p > 0$ and any $p_2 > p$, there is a constant $c_{(3)}$ independent on N such that*

$$\|F^n(\Theta^n) - F(\Theta)\|_{\mathbf{L}_p} \le c_{(3)} \left(\varepsilon_{2p}^{n,(\mathbf{H3})} + [\varepsilon_{\kappa p_2}^{n,(\mathbf{H4}\text{-b})}]^\kappa \right), \quad \forall N \ge 1. \tag{3}$$

Quite intuitively, the global approximation error inherits the rates from that on F and that on Θ modified by the local Hölder regularity of $x \mapsto F(\omega, x)$. Note that there is a second variant of this Theorem (see [13]) which gives an equivalent conclusion but instead of the λ-polynomial dependency of the upper bounds in ($\mathbf{H1} - \mathbf{H2} - \mathbf{H3}$), we consider exponential dependency. This allow us to analyse the approximation of diffusion process in diffusion time $Z_t = X_{|Y_t|}$ see [1] and [12] for this and and for other applications.

From pointwise to locally uniform estimates From a practical point of view the conditions (**H1-H2-H3**) are sometimes very complicated to verify. For example, if x is the time variable, one can verify that (**H1-H2-H3**) are satisfied by using Doob inequalities or other martingale estimates. But in other situations this verification can be complex. One can think of using the Kolmogorov continuity criterion for random fields [16, Theorem 1.4.1 p.31], but it does not yield the quantitative estimates we are looking for. However, there is an interesting result that gives refinement compared to the Kolmogorov criterion. It is the Garsia-Rodemich-Rumsey lemma [11] (see [22, p.353–354]). This result allows us to go from pointwise estimates to locally uniform estimates, by assuming Hölder regularity in $\mathbf{L}_p$. In the literature, this approach has been extensively developed in [4] for studying regularity of local times of continuous martingales w.r.t. the space variable.

The following two results will be used frequently in this paper. The first one is obtained from Garsia-Rodemich-Rumsey Lemma [11], see the proof of [13, Theorem 2], while the second one is a direct application of the first.

Theorem 2 (Gobet-Mrad [13]). *Let $p > d$. Assume that G is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto G(\omega, x) \in \mathcal{E}$, continuous in x for a.e. ω. Assume that $G(x)$ is in $\mathbf{L}_p$ for any x and that there exist constants $C^{(G)} \in [0, +\infty[$, $\beta^{(G)} \in]d/p, 1]$ and $\tau^{(G)} \in [0, +\infty[$ such that*

$$\|G(x) - G(y)\|_{\mathbf{L}_p} \leq C^{(G)} |x - y|^{\beta^{(G)}} (1 + |x| + |y|)^{\tau^{(G)}}, \quad \forall (x, y) \in \mathbb{R}^d \times \mathbb{R}^d. \tag{4}$$

Then, for any $\beta \in]0, \beta^{(G)} - d/p[$, we have

$$\left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} \frac{|G(y) - G(x)|}{|y - x|^{\beta}} \right\|_{\mathbf{L}_p} \leq c_{(5)} C^{(G)} \lambda^{\tau^{(G)} + \beta^{(G)} - \beta}, \quad \forall \lambda \geq 1, \tag{5}$$

where $c_{(5)}$ is a constant depending only on $d, p, \beta, \beta^{(G)}, \tau^{(G)}$.

A similar result is proved in [25, Theorem 2.1, p.26] using the Kolmogorov criterion, with x and y in a compact set, i.e. with $\tau^{(G)} = 0$; the quoted result is not sufficient for our study.

As a consequence, we obtain the following result that may serve to easily check (**H1**).

Corollary 1. *Let us consider the assumptions and notations of Theorem 2. Then we have*

$$\left\| \sup_{|x| \leq \lambda} |G(x)| \right\|_{\mathbf{L}_p} \leq c_{(6)} \lambda^{\tau^{(G)} + \beta^{(G)}}, \quad \forall \lambda \geq 1, \tag{6}$$

where $c_{(6)} := \|G(0)\|_{\mathbf{L}_p} + c_{(5)} C^{(G)}$ where $c_{(5)}$ is defined in Theorem 2 with $\beta = (\beta^{(G)} - d/p)/2$. In particular, the constant $c_{(5)}$ depends only on $d, p, \beta^{(G)}, \tau^{(G)}$.

3 Application to compound schemes

3.1 Problem's setting

For the rest of the paper, T denotes a finite time horizon and the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ is supposed to support two q-dimensional standard Brownian motions $W = (W^1, \ldots, W^q)$ and $B = (B^1, \ldots, B^q)$ on $[0, T]$ and an independent q'-dimensional Poisson random measure N on $[0, \infty[\times \mathbb{R}^{q'}$ with deterministic time dependent intensity measure $\ell(t) \mathrm{d}t \times \nu(\mathrm{d}e)$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. ℓ is the time intensity of jumps with $\ell([0, T]) = [0, \ell_{\max}]$

for some $\ell_{\max} \in]0, \infty[$ and ν is a positive measure on $\mathbb{R}^{q'}$, satisfying standard integrability assumption for Lévy processes:

$$\int_{\mathbb{R}^{q'}} (1 \wedge |e|^2)\nu(\mathrm{d}e) < +\infty. \tag{7}$$

We also denote by $\tilde{N}$ the compensated version of N:

$$\tilde{N}(\mathrm{d}t, \mathrm{d}e) = N(\mathrm{d}t, \mathrm{d}e) - \nu(\mathrm{d}e)\ell(t)\mathrm{d}t.$$

Depending on the potential applications, we may require that B and W are the same, or different, or built from each other, possibly in complicated ways. Actually, observe that we do not assume that the couple (B, W) forms a higher-dimensional Brownian motion: this general setting allows flexibility in further applications. As an example for solving utility-SPDE in Section 5, we need to consider B as the backward Brownian motion of W.

We are also concerned by two $\mathbb{R}^d$-valued stochastic processes X and Y, solutions of the following stochastic differential equations (SDE for short)

$$\mathrm{d}X_t(x) = \mu(t, X_t(x))\mathrm{d}t + \sum_{i=1}^{q} \sigma_i(t, X_t(x))\mathrm{d}W_t^i + \int_{\mathbb{R}^{q'}} h(t, X_{t^-}(y), e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \;\; X_0(x) = x, \tag{8}$$

$$\mathrm{d}Y_t(y) = b(t, Y_t(y))\mathrm{d}t + \sum_{i=1}^{q} \gamma_i(t, Y_t(y))\mathrm{d}B_t^i + \int_{\mathbb{R}^{q'}} g(t, Y_{t^-}(y), e)\tilde{N}(\mathrm{d}t, \mathrm{d}e), \;\; Y_0(y) = y, \tag{9}$$

where $\mu, b, \sigma_i, \gamma_i$ are deterministic functions from $[0, T] \times \mathbb{R}^d$ into $\mathbb{R}^d$ and h, g are deterministic functions from $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{q'}$ into $\mathbb{R}^{d \times q'}$, globally Lipschitz in space to ensure the existence of a unique strong solution.

Denote by $X_T^n(x)$ (resp. $Y_T^n(y)$) the Euler scheme with time step T/n of $X_T(x)$ (resp. $Y_T(y)$). Using previous results, we aim at establishing a new convergence result of the compound scheme $X_t^n(Y_t^n(y))$ to the compound random field $X_t(Y_t(y))$ as n goes to infinity. This approximation issue is actually motivated by the resolution of some SPDEs (with or without jumps) by composition of stochastic flows. Relating compound SDEs to SPDEs is, in a sense, obvious since it is sufficient to apply an extension of the Itô-Ventzel's formula established by Øksendal and Zhang [23, Theorem 3.1] to the compound process $V(t, y) := X_t(Y_t(y))$. Under good regularity assumptions on (μ, σ, h), the quoted result shows that $V(t, y) := X_t(Y_t(y))$ is still a semimartingale. For simplicity, take $d = q = q' = 1$ and $W = B$, $V(t, y)$ is solution of a second order SPDE, with stochastic coefficients, given by

$$\begin{aligned} dV(t,y) = \Big[&\partial_y V(t,y) \tfrac{b(t,Y_t(y))}{\partial_y Y_t(y)} + \tfrac{1}{2}\Big(\partial_y^2 V(t,y) - \partial_y V(t,y) \tfrac{\partial_y^2 Y_t(y)}{\partial_y Y_t(y)}\Big) \tfrac{\gamma^2(t,Y_t(y))}{(\partial_y Y_t(y))^2} \\ &+ \mu(t, V(t,y)) + \partial_y V(t,y)(\partial_x \sigma)(t, V(t,y)) \tfrac{\gamma(t,Y_t(y))}{\partial_y Y_t(y)} \Big] \mathrm{d}t \\ + \int_{\mathbb{R}} &\Big[\bar{V}(t,y,e) - V(t,y) - h(t, V(t,y), e) - \frac{\partial_y V(t,y)}{\partial_y Y_t(y)} g(t, Y_t(y), e) \Big] \nu(\mathrm{d}e)\lambda(t)\mathrm{d}t \\ + \Big[&\partial_y V(t,y) \tfrac{\gamma(t,Y_t(y))}{\partial_y Y_t(y)} + \sigma(t, V(t,y)) \Big] \mathrm{d}W_t + \int_{\mathbb{R}} \Big[\bar{V}(t,y,e) - V(t^-, y) \Big] \tilde{N}(\mathrm{d}t, \mathrm{d}e). \end{aligned}$$

Note that $V(t^-, z) = X_{t^-}(Y_{t^-}(z))$ and $\bar{V}(t, y, e)$ is defined by

$$\bar{V}(t, y, e) := X_{t^-}\Big(Y_{t^-}(y) + g(t, Y_{t^-}(y), e)\Big) + h\Big(t, Y_{t^-}(y) + g(t, Y_{t^-}(y), e)\Big).$$

$\bar{V}(t,.,e)$ corresponds to $V(t,.)$ after a jump on (t,e) which is the compound of $X_{t^-}(.) + h(t, X_{t^-}(.), e)$ corresponding to X after a jump on (t,e) and $Y_{t^-}(.) + g(t, Y_{t^-}(.), e)$ corresponding to Y after a jump at the same time for the same mark e.

In the reverse direction, i.e., linking SPDE with SDEs is not obvious but it is possible in the cases considered by Kunita [16, Chapter 6] and utility-SPDEs of [9, 18].

Preliminary results In the following we will often need to control in $\mathbf{L}_p$-norms of a d-dimensional semimartingales $Z_t = (Z_t^1, Z_t^2, \dots, Z_t^d)$ solutions of a stochastic differential equation of the following form

$$Z_t^i = z^i + \int_0^t b_s^i ds + \sum_{j=1}^m \int_0^t f_s^{i,j} d\hat{W}_s^j + \int_0^t \int_{\mathbb{R}^{q'}} g^i(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \tag{10}$$

where $\hat{W}$ is a m-dimensional Brownian motion and $Z_0 = z \in \mathbb{R}^d$ [1]. To do, we use the following result established in [17, Theorem 2.11, Corollary 2.12].

Theorem 3. *For any $p \geq 2$, there exists a positive constant $C_{p,\ell_{\max}}$ such that*

$$\begin{aligned}\mathbb{E}\big[\sup_{0<s\leq t} |Z_s|^p\big] \leq C_{p,\ell_{\max}} \Big\{ &|z|^p + \mathbb{E}\Big[\big(\int_0^t |b_s| \mathrm{d}s\big)^p\Big] + \mathbb{E}\Big[\big(\int_0^t |f_s|^2 \mathrm{d}s\big)^{\frac{p}{2}}\Big] \\ &+ \mathbb{E}\Big[\big(\int_0^t \int_{\mathbb{R}^{q'}} |g(s,e)|^2 \nu(\mathrm{d}e)\mathrm{d}s\big)^{\frac{p}{2}}\Big] + \mathbb{E}\Big[\int_0^t \int_{\mathbb{R}^{q'}} |g(s,e)|^p \nu(\mathrm{d}e)\mathrm{d}s\Big]\Big\}.\end{aligned}$$

3.2 Assumptions

For the rest of the paper, the process X plays the role of the random field F and Y that of Θ in Theorem 1. In order to apply this Theorem, we need to show that its assumptions namely (**H1-H2-H3-H4**) are satisfied by X and Y. As we are concerned with solutions of stochastic differential equations it is natural to impose certain conditions on their coefficients and not directly on the solutions. Concerning the $X - SDE$, we impose the following time and space regularities. When we will discuss on approximation of $X(Y)$, similar assumptions will be made on the coefficients b, γ_i and g of Equation (9) for Y.

(HP1) The coefficients μ, σ and h are Lipschitz continuous with respect to the space variable x, uniformly in time. More precisely, there exist positive constants C^X and $C^X(e)$ such that for any $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $e \in \mathbb{R}^{q'}$

$$\begin{cases} |\mu(t,x) - \mu(t,y)| \leq C^X |x-y|, & |\mu(t,0)| \leq C^X, \\ |\sigma(t,x) - \sigma(t,y)| \leq C^X |x-y|, & |\sigma(t,0)| \leq C^X, \\ |h(t,x,e) - h(t,y,e)| \leq C^X(e)|x-y|, & |h(t,0,e)| \leq C^X(e), \end{cases} \tag{\textbf{HP1}}$$

where the constant $C^X(e)$ satisfies $\int_{\mathbb{R}^{q'}} [C^X(e)]^p \nu(\mathrm{d}e) < \infty, \ \forall p \geq 2$.

(HP2$_\delta$) μ, σ and h are continuously differentiable with respect to the space variable x such that their derivatives $\nabla_x \mu := \{\nabla_x \mu(t,x); t \in [0,T], x \in \mathbb{R}^d\}$, $\nabla_x \sigma = \{\nabla_x \sigma_i(t,x); 1 \leq i \leq q, t \in [0,T], x \in \mathbb{R}^d\}$ and $\nabla_x h := \{\nabla_x h(t,x,e); t \in [0,T], x \in \mathbb{R}^d, e \in \mathbb{R}^{q'}\}$ are δ-Hölder

[1]The integrals of the form $\int_0^t \int_{\mathbb{R}^{q'}} g^i(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$ should be understood as $\int_{[0,t]} \int_{\mathbb{R}^{q'}} g^i(s,e)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$.

for a certain exponent $\delta \in]0,1]$. Namely, there exist positive constants $C^{X,\nabla}$ and $C^{X,\nabla}(e)$ such that for any $t \in [0,T]$, $x, y \in \mathbb{R}^d$ and $e \in \mathbb{R}^{q'}$

$$\begin{cases} |\nabla_x \mu(t,x) - \nabla_x \mu(t,y)| \leq C^{X,\nabla}|x-y|^\delta, & |\nabla_x \mu(t,x)| \leq C^{X,\nabla}, \\ |\nabla_x \sigma(t,x) - \nabla_x \sigma(t,y)| \leq C^{X,\nabla}|x-y|^\delta, & |\nabla_x \sigma(t,x)| \leq C^{X,\nabla}, \\ |\nabla_x h(t,x,e) - \nabla_x h(t,y,e)| \leq C^{X,\nabla}(e)|x-y|^\delta, & |\nabla_x h(t,x,e)| \leq C^{X,\nabla}(e), \end{cases} \qquad (\mathbf{HP2}_\delta)$$

with $\int_{\mathbb{R}^{q'}} [C^{X,\nabla}(e)]^p \nu(\mathrm{d}e) < \infty, \ \forall p \geq 2$.

($\mathbf{HP3}_\alpha$) μ, σ and h are Hölder continuous in time, locally in space, i.e. there exists an exponent $\alpha \in]0,1]$ and positive constants C^X and $C^X(e)$ $\big(\int_{\mathbb{R}^{q'}} C^X(e)^p \nu(\mathrm{d}e) < \infty, \ \forall p \geq 2\big)$, such that for any $x \in \mathbb{R}^d, e \in \mathbb{R}^{q'}$ and $s, t \in [0,T]$

$$\begin{cases} |\mu(t,x) - \mu(s,x)| + |\sigma(t,x) - \sigma(s,x)| \leq C^X(1+|x|)|t-s|^\alpha. \\ |h(t,x,e) - h(s,x,e)| \leq C^X(e)(1+|x|)|t-s|^\alpha. \end{cases} \qquad (\mathbf{HP3}_\alpha)$$

($\mathbf{HP4}_\alpha$) $\nabla_x \mu$, $\nabla_x \sigma$ and $\nabla_x h$ are Hölder continuous in time, locally in space, i.e. there exists an exponent $\alpha \in]0,1]$, such that for any $x \in \mathbb{R}^d, e \in \mathbb{R}^{q'}$ and $s, t \in [0,T]$

$$\begin{cases} |\nabla_x \mu(t,x) - \nabla_x \mu(s,x)| + |\nabla_x \sigma(t,x) - \nabla_x \sigma(s,x)| \leq C^{X,\nabla}(1+|x|)|t-s|^\alpha. \\ |\nabla_x h(t,x,e) - \nabla_x h(s,x,e)| \leq C^{X,\nabla}(e)(1+|x|)|t-s|^\alpha. \end{cases} \qquad (\mathbf{HP4}_\alpha)$$

Some Comments Assumptions ($\mathbf{HP3}_\alpha$) and ($\mathbf{HP4}_\alpha$) are not necessary to establish a first convergence result but they ensure an optimal convergence rate, see Theorem 5. In fact, they allow to take $\kappa = 1$ in Theorem 1.

3.3 SDE: differentiability, local and uniform estimates

In this section we recall some key results concerning $\mathbf{L}_p$ estimates of the solution X of a regular SDE in order to verify the assumptions of the general Theorem 1. In view to obtain an optimal convergence rate we also need similar estimates for the space derivative of this solution (if it is differentiable). Such random fields are also called stochastic flows and are the main subject of Kunita's book and papers [16, 17, 10], see also [16, Chapter 3 and 4] for continuous framework. According to Kunita's results, under (**HP1**) the map $(t,x) \mapsto X_t(\omega, x)$ has a good modification, continuous with respect to the spatial parameter [17, Theorems 3.1& 3.2 & 4.1], we are working with. Moreover, under the additional space regularity ($\mathbf{HP2}_\delta$) of the coefficients (μ, σ, h), the strong solution $X_t(x)$ to (8) is continuously differentiable in space and its derivative denoted by $\nabla X_t(x)$ is locally ε-Hölder[2] for any $\varepsilon < \delta$, see [17, Theorem 3.4]. Furthermore, it is a semimartingale solution of a linear equation, with bounded stochastic parameters $(\nabla_x \mu(t, X_t(x)), \nabla_x \sigma(t, X_t(x)), \nabla_x h(t, X_t(x), e))$ given by

$$\begin{cases} \mathrm{d}\nabla X_t(x) = \nabla_x \mu(t, X_t(x)) \nabla X_t(x) \mathrm{d}t + \sum_{i=1}^{q} \nabla_x \sigma_i(t, X_t(x)) \nabla X_t(x) \mathrm{d}W_t^i \\ \qquad\qquad + \int_{\mathbb{R}^{q'}} \nabla_x h(t, X_{t^-}(x), e) \nabla X_{t^-}(x) \tilde{N}(\mathrm{d}t, \mathrm{d}e), \\ \nabla X_0(x) = \mathrm{Id}. \end{cases} \qquad (11)$$

[2]That is for any compact K of $\mathbb{R}^d$ there exists a finite positive random variable $C(K)$ such that for any $x, y \in K$ we have $|\nabla X_t(x,\omega) - \nabla X_t(y,\omega)| \leq C(K,\omega)|x-y|^\varepsilon$ a.s., see [17, Theorem 3.3] for details .

Notations: Throughout this paper, we will make use of different constants that may depend on the integer p of $\mathbf{L}_p$-norm, on the dimensions d and q, on the time horizon T and on the constants from the assumptions. These constants will be called generic constant and will be denoted by the same notation C_p even if their values change from line to line. They will not depend on N.

We now recall some L_p-estimates of $X_t(x)$ and its derivative $\nabla_x X_t(x)$, very useful in the rest of this work, see [20] for the proofs.

Proposition 1. *Assume* (**HP1**)*. For any $p > 0$, there exist generic constants $C_{p,(12)}$ and $C_{p,(13)}$ such that*

$$\|X_t(x)\|_{\mathbf{L}_p} \le C_{p,(12)}(1 + |x|), \tag{12}$$

$$\|X_t(x) - X_t(y)\|_{\mathbf{L}_p} \le C_{p,(13)}|x - y|, \tag{13}$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. In addition under (**HP2**$_\delta$)*, for any $p > 0$ there exist generic constants $C_{p,(14)}$ and $C_{p,(15)}$ such that*

$$\|\nabla X_t(x)\|_{\mathbf{L}_p} \le C_{p,(14)}, \tag{14}$$

$$\|\nabla X_t(x) - \nabla X_t(y)\|_{\mathbf{L}_p} \le C_{p,(15)}|x - y|^\delta \tag{15}$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

In order to put the sup over the space variable inside the expectation, we combine Proposition 1 with Theorem 2 and Corollary 1. This is the following assertion.

Theorem 4. *Let Assumption* (**HP1**) *hold. For any $p > 0$ and any $\beta \in]0, 1[$, there exist generic constants $C_{p,(16)}$ and $C_{p,(17)}$ such that, for any $t \in [0, T]$,*

$$\left\| \sup_{|x|\le\lambda} |X_t(x)| \right\|_{\mathbf{L}_p} \le C_{p,(16)}\lambda, \quad \forall\lambda \ge 1, \tag{16}$$

$$\left\| \sup_{x\ne y, |x|\le\lambda, |y|\le\lambda} \frac{|X_t(x) - X_t(y)|}{|y - x|^\beta} \right\|_{\mathbf{L}_p} \le C_{p,(17)}\lambda^{1-\beta}, \quad \forall\lambda \ge 1. \tag{17}$$

If in addition (**HP2**$_\delta$) *is satisfied, for any $p > 0$ and any $\beta \in]0, \delta[$, there exist generic constants $C_{p,(18)}$, $C_{p,(19)}$ and $C_{p,(20)}$ such that, for any $t \in [0, T]$,*

$$\left\| \sup_{|x|\le\lambda} |\nabla X_t(x)| \right\|_{\mathbf{L}_p} \le C_{p,(18)}\lambda^\delta, \quad \forall\lambda \ge 1, \tag{18}$$

$$\left\| \sup_{x\ne y, |x|\le\lambda, |y|\le\lambda} \frac{|\nabla X_t(x) - \nabla X_t(y)|}{|y - x|^\beta} \right\|_{\mathbf{L}_p} \le C_{p,(19)}\lambda^{\delta-\beta}, \quad \forall\lambda \ge 1, \tag{19}$$

$$\left\| \sup_{x\ne y, |x|\le\lambda, |y|\le\lambda} \frac{|X_t(x) - X_t(y)|}{|y - x|} \right\|_{\mathbf{L}_p} \le C_{p,(20)}\lambda^\delta, \quad \forall\lambda \ge 1. \tag{20}$$

Remark 3.1.

- *Observe that the additional smoothness* (**HP2**$_\delta$) *enables us to improve* (17) *(for $\beta < 1$) to* (20) *(i.e. $\beta = 1$): this improvement will play a key role in the derivation of our main Theorem 5, with the optimal convergence order β.*

- *As a direct consequence of this Theorem we have, under* (**HP1**) *and* ($\mathbf{HP2}_\delta$),
  1. *Assumption* (**H1**) *is satisfied by* X *in place of* F, *with* $C_p^{(\mathbf{H1})} := C_{p,(16)}$ *and* $\alpha_p^{(\mathbf{H1})} := 1$ *in view of Theorem 4.*
  2. *Assumption* (**H2**) *is satisfied by* X *in place of* F, *for any given* $\kappa \in]0,1[$ *with* $C_p^{(\mathbf{H2})} := C_{p,(17)}$ *(depending on* κ*) and* $\alpha_p^{(\mathbf{H2})} := 1-\kappa$.

3.4 Compound Euler schemes: Main result

It is well known that, for a general Lévy measure ν, there is no available algorithm to simulate the increments of the driving Lévy process. Moreover, a large number of jumps of N occurring between two discretization points can lead to a large discretization error, so we cannot apply the Euler scheme. But, one can read in the literature that there are many works that have considered this general framework and propose methods to overcome these difficulties. For example, in [15, 14], the authors use the fact that it is easy to simulate the jumps of N larger in absolute value than a certain $\varepsilon > 0$ and to take into account the jumps smaller than ε, they use the idea of Asmussen and Rosiński [2] and replace all such jumps with $\sigma_\varepsilon \tilde{B}$ where $\tilde{B}$ is an independent Brownian motion and σ_ε is a coefficient chosen to match the variance of the Brownian approximation with the variance of the small jumps which are removed. See also [27, 8, 19, 5]. Inspired by these works we use herein a method based on the cutoff of the small jumps, so we introduce for any $\varepsilon > 0$, the finite measure on $\mathbb{R}^{q'}$:

$$\nu^\varepsilon(\mathrm{d}e) = \mathbf{1}_{|e|\geq\varepsilon}\nu(\mathrm{d}e) \ ;$$

finiteness comes from (7).

Considering a non-finite measure ν certainly impacts the convergence rate but complicates our analysis a lot since we are composing two schemes. The challenge in the rest of this paper is to optimally control the error of the compound approximations as a function of the parameter ε and of the integer n (related to a discretization step that we will see in the next section) and study the limit when $\varepsilon \to 0$, which is not obvious contrary to the framework of [20] where the measure of Lévy is supposed to be finite and then the error depends only on n which simplifies the study a lot.

We now follow the presentation of [19] to expose how to simulate the integral with respect to the Poisson measure ν^ε.

Simulation of the integral with respect to the Poisson random measure ν^ε. Consider a sequence $e_1, e_2, \dots$ of independent random variables with common exponential distribution with parameter 1. Define

$$\Lambda^\varepsilon(t) = \nu^\varepsilon(\mathbb{R}^{q'}) \int_0^t \ell(s)\mathrm{d}s, \ t \in [0,T].$$

The number of jumps of the Poisson random measure $\tilde{N}^\varepsilon(\mathrm{d}t, \mathrm{d}e)$ in an interval $[0,t]$ is determined as

$$J^\varepsilon(t) = \max\{k : \sum_{j=1}^{k} e_j \leq \Lambda^\varepsilon(t)\},$$

and the total number of jumps in $[0,T]$ is denoted by $J^\varepsilon = J^\varepsilon(T)$. Let us emphasize that for general Lévy measure ν this number tends to ∞ as ε goes to zero. Let $\Lambda^{\varepsilon,-1}$ be the right-continuous inverse of Λ^ε, the jump times of the Poisson measure can be defined by $\theta_0 = 0$,

$$\theta_k^\varepsilon = \Lambda^{\varepsilon,-1}\left(\sum_{j=1}^k e_j\right),\ k \in [\![1, J^\varepsilon]\!] := \{1, \ldots, J^\varepsilon\},$$

and can be computed recursively by

$$e_k = \int_{\theta_{k-1}^\varepsilon}^{\theta_k^\varepsilon} \nu^\varepsilon(\mathbb{R}^{q'})\ell(s)\mathrm{d}s,\ k \in [\![1, J^\varepsilon]\!].$$

Once the jump times are computed, we proceed to sample the marks (values of the associated Poisson point process sorted in chronological order) $\{E_k^\varepsilon\}$, that, conditionally on the values of the jumps times, are independent random variables distributed respectively according to $\frac{1}{\nu^\varepsilon(\mathbb{R}^{q'})}\nu^\varepsilon(\mathrm{d}e)$. The random measure with intensity $\ell(t)\mathrm{d}t \times \nu^\varepsilon(\mathrm{d}e)$ can then be constructed as

$$N^\varepsilon(\mathrm{d}t, \mathrm{d}e) = \sum_{k=1}^{J^\varepsilon} \delta_{(\theta_k^\varepsilon, E_k^\varepsilon)}(\mathrm{d}t, \mathrm{d}e),$$

and, consequently, the stochastic integral with respect to the Poisson random measure, i.e. the last term in the SDE (8), can be computed as

$$\int_0^t \int_{\mathbb{R}^{q'}} h(s, X_{s^-}, e) N^\varepsilon(\mathrm{d}s, \mathrm{d}e) = \sum_{k=1}^{J^\varepsilon(t)} h(\theta_k, X_{(\theta_k^\varepsilon)^-}, E_k^\varepsilon),\ t \in [0,T]. \tag{21}$$

Euler Scheme We now need a numerical scheme to approximate the solution X of the SDE(8) and show that it satisfies Assumptions (**H2** $-$ **H3**). Here we consider the Euler scheme for the simple reason that it is the simplest and the best known scheme to discretize an SDE, see for example [24] and [5][3]. As we can read in Theorem 1, the result is true for any approximation considered for F and Θ, but we have to expect that the convergence rate of the compound of the approximations changes; the convergence rate for Euler scheme is different from the one of Milstein for example.

Let $n \geq 1$ and consider the discretization family $\{\bar{t}_i := i\frac{T}{n},\ i \in [\![0, n]\!]\}$ of $[0,T]$. Consider also the jump times $\{\theta_k^\varepsilon, k \in [\![1, J^\varepsilon]\!]\}$ with corresponding marks $\{E_k^\varepsilon,\ k \in [\![1, J^\varepsilon]\!]\}$ as explained in [19]. Consider the augmented partition given by the union

$$\{t_l,\ l \in [\![0, n + J^\varepsilon]\!]\} := \{\bar{t}_i := i\frac{T}{n},\ i \in [\![0, n]\!]\} \cup \{\theta_k^\varepsilon, k \in [\![1, J^\varepsilon]\!]\}. \tag{22}$$

- Set $X_0^{n,\varepsilon}(x) = x$.
- For $k = 0, \ldots, n + J^\varepsilon - 1$ and $t \in (t_k, t_{k+1}]$, set

$$\begin{aligned} X_{t^-}^{n,\varepsilon}(x) &= X_{t_k}^{n,\varepsilon}(x) + \mu(t_k, X_{t_k}^{n,\varepsilon}(x))(t - t_k) + \sum_{i=1}^q \sigma_i(t_k, X_{t_k}^{n,\varepsilon}(x))(W_t^i - W_{t_k}^i). \\ &- (t - t_k)\lambda(t_k)\int_{\mathbb{R}^{q'}} h(t_k, X_{t_k}^{n,\varepsilon}(x), e)\nu^\varepsilon(\mathrm{d}e). \end{aligned}$$

[3] [5] is a survey of strong discrete time approximations of jump-diffusion processes described by SDEs.

- When $t_{k+1} = \theta_l^\varepsilon$, we introduce a correction due to jump discontinuities.

$$X^{n,\varepsilon}_{t_{k+1}}(x) = X^{n,\varepsilon}_{t^-_{k+1}}(x) + h(t_{k+1}, X^{n,\varepsilon}_{t^-_{k+1}}(x), E_l^\varepsilon).$$

It can be equivalently written for any time $t \in [0,T]$: denoting by

$$\tau_t := \max\{t_k : k \in [\![0, n+J^\varepsilon]\!], t_k \le t\}$$

the last discretization-time before t, we have

$$\begin{aligned} X^{n,\varepsilon}_t(x) &= x + \int_0^t \mu(\tau_s, X^{n,\varepsilon}_{\tau_s}(x))\mathrm{d}s + \sum_{i=1}^q \int_0^t \sigma_i(\tau_s, X^{n,\varepsilon}_{\tau_s}(x))\mathrm{d}W^i_s \\ &+ \int_0^t \int_{\mathbb{R}^{q'}} h(\tau_s, X^{n,\varepsilon}_{\tau_s^-}, e)\tilde{N}^\varepsilon(\mathrm{d}s, \mathrm{d}e), \end{aligned} \tag{23}$$

where the last integral is as in (21). Similarly, assume that b, γ and g fulfill (**HP1**), so that the strong solution Y to (9) is well defined, together with its Euler scheme $Y^{n,\varepsilon}$.

Notations: In the rest of this work, we define for $Z = X, Y$, $p \ge 1$ and $\varepsilon > 0$

$$\begin{aligned} \mathcal{E}^{Z,\varepsilon}_p &= \int_{\mathbb{R}^{q'}} (C^Z(e))^p \nu^\varepsilon(\mathrm{d}e) \le \mathcal{E}^Z_p = \int_{\mathbb{R}^{q'}} (C^Z(e))^p \nu(\mathrm{d}e), \\ K^{Z,\varepsilon}_p &= \int_{\mathbb{R}^{q'}} (C^Z(e))^p \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e). \end{aligned}$$

Observe that, from (**HP1**) and Jensen's inequality, for any $p \ge 2$ and $\varepsilon > 0$, $\mathcal{E}^{Z,\varepsilon}_p < +\infty$, $\mathcal{E}^Z_p < +\infty$ and $\lim_{\varepsilon \downarrow 0} K^{X,\varepsilon}_p = 0$. Moreover we assume that for any $\varepsilon > 0$,

$$\mathcal{E}^{X,\varepsilon}_1 + \mathcal{E}^{Y,\varepsilon}_1 = \int_{|e|\ge\varepsilon} C^X(e)\nu(\mathrm{d}e) + \int_{|e|\ge\varepsilon} C^Y(e)\nu(\mathrm{d}e) < +\infty. \tag{\textbf{HP5}}$$

Theorem 5 (Main)**.** *Assume that (μ, σ, h) satisfies Assumptions* (**HP1**) *and* (**HP3**$_\alpha$) *(which α-parameter is denoted by α^X) and that (b, γ, g) satisfies Assumptions* (**HP1**) *and* (**HP3**$_\alpha$) *(which α-parameter is denoted by α^Y). Let $\beta^X = \min(\alpha^X, \frac{1}{2})$ and $\beta^Y = \min(\alpha^Y, \frac{1}{2})$. The compound Euler scheme $X^{n,\varepsilon}_\cdot(Y^{n,\varepsilon}_\cdot)$ converges to $X_\cdot(Y_\cdot)$ in any $\mathbf{L}_p$-norm, w.r.t. n and ε: For any $p > 0$, $\rho > 0$ and $y \in \mathbb{R}^d$, there is a constant $C_{p,\rho}$ such that for any $s, t \in [0,T]$,*

$$\begin{aligned} \|X^{n,\varepsilon}_t(Y^{n,\varepsilon}_s(y)) - X_t(Y_s(y))\|_{\mathbf{L}_p} \le C_{p,\rho} &\Bigg[\frac{1}{n^{\kappa\beta^Y}} + \frac{1}{n^{\beta^X - \rho}} + \left(\frac{\mathcal{E}^{Y,\varepsilon}_1}{n}\right)^\kappa + \left(\frac{\mathcal{E}^{X,\varepsilon}_1}{n}\right)^{\frac{\beta^X - \rho}{\beta^X}} \\ &+ (K^{Y,\varepsilon}_2)^{\frac{\kappa}{2}} + (K^{Y,\varepsilon}_p)^{\frac{\kappa}{p}} + \left(K^{X,\varepsilon}_2\right)^{\frac{\beta^X - \rho}{2\beta^X}} + \left(K^{X,\varepsilon}_p\right)^{\frac{\beta^X - \rho}{p\beta^X}} \Bigg] \\ &:= C_{p,\rho}\mathbf{E}^{n,\varepsilon}_p(\kappa, \rho). \end{aligned}$$

Under (**HP5**)*, the right-hand side is finite.*

Assume additionally that (μ, σ, h) satisfies Assumptions (**HP2**$_\delta$) *and* (**HP4**$_\alpha$) *with the α-parameter equal to α^X. Then the above estimate holds true with $\kappa = 1$ ans $\rho = 0$:*

$$\|X^{n,\varepsilon}_t(Y^{n,\varepsilon}_s(y)) - X_t(Y_s(y))\|_{\mathbf{L}_p} \le C_p \Bigg[\frac{1}{n^{\beta^Y}} + \frac{1}{n^{\beta^X}} + \frac{\mathcal{E}^{Y,\varepsilon}_1}{n} + \frac{\mathcal{E}^{X,\varepsilon}_1}{n} + (K^{Y,\varepsilon}_2)^{\frac{1}{2}}$$

$$+(K_p^{Y,\varepsilon})^{\frac{1}{p}} + (K_{2p}^{Y,\varepsilon})^{\frac{1}{2p}} + \left(K_2^{X,\varepsilon}\right)^{\frac{1}{2}} + \left(K_p^{X,\varepsilon}\right)^{\frac{1}{p}} + \left(K_{2p}^{X,\varepsilon}\right)^{\frac{1}{2p}}\Bigg]$$
$$:= C_p \mathbf{E}_p^{n,\varepsilon}.$$

Let us emphasize that the constants $C_{p,\rho}$ and C_p also depend on $l_{\max}$. Compared to [20], the main novelty is the presence of additional terms in the error of the scheme: $\mathcal{E}_1^{Y,\varepsilon}, \mathcal{E}_1^{X,\varepsilon}, K_2^{Y,\varepsilon}, K_2^{X,\varepsilon}, \ldots$ Evoke that if the measure ν is finite as in [20], the global error is

$$\mathbf{E}_p^n(\kappa,\rho) = \frac{1}{n^{\kappa\beta^Y}} + \frac{1}{n^{\beta^X-\rho}}, \quad \mathbf{E}_p^n = \frac{1}{n^{\beta^Y}} + \frac{1}{n^{\beta^X}}.$$

The proof of this main result is postponed in Section 4.

Remark 3.2. *Note that the cross term* $\frac{\mathcal{E}_1^{X,\varepsilon}}{n}$ *may not appear for some particular cases. For example if the Lévy measure has an even density w.r.t. the Lebesgue measure and if* h *and* g *only depend on* e *and are odd ; thus* (8) *becomes:*

$$\mathrm{d}X_t(x) = \mu(t, X_t(x))\mathrm{d}t + \sum_{i=1}^q \sigma_i(t, X_t(x))\mathrm{d}W_t^i + \int_{\mathbb{R}^{q'}} h(e)N(\mathrm{d}t, \mathrm{d}e), \;\; X_0(x) = x.$$

The dynamics of X *is the same as in [20] plus a pure-jump martingale independent of* X*.*

3.5 Global error and examples

The global error $\mathbf{E}_p^{n,\varepsilon}$ in Theorem 5 greatly depends on the Lévy measure ν and the asymptotic behavior of the functions C^X, C^Y. In this section we are interested in some standard examples of these measures. We study in each case the error of the compound scheme and give under which conditions the rate of convergence is optimal i.e., $\mathbf{E}_p^{n,\varepsilon} = O(n^{-\beta})$, with $\beta = \min(\alpha^X, \alpha^Y, \frac{1}{2})$.

Let us mention that under (**HP5**), for any fixed $\varepsilon_0 > 0$ and $\varepsilon < \varepsilon_0$

$$\frac{\mathcal{E}_1^{X,\varepsilon}}{n} = \frac{1}{n}\int_{|e|\geq\varepsilon_0} C^X(e)\nu(\mathrm{d}e) + \frac{1}{n}\int_{|e|\leq\varepsilon_0} C^X(e)\nu^\varepsilon(\mathrm{d}e) = O(n^{-\beta^X}) + \frac{1}{n}\int_{|e|\leq\varepsilon_0} C^X(e)\nu^\varepsilon(\mathrm{d}e).$$

The same holds for Y. Therefore the global error $\mathbf{E}_p^{n,\varepsilon}$ depends on the behavior of ν on a neighborhood of zero. In other words we only have to precise the structure of ν close to zero.

3.5.1 Stable case.

Suppose that the Lévy measure is of the form

$$\nu(\mathrm{d}e) = \frac{1}{|e|^a}\mathrm{d}e$$

where $1 < a < 3$. The stable (tempered) Lévy processes, the CGMY Lévy process ([6]) or the Meixner process ([28]) have a Lévy measure of this form on a neighborhood of zero.

Evoke that the Blumenthal-Getoor index δ of ν is defined as follows:

$$\delta = \inf\left\{\gamma > 0, \quad \int_{|e|<1} |e|^\gamma \nu(\mathrm{d}e) < +\infty\right\}.$$

In what follows, the aim is to establish the next result:

Proposition 2. *Let $1 < a < 3$ and assume that the Lévy measure is of the form*

$$\nu(\mathrm{d}e) = \frac{1}{|e|^a}\mathrm{d}e, \text{ the Blumenthal-Getoor index is } a-1.$$

If in addition $C^X(e) = C^X(1 \wedge |e|^\zeta)$ and $C^Y(e) = C^Y(1 \wedge |e|^{\zeta'})$ for $\zeta, \zeta' \geq 1$, then

- *If $1 < a \leq \min(\zeta,\zeta') + 1$, $\mathbf{E}_p^{n,\varepsilon} = O(n^{-\beta})$ with $\varepsilon = O(n^{\frac{-2\beta}{2\min(\zeta,\zeta')+1-a}})$.*
- *If $\min(\zeta,\zeta') + 1 < a \leq 1 + \dfrac{2\min(\zeta,\zeta')}{\beta+1}$, $\mathbf{E}_p^{n,\varepsilon} = O(n^{-\beta})$ with $\varepsilon = O(n^{-\frac{2}{a-1}})$.*
- *If $1 + \dfrac{2\min(\zeta,\zeta')}{\beta+1} < a < 3$ we have only that $\mathbf{E}_p^{n,\varepsilon} = O(n^{1-\frac{2\min(\zeta,\zeta')}{a-1}})$ with $\varepsilon = O(n^{-\frac{2}{a-1}})$.*

The condition on $C^X(e)$ and $C^Y(e)$ is a standard assumption (see among others [3, 7]). Remark that the rate of convergence only depends on the minimum between ζ and ζ' and that:

- if $\min(\zeta,\zeta') \geq 2$, only the first case holds: $1 \leq a < 3 \leq 1 + \min(\zeta,\zeta')$;
- and if $\min(\zeta,\zeta') \geq \beta + 1$, then the rate of convergence is of order $n^{-\beta}$, whatever a is, because $a < 3 \leq 1 + \dfrac{2\min(\zeta,\zeta')}{\beta+1}$.

The rate can be deteriorated (compared to $n^{-\beta}$) only if $\min(\zeta,\zeta')$ is sufficiently small. Conversely the larger this minimum is, the slower the convergence of ε to zero is.

Proof. In the proof we write for $\beta = \min(\beta^X, \beta^Y)$

$$\mathbf{E}_p^{X,n,\varepsilon}(\zeta) = \frac{1}{n^\beta} + \frac{\mathcal{E}_1^{X,\varepsilon}}{n} + \left(K_2^{X,\varepsilon}\right)^{\frac{1}{2}} + \left(K_p^{X,\varepsilon}\right)^{\frac{1}{p}} + \left(K_{2p}^{X,\varepsilon}\right)^{\frac{1}{2p}}$$

and

$$\mathbf{E}_p^{Y,n,\varepsilon}(\zeta') = \frac{1}{n^\beta} + \frac{\mathcal{E}_1^{Y,\varepsilon}}{n} + \left(K_2^{Y,\varepsilon}\right)^{\frac{1}{2}} + \left(K_p^{Y,\varepsilon}\right)^{\frac{1}{p}} + \left(K_{2p}^{Y,\varepsilon}\right)^{\frac{1}{2p}}$$

and the global error satisfies

$$\mathbf{E}_p^{n,\varepsilon} \leq \mathbf{E}_p^{X,n,\varepsilon}(\zeta) + \mathbf{E}_p^{Y,n,\varepsilon}(\zeta').$$

Since the two errors are similar, in the rest of the proof we only study $\mathbf{E}_p^{X,n,\varepsilon}(\zeta)$ and for simplicity we remove the superscript X.

Under this setting for any $p \geq 2$

$$\int_{\mathbb{R}} C(e)^p \nu(\mathrm{d}e) = C^p \int_{|e|\leq 1} |e|^{\zeta p - a}\mathrm{d}e + C^p \int_{|e|>1} |e|^{-a}\mathrm{d}e < +\infty.$$

That is, (**HP1**) holds since $p\zeta - a > -1$. Here we can explicitly compute for any $\varepsilon < 1$ and $p \geq 2$

$$\left(K_p^\varepsilon\right)^{\frac{1}{p}} = \left(\int_{0<|e|<\varepsilon} |e|^{\zeta p - a}\mathrm{d}e\right)^{\frac{1}{p}} = \left(\frac{2}{\zeta p + 1 - a}\right)^{\frac{1}{p}} \varepsilon^{\zeta + \frac{1-a}{p}}$$

and

$$\mathcal{E}_1^\varepsilon = \int_{\varepsilon<|e|<1} |e|^{\zeta-a}\mathrm{d}e + \int_{|e|>1} |e|^{-a}\mathrm{d}e = \frac{2}{a-1} + \begin{cases} 2\dfrac{1-\varepsilon^{\zeta+1-a}}{\zeta+1-a} & \text{if } 1<a<\zeta+1 \\ -2\ln(\varepsilon) & \text{if } a=\zeta+1 \\ 2\dfrac{\varepsilon^{\zeta+1-a}-1}{a-\zeta-1} & \text{if } \zeta+1<a<3. \end{cases}$$

In particular (**HP5**) holds and the global error becomes

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) = \frac{1}{n^\beta} + \frac{\mathcal{E}_1^\varepsilon}{n} + (K_2^\varepsilon)^{\frac12} + (K_p^\varepsilon)^{\frac1p} + (K_{2p}^\varepsilon)^{\frac{1}{2p}} \leq \frac{1}{n^\beta} + \frac{\mathcal{E}_1^\varepsilon}{n} + C_{p,\zeta}\varepsilon^{\zeta+\frac{1-a}{2}}.$$

Moreover

$$\lim_{\varepsilon\downarrow 0}\mathcal{E}_1^\varepsilon = \begin{cases} \dfrac{2}{\zeta+1-a} & \text{if } 1<a<\zeta+1 \\ +\infty & \text{if } \zeta+1\leq a<3. \end{cases}$$

Finite variation case. If $\zeta > a-1$ (the Blumenthal-Getoor index), $\mathcal{E}_1^\varepsilon$ is bounded, and since $\beta \leq 1/2$ the error can be controlled by:

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq \frac{1}{n^\beta} + \frac{2}{\zeta+1-a}\frac{1}{n} + C_{p,\zeta}\varepsilon^{\zeta+\frac{1-a}{2}} \leq C_{a,\zeta}\frac{1}{n^\beta} + C_{p,\zeta}\varepsilon^{\zeta+\frac{1-a}{2}}$$

and n and ε can be separately fixed. In some sense, there is a decoupling between n and ε:

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq \eta \Longleftrightarrow n \geq \eta^{-\frac{1}{\beta}} \text{ and } \varepsilon \leq \eta^{\frac{2}{2\zeta+1-a}}.$$

Infinite variation case. If $a=\zeta+1$, we have:

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq \frac{1}{n^\beta} + 2\frac{\ln(1/\varepsilon)}{n} + C_p\varepsilon^{\zeta/2}.$$

If $\varepsilon = n^{-2\beta/\zeta}$, then

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq \frac{1}{n^\beta} + \frac{4\beta}{\zeta}\frac{\ln(n)}{n} + C_p\frac{1}{n^\beta} \leq \frac{C}{n^\beta}.$$

For $\zeta+1<a<3$, things are more nested:

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq \frac{1}{n^\beta} + C_a\frac{\varepsilon^{\zeta+1-a}}{n} + C_p\varepsilon^{\zeta+\frac{1-a}{2}},$$

and let us take $\varepsilon = N^{-\gamma}$ for $\gamma>0$. Then

$$\mathbf{E}_p^{n,\varepsilon}(\zeta) \leq n^{-\beta} + C_a n^{-(1-(a-1-\zeta)\gamma)} + C_p n^{-\gamma\zeta+\gamma\frac{a-1}{2}}.$$

Note that $\zeta - \frac{a-1}{2} > 0$ since $\zeta \geq 1$. Hence we should have $0<\gamma<\dfrac{1}{a-1-\zeta}$ to ensure convergence to zero of the error. Easy computations show that

$$\min\left(1-(a-1-\zeta)\gamma, \gamma\zeta-\gamma\frac{a-1}{2}\right) = \begin{cases} \gamma\zeta-\gamma\frac{a-1}{2} & \text{if } 0\leq\gamma\leq\frac{2}{a-1} \\ 1-(a-1-\zeta)\gamma & \text{if } \frac{2}{a-1}\leq\gamma\leq\frac{1}{a-1-\zeta} \end{cases}$$

Hence the optimal choice is $\gamma^* = \dfrac{2}{a-1}$. Moreover for $\gamma = \gamma^*$,

$$\min\left(1-(a-1-\zeta)\gamma^*, \gamma^*\zeta - \gamma^*\frac{a-1}{2}\right) = \frac{2\zeta}{a-1} - 1.$$

And since $\beta \in (0, 1/2]$,

$$\frac{2\zeta}{a-1} - 1 \leq \beta \Longleftrightarrow a \geq 1 + \frac{2\zeta}{\beta+1} = h(\beta) \geq 1 + \frac{4\zeta}{3}.$$

To summarize, previous calculations lead to the following result.

- If $1 < a < \zeta + 1$, $\mathbf{E}_p^{n,\varepsilon}(\zeta) = O(n^{-\beta}) \Longleftrightarrow \varepsilon \leq cN^{\frac{-2\beta}{2\zeta+1-a}}$, for some constant c.
- If $a = \zeta + 1$, it suffices to take $\varepsilon = n^{-2\beta/\zeta}$ to get an optimal convergence rate, i.e., $\mathbf{E}_p^{n,\varepsilon}(\zeta) = O(n^{-\beta})$.
- If $\zeta + 1 < a < 3$ and $\varepsilon = n^{-\gamma}$, $\gamma > 0$, an optimal choice is to take $\gamma = \frac{2}{a-1}$,
  - when $a \leq 1 + \dfrac{2\zeta}{\beta+1}$ we have $\mathbf{E}_p^{n,\varepsilon}(\zeta) = O(n^{-\beta})$.
  - otherwise, $a > 1 + \dfrac{2\zeta}{\beta+1}$ we have only that $\mathbf{E}_p^{n,\varepsilon}(\zeta) = O(n^{1-\frac{2\zeta}{a-1}})$.

We use this result twice for X and Y with parameters ζ and ζ' and the proof of the proposition is achieved. □

3.6 An example with low convergence rate

We now give an example where the convergence rate is very slow.

Proposition 3. *Assume that on a neighborhood of zero the Lévy measure ν and the jump coefficients of X and Y satisfy*

$$\nu(\mathrm{d}e) = \frac{1}{|e|^3(-\ln(e))^\alpha}\mathrm{d}e, \quad \alpha > 1 \qquad C^X(e) = C^X|e| \text{ and } C^Y(e) = C^Y|e|.$$

Then the global error $\mathbf{E}_p^{n,\varepsilon}$ of Theorem 5 is logarithmic, that is, there exists a constant $C_{p,\alpha}$ s.t.

$$\mathbf{E}_p^{n,\varepsilon} \leq C_{p,\alpha} \ln(n)^{\frac{1-\alpha}{2}}.$$

Proof. To simplify, we omit the superscript, the calculations are identical for C^X and C^Y. In order to get the global error, we have (see Theorem 5) to estimate the quantities $\mathcal{E}_1^\varepsilon, K_2^\varepsilon$ and K_p^ε. We start with $\mathcal{E}_1^\varepsilon$,

$$\begin{aligned}\mathcal{E}_1^\varepsilon &= 2\int_\varepsilon^{1/2} \frac{1}{|e|^2(-\ln(e))^\alpha}\mathrm{d}e = 2\int_{\ln(2)}^{-\ln(\varepsilon)} x^{-\alpha}e^x \mathrm{d}x \\ &= 2\int_{\ln(2)}^{\alpha} x^{-\alpha}e^x\mathrm{d}x + 2\int_\alpha^{-\ln(\varepsilon)} x^{-\alpha}e^x\mathrm{d}x.\end{aligned}$$

The first integral in the last line is a constant which we denote by C_α. The idea is then to majorize the second positive integral. Since $(x^{-\alpha}e^x)' = x^{-\alpha-1}e^x(x-\alpha)$, the function$x \mapsto x^{-\alpha}e^x$ is increasing on $[\alpha, +\infty[$ and decreasing on $[\ln(2), \alpha[$. This implies that

$$
\begin{aligned}
0 \le \int_{\ln(2)}^{-\ln(\varepsilon)} x^{-\alpha} e^{x} \mathrm{d}x &\le C_\alpha + (-\ln(\varepsilon))^{-\alpha} e^{-\ln(\varepsilon)} \int_{\alpha}^{-\ln(\varepsilon)} \mathrm{d}x \\
&\le C_\alpha + (-\ln(\varepsilon))^{-\alpha} \varepsilon^{-1} (-\ln(\varepsilon) - \alpha) \\
&= C_\alpha + \varepsilon^{-1}((-\ln(\varepsilon))^{1-\alpha} - \alpha(-\ln(\varepsilon))^{-\alpha}) \\
&\le C\varepsilon^{-1}((-\ln(\varepsilon))^{1-\alpha} - \alpha(-\ln(\varepsilon))^{-\alpha}) \text{ for } \varepsilon \text{ small enough,}
\end{aligned}
$$

which implies that $\lim_{\varepsilon \to 0} \varepsilon \mathcal{E}_1^\varepsilon = 0$ since $\alpha > 1$.

One can also have a more precise boundaries of the second integral using the mean value theorem: as the functions $x^{-\alpha}$ and e^x keep a constant sign on the interval $[\alpha, -\ln(\varepsilon)]$, there exists a constant $c(\alpha, \varepsilon) \in [\alpha, -\ln(\varepsilon)]$ such that,

$$
\int_{\alpha}^{-\ln(\varepsilon)} x^{-\alpha} e^{x} \mathrm{d}x = c(\alpha, \varepsilon)^{-\alpha} \int_{\alpha}^{-\ln(\varepsilon)} e^{x} \mathrm{d}x = c(\alpha, \varepsilon)^{-\alpha} (\varepsilon^{-1} - e^{\alpha}).
$$

Observe that $c(\alpha, \varepsilon)^{-\alpha}$ is bounded as $\alpha > 0$,

$$
(-\ln(\varepsilon))^{-\alpha} \le c(\alpha, \varepsilon)^{-\alpha} \le \alpha^{-\alpha}.
$$

Thus,

$$
\frac{1}{\varepsilon(-\ln(\varepsilon))^{\alpha}}(1 - \varepsilon e^{\alpha}) \le \int_{\ln(2)}^{-\ln(\varepsilon)} x^{-\alpha} e^{x} \mathrm{d}x \le \frac{C}{\varepsilon(-\ln(\varepsilon))^{\alpha}}(-\ln(\varepsilon) - \alpha).
$$

So, for ε small enough $\mathcal{E}_1^\varepsilon \sim C(\varepsilon)\varepsilon^{-1}$ and from previous calculations $\lim_{\varepsilon \to 0} C(\varepsilon) = 0$.
Let now turn to the quantities K_p^ε. For $p > 2$,

$$
\begin{aligned}
K_p^\varepsilon = 2C \int_0^{\varepsilon} e^{p-3} (-\ln(e))^{-\alpha} \mathrm{d}e &= 2C \int_{-\ln(\varepsilon)}^{+\infty} x^{-\alpha} e^{-(p-2)x} \mathrm{d}x \\
\le 2C(-\ln(\varepsilon))^{-\alpha} \int_{-\ln(\varepsilon)}^{+\infty} e^{-(p-2)x} \mathrm{d}x &= \frac{2C}{p-2}(-\ln(\varepsilon))^{-\alpha} \varepsilon^{p-2}.
\end{aligned}
$$

That is, there exists a constant $C_{p,\alpha}$ s.t. $(K_p^\varepsilon)^{\frac{1}{p}} \le C_p(-\ln(\varepsilon))^{-\frac{\alpha}{p}} \varepsilon^{1-\frac{2}{p}}$, $p > 2$.
Taking $p = 2$, we have

$$
K_2^\varepsilon = 2C \int_{-\ln(\varepsilon)}^{+\infty} x^{-\alpha} \mathrm{d}x = 2C \frac{(-\ln(\varepsilon))^{1-\alpha}}{\alpha - 1}.
$$

Equivalently, there exists a constant C_α $(K_2^\varepsilon)^{\frac{1}{2}} \le C_\alpha(-\ln(\varepsilon))^{\frac{1-\alpha}{2}}$.
To sum up, taking $\varepsilon = n^{-\gamma}$, follows

$$
\begin{aligned}
&\frac{1}{n^{\beta}} + \frac{\mathcal{E}_1^\varepsilon}{n} + (K_2^\varepsilon)^{\frac{1}{2}} + (K_p^\varepsilon)^{\frac{1}{p}} + \left(K_{2p}^\varepsilon\right)^{\frac{1}{2p}} \\
&\ge \frac{1}{n^{\beta}} + C\frac{n^{\gamma}}{n}(\gamma \ln(n))^{-\alpha} + C_\alpha(\gamma \ln(n))^{\frac{1-\alpha}{2}}
\end{aligned}
$$

and

$$
\frac{1}{n^{\beta}} + \frac{\mathcal{E}_1^\varepsilon}{n} + (K_2^\varepsilon)^{\frac{1}{2}} + (K_p^\varepsilon)^{\frac{1}{p}} + \left(K_{2p}^\varepsilon\right)^{\frac{1}{2p}}
$$

$$\leq \frac{1}{n^\beta} + \frac{n^\gamma}{n}(\gamma \ln(n))^{1-\alpha} + C_\alpha(\gamma \ln(n))^{\frac{1-\alpha}{2}}$$

that is, for some positive constant $c_{p,\alpha}$

$$\frac{1}{n^\beta} + C\frac{n^\gamma}{n}(\gamma \ln(n))^{-\alpha} + C_\alpha(\gamma \ln(n))^{\frac{1-\alpha}{2}} \leq c_{p,\alpha}\mathbf{E}_p^{n,\varepsilon} \leq \frac{1}{n^\beta} + \frac{n^\gamma}{n}(\gamma \ln(n))^{1-\alpha} + C_\alpha(\gamma \ln(n))^{\frac{1-\alpha}{2}}.$$

So, for n large enough, there exists a new positive constant (independent of N) such that

$$\mathbf{E}_p^{n,\varepsilon} \leq C_{p,\alpha} \ln(n)^{\frac{1-\alpha}{2}}.$$

The proof is then achieved.

□

4 Proof of Theorem 5

The proof of this result requires several intermediate results, some of them being completely new (Theorems 4 and 7 and Proposition 5). Since Theorem 4 above implies (**H1**)-(**H2**) (see Remark 3.1), we seek to prove that Assumptions (**H3**)-(**H4**-a) are also satisfied if that of Theorem 5 hold true.

In this section, to lighten the notations, we assume w.l.o.g. that $\ell_{\max} = 1$. Indeed since we need to control $\nu(\mathrm{d}e)\ell(t)\mathrm{d}t$, we can rewrite

$$\nu(\mathrm{d}e)\ell(t)\mathrm{d}t = \ell_{\max}\nu(\mathrm{d}e)\frac{\ell(t)}{\ell_{\max}}\mathrm{d}t = \widetilde{\nu}(\mathrm{d}e)\widetilde{\ell}(t)\mathrm{d}t$$

with $0 \leq \widetilde{\ell}(t) \leq 1$.

4.1 Euler scheme: local and uniform estimates

In order to prove Theorem 5, we partly generalize the previous results about the SDE to its Euler approximation. Some derivations are more subtle and require details at some places. Recall the definition of Euler scheme in (23).

First, as for the solution of the SDE (8), some estimates for its approximation scheme are needed. This is the analogue of Proposition 1 given in the first statement of the next Proposition. Second, using the same arguments than for the SDE case (Theorem 4), we can put the sup over the space variable inside the $\mathbf{L}_p$-norm to derive the second statement of the following result.

Proposition 4. *Let* (**HP1**) *hold true.*

(i) For any $p > 0$ there exist generic constants $C_{p,(24)}$ and $C_{p,(25)}$ such that

$$\left\| \sup_{0\leq t\leq T} \left|X_t^{n,\varepsilon}(x)\right| \right\|_{\mathbf{L}_p} \leq C_{p,(24)}(1+|x|), \ \forall x \in \mathbb{R}^d, \tag{24}$$

$$\left\| \sup_{0\leq t\leq T} \left|X_t^{n,\varepsilon}(x) - X_t^{n,\varepsilon}(y)\right| \right\|_{\mathbf{L}_p} \leq C_{p,(25)}|x-y|, \ \forall x, \ y \in \mathbb{R}^d. \tag{25}$$

(ii) The estimates (16) and (17) where we replace X by $X^{n,\varepsilon}$ hold true, up to changing the generic constants.

We omit the proof which is quite standard, see that of Theorem 3.2 of Kunita's paper [17]. Let us now show the following estimates on local increments, it will be needed for the sequel.

Lemma 1. *Let Assumption* (**HP1**) *hold true and let $p > 0$. Then there exist generic constants $C_{p,(26)}$ and $C_{p,(27)}$ such that, for any $x, y \in \mathbb{R}^d$ and any $t \in [0,T]$,*

$$\left\| \sup_{\tau_t \le u \le t} |X_u^{n,\varepsilon}(x) - X_{\tau_u}^{n,\varepsilon}(x)| \right\|_{\mathbf{L}_p} \le C_{p,(26)} \frac{(1+|x|)}{n^{1/2}} \left(1 + \frac{\mathcal{E}_1^{X,\varepsilon}}{n^{1/2}} \right), \tag{26}$$

$$\left\| \sup_{\tau_t \le u \le t} |X_u^{n,\varepsilon}(x) - X_u^{n,\varepsilon}(y) - X_{\tau_u}^{n,\varepsilon}(x) + X_{\tau_u}^{n,\varepsilon}(y)| \right\|_{\mathbf{L}_p} \le C_{p,(27)} \frac{|x-y|}{n^{1/2}} \left(1 + \frac{\mathcal{E}_1^{X,\varepsilon}}{n^{1/2}} \right). \tag{27}$$

Proof. Here again, it is enough to prove the estimates for $p \ge 2$, which we assume from now on. Also we take $d = q = q' = 1$ to simplify the exposition. Since, for any $u \in [0,T]$, τ_u is the last discretization time before u, there is no jump in the interval $[\tau_u, u]$, so we have

$$\begin{aligned} X_u^{n,\varepsilon}(x) - X_{\tau_u}^{n,\varepsilon}(x) = & \int_{\tau_u}^u \mu(\tau_s, X_{\tau_s}^{n,\varepsilon}(x)) \mathrm{d}s + \int_{\tau_u}^u \sigma(\tau_s, X_{\tau_s}^{n,\varepsilon}(x)) \mathrm{d}W_s \\ & - \int_{\tau_u}^u \ell(\tau_s) \int_{\mathbb{R}} h(\tau_s, X_{\tau_s^-}^{n,\varepsilon}(x), e) \nu(\mathrm{d}e) \mathrm{d}s. \end{aligned} \tag{28}$$

Applying Theorem 3, the Jensen inequality, and using the fact that $\tau_s = \tau_t$ for any $s \in [\tau_t, t[$, there exists a constant C_p s.t.

$$\begin{aligned} \mathbb{E}\left(\sup_{\tau_t \le u \le t} |X_u^{n,\varepsilon}(x) - X_{\tau_u}^{n,\varepsilon}(x)|^p \right) \le C_p \Big\{ & \mathbb{E}\left((t-\tau_t)^p |\mu(\tau_t, X_{\tau_t}^{n,\varepsilon}(x))|^p + (t-\tau_t)^{p/2} |\sigma(\tau_t, X_{\tau_t}^{n,\varepsilon}(x))|^p \right) \\ & + \mathbb{E}\left((t-\tau_t)^p \left| \int_{\mathbb{R}} |h(\tau_t, X_{\tau_t^-}^{n,\varepsilon}, e)| \nu^\varepsilon(\mathrm{d}e) \right|^p \right) \Big\}. \end{aligned}$$

According to Assumption (**HP1**), it follows that for any $t \in [0,T]$,

$$|\mu(t,x)| + |\sigma(t,x)| \le C^X (1+|x|) \text{ and } |h(t,x,e)| \le C^X(e)(1+|x|).$$

This combined with (24) and the fact that $0 \le t - \tau_t \le \frac{1}{N}$ for any $t \in [0,T]$, follows

$$\begin{aligned} & \mathbb{E}\left(\sup_{\tau_t \le u \le t} |X_u^{n,\varepsilon}(x) - X_{\tau_u}^{n,\varepsilon}(x)|^p \right) \\ & \le C_p C_{p,(24)}^p \left[(C^X)^p \left(\frac{1}{n^p} + \frac{1}{n^{p/2}} \right) + \frac{1}{n^p} \left(\int_{\mathbb{R}} C^X(e) \nu^\varepsilon(\mathrm{d}e) \right)^p \right] (1+|x|)^p), \end{aligned}$$

which readily leads to the announced estimate (26).

Let us now turn to the second inequality. The same arguments combined with Assumption (**HP1**) and (25) lead, for some positive constant C_p, to

$$\begin{aligned} & \mathbb{E}\big(\sup_{\tau_t \le u \le t} |X_u^{n,\varepsilon}(x) - X_u^{n,\varepsilon}(y) - X_{\tau_u}^{n,\varepsilon}(x) + X_{\tau_u}^{n,\varepsilon}(y)|^p \big) \\ & \le C_p \Big\{ \mathbb{E}\left((t-\tau_t)^{p-1} \int_{\tau_t}^t |\mu(\tau_s, X_{\tau_s}^{n,\varepsilon}(x)) - \mu(\tau_s, X_{\tau_s}^{n,\varepsilon}(y))|^p \mathrm{d}s \right) \end{aligned}$$

$$+\mathbb{E}\left((t-\tau_t)^{p/2-1}\int_{\tau_t}^t|\sigma(\tau_s,X^{n,\varepsilon}_{\tau_s}(x))-\sigma(\tau_s,X^{n,\varepsilon}_{\tau_s}(y))|^p\mathrm{d}s\right)$$
$$+\mathbb{E}\left((t-\tau_t)^{p-1}\int_{\tau_t}^t\left|\int_{\mathbb{R}}|h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(y),e)|\nu^\varepsilon(\mathrm{d}e)\right|^p\mathrm{d}s\right)$$
$$\leq C_pC^p_{p,(25)}\left[(C^X)^p(\frac{1}{n^p}+\frac{1}{n^{p/2}})+\frac{1}{n^p}\left(\int_{\mathbb{R}}C^X(e)\nu^\varepsilon(\mathrm{d}e)\right)^p\right]|x-y|^p,$$

which completes the proof. □

Let us explain here Remark 3.2. If h does not depend on x, we don't need to control the increments w.r.t. x and (27) becomes:

$$\left\|\sup_{\tau_t\leq u\leq t}|X^{n,\varepsilon}_u(x)-X^{n,\varepsilon}_u(y)-X^{n,\varepsilon}_{\tau_u}(x)+X^{n,\varepsilon}_{\tau_u}(y)|\right\|_{\mathbf{L}_p}\leq C_{p,(27)}\frac{|x-y|}{n^{1/2}}$$

as in [20]. If the Lévy measure has an even density and if h only depends on e and is odd, then in (28), the term $\int_{\mathbb{R}}h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)\nu(\mathrm{d}e)$ is equal to zero and we obtain the same estimate as in [20] for (26):

$$\left\|\sup_{\tau_t\leq u\leq t}|X^{n,\varepsilon}_u(x)-X^{n,\varepsilon}_{\tau_u}(x)|\right\|_{\mathbf{L}_p}\leq C_{p,(26)}\frac{(1+|x|)}{n^{1/2}}.$$

4.2 Strong convergence results

In order to derive a sharp convergence result, we must take into account the temporal regularity, Assumption ($\mathbf{HP3}_\alpha$), of the coefficients μ, σ and h.

Theorem 6. *Let Assumptions* (**HP1**) *and* ($\mathbf{HP3}_\alpha$) *hold and set* $\beta=\min(\alpha,\frac{1}{2})$. *Then, for any* $p>0$ *there exists a generic constant* $C_{p,(29)}$ *such that for any* $x\in\mathbb{R}^d$

$$\left\|\sup_{t\leq T}|X_t(x)-X^{n,\varepsilon}_t(x)|\right\|_{\mathbf{L}_p}\leq C_{p,(29)}(1+|x|)\left[\frac{1}{n^\beta}+\frac{\mathcal{E}^{X,\varepsilon}_1}{n}+\left(K^{X,\varepsilon}_2\right)^{1/2}+\left(K^{X,\varepsilon}_p\right)^{1/p}\right].\quad(29)$$

Proof. To simplify, as in the previous lemma, we take $d=q=q'=1$. Since it suffices to prove in the case of $p\geq2$, we assume so. By definition, we have

$$X_t(x)-X^{n,\varepsilon}_t(x)=\int_0^t\left(\mu(s,X_s(x))-\mu(\tau_s,X^{n,\varepsilon}_{\tau_s}(x))\right)\mathrm{d}s+\int_0^t\left(\sigma(s,X_s(x))-\sigma(\tau_s,X^{n,\varepsilon}_{\tau_s}(x))\right)\mathrm{d}W_s$$
$$+\int_0^t\int_{\mathbb{R}}\left(h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)\right)\tilde N^\varepsilon(\mathrm{d}s,\mathrm{d}e)$$
$$+\int_0^t\int_{\mathbb{R}}h(s,X_{s^-}(x),e)(\tilde N-\tilde N^\varepsilon)(\mathrm{d}s,\mathrm{d}e).$$

Using the fact that $\lambda(t)\leq1$, $\forall t\in[0,T]$, Theorem 3 and the Jensen inequality for the first two terms, we have for some positive constant C_p,

$$\mathbb{E}\left(\sup_{t\leq T}|X_t(x)-X^{n,\varepsilon}_t(x)|^p\right)\leq C_p\Bigg\{T^{p-1}\int_0^T\mathbb{E}\left(|\mu(s,X_s(x))-\mu(\tau_s,X^{n,\varepsilon}_{\tau_s}(x))|^p\right)\mathrm{d}s$$
$$+T^{p/2-1}\int_0^T\mathbb{E}\left(|\sigma(s,X_s(x))-\sigma(\tau_s,X^{n,\varepsilon}_{\tau_s}(x))|^p\right)\mathrm{d}s$$

$$
\begin{aligned}
&+\mathbb{E}\Big(\Big[\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^2\mathrm{d}s\nu^\varepsilon(\mathrm{d}e)\Big]^{\frac{p}{2}}\Big)\\
&+\mathbb{E}\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\nu^\varepsilon(\mathrm{d}e)\mathrm{d}s\\
&+\mathbb{E}\Big(\Big[\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)|^2\mathrm{d}s\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\Big]^{\frac{p}{2}}\Big)\\
&+\mathbb{E}\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)|^p\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s\Big\}.
\end{aligned}
$$

From now, all terms will be treated in the same way. Taking the jump's terms, we write for any $p\geq 2$,

$$
\begin{aligned}
&|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\\
=\;&|h(s,X_{s^-}(x),e)-h(s,X^{n,\varepsilon}_{s^-}(x),e)+h(s,X^{n,\varepsilon}_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{s^-}(x),e)\\
&\qquad+h(\tau_s,X^{n,\varepsilon}_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\\
\leq\;&3^{p-1}\Big(|h(s,X_{s^-}(x),e)-h(s,X^{n,\varepsilon}_{s^-}(x),e)|^p+|h(s,X^{n,\varepsilon}_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{s^-}(x),e)|^p\\
&\qquad+|h(\tau_s,X^{n,\varepsilon}_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\Big)\\
\leq\;&3^{p-1}[C^X(e)]^p\Big(|X_{s^-}(x)-X^{n,\varepsilon}_{s^-}(x)|^p+(1+|X^{n,\varepsilon}_{\tau_s^-}(x)|)^p(s-\tau_s)^{p\alpha}+|X^{n,\varepsilon}_{s^-}(x)-X^{n,\varepsilon}_{\tau_s^-}(x)|^p\Big)
\end{aligned}
$$

where we used (**HP1**) and (**HP3**$_\alpha$) for the last inequality. This leads to

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\nu^\varepsilon(\mathrm{d}e)\mathrm{d}s\\
\leq\;&3^{p-1}\int_{\mathbb{R}}[C^X(e)]^p\nu^\varepsilon(\mathrm{d}e)\Big(\int_0^T|X_{s^-}(x)-X^{n,\varepsilon}_{s^-}(x)|^p\mathrm{d}s+\int_0^T(1+|X^{n,\varepsilon}_{\tau_s^-}(x)|)^p(s-\tau_s)^{p\alpha}\mathrm{d}s\\
+\;&\int_0^T|X^{n,\varepsilon}_{s^-}(x)-X^{n,\varepsilon}_{\tau_s^-}(x)|^p\mathrm{d}s\Big),\ \forall p\geq 2. \qquad (30)
\end{aligned}
$$

Using estimates (24) in Proposition 4 and (26) in Lemma 1 with the fact that $|s-\tau_s|\leq\dfrac{1}{n}$ for any $s\in[0,T]$, one can easily deduce, for some generic constant C_p,

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\mathbb{E}\big(|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^p\big)\mathrm{d}s\nu^\varepsilon(\mathrm{d}e)\\
&\leq C_p\mathcal{E}_p^{X,\varepsilon}\left(\int_0^T\mathbb{E}\big(|X_{s^-}(x)-X^{n,\varepsilon}_{s^-}(x)|^p\big)\mathrm{d}s+(1+|x|^p)\left[\frac{1}{n^{p\alpha}}+\frac{1}{n^{p/2}}\left(1+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^{p/2}}\right)\right]\right)\\
&\leq C_p\left(\int_0^T\mathbb{E}\big(|X_{s^-}(x)-X^{n,\varepsilon}_{s^-}(x)|^p\big)\mathrm{d}s+(1+|x|^p)\left[\frac{1}{n^{p\beta}}+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}\right]\right), \qquad (31)
\end{aligned}
$$

where we used that for $p\geq 2$, $\mathcal{E}_p^{X,\varepsilon}\leq\mathcal{E}_p^X<+\infty$. Again using inequality (30) for $p=2$, we have

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{R}}|h(s,X_{s^-}(x),e)-h(\tau_s,X^{n,\varepsilon}_{\tau_s^-}(x),e)|^2\mathrm{d}s\nu^\varepsilon(\mathrm{d}e)\\
\leq\;&3\int_{\mathbb{R}}[C^X(e)]^2\nu^\varepsilon(\mathrm{d}e)\Big(\int_0^T|X_{s^-}(x)-X^{n,\varepsilon}_{s^-}(x)|^2\mathrm{d}s+\int_0^T(1+|X^{n,\varepsilon}_{\tau_s^-}(x)|)^2(s-\tau_s)^{2\alpha}\mathrm{d}s
\end{aligned}
$$

$$+ \quad \int_0^T |X^{n,\varepsilon}_{s^-}(x) - X^{n,\varepsilon}_{\tau_s^-}(x)|^2 \mathrm{d}s\Big),$$

and therefore, by the Jensen inequality

$$\begin{aligned}
&\Big(\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e) - h(\tau_s, X^{n,\varepsilon}_{\tau_s^-}(x), e)|^2 \mathrm{d}s\nu^\varepsilon(\mathrm{d}e)\Big)^{\frac{p}{2}} \\
\leq\ & 3^{p-1}\big(\mathcal{E}_2^{X,\varepsilon}\big)^{\frac{p}{2}} T^{\frac{p}{2}-1}\Big(\int_0^T |X_{s^-}(x) - X^{n,\varepsilon}_{s^-}(x)|^p \mathrm{d}s + \int_0^T (1 + |X^{n,\varepsilon}_{\tau_s^-}(x)|)^p (s-\tau_s)^{p\alpha} \mathrm{d}s \\
+\ & \int_0^T |X^{n,\varepsilon}_{s^-}(x) - X^{n,\varepsilon}_{\tau_s^-}(x)|^p \mathrm{d}s\Big),
\end{aligned}$$

thus we conclude as for (31), that

$$\begin{aligned}
&\mathbb{E}\left(\Big(\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e) - h(\tau_s, X^{n,\varepsilon}_{\tau_s^-}(x), e)|^2 \mathrm{d}s\nu(\mathrm{d}e)\Big)^{\frac{p}{2}}\right) \\
\leq\ & C_p\big(\mathcal{E}_2^{X,\varepsilon}\big)^{\frac{p}{2}} \left(\int_0^T \mathbb{E}\big(|X_s(x) - X^{n,\varepsilon}_s(x)|^p\big)\mathrm{d}s + (1+|x|^p)\left[\frac{1}{n^{p\beta}} + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}\right]\right).
\end{aligned}$$

Again evoke that $\mathcal{E}_2^{X,\varepsilon} \leq \mathcal{E}_2^X < +\infty$. Therefore

$$\begin{aligned}
&\mathbb{E}\left(\Big(\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e) - h(\tau_s, X^{n,\varepsilon}_{\tau_s^-}(x), e)|^2 \mathrm{d}s\nu(\mathrm{d}e)\Big)^{\frac{p}{2}}\right) \\
\leq\ & C_p \left(\int_0^T \mathbb{E}\big(|X_s(x) - X^{n,\varepsilon}_s(x)|^p\big)\mathrm{d}s + (1+|x|^p)\left[\frac{1}{n^{p\beta}} + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}\right]\right).
\end{aligned}$$

for some new constant C_p.

$$\begin{aligned}
&\mathbb{E}\left(\Big[\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e)|^2 \mathrm{d}s \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\Big]^{\frac{p}{2}}\right) \\
&+\mathbb{E}\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e)|^p \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s \\
&\leq \mathbb{E}\int_0^T (1+|X_{s^-}|^p)\mathrm{d}s \left[\left(\int_{\mathbb{R}} (C^X(e))^2 \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\right)^{p/2} + \int_{\mathbb{R}} (C^X(e))^p \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\right].
\end{aligned}$$

Then, using (12),

$$\begin{aligned}
&\mathbb{E}\left(\Big[\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e)|^2 \mathrm{d}s \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\Big]^{\frac{p}{2}}\right) + \mathbb{E}\int_0^T \int_{\mathbb{R}} |h(s, X_{s^-}(x), e)|^p \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s \\
&\leq C_p(1+|x|^p)\left[\left(K_2^{X,\varepsilon}\right)^{p/2} + K_p^{X,\varepsilon}\right].
\end{aligned}$$

By the same reasoning for μ and σ, we deduce the existence of a constant C_p such that

$$\begin{aligned}
\mathbb{E}\left(\sup_{t\leq T} |X_t(x) - X^{n,\varepsilon}_t(x)|^p\right) &\leq C_p\left(\int_0^T \mathbb{E}\big(\sup_{u\leq s} |X_u(x) - X^{n,\varepsilon}_u(x)|^p\big)\mathrm{d}s\right. \\
&\left. + (1+|x|^p)\left[\frac{1}{n^{p\beta}} + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p} + \left(K_2^{X,\varepsilon}\right)^{p/2} + K_p^{X,\varepsilon}\right]\right).
\end{aligned}$$

The proof is then achieved by applying the Gronwall's lemma and Jensen's inequality. □

As mentioned in Remark 3.2, in Estimate (29), the cross term $\mathcal{E}_1^{X,\varepsilon}/n$ can be removed for some cases.

This result will be useful in the proof of our main result Theorem 5 (see Section 4.3), but unfortunately it is not sufficient at all. In view of Theorem 1 and its assumptions (in particular (**H3**)), one should have a sup over $|x| \leq \lambda$ inside the $\mathbf{L}_p$-norm. This is the purpose of the next derivations.

We now aim at obtaining uniform in space convergence results. Let us start with an easy result.

Proposition 5. *Assume* (**HP1**), (**HP3**$_\alpha$) *and let* $\beta = \min(\alpha, \frac{1}{2})$. *For any* $p > 0$ *and any* $\rho \in [0, \beta]$, *there exists a generic constant* $C_{p,\rho,(32)}$ *such that*

$$\|X_t(x) - X_t^{n,\varepsilon}(x) - X_t(y) + X_t^{n,\varepsilon}(y)\|_{\mathbf{L}_p} \leq C_{p,\rho,(32)}|x-y|^{\rho/\beta}(1+|x|+|y|)^{1-\rho/\beta} \times \left[\frac{1}{n^{\beta-\rho}} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n}\right)^{\frac{\beta-\rho}{\beta}} + \left(K_2^{X,\varepsilon}\right)^{\frac{\beta-\rho}{2\beta}} + \left(K_p^{X,\varepsilon}\right)^{\frac{\beta-\rho}{p\beta}}\right] \tag{32}$$

for all $x, y \in \mathbb{R}^d$ *and* $t \in [0,T]$. *Furthermore, for any* $p > 0$ *and any* $\rho > 0$, *there exists a generic constant* $C_{p,\rho,(33)}$ *such that, for any* $t \in [0,T]$,

$$\left\|\sup_{|x|\leq\lambda} |X_t(x) - X_t^{n,\varepsilon}(x)|\right\|_{\mathbf{L}_p} \leq \lambda C_{p,\rho,(33)} \left[\frac{1}{n^{\beta-\rho}} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n}\right)^{\frac{\beta-\rho}{\beta}} + \left(K_2^{X,\varepsilon}\right)^{\frac{\beta-\rho}{2\beta}} + \left(K_p^{X,\varepsilon}\right)^{\frac{\beta-\rho}{p\beta}}\right]. \tag{33}$$

Here again Remark (3.2) holds. To prove this result we can adapt the one of [13, Theorem 6] except that we have to pay attention to the additional terms $\mathcal{E}_1^{X,\varepsilon}$, $K_2^{X,\varepsilon}$ and $K_p^{X,\varepsilon}$.

Proof. Let $\Gamma(t,x,y,n) := \||X_t(x) - X_t^{n,\varepsilon}(x) - X_t(y) + X_t^{n,\varepsilon}(y)|\|_{\mathbf{L}_p}$ and consider $p \geq 1$ without loss of generality. Using triangular inequality in two ways and inequalities (13)-(25)-(29), we get

$$\begin{cases} \Gamma(t,x,y,n) \leq \left(C_{p,(13)} + C_{p,(25)}\right)|x-y|, \\ \Gamma(t,x,y,n) \leq 2C_{p,(29)}(1+|x|+|y|)\left[\frac{1}{n^\beta} + \frac{\mathcal{E}_1^{X,\varepsilon}}{n} + \left[\left(K_2^{X,\varepsilon}\right)^{p/2} + K_p^{X,\varepsilon}\right]^{1/p}\right]. \end{cases}$$

Let $C_{p,\rho,(32)} := \left(C_{p,(13)} + C_{p,(25)}\right)^{\rho/\beta}\left(2C_{p,(29)}\right)^{1-\rho/\beta}$, by interpolation between the two estimates, it readily follows that

$$\begin{aligned}\Gamma(t,x,y,n) &\leq C_{p,\rho,(32)}|x-y|^{\rho/\beta}(1+|x|+|y|)^{1-\rho/\beta}\left[\frac{1}{n^\beta} + \frac{\mathcal{E}_1^{X,\varepsilon}}{n} + \left[\left(K_2^{X,\varepsilon}\right)^{p/2} + K_p^{X,\varepsilon}\right]^{1/p}\right]^{1-\rho/\beta} \\ &\overset{\text{Jensen}}{\leq} C_{p,\rho,(32)}|x-y|^{\rho/\beta}(1+|x|+|y|)^{1-\rho/\beta} \times \\ &\left[\frac{1}{n^{\beta-\rho}} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n}\right)^{\frac{\beta-\rho}{\beta}} + \left(K_2^{X,\varepsilon}\right)^{\frac{\beta-\rho}{2\beta}} + \left(K_p^{X,\varepsilon}\right)^{\frac{\beta-\rho}{p\beta}}\right]\end{aligned}$$

therefore (32) is proved.

To get (33) it is enough to assume $\rho \in (0, \beta]$; we apply Corollary 1 by checking the assumptions of Theorem 2 applied to $G(x) := X_t(x) - X_t^{n,\varepsilon}(x)$. From (32) we can set

$$C^{(G)} = C_{p,\rho,(32)} \left[\frac{1}{n^{\beta-\rho}} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n} \right)^{\frac{\beta-\rho}{\beta}} + \left(K_2^{X,\varepsilon} \right)^{\frac{\beta-\rho}{2\beta}} + \left(K_p^{X,\varepsilon} \right)^{\frac{\beta-\rho}{p\beta}} \right],$$

$\tau^{(G)} = 1 - \rho/\beta$ and $\beta^{(G)} = \rho/\beta$ provided that $\rho/\beta \in (d/p, 1]$, which is true for p large enough (recall that $\rho > 0$). Therefore for such p, the estimate (6) holds true, which yields the announced inequality (33). The estimate for smaller values of p are automatically satisfied invoking once again the stability of $\mathbf{L}_p$ norms as p decreases. □

As it is the case for Theorem 6, this result is not sufficient to derive Theorem 5. The next step is to generalize these two results. By making the best use of the regularity assumptions made on $\nabla_x \mu$ and $\nabla_x \sigma_i$ (see ($\mathbf{HP2}_\delta$) and ($\mathbf{HP4}_\alpha$)) we now obtain, in the following crucial Theorem, improved dependency in n by allowing the case $\rho = 0$.

In the rest of the paper,

$$K^{X,\varepsilon} = \left(K_2^{X,\varepsilon} \right)^{p/2} + K_p^{X,\varepsilon} + (K_{2p}^{X,\varepsilon})^{\frac{1}{2}}. \tag{34}$$

Theorem 7. *Let* ($\mathbf{HP1}$), ($\mathbf{HP2}_\delta$), ($\mathbf{HP3}_\alpha$), ($\mathbf{HP4}_\alpha$) *hold and let* $\beta = \min(\alpha^X, \frac{1}{2})$. *For any* $p > 0$, *there exists a generic constant* $C_{p,(35)}$ *such that*

$$\begin{aligned} &\left\| \sup_{u \le t} |X_u(x) - X_u^{n,\varepsilon}(x) - X_u(y) + X_u^{n,\varepsilon}(y)| \right\|_{\mathbf{L}_p} \\ &\le C_{p,(35)}(1 + |x| + |y|) \left[|x - y| + |x - y|^\delta \right] \left[\frac{1}{n^\beta} + \frac{\mathcal{E}_1^{X,\varepsilon}}{n} + (K^{X,\varepsilon})^{\frac{1}{p}} \right] \end{aligned} \tag{35}$$

for all $x, y \in \mathbb{R}^d$ *and* $t \in [0, T]$. *Furthermore, for any* $p > 0$ *there exists a generic constant* $C_{p,(36)}$ *such that, for any* $t \in [0, T]$,

$$\left\| \sup_{|x| \le \lambda} |X_t(x) - X_t^{n,\varepsilon}(x)| \right\|_{\mathbf{L}_p} \le \lambda^2 C_{p,(36)} \left(\frac{1}{n^\beta} + \frac{\mathcal{E}_1^{X,\varepsilon}}{n} + (K^{X,\varepsilon})^{\frac{1}{p}} \right), \quad \forall \lambda \ge 1. \tag{36}$$

Proof. As in the previous proofs, we argue that it is enough to assume $p \ge 2$. To alleviate the presentation, we additionally assume $d = q = q' = 1$, the derivation in the general case being similar. From the dynamics of X and $X^{n,\varepsilon}$, we write

$$\begin{aligned} &X_t(x) - X_t^{n,\varepsilon}(x) - X_t(y) + X_t^{n,\varepsilon}(y) \\ &= \int_0^t \Big(\mu(s, X_s(x)) - \mu(\tau_s, X_{\tau_s}^{n,\varepsilon}(x)) - \mu(s, X_s(y)) + \mu(\tau_s, X_{\tau_s}^{n,\varepsilon}(y)) \Big) \mathrm{d}s \\ &+ \int_0^t \Big(\sigma(s, X_s(x)) - \sigma(\tau_s, X_{\tau_s}^{n,\varepsilon}(x)) - \sigma(s, X_s(y)) + \sigma(\tau_s, X_{\tau_s}^{n,\varepsilon}(y)) \Big) \mathrm{d}W_s \\ &+ \int_0^t \int_{\mathbb{R}} \Big(h(s, X_{s^-}(x), e) - h(\tau_s, X_{\tau_s^-}^{n,\varepsilon}(x), e) - h(s, X_{s^-}(y), e) + h(\tau_s, X_{\tau_s^-}^{n,\varepsilon}(y), e) \Big) \tilde{N}^\varepsilon(\mathrm{d}s, \mathrm{d}e) \\ &+ \int_0^t \int_{\mathbb{R}} \Big(h(s, X_{s^-}(x), e) - h(s, X_{s^-}(y), e), e) \Big) (\tilde{N} - \tilde{N}^\varepsilon)(\mathrm{d}s, \mathrm{d}e). \end{aligned}$$

The same reasoning as in the proof of Theorem 6 leads, for some generic constant C_p, to

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{u\le t}|X_u(x)-X_u^{n,\varepsilon}(x)-X_u(y)+X_u^{n,\varepsilon}(y)|^p\right)\\
&\le C_p\Bigg\{t^{p-1}\int_0^t\mathbb{E}\left(\left|\mu(s,X_s(x))-\mu(\tau_s,X_{\tau_s}^{n,\varepsilon}(x))-\mu(s,X_s(y))+\mu(\tau_s,X_{\tau_s}^{n,\varepsilon}(y))\right|^p\right)\mathrm{d}s\\
&+t^{p/2-1}\int_0^t\mathbb{E}\left(\left|\sigma(s,X_s(x))-\sigma(\tau_s,X_{\tau_s}^{n,\varepsilon}(x))-\sigma(s,X_s(y))+\sigma(\tau_s,X_{\tau_s}^{n,\varepsilon}(y))\right|^p\right)\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}}\mathbb{E}\left(\left|h(s,X_s(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y))\right|^p\right)\nu^\varepsilon(\mathrm{d}e)\mathrm{d}s\\
&+\mathbb{E}\left(\Big[\int_0^t\int_{\mathbb{R}}\left|h(s,X_s(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y),e)\right|^2\nu^\varepsilon(\mathrm{d}e)\mathrm{d}s\Big]^{\frac{p}{2}}\right)\\
&+\int_0^t\int_{\mathbb{R}}\mathbb{E}\left(\left|h(s,X_s(x),e)-h(s,X_s(y),e)\right|^p\right)\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s\\
&+\mathbb{E}\left(\Big[\int_0^t\int_{\mathbb{R}}\left|h(s,X_s(x),e)-h(s,X_s(y),e)\right|^2\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s\Big]^{\frac{p}{2}}\right)\Bigg\}. \qquad (37)
\end{aligned}
$$

1. The first four terms of the right side of above inequality can be treated in the same way, thus we only detail the computations for the last integral. First write that

$$
\begin{aligned}
&h(s,X_s(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y),e)\\
&=h(s,X_s(x),e)-h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(s,X_s^{n,\varepsilon}(y),e)\\
&+h(s,X_s^{n,\varepsilon}(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_s^{n,\varepsilon}(y),e)+h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y),e). \qquad (38)
\end{aligned}
$$

Now, we treat the two lines above separately.
1.a) Denoting by $X_s^{n,\varepsilon,\lambda}(x):=X_s(x)+\lambda\big(X_s^{n,\varepsilon}(x)-X_s(x)\big)$ for $\lambda\in[0,1]$, we have

$$
\begin{aligned}
&h(s,X_s(x),e)-h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(s,X_s^{n,\varepsilon}(y),e)\\
&=\big(X_s(x)-X_s^{n,\varepsilon}(x)-X_s(y)+X_s^{n,\varepsilon}(y)\big)\int_0^1\nabla_xh\big(s,X_s^{n,\varepsilon,\lambda}(x),e\big)\mathrm{d}\lambda\\
&+\big(X_s(y)-X_s^{n,\varepsilon}(y)\big)\int_0^1\big(\nabla_xh(s,X_s^{n,\varepsilon,\lambda}(x),e)-\nabla_xh(s,X_s^{n,\varepsilon,\lambda}(y),e)\big)\mathrm{d}\lambda.
\end{aligned}
$$

By definition of $X^{n,\varepsilon,\lambda}$, using the fact that $|\nabla_xh(t,x,e)|\le C^{X,\nabla}(e)$ and $|\nabla_xh(t,x)-\nabla_xh(t,y)|\le C^{X,\nabla}(e)|x-y|^\delta$ with $\int_{\mathbb{R}}(C^{X,\nabla}(e))^p\nu^\varepsilon(de)<\infty$, $\forall p\ge 2$; there exists a generic constant C_p (whose values may change from line to line)

$$
\begin{aligned}
&\int_{\mathbb{R}}|h(s,X_s(x),e)-h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(s,X_s^{n,\varepsilon}(y),e)|^p\nu^\varepsilon(\mathrm{d}e)\\
&\le C_p\Big(|X_s(x)-X_s^{n,\varepsilon}(x)-X_s(y)+X_s^{n,\varepsilon}(y)|^p\\
&\qquad+|X_s(y)-X_s^{n,\varepsilon}(y)|^p\int_0^1\big|(1-\lambda)\big(X_s(x)-X_s(y)\big)+\lambda\big(X_s^{n,\varepsilon}(x)-X_s^{n,\varepsilon}(y)\big)\big|^{\delta p}\mathrm{d}\lambda\Big)\\
&\le C_p\Big[|X_s(x)-X_s^{n,\varepsilon}(x)-X_s(y)+X_s^{n,\varepsilon}(y)|^p\\
&\qquad+|X_s(y)-X_s^{n,\varepsilon}(y)|^p\Big(|X_s(x)-X_s(y)|^{\delta p}+|X_s^{n,\varepsilon}(x)-X_s^{n,\varepsilon}(y)|^{\delta p}\Big)\Big],
\end{aligned}
$$

where we have used the Minkowsky inequality to handle the dλ-integral and also used the inequality

$$(a+b)^\gamma \le 2^{(\gamma-1)_+}\left(a^\gamma+b^\gamma\right) \le 2^\gamma\left(a^\gamma+b^\gamma\right),\ \forall a,b,\gamma\ge 0,$$

where $(\gamma-1)_+ = \begin{cases} \gamma-1 & \text{if } \gamma-1\ge 0 \\ 0 & \text{if not} \end{cases}$.

Now, we integrate over (s,ω) and apply the Cauchy-Schwarz inequality, to obtain

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}} |h(s,X_s(x),e)-h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(s,X_s^{n,\varepsilon}(y),e)|^p\nu^\varepsilon(\mathrm{d}e)\mathrm{d}s\Big]\\
&\le C_p\Big[\int_0^t \mathbb{E}\left(|X_s(x)-X_s^{n,\varepsilon}(x)-X_s(y)+X_s^{n,\varepsilon}(y)|^p\right)\mathrm{d}s\\
&+\int_0^t\sqrt{\mathbb{E}\left(|X_s(y)-X_s^{n,\varepsilon}(y)|^{2p}\right)}\sqrt{\mathbb{E}\left(|X_s(x)-X_s(y)|^{2\delta p}+|X_s^{n,\varepsilon}(x)-X_s^{n,\varepsilon}(y)|^{2\delta p}\right)}\mathrm{d}s\Big]
\end{aligned}$$

which rewrites, owing to (13)-(25) and (29),

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t\int_{\mathbb{R}} |h(s,X_s(x),e)-h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_s(y),e)+h(s,X_s^{n,\varepsilon}(y),e)|^p\nu^\varepsilon(de)\mathrm{d}s\Big]\\
&\le C_p\Bigg(\int_0^t \mathbb{E}\left(|X_s(x))-X_s^{n,\varepsilon}(x)-X_s(y)+X_s^{n,\varepsilon}(y)|^p\right)\mathrm{d}s\\
&\qquad+(1+|y|)^p|x-y|^{\delta p}\left[\frac{1}{n^{p\beta}}+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}+\left(K_2^{X,\varepsilon}\right)^{p/2}+(K_{2p}^{X,\varepsilon})^{\frac12}\right]\Bigg),
\end{aligned}\tag{39}$$

for a new generic constant C_p.

1.b) Now we focus on the second line of identity (38). Similarly to before, we introduce for simplicity the notations

$$\begin{cases} \tilde{X}_s^{n,\varepsilon,\lambda}(x) := X_s^{n,\varepsilon}(x)+\lambda(X_{\tau_s}^{n,\varepsilon}(x)-X_s^{n,\varepsilon}(x))\\ X_{\tau_s}^{n,\varepsilon,\lambda}(x,y) := X_{\tau_s}^{n,\varepsilon}(x)+\lambda(X_{\tau_s}^{n,\varepsilon}(y)-X_{\tau_s}^{n,\varepsilon}(x)) \end{cases}$$

$$\begin{aligned}
&h(s,X_s^{n,\varepsilon}(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_s^{n,\varepsilon}(y),e)+h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y),e)\\
&=h(s,X_s^{n,\varepsilon}(x),e)-h(s,X_{\tau_s}^{n,\varepsilon}(x),e)-\big(h(s,X_s^{n,\varepsilon}(y),e)-h(s,X_{\tau_s}^{n,\varepsilon}(y),e)\big)\\
&+h(s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(s,X_{\tau_s}^{n,\varepsilon}(y),e)-\big(h(\tau_s,X_{\tau_s}^{n,\varepsilon}(x),e)-h(\tau_s,X_{\tau_s}^{n,\varepsilon}(y),e)\big)\\
&=\int_0^1\nabla_x h\big(s,\tilde{X}_s^{n,\varepsilon,\lambda}(x),e\big)\mathrm{d}\lambda(X_s^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(x))-\int_0^1\nabla_x h\big(s,\tilde{X}_s^{n,\varepsilon,\lambda}(y),e\big)\mathrm{d}\lambda(X_s^{n,\varepsilon}(y)-X_{\tau_s}^{n,\varepsilon}(y))\\
&+\int_0^1\nabla_x h\big(s,X_{\tau_s}^{n,\varepsilon,\lambda}(x,y),e\big)\mathrm{d}\lambda(X_{\tau_s}^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(y))-\int_0^1\nabla_x h\big(\tau_s,X_{\tau_s}^{n,\varepsilon,\lambda}(x,y),e\big)\mathrm{d}\lambda(X_{\tau_s}^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(y))\\
&=\int_0^1\Big(\nabla_x h\big(s,\tilde{X}_s^{n,\varepsilon,\lambda}(x),e\big)-\nabla_x h\big(s,\tilde{X}_s^{n,\varepsilon,\lambda}(y),e\big)\Big)\mathrm{d}\lambda(X_s^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(x))\\
&+\int_0^1\nabla_x h\big(s,\tilde{X}_s^{n,\varepsilon,\lambda}(y),e\big)\mathrm{d}\lambda\times(X_s^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(x)-X_s^{n,\varepsilon}(y)+X_{\tau_s}^{n,\varepsilon}(y))\\
&+\int_0^1\Big(\nabla_x h\big(s,X_{\tau_s}^{n,\varepsilon,\lambda}(x,y),e\big)-\nabla_x h\big(\tau_s,X_{\tau_s}^{n,\varepsilon,\lambda}(x,y),e\big)\Big)\mathrm{d}\lambda(X_{\tau_s}^{n,\varepsilon}(x)-X_{\tau_s}^{n,\varepsilon}(y)).
\end{aligned}$$

As in the first step of this proof, taking the power p using the fact that $|\nabla_x h(t,x,e)| \leq C^{X,\nabla}(e)$ and $|\nabla_x h(t,x) - \nabla_x h(t,y)| \leq C^{X,\nabla}(e)|x-y|^\delta$ with $\int_{\mathbb{R}} (C^{X,\nabla}(e))^p \nu(\mathrm{d}e) < \infty,\ \forall p \geq 2$; there exists a generic constant C_p such that

$$\begin{aligned}
&\int_{\mathbb{R}} |h(s, X_s^{n,\varepsilon}(x), e) - h(\tau_s, X_{\tau_s}^{n,\varepsilon}(x), e) - h(s, X_s^{n,\varepsilon}(y), e) + h(\tau_s, X_{\tau_s}^{n,\varepsilon}(y), e)|^p \nu^\varepsilon(\mathrm{d}e) \\
&\leq C_p \Big[\int_0^1 \Big| (1-\lambda)(X_s^{n,\varepsilon}(x) - X_s^{n,\varepsilon}(y)) + \lambda (X_{\tau_s}^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(y)) \Big|^{p\delta} \mathrm{d}\lambda |X_s^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(x)|^p \\
&\quad + \big| X_s^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(x) - X_s^{n,\varepsilon}(y) + X_{\tau_s}^{n,\varepsilon}(y) \big|^p \\
&\quad + |s - \tau_s|^{p\alpha} \int_0^1 \Big(1 + \big| X_{\tau_s}^{n,\varepsilon}(x) + \lambda (X_{\tau_s}^{n,\varepsilon}(y) - X_{\tau_s}^{n,\varepsilon}(x)) \big| \Big)^p \mathrm{d}\lambda \big| X_{\tau_s}^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(y) \big|^p \Big].
\end{aligned}$$

Integrating w.r.t. (s,ω), one can easily get, for some new generic constant C_p

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \mathbb{E} \left(|h(s, X_s^{n,\varepsilon}(x), e) - h(\tau_s, X_{\tau_s}^{n,\varepsilon}(x), e) - h(s, X_s^{n,\varepsilon}(y), e) + h(\tau_s, X_{\tau_s}^{n,\varepsilon}(y), e)|^p \right) \nu^\varepsilon(\mathrm{d}e)\mathrm{d}s \\
&\leq C_p \Bigg[\int_0^t \left(\sqrt{\mathbb{E}\left(|X_s^{n,\varepsilon}(x) - X_s^{n,\varepsilon}(y)|^{2p\delta}\right)} + \sqrt{\mathbb{E}\left(|X_{\tau_s}^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(y))|^{2p\delta}\right)} \right) \\
&\qquad\qquad \times \sqrt{\mathbb{E}\left(|X_s^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(x)|^{2p}\right)} \mathrm{d}s \\
&\quad + \int_0^t \mathbb{E}\left(|X_s^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(x) - X_s^{n,\varepsilon}(y) + X_{\tau_s}^{n,\varepsilon}(y)|^p \right) \mathrm{d}s \\
&+ \frac{1}{n^{\alpha p}} \int_0^t \left(1 + \sqrt{\mathbb{E}\left(|X_{\tau_s}^{n,\varepsilon}(x)|^{2p}\right)} + \sqrt{\mathbb{E}\left(|X_{\tau_s}^{n,\varepsilon}(y)|^{2p}\right)} \right) \sqrt{\mathbb{E}\left(\big| X_{\tau_s}^{n,\varepsilon}(x) - X_{\tau_s}^{n,\varepsilon}(y) \big|^{2p} \right)} \mathrm{d}s \Bigg].
\end{aligned}$$

From results of Proposition 4 and Lemma 1, there is a new constant C_p such that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} |h(s, X_s^{n,\varepsilon}(x), e) - h(\tau_s, X_{\tau_s}^{n,\varepsilon}(x), e) - h(s, X_s^{n,\varepsilon}(y), e) + h(\tau_s, X_{\tau_s}^{n,\varepsilon}(y), e)|^p \nu(\mathrm{d}e)\mathrm{d}s \Big] \\
&\leq C_p \Bigg(|x-y|^{p\delta}(1+|x|)^p \frac{1}{n^{p/2}} \left(1 + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^{p/2}} \right) + \frac{|x-y|^p}{n^{p/2}} \left(1 + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^{p/2}} \right) \\
&\qquad + \frac{|x-y|^p}{n^{\alpha p}} (1 + |x|^p + |y|^p) \Bigg) \\
&\leq C_p (1 + |x| + |y|)^p \left(|x-y|^p + |x-y|^{\delta p} \right) \left(\frac{1}{n^{\beta p}} + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p} \right).
\end{aligned} \tag{40}$$

Thus, (39) and (40) combined with (38) lead to

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}} |h(s, X_s^{n,\varepsilon}(x), e) - h(\tau_s, X_{\tau_s}^{n,\varepsilon}(x), e) - h(s, X_s^{n,\varepsilon}(y), e) + h(\tau_s, X_{\tau_s}^{n,\varepsilon}(y), e)|^p \nu(\mathrm{d}e)\mathrm{d}s \Big] \\
&\leq C_p \Bigg[\int_0^t \mathbb{E}\left(|X_s(x) - X_s^{n,\varepsilon}(x) - X_s(y) + X_s^{n,\varepsilon}(y)|^p \right) \mathrm{d}s \\
&+ (1 + |x| + |y|)^p \left(|x-y|^p + |x-y|^{\delta p} \right) \left(\frac{1}{n^{\beta p}} + \frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p} + (K_2^{X,\varepsilon})^{p/2} + (K_{2p}^{X,\varepsilon})^{1/2} \right) \Bigg]
\end{aligned} \tag{41}$$

where C_p is a new constant. The same estimates, without the terms $K^{X,\varepsilon}$ and $\mathcal{E}^{X,\varepsilon}$, hold for μ and σ instead of h.

2. For the last two remaining terms in (37), using (**HP1**) and Proposition 4, we have

$$\mathbb{E}\left(\int_0^t\int_{\mathbb{R}}\Big|h(s,X_s(x))-h(s,X_s(y))\Big|^p\right)\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s\le C_p|x-y|^pK_p^{X,\varepsilon}$$

$$\mathbb{E}\left(\Big(\int_0^t\int_{\mathbb{R}}\Big|h(s,X_s(x))-h(s,X_s(y))\Big|^2\mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)\mathrm{d}s\Big)^{\frac{p}{2}}\right)\le C_p|x-y|^p(K_2^{X,\varepsilon})^{p/2}.$$

Hence, plugging the above into (37), we obtain the existence of generic constants C_p such that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{u\le t}|X_u(x)-X_u^{n,\varepsilon}(x)-X_u(y)+X_u^{n,\varepsilon}(y)|^p\right)\\
&\le C_p\Big[\int_0^t\mathbb{E}\left(\sup_{u\le s}|X_u(x)-X_u^{n,\varepsilon}(x)-X_u(y)+X_u^{n,\varepsilon}(y)|^p\right)\mathrm{d}s\\
&\qquad+(1+|x|+|y|)^p\big(|x-y|^p+|x-y|^{\delta p}\big)\left(\frac{1}{n^{\beta p}}+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}+K_p^{X,\varepsilon}+(K_2^{X,\varepsilon})^{p/2}+(K_{2p}^{X,\varepsilon})^{\frac12}\right)\Big]\\
&\le C_p(1+|x|+|y|)^p\big(|x-y|^p+|x-y|^{\delta p}\big)\left(\frac{1}{n^{\beta p}}+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}+K_p^{X,\varepsilon}+(K_2^{X,\varepsilon})^{\frac{p}{2}}+(K_{2p}^{X,\varepsilon})^{\frac12}\right),
\end{aligned}$$

where the last inequality follows from Gronwall's Lemma and Jensen's inequality, we recall that $K^{X,\varepsilon}$ is defined by (34); the proof of (35) is completed.
Let us now deduce (36) from (35) by applying Corollary 1 with $G(x):=X_t(x)-X_t^{n,\varepsilon}(x)$. From (35) we have

$$\begin{aligned}
\|G(x)-G(y)\|_{\mathbf{L}_p}&\le C_{p,(35)}C_p(1+|x|+|y|)^p\big(|x-y|+|x-y|^\delta\big)\left(\frac{1}{n^{p\beta}}+\frac{(\mathcal{E}_1^{X,\varepsilon})^p}{n^p}+K^{X,\varepsilon}\right)^{\frac1p}\\
&\le 2C_{p,(35)}(1+|x|+|y|)^{2-\delta}|x-y|^\delta\left(\frac{1}{n^\beta}+\frac{\mathcal{E}_1^{X,\varepsilon}}{n}+(K^{X,\varepsilon})^{\frac1p}\right),
\end{aligned}$$

using $|x-y|+|x-y|^\delta=|x-y|^\delta(1+|x-y|^{1-\delta})\le 2|x-y|^\delta(1+|x|+|y|)^{1-\delta}$. Thus, we can take $C^{(G)}=2C_{p,(35)}\left(\frac{1}{n^\beta}+\frac{\mathcal{E}_1^{X,\varepsilon}}{n}+(K^{X,\varepsilon})^{\frac1p}\right)$, $\tau^{(G)}=2-\delta$ and $\beta^{(G)}=\delta$ provided that $\delta\in(d/p,1]$, which is true for p large enough. We then conclude by applying Corollary 1. □

We now have all the necessary elements to finalize the proof of our main result.

4.3 Proof of Theorem 5

We now carefully apply Theorem 1 with $F(\omega,x):=X_t(\omega,x)$, $F^n(\omega,x):=X_t^n(\omega,x)$, $\Theta:=Y_s(\omega,y)$ and $\Theta^n:=Y_s^n(\omega,y)$.

(a) If the coefficients of X and Y satisfy (**HP1**) and (**HP3**$_\alpha$).

1. As it is highlighted in Remark 3.1, Assumption (**H1**) is satisfied with $C_p^{(\mathbf{H1})}:=C_{p,(16)}$ and $\alpha_p^{(\mathbf{H1})}:=1$ (in view of Theorem 4) and (**H2**) is also satisfied for any given $\kappa\in(0,1)$ with $C_p^{(\mathbf{H2})}:=C_{p,(17)}$ (depending on κ) and $\alpha_p^{(\mathbf{H2})}:=1-\kappa$.

2. (**H3**) is valid owing to Proposition 5 where, for any given $\rho > 0$, we take $\alpha_p^{(\mathbf{H3})} := 1$ and

$$\varepsilon_p^{n,(\mathbf{H3})} := C_{p,\rho,(33)} \left[\frac{1}{n^{\beta^X - \rho}} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n} \right)^{\frac{\beta^X - \rho}{\beta^X}} + \left(K_2^{X,\varepsilon} \right)^{\frac{\beta^X - \rho}{2\beta^X}} + \left(K_p^{X,\varepsilon} \right)^{\frac{\beta^X - \rho}{p\beta^X}} \right]$$

with $\beta^X := \min(\alpha^X, \frac{1}{2})$.

3. Finally, (**H4**) is clearly true using Propositions 1 and 4 applied to Y instead of X, which yields $C_p^{(\mathbf{H4}\text{-a})} := \max(C_{p,(12)}, C_{p,(24)})(1 + |y|)$, and Theorem 6, applied to Y and Y^N, which gives

$$\varepsilon_p^{n,(\mathbf{H4}\text{-b})} := C_{p,(29)}(1 + |y|) \left[\frac{1}{n^{\beta^Y}} + \frac{\mathcal{E}_1^{Y,\varepsilon}}{n} + (K_2^{Y,\varepsilon})^{\frac{1}{2}} + (K_p^{Y,\varepsilon})^{\frac{1}{p}} \right]$$

with $\beta^Y := \min(\alpha^Y, \frac{1}{2})$, $\mathcal{E}_1^{Y,\varepsilon} = \int_{\mathbb{R}} C^Y(e)\nu^\varepsilon(\mathrm{d}e)$ and $K_p^{Y,\varepsilon} = \int_{\mathbb{R}} (C^Y(e))^p \mathbf{1}_{B_\varepsilon}(e)\nu(\mathrm{d}e)$.

Thus, for any given $\kappa \in (0,1)$ and $\rho > 0$, Theorem 1 gives, for any y

$$\|X_t^{n,\varepsilon}(Y_s^{n,\varepsilon}(y)) - X_t(Y_s(y))\|_{\mathbf{L}_p} \leq C_p \left[\frac{1}{n^{\kappa\beta^Y}} + \frac{1}{n^{\beta^X - \rho}} + \left(\frac{\mathcal{E}_1^{Y,\varepsilon}}{n} \right)^{\kappa} + \left(\frac{\mathcal{E}_1^{X,\varepsilon}}{n} \right)^{\frac{\beta^X - \rho}{\beta^X}} \right.$$
$$\left. + (K_2^{Y,\varepsilon})^{\frac{\kappa}{2}} + (K_p^{Y,\varepsilon})^{\frac{\kappa}{p}} + \left(K_2^{X,\varepsilon} \right)^{\frac{\beta^X - \rho}{2\beta^X}} + \left(K_p^{X,\varepsilon} \right)^{\frac{\beta^X - \rho}{p\beta^X}} \right],$$

thus the announced result.

(b) If in addition to the first case, the coefficients of X satisfy ($\mathbf{HP2}_\delta$) and ($\mathbf{HP4}_\alpha$). Assumption (**H1**) and (**H4**) are still valid. Assumption (**H2**) is satisfied with $\kappa = 1$ thanks to (20). Assumptions (**H3**) holds as in the previous case, but now with $\rho = 0$ owing to Theorem 7. The rest of the proof is unchanged, we are done.

5 Approximation of utility-SPDE

In this section, we apply the previous results to numerically solve the utility-SPDE (1). Let us specify that in this context $d = q' = 1$ and the marginal utility is, according to (2), represented as follows

$$U_z(t, z) = X_t\big(u_z(\xi_t(z))\big), \quad U(0, z) = u(z) \tag{42}$$

where we recall that $\xi_t(z) := Y_t^{-1}(z)$ is the inverse flow of $y \mapsto Y_t(y)$. Moreover, X and Y are solutions to two scalar SDEs with coefficients (μ, σ, h) and (b, γ, g) respectively, driven by the same q-dimensional Brownian motion B and the same Poisson random measure. To simplify, we take $\lambda(t) = 1$ for any t.
However, it is obvious that Theorem 5 is not directly applicable since at this step remains the question how to invert $y \mapsto Y_t(y)$ for all t?

Different approaches to answer this question have been discussed in detail in [13, Section 4] and it turned out that the best way is to compute the inverse flow in a backward way, because in this case it is a solution of a SDE to which we can consider an Euler scheme as before and thus easily apply the Theorem 5.

To be more precise, we denote by $Y_{s,t}(y)$ the solution, starting from y at time s, of the SDE with coefficients (b,γ,g). Let $\xi_{s,t}$ be the inverse map of $Y_{s,t}$. Our idea consists on considering the dynamics of the process $\xi_{s,t}(x)$ in the variable s instead of t, this relies on the following key result.

Theorem 8 ([17, Theorems 3.11 and 3.13])**.** *Suppose the coefficients b, γ and g of the SDE (9) satisfy* (**HP1**)*,* (**HP2**$_\delta$)*. Assume further that the maps $\phi(t,.,e): y \mapsto y+g(t,y,e); \mathbb{R}\to\mathbb{R}$ are homeomorphic with $1+\partial_y g(t,y,e)$ being invertible in y for a.e (t,e).*

(i) *Then the solution Y defines a stochastic flow of $\mathcal{C}^1$-diffeomorphisms.*

(ii) *Let $\psi(t,.,e)$ be the inverse maps of $\phi(t,.,e)$ and $k(t,z,e)=z-\psi(t,z,e)$. Assume $\gamma\in\mathcal{C}^{1,2}$ in (t,x) and $\int_{\mathbb{R}}|k(t,z,e)-g(t,z,e)|\nu(\mathrm{de})$ is bounded. Then the inverse flow ξ satisfies the following backward SDE*

$$\begin{cases} \mathrm{d}\xi_{s,t}(z) = \big[b(s,\xi_{s,t}(z)) - \partial_x\gamma(s,\xi_{s,t}(z))\cdot\gamma(s,\xi_{s,t}(z) - \int_{\mathbb{R}} g(s,\xi_{s,t}(z),e)\nu(\mathrm{de})\big]\hat{\mathrm{d}s} \\ \qquad +\gamma(t,\xi_{s,t}(z))\cdot \mathrm{d}\overleftarrow{B}_s + \int_{\mathbb{R}} k(s,\xi_{s,t}(z),e)N(\hat{\mathrm{d}s},\mathrm{de}),\ \ s\le t \\ \xi_{t,t}(z) = z. \end{cases} \tag{43}$$

The notations $\hat{\mathrm{d}s}$ and $\overleftarrow{B}_s$ are to remind that the integrals must be considered in a backward way:

$$\begin{cases} \int_s^t \mu(s,X_u)\hat{du} := \lim_{|\Delta|\to 0}\sum_{k=0}^{n-1}\mu(t_{k+1},X_{t_{k+1}})(t_{k+1}-t_k) \\ \int_s^t \sigma(s,X_u)\mathrm{d}\overleftarrow{B}_u := \lim_{|\Delta|\to 0}\sum_{k=0}^{n-1}\sigma(t_{k+1},X_{t_{k+1}})\big(B_{t_{k+1}}-B_{t_k}\big), \\ \text{where } \Delta=\{s=t_0<t_2<\cdots<t_n=t\} \text{ and} |\Delta| = \max_{0\le k\le n-1}(t_{k+1}-t_k) \end{cases}$$

With this result in hand, the approximation of $\xi_{s,t}$ is made possible simply using a standard Euler scheme like (23) ($W\equiv B$) for X. Using the same notations of Section 3.4, the backward Euler scheme $\xi^{n,\varepsilon}_{.,t}$ is defined as follows:

- Set $\xi^{n,\varepsilon}_{t,t}(z) = z$. If $t=\theta_k$ for some $k\in [\![1,J(\varepsilon)]\!]$, set $\xi^{n,\varepsilon}_{t^-,t}(z) = z-k(t,z,E^\varepsilon_k)$ else $\xi^{n,\varepsilon}_{t^-,t}(z)=z$.

- For $s\in]\tau_t,t]$, set

$$\begin{aligned} \xi^{n,\varepsilon}_{s,t}(z) &= \xi^{n,\varepsilon}_{t^-,t} - \gamma(t,\xi^{n,\varepsilon}_{t^-,t}(z))\cdot(B_t-B_s) \\ &- \big[b(t,\xi^{n,\varepsilon}_{t^-,t}(z)) - \partial_z\gamma(t,\xi^{n,\varepsilon}_{t^-,t}(z))\cdot\gamma(t,\xi^{n,\varepsilon}_{t^-,t}(z)) - \int_{\mathbb{R}} g(t,\xi^{n,\varepsilon}_{t^-,t}(z),e)\nu^\varepsilon(\mathrm{de})\big](t-s). \end{aligned} \tag{44}$$

- For $l\in [\![0,n+J^\varepsilon]\!]$ satisfying $t_l<\tau_t$,

- if $t_l=\theta_k$ for some $k\in [\![1,J^\varepsilon]\!]$, set $\xi^{n,\varepsilon}_{t_l^-,t}(z) = \xi^{n,\varepsilon}_{t_l,t}(z) - k(t_l,\xi^{n,\varepsilon}_{t_l,t}(z),E^\varepsilon_k)$ else $\xi^{n,\varepsilon}_{t_l^-,t}(z) = \xi^{n,\varepsilon}_{t_l,t}(z)$

- and for $s \in [t_{(l-1)}, t_l[$

$$\xi^{n,\varepsilon}_{s,t}(z) = \xi^{n,\varepsilon}_{t_l^-,t}(z) - \gamma(t_l, \xi^{n,\varepsilon}_{t_l^-,t}(z)) \cdot (B_{t_l} - B_s) \tag{45}$$
$$- \big[b(t_l, \xi^{n,\varepsilon}_{t_l^-,t}(z)) - \partial_x\gamma(t_l, \xi^{n,\varepsilon}_{t_l^-,t}(z)) \cdot \gamma(t_l, \xi^{n,\varepsilon}_{t_l^-,t}(z)) - \int_{\mathbb{R}} g(t_l, \xi^{n,\varepsilon}_{t_l^-,t}(z), e)\nu^\varepsilon(\mathrm{de})\big](t_l - s).$$

Theorem 9. *Assume that the coefficients (μ, σ, h) of X satisfy* (**HP1**)-(**HP2**$_\delta$)-(**HP3**$_\alpha$)-(**HP4**$_\alpha$) *(which α-parameter is denoted by α^X) and the coefficients $(b - \partial_x\gamma \cdot \gamma - \int g\nu(\mathrm{de}), \gamma, k)$ of $\xi_{.,t}$ satisfy Assumptions* (**HP1**)-(**HP3**$_\alpha$) *(which α-parameter is denoted by α^Y). Denote by $X^{n,\varepsilon}_{0,.}$ the Euler approximation* (23) *($W \equiv B$) associated to $X_{0,.}$, and by $\xi^{n,\varepsilon}_{.,t}$ the Euler approximation of the inverse flow $\xi_{.,t}$ of $Y_{.,t}$, according to* (44)-(45).
Then, for any concave function u with Lipschitz marginal utility u_z, the compound Euler scheme $X^{n,\varepsilon}_{.}(u_z(\xi^{n,\varepsilon}_{.}))$ converges to $U_z(.,.)$ (solution to the SPDE of the form (1)*) in any $\mathbf{L}_p$-norm w.r.t. n and ε: For any $p > 0$, $t \in [0,T]$ and any z,*

$$\left\| X^{n,\varepsilon}_{0,t}(u_z(\xi^{n,\varepsilon}_{0,t}(z))) - U_z(t,z) \right\|_{\mathbf{L}_p} = O(\mathbf{E}^{n,\varepsilon}_p).$$

From this accurate approximation of $U_z(t,z)$ using two Euler schemes, we can easily retrieve $U(t,z)$ by standard numerical integration, using that in general the utility of a zero wealth is equal to zero at any time, i.e., $U(t,0) = 0$.
Note that the fact that the two Euler schemes are built with the same Brownian motion (actually one is the time-reversal of the other) justifies the need for a general result like Theorems 1 and 5, available for arbitrary dependency in F and Θ.
Concerning the proof of this result it is obvious: without the function u_z, it would be a direct application of Theorem 5. However, since u_z is Lipschitz, one can easily check that the estimates are unchanged.

Conclusion. As explained in [20] and in [13], the solution of a SPIDE can be represented in some cases as the compound of two random fields. Therefore the approximation of this compound provides a numerical approximation for some SPIDE. When the Lévy measure is finite, it is proved in [20] that the rate of convergence is of order $n^{-1/2}$.

In this paper, we generalize this result for non-finite Lévy measure, by truncation of the small jumps. We show that there is a balance to find between the time discretization and this truncation threshold ; in general it is not optimal to fix them independently. As a consequence, in some cases the rate of convergence can be very deteriorated (compared to $n^{-1/2}$).

Improvements using other approximations for small jumps (Asmussen-Rosinksy technics or series representation) are left for further research.

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