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Backward stochastic differential equations with singular terminal condition

A. Popier*

L.A.T.P., 39 rue F. Joliot Curie, 13453 Marseille cedex 13, France

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Abstract

In this paper, we are concerned with backward stochastic differential equations (BSDE for short) of the following type:

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q \, \mathrm{d}r - \int_t^T Z_r \, \mathrm{d}B_r,$$

where q is a positive constant and ξ is a random variable such that $\mathbb{P}(\xi = +\infty) > 0$. We study the link between these BSDE and the associated Cauchy problem with terminal data g, where $g = +\infty$ on a set of positive Lebesgue measure.

Keywords: Backward stochastic differential equation; Non-integrable data; Viscosity solutions of partial differential equations

Contents

* Corresponding address: Ecole Polytechnique, CMAP, Route de Palaiseau, 91128 Palaiseau, France. Tel.: +33 1 69 33 46 48; fax: +33 1 69 33 30 11.
E-mail address: popier@cmap.polytechnique.fr.

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0. Introduction and main results

Backward stochastic differential equations (BSDE for short in the remainder) are equations of the following type:

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}B_r, \quad 0 \le t \le T,$$

where $(B_t)_{0\le t\le T}$ is a standard d-dimensional Brownian motion on a probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}\right)$, with $(\mathcal{F}_t)_{0\le t\le T}$ the standard Brownian filtration. The function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n$ is called the *generator*, T the *terminal time*, and the $\mathbb{R}^n$-valued $\mathcal{F}_T$-adapted random variable ξ a *terminal condition*.

The unknowns are the processes $\{Y_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$, which are required to be adapted with respect to the filtration of the Brownian motion: this is a crucial point.

Such equations, in the non-linear case, were introduced by Pardoux and Peng in 1990 in [1]. They gave the first existence and uniqueness result for n-dimensional BSDE under the following assumptions: f is Lipschitz continuous in both variables y and z and the data, ξ, and the process, $\{f(t, 0, 0)\}_{t\in[0,T]}$, are square integrable. Since then, BSDE have been studied with great interest. In particular, many efforts have been made to relax the assumptions on the generator and the terminal condition. For instance Briand et al. in [2] proved an existence and uniqueness result under the following assumptions: f is Lipschitz in z, continuous and monotone in y, the data, ξ, and the process, $\{f(t, 0, 0)\}_{t\in[0,T]}$, are in L^p for $p > 1$. The result is still true for $p = 1$ with another technical condition.

The results of [2] are the starting point of this work, where we consider a one-dimensional BSDE with a non-linear generator:

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}B_r \quad \text{with } q \in \mathbb{R}_+^*. \tag{1}$$

The generator $f(y) = -y|y|^q$ satisfies all the assumptions of the Theorems 4.2 and 6.2–6.3 of [2]: f is continuous on $\mathbb{R}$, does not depend on z, and is monotone:

$$\forall (y, y') \in \mathbb{R}^2, \quad (y - y')(f(y) - f(y')) \le 0. \tag{2}$$

Therefore there exists a unique solution (Y, Z) for $\xi \in L^p(\Omega)$ for $p \geq 1$ (we do not make precise the class of (Y, Z) in which uniqueness holds). The solution of the related ordinary differential equation, namely $y' = y|y|^q$, $y_T = x$, is given by the formula:

$$\operatorname{sign}(x) \left(\frac{1}{q(T-t) + \frac{1}{|x|^q}} \right)^{\frac{1}{q}}$$

where $\operatorname{sign}(x) = -1$ if $x < 0$ and $\operatorname{sign}(x) = 1$ if $x > 0$. We remark that, even if x is equal to $+\infty$ or $-\infty$, y is finite on $[0, T[$.

Numerous theorems (see for instance [3,4] and [5]) show the connections between BSDE associated with some forward classical stochastic differential equation (SDE for short) (or forward–backward system) and solutions of a large class of semi-linear and quasilinear parabolic and elliptic partial differential equations. Those results may be seen as a non-linear generalization of the celebrated Feynman–Kac formula.

The BSDE (1) is connected with the following type of PDE (see [5]):

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^q = 0, & (t,x) \in [0,T[\times\mathbb{R}^m; \\ u(T,x) = g(x), & x \in \mathbb{R}^m. \end{cases} \tag{3}$$

where $\mathcal{L}$ is the infinitesimal generator:

$$\mathcal{L} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} = \frac{1}{2}\operatorname{Trace}\left(\sigma\sigma^* D^2\right) + b\nabla; \tag{4}$$

where in the rest of the paper, ∇ and D^2 will always denote respectively the gradient and the Hessian matrix w.r.t. the space variable.

Indeed Baras and Pierre [6], Marcus and Veron [7] have given existence and uniqueness results for this PDE. In [7] it is shown that every positive solution of (3) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^m$ and μ a non-negative Radon measure on $\mathcal{R} = \mathbb{R}^m \setminus \mathcal{S}$. The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^m,\ A \text{ Borel},\quad \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view Dynkin and Kuznetsov [8] and Le Gall [9] have proved similar results for the PDE (3) in the case $0 < q \leq 1$: they use the theory of superprocesses.

In this paper we are concerned with a real $\mathcal{F}_T$-measurable random variable such that:

$$\mathbb{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0. \tag{5}$$

Thus ξ is not in $L^1(\Omega)$. We give a new definition of a solution of the BSDE.

Definition 1. Let us have $q > 0$ and ξ an $\mathcal{F}_T$-measurable random variable. We say that the process (Y, Z) is a solution of the BSDE

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q \,\mathrm{d}r - \int_t^T Z_r \,\mathrm{d}B_r$$

if (Y, Z) verifies:

(D1) for all $0 \leq s \leq t < T$: $Y_s = Y_t - \int_s^t Y_r |Y_r|^q \,\mathrm{d}r - \int_s^t Z_r \,\mathrm{d}B_r$;

(D2) for all $t \in [0, T[$, $\mathbb{E}\left(\sup_{0\leq s\leq t} |Y_s|^2 + \int_0^t \|Z_r\|^2 \,\mathrm{d}r\right) < +\infty$;

(D3) $\mathbb{P}$-a.s. $\lim_{t\to T} Y_t = \xi$.

The outline of the paper is as follows. Except in Section 5, ξ is supposed to be non-negative. In the first section, without any further assumptions on ξ, we construct a process (Y, Z) which satisfies all conditions for being a solution in the sense of the previous definition, except the last one. More precisely we establish in Section 1 the

Theorem 2. *Let $\xi \geq 0$ a.s. There exists a progressively measurable process (Y, Z), with values in $\mathbb{R}_+ \times \mathbb{R}^d$, such that:*

(1) (D1) *and* (D2) *are satisfied:*

(a) *for all $t \in [0, T[$, and all $0 \leq s \leq t$:*

$$Y_s = Y_t - \int_s^t (Y_r)^{1+q} \,\mathrm{d}r - \int_s^t Z_r \,\mathrm{d}B_r, \tag{i}$$

(b) *for all $t \in [0, T[$,*

$$0 \leq Y_t \leq \frac{1}{(q(T-t))^{\frac{1}{q}}}, \quad \textit{and} \quad \mathbb{E}\int_0^t \|Z_r\|^2 \,\mathrm{d}r \leq \frac{1}{(q(T-t))^{\frac{2}{q}}}; \tag{ii}$$

(2) *Y is continuous on $[0, T[$, the limit of Y_t, when t goes to T, exists and:*

$$\lim_{t\to T} Y_t \geq \xi, \quad \mathbb{P}-a.s.; \tag{iii}$$

(3) *Z satisfies also:*

$$\mathbb{E}\left(\int_0^T (T-r)^{2/q} \|Z_r\|^2 \,\mathrm{d}r\right) \leq 8\left(\frac{1}{q}\right)^{\frac{2}{q}}. \tag{iv}$$

Note that this result does not specify whether Y satisfies $\lim_{t\to T} Y_t = \xi$. The existence of this process (Y, Z) is obtained by approximation. For every integer n, let (Y^n, Z^n) be the solution of the BSDE (1) with terminal condition $\xi \wedge n \in L^\infty(\Omega)$. (Y, Z) is the limit of this sequence $(Y^n, Z^n)_{n\in\mathbb{N}}$.

In Section 2, we study our process Y in the neighbourhood of T. In a first part we make precise the asymptotic behaviour of Y on the "blow-up" set.

Proposition 3. *On the set $\{\xi = +\infty\}$*

$$\lim_{t\to T} (T-t)^{1/q} Y_t = \left(\frac{1}{q}\right)^{1/q} \quad a.s. \tag{6}$$

In the second part we will prove the continuity of Y under stronger conditions on ξ. So far we only have the inequality:

$$\lim_{t\to T} Y_t \geq \xi = Y_T.$$

Without additional assumption, we were unable to prove the converse inequality. The first hypothesis on ξ is the following:

$$\xi = g(X_T), \tag{H1}$$

where g is a measurable function defined on $\mathbb{R}^m$ with values in $\overline{\mathbb{R}^+}$ such that the set $F_\infty = \{g = +\infty\}$ is closed; and where X_T is the value at $t = T$ of a diffusion process or more precisely the solution of a stochastic differential equation (for short SDE):

$$X_t = x + \int_0^t b(r, X_r)\,\mathrm{d}r + \int_0^t \sigma(r, X_r)\,\mathrm{d}B_r, \quad \text{for } t \in [0, T]. \tag{7}$$

We will always assume that b and σ are defined on $[0, T] \times \mathbb{R}^m$, with values respectively in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, are measurable w.r.t. the Borelian σ-algebras, and that there exists a constant $K > 0$ s.t. for all $t \in [0, T]$ and for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^m$:

(1) Lipschitz condition:

$$|b(t, x) - b(t, y)| + \|\sigma(t, x) - \sigma(t, y)\| \leq K|x - y|; \tag{L}$$

(2) Growth condition:

$$|b(t, x)| + \|\sigma(t, x)\| \leq K(1 + |x|). \tag{G}$$

The second hypothesis on ξ is: for all compact sets $\mathcal{K} \subset \mathbb{R}^m \setminus F_\infty$

$$g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}). \tag{H2}$$

Moreover in the case $q \leq 2$ we will add the following conditions:

(1) σ and b are bounded: there exists a constant K s.t.

$$\forall (t, x) \in [0, T] \times \mathbb{R}^m, \qquad |b(t, x)| + \|\sigma(t, x)\| \leq K; \tag{B}$$

(2) the second derivatives of $\sigma\sigma^*$ belongs to L^∞:

$$\frac{\partial^2 \sigma\sigma^*}{\partial x_i \partial x_j} \in L^\infty([0, T] \times \mathbb{R}^m). \tag{D}$$

(3) $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t, x) \in [0, T] \times \mathbb{R}^m$:

$$\forall y \in \mathbb{R}^m, \quad \sigma\sigma^*(t, x)y.y \geq \lambda|y|^2. \tag{E}$$

(4) g is continuous from $\mathbb{R}^m$ to $\overline{\mathbb{R}_+}$ and:

$$\forall M \geq 0, \quad g \text{ is a Lipschitz function on the set } \mathcal{O}_M = \{|g| \leq M\}. \tag{H3}$$

Theorem 4 (*Continuity of Y at T*). *Under the assumptions (H1), (H2), (L) and (G), and with either $q > 2$ or (H3), (B), (D) and (E), Y is continuous at time T*

$$\lim_{t \to T} Y_t = \xi, \quad \mathbb{P} - a.s.$$

In Section 3, we prove that our solution is the minimal solution.

Theorem 5 (*Minimal Solution*). *The solution (Y, Z) obtained in Theorems 2 and 4 is minimal: if $(\bar{Y}, \bar{Z})$ is another non-negative solution in the sense of Definition 1, then for all $t \in [0, T]$, $\mathbb{P}$-a.s.: $\bar{Y}_t \geq Y_t$.*

Moreover we prove that:

$$\bar{Y}_t \leq \left(\frac{1}{q(T - t)} \right)^{1/q}.$$

The fourth section provides connections between this constructed solution of the BSDE (1) and viscosity solutions of related semi-linear PDE (3). For all $(t, x) \in [0, T] \times \mathbb{R}^m$, we denote by $X^{t,x}$ the solution of the SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}B_r, \quad \text{for } s \in [t, T], \tag{8}$$

and $X_s^{t,x} = x$ for $s \in [0, t]$. The coefficients b and σ verify always the assumptions of the second section. As a final condition of the BSDE, we take $g(X_T^{t,x})$, where g is a function defined from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$ such that the set $F_\infty = \{g = +\infty\}$ is closed and such that the condition (H2) is verified. Moreover g is supposed to be continuous from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$.

Theorem 6 (*Viscosity Solution*). *The minimal solution of the BSDE* (1) *with* $\xi = g(X_T^{t,x})$ *is denoted by* $Y^{t,x}$. *Then* $Y_t^{t,x}$ *is a deterministic number and if we set* $u(t, x) = Y_t^{t,x}$, *then* u *is lower semi-continuous from* $[0, T] \times \mathbb{R}^m$ *to* $\overline{\mathbb{R}^+}$ *with* $u(t, .) < +\infty$ *whenever* $t < T$ *and* u *is a (discontinuous) viscosity solution of the PDE* (3).

We prove that the previous solution is minimal among all non-negative viscosity solutions.

Theorem 7 (*Minimal Viscosity Solution*). *If* v *is a non-negative viscosity solution of the PDE* (3), *then for all* $(t, x) \in [0, T] \times \mathbb{R}^m$:

$$u(t, x) \leq v(t, x).$$

We give sufficient conditions to have u continuous on $[0, T] \times \mathbb{R}^m$.

In Section 5, we extend our results when there is no sign assumption on ξ.

Theorem 8. *Let* ξ *be an* $\mathcal{F}_T$*-measurable random variable, possibly negative, such that:*

$$\mathbb{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0. \tag{5}$$

Moreover ξ *satisfies*

$$\xi = g(X_T), \tag{H1}$$

with $g : \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$ *such that:*

(1) *the two sets* $\{g = +\infty\}$ *and* $\{g = -\infty\}$ *are closed;*

(2) *the condition* (H2) *is verified: for all compact sets* $\mathcal{K} \subset \mathbb{R}^m \setminus \{g = \pm\infty\}$

$$g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in L^1(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R});$$

(3) *if* $q \leq 2$, *(H3) holds.*

The coefficients of the SDE (7) *satisfy (L), (G) and in the case* $0 < q \leq 2$, *they also verify (B), (D) and (E).*

There exists a process (Y, Z) *which is a solution of the BSDE*

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}B_r$$

in the sense of Definition 1.

In the continuation, unimportant constants will be denoted by C.

1. Approximation and construction of a solution

From now and in Sections 2–4, ξ satisfies:

$$\mathbb{P}(\xi \geq 0) = 1 \quad \text{and} \quad \mathbb{P}(\xi = +\infty) > 0.$$

In this section we prove Theorem 2. For $q > 0$, let us consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(y) = -y|y|^q$. f is continuous and monotone (inequality (2)). By Theorem 2.2 and Example 3.9 in [5], for $\zeta \in L^2(\mathcal{F}_T)$, the BSDE (1) with ζ as terminal condition has a unique solution (Y, Z) with values in $\mathbb{R} \times \mathbb{R}^d$, such that Y is continuous on $[0, T]$ and that:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T \|Z_t\|^2 \,\mathrm{d}t\right) < \infty. \tag{9}$$

Remark that a straightforward application of the Tanaka formula (see [10]) and of the comparison Theorem 2.4 in [5] shows that:

$$\forall t \in [0, T], \quad |Y_t| \leq \left(\frac{1}{q(T-t)}\right)^{1/q}. \tag{10}$$

For every $n \in \mathbb{N}^*$, we introduce $\xi_n = \xi \wedge n$. ξ_n belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We apply the previous result with ξ_n as the final data, and we build a sequence of random processes (Y^n, Z^n) which satisfy (1) and (9).

From the comparison Theorem 2.4 in [5], if $n \leq m$, $0 \leq \xi_n \leq \xi_m \leq m$, which implies for all t in $[0, T]$, a.s.,

$$0 \leq Y_t^n \leq Y_t^m \leq \left(\frac{1}{q(T-t) + \frac{1}{m^q}}\right)^{\frac{1}{q}} \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}. \tag{11}$$

We define the progressively measurable $\mathbb{R}$-valued process Y, as the increasing limit of the sequence $(Y_t^n)_{n\geq 1}$:

$$\forall t \in [0, T], \quad Y_t = \lim_{n\to+\infty} Y_t^n. \tag{12}$$

Then we obtain for all $0 \leq t \leq T$

$$0 \leq Y_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}. \tag{13}$$

In particular Y is finite on the interval $[0, T[$ and bounded on $[0, T - \delta]$ for all $\delta > 0$.

1.1. Proof of (i) and (ii) from Theorem 2

Here we will prove the properties (i) and (ii). Let $\delta > 0$ and $s \in [0, T - \delta]$. For all $0 \leq t \leq s$, Itô's formula leads to the equality:

$$\begin{aligned}|Y_t^n - Y_t^m|^2 + \int_t^s \|Z_r^n - Z_r^m\|^2 \,\mathrm{d}r &= |Y_s^n - Y_s^m|^2 \\ - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)\,\mathrm{d}B_r &+ 2\int_t^s (Y_r^n - Y_r^m)\left(f(Y_r^n) - f(Y_r^m)\right)\mathrm{d}r\end{aligned}$$

$$\leq |Y_s^n - Y_s^m|^2 - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)\, \mathrm{d}B_r,$$

from the monotonicity of f (inequality (2)). Thanks to (9):

$$\mathbb{E}\left(\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m)\, \mathrm{d}B_r\right) = 0.$$

From the Burkholder–Davis–Gundy inequality, we deduce the existence of a universal constant C with:

$$\mathbb{E}\left(\sup_{0\leq t\leq s} |Y_t^n - Y_t^m|^2 + \int_0^s \|Z_r^n - Z_r^m\|^2\, \mathrm{d}r\right) \leq C\,\mathbb{E}\left(|Y_s^n - Y_s^m|^2\right). \tag{14}$$

From the estimate (13), for $s \leq T - \delta$, $Y_s^n \leq \frac{1}{(q\delta)^{1/q}}$ and $Y_s \leq \frac{1}{(q\delta)^{1/q}}$. Since Y_s^n converges to Y_s a.s., the dominated convergence theorem and the previous inequality (14) imply:

(1) for all $\delta > 0$, $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$, and converges to $Z \in L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$,
(2) $(Y^n)_{n\geq 1}$ converges to Y uniformly in mean square on the interval $[0, T-\delta]$; in particular Y is continuous on $[0, T[$,
(3) (Y, Z) satisfies for every $0 \leq s < T$, for all $0 \leq t \leq s$:

$$Y_t = Y_s - \int_t^s (Y_r)^{1+q}\, \mathrm{d}r - \int_t^s Z_r\, \mathrm{d}B_r. \tag{i}$$

The relation (i) is proved. Since Y_t is smaller than $1/(q(T-t))^{1/q}$ by (13), and since $Z \in L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$, applying the Itô formula to $|Y|^2$, with $s < T$ and $0 \leq t \leq s$, we obtain:

$$\begin{aligned} |Y_t|^2 + \int_t^s \|Z_r\|^2\, \mathrm{d}r &= |Y_s|^2 - 2\int_t^s Y_r Z_r\, \mathrm{d}B_r + 2\int_t^s Y_r f(Y_r)\, \mathrm{d}r \\ &\leq \frac{1}{(q(T-s))^{\frac{2}{q}}} - 2\int_t^s Y_r Z_r\, \mathrm{d}B_r, \end{aligned}$$

again thanks to the monotonicity of f (inequality (2)). From (13), since $Z \in L^2([0, s] \times \Omega)$, we have: $\mathbb{E}\int_t^s Y_r Z_r\, \mathrm{d}B_r = 0$. Therefore, we deduce:

$$\forall s \in [0, T[, \quad \mathbb{E}\int_0^s \|Z_r\|^2\, \mathrm{d}r \leq \frac{1}{(q(T-s))^{\frac{2}{q}}}. \tag{ii}$$

1.2. *Proof of (iii)*

From now, the process Y is continuous on $[0, T[$ and we define $Y_T = \xi$. The main difficulty will be to prove the continuity at time T. It is easy to show that:

$$\xi \leq \liminf_{t\to T} Y_t. \tag{15}$$

Indeed, for all $n \geq 1$ and all $t \in [0, T]$, $Y_t^n \leq Y_t$, therefore:

$$\xi \wedge n = \liminf_{t\to T} Y_t^n \leq \liminf_{t\to T} Y_t. \tag{16}$$

Thus, Y is lower semi-continuous on $[0, T]$ (this is clear since Y is the supremum of continuous functions). Without other assumptions on ξ, we are unable to prove the continuity of Y at $t = T_-$. But now we will show that Y has a limit on the left at time T. We will distinguish the case when ξ is greater than a positive constant from the case ξ non-negative.

1.2.1. The case ξ bounded away from zero

We can show that Y has a limit on the left at T, by using Itô's formula applied to the process $1/(Y^n)^q$. We prove the following result:

Proposition 9. *Suppose there exists a real $\alpha > 0$ such that $\xi \geq \alpha > 0$, $\mathbb{P}$-a.s. Then*

$$Y_t = \left(q(T-t) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right) - \Phi_t\right)^{-\frac{1}{q}}, \quad 0 \leq t \leq T, \tag{17}$$

where Φ is a non-negative supermartingale.

Proof. From the comparison result 2.4 of [5], for every $n \in \mathbb{N}^*$ and every $0 \leq t \leq T$:

$$n \geq Y_t^n \geq \left(\frac{1}{q(T-t) + 1/\alpha^q}\right)^{1/q} \geq \left(\frac{1}{qT + 1/\alpha^q}\right)^{1/q} > 0.$$

By the Itô formula

$$\begin{aligned}\frac{1}{(Y_t^n)^q} &= \frac{1}{(\xi \wedge n)^q} + q(T-t) - \frac{q(q+1)}{2}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}} \mathrm{d}s + \int_t^T \frac{qZ_s^n}{\left(Y_s^n\right)^{q+1}} \mathrm{d}B_s \\ &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge n)^q}\right) + q(T-t) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}} \mathrm{d}s.\end{aligned}$$

The process:

$$\left(\frac{1}{(Y_t^n)^q}, \frac{-qZ_s^n}{\left(Y_s^n\right)^{q+1}}\right)$$

is the maximal solution in $L^\infty([0, T] \times \Omega) \times L^2\left([0, T] \times \Omega; \mathbb{R}^d\right)$ of the BSDE:

$$Y_t = \frac{1}{(\xi \wedge n)^q} + \int_t^T \left(q - \frac{q+1}{2q}\frac{\|Z_s\|^2}{Y_s \vee 1/n^q}\right) \mathrm{d}s - \int_t^T Z_s \, \mathrm{d}B_s.$$

Indeed, for this BSDE there exists a maximal solution (U, V) (see [11], Theorem 2.3) and since $U \geq 1/(Y^n)^q$, we can apply Itô's formula to the process $U^{-1/q}$. We find that $(U^{-1/q}, (1/q)V/U^{1+1/q})$ satisfies the BSDE (1) with terminal value $\xi \wedge n$. Thus $U^{-1/q} = Y^n$ and the conclusion follows.

Let $n \geq m$. Since $\xi \wedge n \geq \xi \wedge m$, we obtain for all $0 \leq t \leq T$:

$$\begin{aligned}0 \leq \frac{1}{(Y_t^m)^q} - \frac{1}{(Y_t^n)^q} &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge m)^q} - \frac{1}{(\xi \wedge n)^q}\right) \\ &\quad - \frac{q(q+1)}{2}\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^m\|^2}{\left(Y_s^m\right)^{q+2}} \mathrm{d}s - \mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}} \mathrm{d}s\right).\end{aligned}$$

Now:

$$\frac{q(q+1)}{2}\left|\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^m\|^2}{\left(Y_s^m\right)^{q+2}}\,\mathrm{d}s - \mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s\right|$$

$$\leq \left[\mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge m)^q} - \frac{1}{(\xi\wedge n)^q}\right)\right] \vee \left[\frac{1}{(Y_t^m)^q} - \frac{1}{(Y_t^n)^q}\right].$$

For a fixed $t \in [0,T]$, the sequences $\left(\mathbb{E}^{\mathcal{F}_t}\frac{1}{(\xi\wedge n)^q}\right)_{n\geq 1}$ and $\left(\frac{1}{(Y_t^n)^q}\right)_{n\geq 1}$ converge a.s. and in L^1 (dominated convergence theorem). Then $\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}\,\mathrm{d}s\right)_{n\geq 1}$ converges a.s. and in L^1 and we denote by Φ_t the limit:

$$\Phi_t = \lim_{n\to+\infty}\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s.$$

We can also remark that:

$$\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s = \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) + q(T-t) - \frac{1}{(Y_t^n)^q},$$

with $Y_t^n \leq 1/(q(T-t))^{1/q}$, so

$$0 \leq \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s \leq \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) \leq \frac{1}{\alpha^q}.$$

Therefore,

$$0 \leq \Phi_t \leq \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right).$$

For $r \leq t$,

$$\int_r^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s \geq \int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s$$

$$\Longrightarrow \mathbb{E}^{\mathcal{F}_r}\int_r^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s \geq \mathbb{E}^{\mathcal{F}_r}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s$$

$$\Longrightarrow \Phi_r \geq \mathbb{E}^{\mathcal{F}_r}\Phi_t.$$

We deduce that $(\Phi_t)_{0\leq t<T}$ is a non-negative bounded supermartingale. Now for all $n \in \mathbb{N}^*$,

$$\frac{1}{(Y_t^n)^q} = q(T-t) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{\left(Y_s^n\right)^{q+2}}\,\mathrm{d}s.$$

Fix $t < T$. Taking the limit as $n \to +\infty$, we deduce:

$$\frac{1}{(Y_t)^q} = q(T-t) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right) - \Phi_t.$$

From the above expression, $(\Phi_t)_{0\leq t<T}$ is right-continuous. $\square$

Φ being a right-continuous non-negative supermartingale, the limit of Φ_t as t goes to T exists $\mathbb{P}$-a.s. and this limit Φ_T is finite $\mathbb{P}$-a.s., since it is bounded by $1/\alpha^q$. The L^1-bounded martingale $\mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right)$ converges a.s. to $1/\xi^q$, as t goes to T; then the limit of Y_t as $t \to T$ exists and is equal to:

$$\lim_{t \to T} Y_t = \frac{1}{(\frac{1}{\xi^q} - \Phi_T)^{1/q}}.$$

If we were able to prove that Φ_T is zero a.s., we would have shown that $Y_T = \xi$.

1.2.2. The case of ξ non-negative

Now we just assume that $\xi \geq 0$. We cannot apply the Itô formula to $1/(Y^n)^q$ because we have no positive lower bound for Y^n. We will approach Y^n in the following way. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}.$$

This random variable is in L^2 and is greater than or equal to $1/m$ a.s. The BSDE (1), with $\xi^{n,m}$ as terminal condition, has a unique solution $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m})$. It is immediate that if $m \leq m'$ and $n \leq n'$ then:

$$\widetilde{Y}^{n,m'} \leq \widetilde{Y}^{n',m}.$$

As for the sequence Y^n, we can define $\widetilde{Y}^m$ as the limit when n grows to $+\infty$ of $\widetilde{Y}^{n,m}$. That limit $\widetilde{Y}^m$ is greater than $Y = \lim_{n \to +\infty} Y^n$. But for $m \leq m'$, for $t \in [0, T]$:

$$\begin{aligned}
\widetilde{Y}^{n,m}_t - \widetilde{Y}^{n,m'}_t &= \xi^{n,m} - \xi^{n,m'} - \int_t^T \left[\left(\widetilde{Y}^{n,m}_r\right)^{q+1} - \left(\widetilde{Y}^{n,m'}_r\right)^{q+1} \right] \mathrm{d}r \\
&\quad - \int_t^T \left[\widetilde{Z}^{n,m}_r - \widetilde{Z}^{n,m'}_r \right] \mathrm{d}B_r \\
&\leq \xi^{n,m} - \xi^{n,m'} - \int_t^T \left[\widetilde{Z}^{n,m}_r - \widetilde{Z}^{n,m'}_r \right] \mathrm{d}B_r,
\end{aligned}$$

and taking the conditional expectation given $\mathcal{F}_t$:

$$0 \leq \widetilde{Y}^{n,m}_t - \widetilde{Y}^{n,m'}_t \leq \mathbb{E}^{\mathcal{F}_t}\left(\xi^{n,m} - \xi^{n,m'}\right) \leq \frac{1}{m}. \tag{18}$$

Recall that (Y^n, Z^n) is the solution of the BSDE (1) with $\xi^n = \xi \wedge n$ as terminal data. Thus we also have:

$$0 \leq \widetilde{Y}^{n,m'}_t - Y^n_t \leq \mathbb{E}^{\mathcal{F}_t}\left(\xi^{n,m'} - \xi^n\right) \leq \frac{1}{m'}.$$

Letting $m' \to +\infty$ in the last estimate leads to $\lim_{m' \to +\infty} \widetilde{Y}^{n,m'}_t = Y^n_t$ a.s. and using the inequality (18):

$$0 \leq \widetilde{Y}^{n,m}_t - Y^n_t \leq \frac{1}{m}.$$

Therefore $\mathbb{P}$-a.s.:

$$\sup_{t\in[0,T]} \left|\widetilde{Y}_t^m - Y_t\right| \leq \frac{1}{m}.$$

Since $\widetilde{Y}^m$ has a limit on the left at T, so does Y.

1.3. *Proof of* (iv)

In order to complete the proof of Theorem 2, we need to establish the statement (iv).

Proposition 10. *Suppose there exists a constant $\alpha > 0$ such that $\mathbb{P}$-a.s. $\xi \geq \alpha$. In this case:*

$$\mathbb{E}\int_0^T (T-s)^{2/q}\|Z_s\|^2\,\mathrm{d}s \leq 8\left(\frac{1}{q}\right)^{2/q}. \tag{iv}$$

Proof. If $\xi \geq \alpha$, by comparison, for all integers n and all $t \in [0,T]$:

$$Y_t^n \geq \left(\frac{1}{qT + 1/\alpha^q}\right)^{1/q} > 0.$$

Let $\delta > 0$ and $\theta : \mathbb{R} \to \mathbb{R}$, $\theta_q : \mathbb{R} \to \mathbb{R}$ defined by:

$$\begin{cases}\theta(x) = \sqrt{x} & \text{on } [\delta, +\infty[,\\ \theta(x) = 0 & \text{on }]-\infty, 0],\end{cases} \quad \text{and} \quad \begin{cases}\theta_q(x) = x^{\frac{1}{2q}} & \text{on } [\delta, +\infty[,\\ \theta_q(x) = 0 & \text{on }]-\infty, 0],\end{cases}$$

and such that θ and θ_q are non-negative, non-decreasing and in respectively $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$. We apply the Itô formula on $[0, T-\delta]$ to the process $\theta_q(T-t)\theta(Y_t^n)$, with $\delta < (qT+1/\alpha^q)^{-1/q}$:

$$\begin{aligned}\theta_q(\delta)\theta(Y_{T-\delta}^n) - \theta_q(T)\theta(Y_0^n) = {} & \frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y_s^n)^{-1/2}Z_s^n\,\mathrm{d}B_s\\ & + \frac{1}{2}\int_0^{T-\delta}(T-s)^{\frac{1}{2q}}(Y_s^n)^{\frac{1}{2}}\left((Y_s^n)^q - \frac{1}{q(T-s)}\right)\mathrm{d}s\\ & - \frac{1}{8}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}}\,\mathrm{d}s\end{aligned}$$

so:

$$\begin{aligned}&\frac{1}{8}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}}\,\mathrm{d}s \leq T^{1/2q}\theta(Y_0^n) + \frac{1}{2}\int_0^{T-\delta}\frac{(T-s)^{1/2q}}{(Y_s^n)^{1/2}}Z_s^n\,\mathrm{d}B_s\\ &\quad + \frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y_s^n)^{1/2}\left((Y_s^n)^q - \frac{1}{q(T-s)}\right)\mathrm{d}s\end{aligned}$$

and since $Y_s^n \leq 1/(q(T-s))^{1/q}$ and $T^{1/q}Y_0^n \leq q^{-1/q}$, taking the expectation we obtain:

$$\frac{1}{8}\mathbb{E}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}}\,\mathrm{d}s \leq \theta_q(T)\theta(Y_0^n) \leq (1/q)^{1/2q},$$

that is for all n and all $\delta > 0$:

$$\mathbb{E}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z_s^n\|^2}{(Y_s^n)^{3/2}}\,\mathrm{d}s \leq 8(1/q)^{1/2q}.$$

Now, since $1/Y_s^n \geq (q(T-s))^{1/q}$, $\mathbb{E}\int_0^{T-\delta}(T-s)^{2/q}\|Z_s^n\|^2\,\mathrm{d}s \leq 8(1/q)^{2/q}$, and letting $\delta \to 0$ and with the Fatou lemma, we deduce that

$$\mathbb{E}\int_0^T (T-s)^{2/q}\|Z_s\|^2\,\mathrm{d}s \leq 8(1/q)^{2/q}.$$

This achieves the proof of the proposition. □

Now we come back to the case $\xi \geq 0$. We cannot apply the Itô formula because we do not have any positive lower bound for Y^n. But we can approximate the process (Y^n, Z^n) by a sequence of processes $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m})$ as in the proof of the existence of a limit for Y at time T. Let us recall that we solve the BSDE (1) with $\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}$ as terminal condition. For all $\delta > 0$, the Itô formula, applied to the process $(T-.)^{2/q}\left|\widetilde{Y}^{n,m} - Y^n\right|^2$, leads to the inequality:

$$\begin{aligned}\mathbb{E}\int_0^{T-\delta}(T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|^2\,\mathrm{d}r &\leq (\delta)^{2/q}\mathbb{E}\left|\widetilde{Y}_{T-\delta}^{n,m} - Y_{T-\delta}^n\right|^2 \\ &\quad + \frac{2}{q}\mathbb{E}\int_0^{T-\delta}(T-s)^{(2/q)-1}\left|\widetilde{Y}_s^{n,m} - Y_s^n\right|^2\,\mathrm{d}s.\end{aligned}$$

Let δ go to 0 in the previous inequality. We can do that because for all $t \in [0,T]$, $0 \leq \widetilde{Y}_t^{n,m} - Y_t^n$, which implies $\left|\widetilde{Y}_t^{n,m} - Y_t^n\right|^2 \leq n^2$, and because $(T-.)^{(2/q)-1}$ is integrable on the interval $[0,T]$. Finally, using the inequality: $0 \leq \widetilde{Y}_s^{n,m} - Y_s^n \leq \frac{1}{m}$, we have:

$$\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|^2\,\mathrm{d}r \leq \frac{2}{q}\frac{1}{m^2}\int_0^T (T-s)^{(2/q)-1}\,\mathrm{d}s = \frac{T^{2/q}}{m^2}.$$

Therefore, for all $\varepsilon > 0$:

$$\begin{aligned}\mathbb{E}\int_0^T (T-r)^{2/q}\left\|Z_r^n\right\|^2\,\mathrm{d}r &\leq \mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}\right\|^2\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|^2\,\mathrm{d}r \\ &\quad + 2\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}\right\|\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|\,\mathrm{d}r \\ &\leq (1+\varepsilon)\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m}\right\|^2\,\mathrm{d}r \\ &\quad + \left(1+\frac{1}{\varepsilon}\right)\mathbb{E}\int_0^T (T-r)^{2/q}\left\|\widetilde{Z}_r^{n,m} - Z_r^n\right\|^2\,\mathrm{d}r \\ &\leq 8(1+\varepsilon)(1/q)^{2/q} + \frac{T^{2/q}}{m^2}\left(1+\frac{1}{\varepsilon}\right).\end{aligned}$$

We have applied the previous result to $\widetilde{Z}^{n,m}$. Now we let first m go to $+\infty$ and then ε go to 0, we have:

$$\mathbb{E}\int_0^T (T-r)^{2/q}\left\|Z_r^n\right\|^2\,\mathrm{d}r \leq 8(1/q)^{2/q}. \tag{19}$$

The result follows by letting finally n go to ∞.

2. Continuity of Y at T

In this section, ξ is still supposed non-negative. We make precise the behaviour of Y in a neighbourhood of T (Proposition 3) and we show the continuity of Y at T under stronger assumptions (Theorem 4).

2.1. Lower bound and asymptotic behaviour in a neighbourhood of T

Now we construct an adapted process which is smaller than Y.

Lemma 11. *For $0 \leq t < T$, $\mathbb{P}$-a.s.*

$$Y_t \geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right].$$

Remark 12. The right hand side is obtained through the following operation: first, we solve the ordinary differential equation $y' = y^{1+q}$ with ξ as terminal condition; then we project this solution on the σ-algebra $\mathcal{F}_t$.

Proof. Let $n \in \mathbb{N}^*$ and consider for $0 \leq t \leq T$:

$$\Gamma_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{(\xi\wedge n)^q}}\right)^{1/q}\right].$$

Γ^n is well defined because the term in the conditional expectation is bounded by n. We have:

$$\left(\frac{1}{q(T-t)+\frac{1}{(\xi\wedge n)^q}}\right)^{1/q} = \xi\wedge n - \int_t^T \left[\frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1}{q}}}\right]^{1+q} \mathrm{d}r.$$

So Γ^n verifies:

$$\begin{aligned}\Gamma_t^n &= \mathbb{E}^{\mathcal{F}_t}\left(\xi\wedge n - \int_t^T \frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1+q}{q}}}\,\mathrm{d}r\right)\\ &= \mathbb{E}^{\mathcal{F}_t}\left(\xi\wedge n - \int_t^T (\Gamma_r^n)^{1+q}\,\mathrm{d}r - \int_t^T U_r^n\,\mathrm{d}r\right),\end{aligned}$$

with U^n the adapted and bounded (by n^{1+q}) process:

$$U_r^n = \mathbb{E}^{\mathcal{F}_r}\left(\frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1+q}{q}}}\right) - (\Gamma_r^n)^{1+q};$$

the Jensen inequality ($1+q > 1$) showing that $U_r^n \geq 0$, for $0 \leq r \leq T$. Then, the comparison Theorem 2.4 in [5] allows us to conclude that, for all $t \in [0, T]$, a.s.

$$\Gamma_t^n \leq Y_t^n \leq Y_t.$$

We then deduce from the monotone convergence theorem:

$$\lim_{n\to+\infty} \Gamma_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right] = \Gamma_t. \quad \square$$

We now establish the:

Proposition 13. *On the set* $\{\xi = +\infty\}$,

$$\lim_{t\uparrow T}(T-t)^{1/q} Y_t = \left(\frac{1}{q}\right)^{1/q}, \ a.s. \tag{6}$$

Proof. Indeed,

$$\begin{aligned}\Gamma_t &= \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+q(T-t)\xi^q}\right)^{1/q}\right] + \frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right)\\ &\geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+qT\xi^q}\right)^{1/q}\right] + \frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).\end{aligned}$$

Then,

$$(T-t)^{1/q}\Gamma_t \geq (T-t)^{1/q}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+qT\xi^q}\right)^{1/q}\right] + \left(\frac{1}{q}\right)^{1/q}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).$$

The first term, in the right hand side, converges to 0 on the set $\{\xi = +\infty\}$ and the second converges to $(1/q)^{1/q}$. Indeed, we have:

$$0 \leq \frac{\xi^q}{1+qT\xi^q} \leq \frac{1}{qT}$$

and we can apply the martingale convergence theorem. Since Y is bounded from above by $1/(q(T-t))^{1/q}$, this achieves the proof. $\square$

2.2. *Continuity at time T: The first step*

We now want to prove Theorem 4, i.e. $\xi \geq \limsup_{t\to T} Y_t$. Recall that we already know that the limit of Y_t as $t \to T$ exists a.s. From the inequality (15), we just have to show that on the set $\{\xi < +\infty\}$,

$$\xi \geq \lim_{t\to T} Y_t = \liminf_{t\to T} Y_t.$$

The main difficulty here is to find a "good" upper bound of Y_t. We shall use a method widely inspired by the article of Marcus and Véron [7] and more precisely by the proof of Lemma 2.2 page 1450. We try to adapt this method to our case.

We make stronger assumptions on ξ. From now and for the rest of this paper, we suppose that the conditions (H1), (H2), (L) and (G) hold.

Let φ be a function in the class $C^2(\mathbb{R}^m)$ with a compact support. Let (Y, Z) be the solution of the BSDE (1) with the final condition $\zeta \in L^2(\Omega)$. For any $t \in [0, T]$:

$$\begin{aligned} Y_t\varphi(X_t) &= Y_0\varphi(X_0) + \int_0^t \varphi(X_r)\left[Y_r|Y_r|^q \,\mathrm{d}r + Z_r.\mathrm{d}B_r\right] + \int_0^t Y_r \,\mathrm{d}(\varphi(X_r)) \\ &\quad + \int_0^t Z_r.\nabla\varphi(X_r)\sigma(r, X_r)\,\mathrm{d}r \\ &= Y_0\varphi(X_0) + \int_0^t \varphi(X_r)Y_r|Y_r|^q \,\mathrm{d}r + \int_0^t Z_r.\nabla\varphi(X_r)\sigma(r, X_r)\,\mathrm{d}r \\ &\quad + \int_0^t Y_r\mathcal{L}\varphi(X_r)\,\mathrm{d}r + \int_0^t (Y_r\nabla\varphi(X_r)\sigma(r, X_r) + \varphi(X_r)Z_r).\mathrm{d}B_r \end{aligned}$$

where $\mathcal{L}$ is the operator defined by (4). Taking the expectation:

$$\begin{aligned} \mathbb{E}(Y_t\varphi(X_t)) &= \mathbb{E}(Y_0\varphi(X_0)) + \mathbb{E}\int_0^t \varphi(X_r)Y_r|Y_r|^q \,\mathrm{d}r \\ &\quad + \mathbb{E}\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r, X_r)\,\mathrm{d}r + \mathbb{E}\int_0^t Y_r\mathcal{L}\varphi(X_r)\,\mathrm{d}r. \end{aligned} \tag{20}$$

The idea is to use the relation (20) with a suitable function φ. The set $F_\infty^c = \{g < +\infty\}$ is open in $\mathbb{R}^m$. Let U be a bounded open set with a regular boundary and such that the compact set $\overline{U}$ is included in F_∞^c. We denote by $\Phi = \Phi_U$ a function which is supposed to belong to $C^2(\mathbb{R}^m; \mathbb{R}_+)$ and such that Φ is equal to zero on $\mathbb{R}^m \setminus U$, is positive on U. Let α be a real number such that

$$\alpha > 2(1 + 1/q).$$

For $n \in \mathbb{N}$, let (Y^n, Z^n) be the solution of the BSDE (1) with the final condition $(g \wedge n)(X_T)$. The equality (20) becomes for $0 \le t \le T$:

$$\begin{aligned} \mathbb{E}(Y_T^n\Phi^\alpha(X_T)) &= \mathbb{E}(Y_t^n\Phi^\alpha(X_t)) + \mathbb{E}\int_t^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r + \mathbb{E}\int_t^T Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)\,\mathrm{d}r. \end{aligned} \tag{21}$$

Using (21), we will first prove that for every real $\alpha > 2(1 + 1/q)$ and for every n:

$$\mathbb{E}\int_0^T (Y_r^n)^{1+q}\Phi^\alpha(X_r)\,\mathrm{d}r \le C < \infty,$$

where the constant C is independent of n. In particular, we will have to find a bound for the term containing Z in the right hand side of the equation (21). We know how to control this term in the two cases of Theorem 4: we suppose that either $q > 2$ or (H3), (B), (D) and (E) are satisfied. Thanks to the Fatou lemma, $Y^{1+q}\Phi^\alpha(X)$ will belong to $L^1(\Omega \times [0, T])$. Then, we will deduce that the limit as t goes to T of Y_t is less than or equal to ξ a.s. on the set $\{\xi < +\infty\}$.

2.3. Continuity when $q > 2$

In this section, we will suppose that $q > 2$. In that case, we can easily control the term

$$\mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\,\mathrm{d}r.$$

Using the Cauchy–Schwarz inequality, we obtain:

$$\left|\mathbb{E}\int_0^t Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\,\mathrm{d}r\right|$$
$$\leq \left(\mathbb{E}\int_0^t \|Z_r^n\|^2(T-r)^{2/q}\,\mathrm{d}r\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^t \frac{\|\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\|^2}{(T-r)^{2/q}}\,\mathrm{d}r\right)^{\frac{1}{2}}.$$

From the inequality (19) of the first section:

$$\mathbb{E}\int_0^T \|Z_r^n\|^2(T-r)^{2/q}\,\mathrm{d}r \leq 8\left(\frac{1}{q}\right)^{\frac{2}{q}}.$$

And the second term

$$\mathbb{E}\int_0^T \frac{\|\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\|^2}{(T-r)^{2/q}}\,\mathrm{d}r$$

is finite if $q > 2$. Indeed, recall that Φ is compactly supported and $\alpha > 2$. Hence the numerator is bounded for all $(t, x) \in [0, T] \times \mathbb{R}^m$:

$$\|\nabla(\Phi^\alpha)(x)\sigma(t, x)\|^2 \leq \alpha^2\Phi^{2(\alpha-1)}(x)\|\nabla\Phi(x)\|^2(K(1+|x|))^2 \leq C.$$

Therefore, there exists a constant $C = C(q, \Phi, \alpha, \sigma)$ such that for all $t \in [0, T]$ and $n \in \mathbb{N}$:

$$\left|\mathbb{E}\int_0^t Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r, X_r)\,\mathrm{d}r\right| \leq C. \tag{22}$$

The term

$$\mathbb{E}\int_0^T Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)\,\mathrm{d}r$$

can be bounded using the Hölder inequality. Let p be such that $1/p + 1/(1+q) = 1$ or $p = 1 + 1/q$ and $p/(1+q) = p - 1$.

We will prove that

$$\Phi^{-\alpha(p-1)}\left|\mathcal{L}(\Phi^\alpha)\right|^p \in L^\infty([0, T] \times \mathbb{R}^m).$$

Using the growth condition (G) on σ, we have:

$$\Phi^{-\alpha(p-1)}\left|\mathrm{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t, .))\right|^p \leq C\Phi^{-\alpha(p-1)}\left\|D^2(\Phi^\alpha)\right\|^p(1+|x|^2)^p,$$

and

$$D^2(\Phi^\alpha) = \alpha\Phi^{\alpha-1}D^2\Phi + \alpha(\alpha-1)\Phi^{\alpha-2}\nabla\Phi\otimes\nabla\Phi$$
$$\Phi^{-\alpha(p-1)}\left\|D^2(\Phi^\alpha)\right\|^p \leq 2^{p-1}\left(\alpha^p\Phi^{\alpha-p}\left\|D^2\Phi\right\|^p + (\alpha(\alpha-1))^p\Phi^{\alpha-2p}|\nabla\Phi|^{2p}\right).$$

Hence:

$$\Phi^{-\alpha(p-1)} \left|\mathrm{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p \leq C\,\Phi^{\alpha-2p}\,\Psi$$

where Ψ is the continuous function on $\mathbb{R}^m$:

$$\Psi = \left[\alpha^p \Phi^p \left\|D^2\Phi\right\|^p + (\alpha(\alpha-1))^p|\nabla\Phi|^{2p}\right]\left[1+|x|^{2p}\right].$$

Since $\alpha > 2(1+1/q) = 2p$ and since Φ has a compact support,

$$\Phi^{-\alpha(p-1)} \left|\mathrm{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p \in L^\infty([0,T]\times\mathbb{R}^m). \tag{23}$$

For the second term $\Phi^{-\alpha(p-1)}\,|\nabla(\Phi^\alpha).b(t,.)|^p$, we have:

$$\nabla(\Phi^\alpha) = \alpha\,\Phi^{\alpha-1}\nabla\Phi \Rightarrow \Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha)\right|^p = \alpha^p\,\Phi^{\alpha-p}\,|\nabla\Phi|^p.$$

Since $\alpha > 2p$ and $|b(t,x)| \leq K(1+|x|)$,

$$\begin{aligned}\Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha).b(t,.)\right|^p &\leq 2^{p-1}K^p\,\Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha)\right|^p(1+|x|^p)\\ &= 2^{p-1}(K\alpha)^p\,\Phi^{\alpha-p}\,|\nabla\Phi|^p\,(1+|x|^p).\end{aligned}$$

Therefore,

$$\Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha).b(t,.)\right|^p \in L^\infty([0,T]\times\mathbb{R}^m). \tag{24}$$

Since

$$\left|\mathcal{L}(\Phi^\alpha)\right|^p \leq 2^{p-1}\left|\nabla(\Phi^\alpha).b(t,.)\right|^p + 2^{p-1}\left|\mathrm{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p,$$

we apply the Hölder inequality:

$$\begin{aligned}\mathbb{E}\left|Y_t^n\mathcal{L}(\Phi^\alpha)(X_t)\right| &\leq \left[\mathbb{E}\left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right)\right]^{\frac{1}{1+q}}\left[\mathbb{E}\left(\Phi^{-\alpha(p-1)}(X_t)\left|\mathcal{L}(\Phi^\alpha)(X_t)\right|^p\right)\right]^{1/p}\\ &\leq C\left[\mathbb{E}\left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right)\right]^{\frac{1}{1+q}},\end{aligned}$$

and the constant C depends only on q, b, σ, Φ and α, not on n, or on t. Finally, we have:

$$\begin{aligned}\mathbb{E}\int_0^T\left|Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)\right|\,\mathrm{d}r &\leq C\int_0^T\left[\mathbb{E}\left(\Phi^\alpha(X_r)(Y_r^n)^{1+q}\right)\right]^{\frac{1}{1+q}}\,\mathrm{d}r\\ &\leq C\left[\mathbb{E}\int_0^T\Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r\right]^{\frac{1}{1+q}}.\end{aligned} \tag{25}$$

We come back to the Eq. (21) for $t=0$:

$$\begin{aligned}\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) = \mathbb{E}(Y_0^n\Phi^\alpha(X_0)) &+ \mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)\,\mathrm{d}r\\ &+\mathbb{E}\int_0^T\Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r + \mathbb{E}\int_0^T Y_r^n\mathcal{L}(\Phi^\alpha)(X_r)\,\mathrm{d}r.\end{aligned}$$

Recall that $Y_T^n\Phi^\alpha(X_T) \leq g(X_T)\Phi^\alpha(X_T)$; since Φ is equal to zero outside a compact set included in $F_\infty^c = \{g < +\infty\}$, using the condition (H2) the left hand side of the previous equality is bounded by a constant independent of n.

In the right hand side, using the inequality (22), we deduce that the first two terms are also bounded. Thus, we have:

$$\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r + \mathbb{E}\int_0^T Y_r^n \mathcal{L}(\Phi^\alpha)(X_r)\,\mathrm{d}r \le C,$$

which implies with the inequality (25):

$$\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r - C\left[\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r\right]^{\frac{1}{1+q}} \le C. \tag{26}$$

The set $\left\{x \in \mathbb{R}_+,\ x - Cx^{\frac{1}{1+q}} \le C\right\}$ is bounded. Therefore, we deduce that:

$$\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r \le C.$$

Hence, we have proved:

Lemma 14. *The sequence $\Phi^\alpha(X)(Y^n)^{1+q}$ is a bounded sequence in $L^1(\Omega \times [0,T])$ and with the Fatou lemma, $Y^{1+q}\Phi^\alpha(X)$ belongs to $L^1(\Omega \times [0,T])$.*

This inequality allows us to show that

$$\liminf_{t\to T} Y_t \le \xi.$$

Indeed, let θ be a function of class $C^2(\mathbb{R}^m;\mathbb{R}^+)$ with a compact support strictly included in $F_\infty^c = \{g < +\infty\}$. There exists an open set U s.t. the support of θ is included in U and $\overline{U} \subset F_\infty^c$. Let $\Phi = \Phi_U$ be the previously used function. Let us recall that α is strictly greater than $2(1+1/q) > 2$. Thanks to a result in the proof of the lemma 2.2 of [7], there exists a constant $C = C(\theta,\alpha)$ such that:

$$|\theta| \le C\Phi^\alpha, \quad |\nabla\theta| \le C\Phi^{\alpha-1} \quad \text{and} \quad \left\|D^2\theta\right\| \le C\Phi^{\alpha-2}.$$

We write again the Eq. (21) for θ, $n \in \mathbb{N}^*$ and $0 \le t \le T$:

$$\begin{aligned}\mathbb{E}(Y_T^n\theta(X_T)) &= \mathbb{E}(Y_t^n\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_t^T Y_r^n\mathcal{L}\theta(X_r)\,\mathrm{d}r + \mathbb{E}\int_t^T Z_r^n\nabla\theta(X_r)\sigma(r,X_r)\,\mathrm{d}r.\end{aligned} \tag{27}$$

In the left hand side, we use the assumption (H2) to pass to the limit as n tends to ∞. We just have to control the right hand side as n tends to infinity. For the first term, there is no problem: we use the dominated convergence theorem. For the second, we apply the monotone convergence theorem. For the third one, we can do the same calculations using the previously given estimations on θ, $\nabla\theta$ and $D^2\theta$ in terms of power of Φ^α and Hölder's inequality:

$$\Phi^{-\alpha(p-1)}|\mathcal{L}\theta|^p \in L^\infty([0,T]\times\mathbb{R}^m). \tag{28}$$

Now we can write:

$$Y_r^n\mathcal{L}\theta(X_r) = \left(Y_r^n\Phi^{\alpha/(1+q)}\right)\left(\Phi^{-\alpha/(1+q)}\mathcal{L}\theta(X_r)\right).$$

The sequence $Y^n \Phi^{\alpha/(1+q)} = Y^n \Phi^{\alpha(1-1/p)}$ is a bounded sequence in $L^{1+q}(\Omega \times [0,T])$ (see Lemma 14). With (28), using a weak convergence result and extracting a subsequence if necessary, we can pass to the limit in the term $\mathbb{E}\int_t^T Y_r^n \mathcal{L}\theta(X_r)\,\mathrm{d}r$. Moreover:

$$\forall n \in \mathbb{N}, \mathbb{E}\int_0^T \left|Y_r^n \mathcal{L}\theta(X_r)\right| \mathrm{d}r \leq C. \tag{29}$$

For the remaining term

$$\mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r, X_r)\,\mathrm{d}r$$

recall that from Section 1.3, there exists a constant $C = C(q)$ for all $n \in \mathbb{N}$:

$$\mathbb{E}\int_0^T \|Z_r^n\|^2 (T-r)^{\frac{2}{q}}\,\mathrm{d}r \leq C.$$

Hence, there exists a subsequence, which we still denote as $Z^n(T-r)^{1/q}$, and which converges weakly in the space $L^2\left(\Omega \times (0,T), d\mathbb{P} \times dt; \mathbb{R}^d\right)$ to a limit, and the limit is $Z(T-r)^{1/q}$, because we already know that Z^n converges to Z in $L^2(\Omega \times (0, T-\delta))$ for all $\delta > 0$. $\nabla\theta(X)\sigma(.,X)(T-\ .)^{-1/q}$ is $L^2(\Omega \times (0,T))$, because θ is compactly supported and $q > 2$. Therefore,

$$\lim_{n\to+\infty} \mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)\,\mathrm{d}r = \mathbb{E}\int_t^T Z_r.\nabla\theta(X_r)\sigma(r,X_r)\,\mathrm{d}r$$

and

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\int_0^T \left|Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)\,\mathrm{d}r\right| \mathrm{d}r \leq C. \tag{30}$$

Passing to the limit as $n \to +\infty$ in (27):

$$\begin{aligned}\mathbb{E}(\xi\theta(X_T)) &= \mathbb{E}(Y_t\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r)^{1+q}\,\mathrm{d}r + \mathbb{E}\int_t^T Y_r\mathcal{L}\theta(X_r)\,\mathrm{d}r \\ &\quad + \mathbb{E}\int_t^T Z_r.\nabla\theta(X_r)\sigma(r,X_r)\,\mathrm{d}r.\end{aligned}$$

We let t go to T and we apply Lemma 14, inequalities (29) and (30), and Fatou's lemma:

$$\begin{aligned}\mathbb{E}[\xi\theta(X_T)] = \lim_{t\to T}\mathbb{E}[Y_t\theta(X_t)] &\geq \mathbb{E}\left[\left(\liminf_{t\to T} Y_t\right)\theta(X_T)\right] \\ &= \mathbb{E}\left[\left(\lim_{t\to T} Y_t\right)\theta(X_T)\right].\end{aligned} \tag{31}$$

But recall that we already know (iii): $\lim_{t\to T} Y_t \geq g(X_T)$. Hence, the inequality in (31) is in fact a equality, i.e.

$$\mathbb{E}[g(X_T)\theta(X_T)] = \mathbb{E}\left[\theta(X_T)\left(\lim_{t\to+\infty} Y_t\right)\right].$$

And using again (iii), we conclude that:

$$\lim_{t\to+\infty} Y_t = g(X_T), \quad \mathbb{P}\text{-a.s. on } \{g(X_T) < \infty\}.$$

Therefore, we have proved the continuity of Y on $[0, T]$ in the case $q > 2$.

Remark 15. A little modification in the previous arguments shows that the sequences $Y^n\mathcal{L}(\Phi^\alpha)(X)$ and $Y^n\mathcal{L}\theta(X)$ are bounded in $L^r(\Omega \times [0, T])$ for $1 \le r < 1 + q$. Therefore the sequence $Y^n\mathcal{L}\theta(X)$ is uniformly integrable and the passage to the limit and the estimate (29) follow.

2.4. When assumptions (H3), (B), (D) and (E) are satisfied

If we just assume $q > 0$, our previous control on the term containing Z in (21) fails. But with the assumptions (H3), (B), (D) and (E), we are able to prove that there exists a function ψ such that for $0 < t \le T$:

$$\mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r, X_r)\,\mathrm{d}r = \mathbb{E}\int_t^T Y_r^n\psi(r, X_r)\,\mathrm{d}r,$$

and then, we apply again the Hölder inequality in order to control

$$\mathbb{E}\int_t^T Y_r^n\psi(r, X_r)\,\mathrm{d}r \quad \text{by } \mathbb{E}\int_t^T (Y_r^n)^{1+q}\,\Phi^\alpha(X_r)\,\mathrm{d}r.$$

We need the existence of a regular density for the process X solution of the SDE (7). According to the article of Aronson [12], Theorems 7 and 10, there exists a density (Green's function) for X, $p(x; ., .) \in L^2\left(\delta, T; H^2\right)$ for all $\delta > 0$.

Moreover, from the Theorem 7 of [12] and the Theorem II.3.8 of [13], the density is Hölder continuous in x and satisfies the following inequality for $s \in]0, T]$:

$$\frac{\exp\left(-C\frac{|y-x|^2}{s}\right)}{Cs^{m/2}} \le p(x; s, y) \le \frac{C\exp\left(-\frac{|y-x|^2}{Cs}\right)}{s^{m/2}}, \tag{32}$$

C depending only on T, on the bounds K and λ in (B) and (E), and C is independent of the regularity of these functions.

From now on, we omit the variable x in $p(x; ., .)$.

2.4.1. A preliminary result

We now prove the following result for the solution (Y, Z) of the BSDE:

$$Y_t = h(X_T) + \int_t^T f(Y_r)\,\mathrm{d}r - \int_t^T Z_r\,\mathrm{d}B_r,$$

where $h : \mathbb{R}^m \to \mathbb{R}$ is a bounded and Lipschitz function.

Proposition 16. *Under the assumptions (B), (D) and (E), for each function φ in the class $C^2(\mathbb{R}^m)$ with a compact support, there exists a real Borel function ψ defined on $]0, T] \times \mathbb{R}^m$ s.t. for all $t > 0$, $\mathbb{E}(|Y_t\psi(t, X_t)|) < +\infty$ and*

$$\mathbb{E}[Z_t.\nabla\varphi(X_t)\sigma(t, X_t)] = -\mathbb{E}[Y_t\psi(t, X_t)].$$

The function ψ is given by the following formula:

$$\psi(t,x) = \sum_{i=1}^{d} (\nabla\varphi\sigma)_i(x) \frac{\mathrm{div}(p\sigma^i)(t,x)}{p(t,x)} + \mathrm{Trace}\left(D^2\varphi(x)\sigma\sigma^*(t,x)\right) + \sum_{i=1}^{d} \nabla\varphi(x).[(\nabla\sigma^i)\sigma^i](t,x); \tag{33}$$

where σ^i is the i-th column of the matrix σ and p is the density of the process X.

Proof. To find this function ψ, we use the following result:

Proposition 17. *For all $1 \le i \le d$, $\left\{D^i_s Y_s, 0 \le s \le T\right\}$ is a version of $(Z)_i$.*

$(Z)_i = \{(Z_s)_i, 0 \le s \le T\}$ denotes the i-th component of Z. This result comes from the Proposition 5.3 in the article of El Karoui et al. [14]. Here, $D^i_s Y_s$ has the following sense:

$$D^i_s Y_s = \lim_{\substack{r\to s\\ r<s}} D^i_r Y_s.$$

From the conditions (L) and (G) and the Theorem 2.2.1 of [15], we know that X_T belongs to $\mathbb{D}^{1,\infty}$, and since h is Lipschitz, with the Proposition 1.2.3 of [15], $\xi = h(X_T) \in \mathbb{D}^{1,2}$. Moreover, since h is bounded (by M), Y is also bounded (by M) and f is a C^1-function. Hence, the conclusion of the Proposition 5.3 of [14] holds.

We must calculate:

$$\mathbb{E}[Z_t.\nabla\varphi(X_t)\sigma(t,X_t)] = \mathbb{E}[D_tY_t.\nabla\varphi\sigma(X_t)] = \sum_{i=1}^{d} \mathbb{E}\left[D^i_t Y_t.(\nabla\varphi\sigma)_i(X_t)\right],$$

where $\nabla\varphi\sigma(X_t) = (\nabla\varphi)(X_t)\sigma(t,X_t)$ and $(\nabla\varphi\sigma)_i(X_t)$ denotes the i-th component of $\nabla\varphi\sigma(X_t)$.

Let ν^i_j, $j \in \mathbb{N}^*$, be the following function:

$$\nu^i_j(r) = j\mathbf{1}_{[t-1/j,t]}(r)e_i,$$

with $r \in [0,T]$ and with $(e_1,\ldots,e_d)$ the canonical basis of $\mathbb{R}^d$. Here, we need $t > 0$. We define

$$D_{\nu^i_j} Y_t = \langle DY_t, \nu^i_j\rangle_H,$$

H being the Hilbert space $L^2([0,T];\mathbb{R}^d)$. The integration by parts formula for the Malliavin derivative is the following:

$$\mathbb{E}\left[D_{\nu^i_j} Y_t(\nabla\varphi\sigma)_i(X_t)\right] = \mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)\int_0^T \nu^i_j(r).\,\mathrm{d}B_r\right] - \mathbb{E}\left[Y_t D_{\nu^i_j}((\nabla\varphi\sigma)_i(X_t))\right]. \tag{34}$$

Now we calculate the first term of the right hand side.

$$\begin{aligned}\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)\int_0^T \nu^i_j(r).\,\mathrm{d}B_r\right] &= j\,\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)(B^i_t - B^i_{t-1/j})\right]\\ &= -j\,\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)\int_{t-1/j}^t \frac{\mathrm{div}(p\sigma^i)(u,X_u)}{p(u,X_u)}\,\mathrm{d}u\right],\end{aligned} \tag{35}$$

where p is the density of X and σ^i is the i-th column of the matrix σ.

The last equality is justified by the Lemmas 3.1 and 4.1 of the article of Pardoux [16]. We must show that the assumptions of these lemmas are satisfied. First, Y is a function of X: there exists a continuous function $u : [0, T] \times \mathbb{R}^m \to \mathbb{R}$ such that $Y_t = u(t, X_t)$. Hence, we can write:

$$\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)(B^i_t - B^i_{t-1/m})\right] = \mathbb{E}\left[v(t, X_t)(B^i_t - B^i_{t-1/m})\right],$$

with $v(t, x) = u(t, x)(\nabla\varphi\sigma)_i(x)$. Since φ has a compact support, so does v. And v is measurable and bounded. So the conditions of the Lemma 4.1 are verified, the time dependence of g playing no role in the demonstration. (L), (B), (E) are not exactly the assumptions of the Lemmas 3.1 and 4.1, but with these conditions, the conclusions are the same. In fact, the main problem is to give a sense to the fraction $\mathrm{div}(p\sigma^i)/p$. From the lower bound in (32), the set $\{(s, y) \in]0, T] \times \mathbb{R}^m;\ p(s, y) = 0\}$ is empty. Thus the conclusion of the Lemma 3.1 holds. Moreover the property $p \in L^2\left(\delta, T; H^2\right), \forall\delta > 0$, implies that $\mathrm{div}(p\sigma^i)$ belongs to $L^2([\delta, T] \times \mathbb{R}^m)$ for all $\delta > 0$ (Lemma 2.1 in [16] with $m = 0$ and δ instead of 0). Since $t > 0$ is fixed here, the proof of the Theorem 2.2 in [16] (see page 53) gives us the equality (35).

We have an additional regularity property on div p. This property is given by the Theorem 12.1, Section 3, page 223 of [17]. Since $p(., .)$ is solution of the PDE:

$$\partial_t p = \mathcal{L}^* p,$$

and since the coefficients of $\mathcal{L}$ are bounded and Lipschitz in x, the first derivatives of $\sigma\sigma^*$ belong to $L^\infty([0, T] \times \mathbb{R}^m)$. Therefore $\mathcal{L}^*$ is uniformly elliptic and can be written in divergence form. From (B) and (L), b and ∇b are bounded. From (B), (L) and (D), all coefficients of $\mathcal{L}^*$ are bounded. Hence the conclusions of the Theorem III.12.1 are valid: $\partial p/\partial x_i$ satisfies a Hölder condition in x.

From the lower bound in (32), $1/p$ is a continuous function. Moreover, $\mathrm{div}(p\sigma^i)$ is also continuous. Let j goes to $+\infty$ in the identity (34):

$$\begin{aligned}\mathbb{E}\left[D^i_t Y_t(\nabla\varphi\sigma)_i(X_t)\right] = &-\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)\frac{\mathrm{div}(p\sigma^i)(t, X_t)}{p(t, X_t)}\right] \\ &-\mathbb{E}\left[Y_t D^i_t\left((\nabla\varphi\sigma)_i(X_t)\right)\right].\end{aligned}$$

To find $D^i_t\left((\nabla\varphi\sigma)_i(X_t)\right)$, recall the following result (see Proposition 1.2.3 and Theorem 2.2.1 in [15]): since the process X has a density, if μ is a Lipschitz function, then: $D^i_t\mu(X_t) = \nabla\mu(X_t)D^i_t(X_t) = \nabla\mu(X_t)\sigma^i(X_t)$. Applying this result with $\mu = (\nabla\varphi\sigma)_i$, we obtain:

$$D^i_t\left((\nabla\varphi\sigma)_i(X_t)\right) = (\sigma^* D^2\varphi\sigma)_{ii}(X_t) + \nabla\varphi.[\nabla\sigma^i.\sigma^i](X_t).$$

Finally,

$$\begin{aligned}\mathbb{E}\left[Z_t.\nabla\varphi(X_t)\sigma(t, X_t)\right] &= \sum_{i=1}^d \mathbb{E}\left[D^i_t Y_t.(\nabla\varphi\sigma)_i(X_t)\right] = -\mathbb{E}\left[Y_t\psi(t, X_t)\right] \\ &= -\mathbb{E}\left[Y_t\left(\sum_{i=1}^d(\nabla\varphi\sigma)_i(X_t)\frac{\mathrm{div}(p\sigma^i)(t, X_t)}{p(t, X_t)}\right)\right] \\ &\quad -\mathbb{E}\left[Y_t\left(\mathrm{Trace}(D^2\varphi\sigma\sigma^*)(X_t) + \sum_{i=1}^d\nabla\varphi.[(\nabla\sigma^i)\sigma^i](X_t)\right)\right]. \quad \square\end{aligned}$$

The hypothesis (H3) implies that $g \wedge n$ is a Lipschitz function on $\mathbb{R}^m$. Indeed if we define

$$K_n = \sup \left\{ \frac{|g(x) - g(y)|}{|x - y|};\ g(x) \vee g(y) \leq n \right\},$$

then the assumption (H3) implies that (K_n) is a non-decreasing sequence of real positive numbers. Moreover, if x and y satisfy $g(x) \vee g(y) \leq n$ or $g(x) \wedge g(y) \geq n$, then $|(g \wedge n)(x) - (g \wedge n)(y)| \leq K_n |x - y|$. And if $g(y) < n \leq g(x)$, then the continuity of g leads to:

$$\begin{aligned} |(g \wedge n)(x) - (g \wedge n)(y)| = n - g(y) &\leq K_{n+1} \mathrm{dist}(y, \left\{z \in \mathbb{R}^m; g(z) \geq n\right\}) \\ &\leq K_{n+1}|y - x|. \end{aligned}$$

Finally $g \wedge n$ has a Lipschitz norm smaller than K_{n+1}.

We can apply Proposition 16 with Y^n, Z^n, φ. Coming back to the Eq. (21) for $t > 0$:

$$\begin{aligned} \mathbb{E}(Y_T^n \varphi(X_T)) &= \mathbb{E}(Y_t^n \varphi(X_t)) + \mathbb{E} \int_t^T \varphi(X_r)(Y_r^n)^{1+q} \,\mathrm{d}r \\ &\quad - \mathbb{E} \int_t^T Y_r^n \psi(r, X_r) \,\mathrm{d}r + \mathbb{E} \int_t^T Y_r^n \nabla\varphi(X_r) b(r, X_r) \,\mathrm{d}r \\ &\quad + \frac{1}{2} \mathbb{E} \int_t^T Y_r^n \mathrm{Trace}(D^2 \varphi(X_r) \sigma\sigma^*(r, X_r)) \,\mathrm{d}r \end{aligned} \tag{36}$$

where ψ is given by the formula (33) in Proposition 16.

2.4.2. *Continuity with (H3), (B), (E) and (D)*

Recall that U is a bounded open set such that $\overline{U} \subset F_\infty^c = \{g < \infty\}$, that $\Phi = \Phi_U$ is a function which is supposed to belong to $C^2(\mathbb{R}^m; \mathbb{R}_+)$ and such that Φ is equal to zero on $\mathbb{R}^m \setminus U$, is strictly positive on U. α is a real such that $\alpha > 2(1 + 1/q)$. For $n \in \mathbb{N}$, let (Y^n, Z^n) be the solution of the BSDE (1) with the final condition $(g \wedge n)(X_T)$. For $0 < t \leq T$, the relation (36) is:

$$\begin{aligned} \mathbb{E}(Y_T^n \Phi^\alpha(X_T)) &= \mathbb{E}(Y_t^n \Phi^\alpha(X_t)) + \mathbb{E} \int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q} \,\mathrm{d}r \\ &\quad + \mathbb{E} \int_t^T Y_r^n \Psi_\alpha(r, X_r) \,\mathrm{d}r, \end{aligned} \tag{37}$$

with Ψ_α the following function: for $t \in \,]0, T]$ and $x \in \mathbb{R}^m$

$$\begin{aligned} \Psi_\alpha(t, x) &= \nabla(\Phi^\alpha)(x).b(t, x) - \frac{1}{2} \mathrm{Trace}(D^2(\Phi^\alpha)(x) \sigma\sigma^*(t, x)) \\ &\quad - \sum_{i=1}^d \left((\nabla(\Phi^\alpha)(x)\sigma(t, x))_i \, \frac{\mathrm{div}(p(t, x)\sigma^i(t, x))}{p(t, x)} \right) \\ &\quad - \sum_{i=1}^d \left(\nabla(\Phi^\alpha)(x).[\nabla\sigma^i(t, x)\sigma^i(t, x)] \right). \end{aligned}$$

Our goal now is to prove that for a fixed $\varepsilon > 0$ and $p = 1 + 1/q$:

$$\Phi^{-\alpha(p-1)} \,|\, \Psi_\alpha|^p \in L^\infty([\varepsilon, T] \times \mathbb{R}^m).$$

If it is true, then the last term in (37) satisfies:

$$\mathbb{E}\int_t^T \left|Y_r^n \Psi_\alpha(r, X_r)\right| \mathrm{d}r \le C \left(\mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r\right)^{\frac{1}{1+q}}$$

and the end of the proof will be the same as in the case $q > 2$.

From the case $q > 2$ (see (23) and (24)), we already know that

$$\Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha).b(t,.)\right|^p \quad \text{and} \quad \Phi^{-\alpha(p-1)}\left|\frac{1}{2}\mathrm{Trace}\left(D^2(\Phi^\alpha)\sigma\sigma^*(t,.)\right)\right|^p$$

are in $L^\infty([0,T]\times\mathbb{R}^m)$. The next term is:

$$\Phi^{-\alpha(p-1)}\left|\sum_{i=1}^d \nabla(\Phi^\alpha).[\nabla\sigma^i(t,.)\sigma^i(t,.)]\right|^p.$$

σ satisfies the conditions (L) and (B). We use again the calculation done for the gradient of Φ^α (see the proof of (24)) to deduce that if $\alpha > 2p$ this term is in $L^\infty([0,T]\times\mathbb{R}^m)$. Now, we come to the last term which involves the density of X:

$$\begin{aligned}
&\Phi^{-\alpha(p-1)}\left|\sum_{i=1}^d (\nabla(\Phi^\alpha)\sigma(t,.))_i\,\frac{\mathrm{div}(p\sigma^i)(t,.)}{p(t,.)}\right|^p\\
&\quad = \alpha^p\,\Phi^{\alpha-p}\left|\sum_{i=1}^d (\nabla\Phi\sigma(t,.))_i\,\frac{\mathrm{div}(p\sigma^i)(t,.)}{p(t,.)}\right|^p.
\end{aligned}$$

We split this term into two parts:

$$\begin{aligned}
&\Phi^{\alpha-p}\left|\sum_{i=1}^d\left((\nabla\Phi\sigma(t,.))_i\,\left((\mathrm{div}\sigma^i)+\sigma^i\frac{(\nabla p)}{p}\right)(t,.)\right)\right|^p\\
&\quad \le (2d)^{p-1}\sum_{i=1}^d \Phi^{\alpha-p}\left|(\nabla\Phi\sigma(t,.))_i\,(\mathrm{div}\,\sigma^i)(t,.)\right|^p\\
&\qquad + (2d)^{p-1}\sum_{i=1}^d \Phi^{\alpha-p}\left|(\nabla\Phi\sigma(t,.))_i\sigma^i(t,.)\right|^p\frac{|(\nabla p)(t,.)|^p}{|p(t,.)|^p}.
\end{aligned}$$

For the first part, there is no problem because $\alpha - p > 0$, so $\Phi^{\alpha-p}$ is continuous and compactly supported and $(\nabla\Phi\sigma)_i$ $(\mathrm{div}\,\sigma^i)$ is bounded because of conditions (L) and (B). For the second part, we use the inequality (32) and the fact that the support of $\Phi^{\alpha-p}$ is a compact set $\mathcal{K}$. So the minimum of $p(.,.)$ exists on the set $[\varepsilon,T]\times\mathcal{K}$ and is positive. Therefore, we control the denominator. For the numerator, we already know that $\partial p/\partial x_i$ satisfies a Hölder condition in x. We can now conclude that the second part is bounded by a constant K independent of n and t.

Finally, we have:

$$\Phi^{-\alpha(p-1)}|\Psi_\alpha|^p \in L^\infty([\varepsilon,T]\times\mathbb{R}^m),$$

and thus, for $t \ge \varepsilon$:

$$\left|\mathbb{E}\left[Y_t^n\Psi_\alpha(t,X_t)\right]\right| \le C\mathbb{E}\left[\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right]^{\frac{1}{1+q}}, \tag{38}$$

where C is a constant independent of t and n. The Eq. (37) is:

$$\mathbb{E}(Y_T^n \Phi^\alpha(X_T)) - \mathbb{E}(Y_\varepsilon^n \Phi^\alpha(X_\varepsilon)) = \mathbb{E}\int_\varepsilon^T \left[\Phi^\alpha(X_r)(Y_r^n)^{1+q}\right] \mathrm{d}r + \mathbb{E}\int_\varepsilon^T \left[Y_r^n \Psi_\alpha(r, X_r)\right] \mathrm{d}r.$$

We have: $Y_T^n \Phi^\alpha(X_T) \leq g(X_T)\Phi^\alpha(X_T)$; since Φ is equal to zero outside a compact set included in $F_\infty^c = \{g < +\infty\}$, using the condition (H2) the left hand side of the previous equality is bounded by a constant independent of n. Therefore, we obtain:

$$\mathbb{E}\int_\varepsilon^T \left[\Phi^\alpha(X_r)(Y_r^n)^{1+q}\right] \mathrm{d}r - C\left[\mathbb{E}\int_\varepsilon^T \left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right) \mathrm{d}t\right]^{\frac{1}{1+q}} \leq C.$$

Since the set $\{x \in \mathbb{R}_+, x - Cx^{\frac{1}{1+q}} \leq C\}$ is bounded, we immediately deduce that $\Phi^\alpha(X)(Y^n)^{1+q}$ is a bounded sequence in $L^1(\Omega \times [\varepsilon, T])$. With the Fatou lemma, for $0 < \varepsilon \leq T$, $Y^{1+q}\Phi^\alpha(X)$ belongs to $L^1(\Omega \times [\varepsilon, T])$.

As Y is bounded on the interval $[0, \varepsilon]$, $\varepsilon < T$, by $1/(q(T-\varepsilon))^{1/q}$, we have proved:

$$\mathbb{E}\int_0^T Y_r^{1+q}\Phi^\alpha(X_r)\,\mathrm{d}r < +\infty. \tag{39}$$

As in the case $q > 2$, this inequality allows us to show that: $\liminf_{t\to T} Y_t \leq \xi$. Indeed, let θ be a function of class $C^2(\mathbb{R}^d; \mathbb{R}^+)$ with a compact support strictly included in $F_\infty^c = \{g < +\infty\}$. We write again the Eq. (36) for θ, $n \in \mathbb{N}^*$, $\varepsilon > 0$ and $t \geq \varepsilon$:

$$\begin{aligned}\mathbb{E}((\xi \wedge n)\theta(X_T)) &= \mathbb{E}(Y_T^n\theta(X_T)) = \mathbb{E}(Y_t^n\theta(X_t)) \\ &\quad + \mathbb{E}\int_t^T \theta(X_r)(Y_r^n)^{1+q}\,\mathrm{d}r + \mathbb{E}\int_t^T Y_r^n\,\Theta(r, X_r)\,\mathrm{d}r,\end{aligned} \tag{40}$$

with

$$\begin{aligned}\Theta(t,x) &= \nabla\theta(x).b(t,x) - \sum_{i=1}^d \left((\nabla\theta(x)\sigma(t,x))_i \frac{\operatorname{div}(p(t,x)\sigma^i(t,x))}{p(t,x)}\right) \\ &\quad - \frac{1}{2}\mathrm{Trace}(D^2\theta(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^d \left(\nabla\theta(x).[\nabla\sigma^i(t,x).\sigma^i(t,x)]\right).\end{aligned}$$

With the same arguments as in the case $q > 2$, we can prove that

$$\xi = \lim_{t\to T} Y_t, \quad \mathbb{P}\text{-a.s. on } \{\xi < +\infty\}.$$

3. Minimal solution

In this section we prove Theorem 5: the solution constructed in Sections 1 and 2 is the minimal one. Before we obtain the following estimate:

Proposition 18. *With the assumptions of Theorem 5, we prove:*

$$\forall t \in [0, T],\ \bar{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

Proof. For every $0 < h < T$, we define on $[0, T-h]$

$$\Lambda_h(t) = \left(\frac{1}{q(T-h-t)}\right)^{\frac{1}{q}}.$$

Λ_h is the solution of the ordinary differential equation: $\Lambda_h'(t) = (\Lambda_h(t))^{1+q}$, with final condition $\Lambda_h(T-h) = +\infty$. But on the interval $[0, T-h]$, $(\bar{Y}, \bar{Z})$ is a solution of the BSDE (1) with final condition $\bar{Y}_{T-h}$. From the assumptions $\bar{Y}_{T-h}$ is in $L^2(\Omega)$, so is finite a.s. Now we take the difference between $\bar{Y}$ and Λ_h for all $0 \leq t \leq s < T-h$:

$$\begin{aligned}\Lambda_h(t) - \bar{Y}_t &= \Lambda_h(s) - \bar{Y}_s - \int_t^s \left(\Lambda_h(r)^{1+q} - \left(\bar{Y}_r\right)^{1+q}\right) \mathrm{d}r - \int_t^s \bar{Z}_r \,\mathrm{d}B_r \\ &= \Lambda_h(s) - \bar{Y}_s - \int_t^s \alpha_r \left(\Lambda_h(r) - \bar{Y}_r\right) \mathrm{d}r - \int_t^s \bar{Z}_r \,\mathrm{d}B_r\end{aligned}$$

with

$$\alpha_r = \begin{cases} \dfrac{(\Lambda_h(r))^{1+q} - (\bar{Y}_r)^{1+q}}{\Lambda_h(r) - \bar{Y}_r} & \text{for } \bar{Y}_r \neq \Lambda_h(r) \\ 0 & \text{if } \bar{Y}_r = \Lambda_h(r). \end{cases}$$

So α is a non-negative progressively measurable process. Then we deduce:

$$\Lambda_h(t) - \bar{Y}_t = \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \bar{Y}_s\right) \exp\left(\int_t^s -\alpha_r \,\mathrm{d}r\right)\right].$$

Moreover we know that: $\bar{Y}_s \leq \mathbb{E}^{\mathcal{F}_s}\left(\bar{Y}_{T-h}\right)$. Therefore

$$\begin{aligned}\Lambda_h(t) - \bar{Y}_t &\geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \mathbb{E}^{\mathcal{F}_s}\left(\bar{Y}_{T-h}\right)\right) \exp\left(\int_t^s -\alpha_r \,\mathrm{d}r\right)\right] \\ &= \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \bar{Y}_{T-h}\right) \exp\left(\int_t^s -\alpha_r \,\mathrm{d}r\right)\right].\end{aligned}$$

Now Fatou's lemma leads, as s goes to $T-h$, to: $\Lambda_h(t) - \bar{Y}_t \geq 0$. This inequality is true for all $t \in [0, T-h]$ and for all $0 < h < T$. So it is clear that for every $t \in [0, T]$:

$$\bar{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

This achieves the proof of the proposition. □

In the case where $\xi = +\infty$ a.s. this inequality and Lemma 11 give the uniqueness of the solution. If $\xi = +\infty$, there is a unique solution, namely

$$Y_t = \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \quad \text{and} \quad Z_t = 0.$$

Proof of Theorem 5. We will prove that $\bar{Y}$ is greater than Y^n for all $n \in \mathbb{N}$, which implies that Y is the minimal solution.

Let (Y^n, Z^n) be the solution of the BSDE (1) with $\xi \wedge n$ as terminal condition. By comparison with the solution of the same BSDE with the deterministic terminal data n:

$$Y_t^n \leq \left(\frac{1}{q(T-t)+1/n^q}\right)^{1/q} \leq n.$$

Between the instants $0 \leq t \leq s < T$:

$$\begin{aligned} \bar{Y}_t - Y_t^n &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \left((\bar{Y}_r)^{1+q} - (Y_r^n)^{1+q}\right) \mathrm{d}r - \int_t^s \left(\bar{Z}_r - Z_r^n\right) \mathrm{d}B_r \\ &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \left(\frac{(\bar{Y}_r)^{1+q} - (Y_r^n)^{1+q}}{\bar{Y}_r - Y_r^n}\right) \left(\bar{Y}_r - Y_r^n\right) \mathrm{d}r - \int_t^s \left(\bar{Z}_r - Z_r^n\right) \mathrm{d}B_r \\ &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \alpha_r^n \left(\bar{Y}_r - Y_r^n\right) \mathrm{d}r - \int_t^s \left(\bar{Z}_r - Z_r^n\right) \mathrm{d}B_r \end{aligned} \tag{41}$$

with

$$\alpha_r^n = \begin{cases} \dfrac{(\bar{Y}_r)^{1+q} - (Y_r^n)^{1+q}}{\bar{Y}_r - Y_r^n} & \text{for } \bar{Y}_r \neq Y_r^n \\ 0 & \text{if } \bar{Y}_r = Y_r^n. \end{cases}$$

The process α^n is well defined, progressively measurable and verifies:

$$0 \leq \alpha_r^n \leq (1+q)(\bar{Y}_r \vee Y_r^n)^q.$$

We deduce that:

$$\bar{Y}_t - Y_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\bar{Y}_s - Y_s^n\right) \exp\left(-\int_t^s \alpha_r^n \,\mathrm{d}r\right)\right]$$

using the linearity of the BSDE (41) and the fact that the generator of this BSDE is monotone. Then with Fatou's lemma:

$$\begin{aligned} \bar{Y}_t - Y_t^n &= \liminf_{s \to T} \left\{\mathbb{E}^{\mathcal{F}_t}\left[\left(\bar{Y}_s - Y_s^n\right) \exp\left(-\int_t^s \alpha_r^n \,\mathrm{d}r\right)\right]\right\} \\ &\geq \mathbb{E}^{\mathcal{F}_t}\left\{\liminf_{s \to T}\left[\left(\bar{Y}_s - Y_s^n\right) \exp\left(-\int_t^s \alpha_r^n \,\mathrm{d}r\right)\right]\right\}. \end{aligned}$$

It is legal to apply Fatou's lemma because what is inside the conditional expectation has a lower bound equal to $-n$: $\bar{Y}_s \geq 0$ (this belongs to the hypothesis) and $-Y_s^n \geq -n$ and

$$1 \geq \exp\left(-\int_t^s \alpha_r^n \,\mathrm{d}r\right).$$

Finally $\bar{Y}_t - Y_t^n \geq 0$. As it is true for every $n \in \mathbb{N}^*$ and every $t \in [0, T]$, we have $\bar{Y}_t \geq Y_t$. □

4. Parabolic PDE, viscosity solutions

In the introduction, we have said that there is a connection between BSDE whose terminal data is a function of the value at time T of a solution of a SDE (or forward–backward system), and solutions of a large class of semi-linear parabolic PDE. Let us make this connection precise in our case.

To begin with, we modify the Eq. (7). For all $(t, x) \in [0, T] \times \mathbb{R}^m$, we denote by $X^{t,x}$ the solution of the following SDE:

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})\,\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x})\,\mathrm{d}B_r, \quad \text{for } s \in [t, T], \tag{8}$$

and $X_s^{t,x} = x$ for $s \in [0, t]$. b and σ satisfy the assumptions (L) and (G), and we add that b and σ are jointly continuous in (t, x). We consider the following BSDE for $t \leq s \leq T$:

$$Y_s^{t,x} = h(X_T^{t,x}) - \int_s^T Y_r^{t,x}|Y_r^{t,x}|^q\,\mathrm{d}r - \int_s^T Z_r^{t,x}\,\mathrm{d}B_r, \tag{42}$$

where h is a function defined on $\mathbb{R}^m$ with values in $\mathbb{R}^+$ such that h is continuous and bounded. The two Eqs. (8) and (42) are called a forward–backward system. This system is connected with the PDE (3) with terminal condition h. This result is proved in the Theorem 3.2 of the article [5]:

Theorem 19 (*Theorem 3.2 of [5]*). *If we solve the Eqs.* (8) *and* (42) *and if we define $u(t, x)$ for $(t, x) \in [0, T] \times \mathbb{R}^m$ by $u(t, x) = Y_t^{t,x}$, then u is a continuous function and it is a viscosity solution of the PDE* (3).

Let us recall the definition of a viscosity solution (see [18,19] pages 80 and 99 or [20] for v continuous). For $v : [0, T] \times \mathbb{R}^m \to \mathbb{R}$, we define the upper and lower semi-continuous envelope of v, namely:

$$v^*(t, x) = \limsup_{(t',x')\to(t,x)} v(t', x') \quad \text{and} \quad v_*(t, x) = \liminf_{(t',x')\to(t,x)} v(t', x').$$

Definition 20. In this definition, h is continuous and bounded on $\mathbb{R}^m$.

1. We say that v is a subsolution of (3) on $[0, T] \times \mathbb{R}^m$ if $v^* < +\infty$, if $v^*(T, x_0) \leq h(x_0)$, and if, for every function $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^m)$ and local maximum (t_0, x_0) of $v^* - \varphi$,
$$-\frac{\partial\varphi}{\partial t}(t_0, x_0) - \mathcal{L}\varphi(t_0, x_0) + v^*(t_0, x_0)|v^*(t_0, x_0)|^q \leq 0.$$
2. We say that v is a supersolution of (3) on $[0, T] \times \mathbb{R}^m$ if $v_* > -\infty$, if $v_*(T, x_0) \geq h(x_0)$, and if, for every function $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^m)$ and local minimum (t_0, x_0) of $v_* - \varphi$,
$$-\frac{\partial\varphi}{\partial t}(t_0, x_0) - \mathcal{L}\varphi(t_0, x_0) + v_*(t_0, x_0)|v_*(t_0, x_0)|^q \geq 0.$$
3. A function v is a viscosity solution if it is both a viscosity subsolution and supersolution.

Now, in our case, the function h is replaced by the function $g : \mathbb{R}^m \to \overline{\mathbb{R}_+}$ which is supposed to be continuous from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$ and such that the set $F_\infty = \{g = +\infty\}$ is closed and non-empty. We cannot apply the Theorem 3.2 in [5], or the previous definition, because g is unbounded on $\mathbb{R}^m$.

Definition 21 (*Viscosity Solution with Unbounded Data*). We say that v is a viscosity solution of the PDE (3) with terminal data g if v is a viscosity solution on $[0, T[\times\mathbb{R}^m$ and satisfies:

$$\lim_{(t,x)\to(T,x_0)} v(t, x) = g(x_0).$$

We take the notation of the construction of the minimal solution. For all $n \in \mathbb{N}$ and $(t, x) \in [0, T] \times \mathbb{R}^m$, we obtain a sequence of random variables $(Y^{t,x,n}, Z^{t,x,n})$ satisfying (1):

$$Y_s^{t,x,n} = \left(g(X_T^{t,x}) \wedge n\right) - \int_s^T Y_r^{t,x,n}|Y_r^{t,x,n}|^q \,\mathrm{d}r - \int_s^T Z_r^{t,x,n}\,\mathrm{d}B_r,$$

and (9). We know now that this sequence converges to $(Y^{t,x}, Z^{t,x})$, which is the minimal solution verifying the conclusions of Theorems 2 and 4. In particular, this means that either $q > 2$ or else (H3), (B), (D) and (E) are satisfied by σ and b. We want to prove Theorem 6.

If we define the function u_n by $u_n(t, x) = Y_t^{t,x,n}$, then from Theorem 3.2 in [5], we know that u_n is jointly continuous in (t, x) and is a viscosity solution of the parabolic PDE (3) with terminal value $g \wedge n$. The fact that g is supposed to be continuous implies that $g \wedge n$ is bounded and continuous on $\mathbb{R}^m$.

By the comparison theorem for BSDE, $(Y_t^{t,x,n} = u_n(t, x))_{n\in\mathbb{N}}$ is a non-decreasing sequence, and hence it converges to $Y_t^{t,x} = u(t, x)$ when n tends to infinity. Some remarks about the function u. It is a non-negative function satisfying the following bound:

$$\forall (t, x) \in [0, T] \times \mathbb{R}^m, \quad 0 \le u(t, x) \le \frac{1}{(q(T-t))^{\frac{1}{q}}}. \tag{43}$$

Moreover, $u(T, x) = g(x)$ for all $x \in \mathbb{R}^m$. At least, u is lower semi-continuous on $[0, T] \times \mathbb{R}^m$ as the supremum of continuous functions (the sequence (u_n) is a non-decreasing sequence), and for all $x_0 \in \mathbb{R}^m$:

$$\liminf_{(t,x)\to(T,x_0)} u(t, x) \ge g(x_0).$$

Proof of Theorem 6. The main tool is the half-relaxed upper and lower limit of the sequence of functions $\{u_n\}$, i.e.

$$\overline{u}(t, x) = \limsup_{\substack{n\to+\infty \\ (t',x')\to(t,x)}} u_n(t', x') \quad \text{and} \quad \underline{u}(t, x) = \liminf_{\substack{n\to+\infty \\ (t',x')\to(t,x)}} u_n(t', x').$$

In our case, $\underline{u} = u \le \overline{u} = u^*$ because the sequence $\{u_n\}$ is non-decreasing and u_n is continuous for all $n \in \mathbb{N}^*$.

First, u is a supersolution of the PDE (3) on $[0, T[\times\mathbb{R}^m$. $u = u_* = \underline{u}$ is lower semi-continuous on $[0, T[\times\mathbb{R}^m$. From the estimate (43), for all $\delta > 0$, $n \in \mathbb{N}^*$ and all $(t, x) \in [0, T-\delta] \times \mathbb{R}^m$,

$$u_n(t, x) \le u(t, x) \le \left(\frac{1}{q\delta}\right)^{1/q}.$$

Since u_n is a supersolution of the PDE (3), passing to the limit with the Lemma 6.1, page 33, of [20], we obtain that u is a supersolution of (3) on $[0, T[\times\mathbb{R}^m$.

The same argument shows that u^* is a subsolution on $[0, T[\times\mathbb{R}^m$. As in the case of the BSDE (1), the main difficulty is in showing that

$$\limsup_{(t,x)\to(T,x_0)} u(t, x) \le g(x_0) = u(T, x_0).$$

We will prove that u^* is locally bounded on a neighbourhood of T on the set $\{g < +\infty\}$. Then, we deduce u^* is a subsolution and we apply this to demonstrate that $u^*(T, x) \le g(x)$ if $x \in \{g < +\infty\}$, which shows the wanted inequality on u.

We make the same calculation as in the proof of the continuity of Y at T. Let θ be a function of class $C^2(\mathbb{R}^m;\mathbb{R}^+)$ with a compact support included in $\{g < +\infty\}$. We will prove that $u_n\theta$ is uniformly bounded on $[0,T]\times\mathbb{R}^m$. On $[0,T-\delta]\times\mathbb{R}^m$ the bound (43) gives immediately the result. It remains to treat the problem on a neighbourhood of T.

First case: $q > 2$: We write the equality (20) between t and T, for $x \in \mathbb{R}^m$;

$$u_n(t,x)\theta(x) = \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})) - \mathbb{E}\int_t^T \left[\theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\right] \mathrm{d}r - \mathbb{E}\int_t^T Y_r^{t,x,n}\mathcal{L}\theta(X_r^{t,x})\,\mathrm{d}r - \mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})\,\mathrm{d}r.$$

The last term is controlled by:

$$\left|\mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})\,\mathrm{d}r\right| \le \left(\mathbb{E}\int_t^T \|Z_r^{t,x,n}\|^2(T-r)^{2/q}\,\mathrm{d}r\right)^{1/2} \times \left(\mathbb{E}\int_t^T \frac{\|\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})\|^2}{(T-r)^{2/q}}\,\mathrm{d}r\right)^{1/2} \le C = C(q,\theta,\sigma).$$

Here, we use the fact that $q > 2$, θ is compactly supported, and the condition (G). Thus, we have:

$$\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\,\mathrm{d}r + \mathbb{E}\int_t^T Y_r^{t,x,n}\mathcal{L}\theta(X_r^{t,x})\,\mathrm{d}r \le \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})) - \mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})\,\mathrm{d}r.$$

The right hand side is bounded by the supremum of $g\theta$ and C. In the left hand side, the second term is controlled by the first one raised to a power strictly smaller than 1 using Hölder's inequality (see (25) and (28)). Therefore, there exists a constant C independent of n, t, x:

$$\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\,\mathrm{d}r \le C.$$

We deduce that: $u_n(t,x)\theta(x) \le C = C(T,g,\theta,q)$.

Second case: the assumptions (H3), (B), (D) and (E) are satisfied: For $n \in \mathbb{N}^*$, $(t,x) \in [0,T]\times\mathbb{R}^m$, $t > 0$, the Eq. (40) becomes:

$$u_n(t,x)\theta(x) = \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})) - \mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\,\mathrm{d}r - \mathbb{E}\int_t^T Y_r^{t,x,n}\,\Theta(r,X_r^{t,x})\,\mathrm{d}r. \tag{44}$$

From Proposition 16 (or formula (36)), the function Θ is defined by:

$$\Theta(t,x) = \nabla\theta(x).b(t,x) - \sum_{i=1}^d \left((\nabla\theta(x)\sigma(t,x))_i\,\frac{\mathrm{div}(p(t,x)\sigma^i(t,x))}{p(t,x)}\right) - \frac{1}{2}\mathrm{Trace}(D^2\theta(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^d \left(\nabla\theta(x).[\nabla\sigma^i(t,x)\sigma^i(t,x)]\right).$$

As u_n and θ are two non-negative functions,

$$\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\,\mathrm{d}r + \mathbb{E}\int_t^T Y_r^{t,x,n}\,\Theta(r, X_r^{t,x})\,\mathrm{d}r \le \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})).$$

The right hand side in the previous inequality is bounded by the supremum of $g\theta$; this supremum is well defined because $g\theta$ is continuous with compact support. And from the calculations made on the BSDE, we know that the absolute value of the second term in the left hand side is controlled by the first term (which is non-negative) raised to a power strictly smaller than 1. Thus, we deduce:

$$\mathbb{E}\int_0^T \theta(B_r^{t,x})(Y_r^{t,x,n})^{1+q}\,\mathrm{d}r \le C = C(T, g, \theta, q).$$

It is important to note that this constant is independent of n, t, x. If we come back to the inequality (44), we deduce that for all $(t, x) \in [0, T] \times \mathbb{R}^m$:

$$u_n(t, x)\theta(x) \le C = C(T, g, \theta, q).$$

Let U be an open subset s.t. $\overline{U} \subset F_\infty^c = \{g < +\infty\}$. Thus, u_n is uniformly bounded on $[0, T] \times U$ w.r.t. to n. Therefore, u^* is bounded on $[0, T] \times U$. We know that u_n is a subsolution of the PDE (3) restricted to $[0, T] \times U$, i.e.

$$\begin{cases} -\dfrac{\partial u_n}{\partial t}(t, x) - \mathcal{L}u_n(t, x) + u_n(t, x)|u_n(t, x)|^q = 0, & (t, x) \in [0, T[\times U; \\ u_n(T, x) = (g \wedge n)(x), & x \in U. \end{cases}$$

We can apply Theorem 4.1 in [19] (see also Section 4.4.5 in [19]). Since g is continuous,

$$g(x) = \overline{g}(x) = \limsup_{\substack{n\to+\infty \\ (x')\to(x)}} (g \wedge n)(x').$$

Thus u^* is a subsolution of the PDE:

$$\begin{cases} -\dfrac{\partial u^*}{\partial t} - \mathcal{L}u^* + u^*|u^*|^q = 0, & \text{in } [0, T[\times U; \\ \min\left[-\dfrac{\partial u^*}{\partial t} - \mathcal{L}u^* + u^*|u^*|^q; u^* - g\right] \le 0, & \text{in } \{T\} \times U. \end{cases}$$

Now the Theorem 4.7 (with straightforward modifications) shows that $u^* \le g$ in $\{T\} \times U$. □

This achieves the proof of Theorem 6. The next proposition makes precise the behaviour of the solution u on a neighbourhood of T.

Proposition 22. *The previously defined solution u satisfies for all x in the interior of $\{g = +\infty\}$:*

$$\lim_{t\to T} [q(T-t)]^{1/q}\, u(t, x) = 1.$$

Proof. We take the notation of Lemma 11. For all $(t, x) \in [0, T[\ \times \mathbb{R}^m$,

$$Y_t^{t,x,n} \ge \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{q(T-t) + 1/(\xi \wedge n)^q}\right)^{1/q}.$$

Thus, for all integers n:

$$\begin{aligned}[q(T-t)]^{1/q}\,u(t,x) &\geq [q(T-t)]^{1/q}\,u_n(t,x)\\ &\geq \left(\frac{q(T-t)}{q(T-t)+1/n^q}\right)^{1/q}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right)\\ &\quad+\mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{1/q}.\end{aligned}$$

Therefore,

$$[q(T-t)]^{\frac{1}{q}}\,u(t,x)\geq \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right)+\mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{\frac{1}{q}}.$$

The last term is bounded by

$$\mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{1/q}\leq [q(T-t)]^{1/q}\left[\frac{1}{qT}\right]^{1/q}. \tag{45}$$

Hence:

$$\liminf_{t\to T}\,[q(T-t)]^{1/q}\,u(t,x)\geq \lim_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right);$$

this limit is equal to 1 for x in the interior of $\{g=+\infty\}$. We conclude using the bound (43). □

4.1. Minimal solution

The goal of this paragraph is to demonstrate that the viscosity solution obtained with the BSDE is minimal among all non-negative viscosity solutions (Theorem 7). We compare a viscosity solution v (in the sense of Definition 21) with u_n, for all integer n: for all $(t,x)\in[0,T]\times\mathbb{R}^m$, $u_n(t,x)\leq v_*(t,x)$. We deduce that $u\leq v_*\leq v$. Remark that the only used assumptions in the proof will be (L) and (G). Recall that $g:\mathbb{R}^m\to\overline{\mathbb{R}^+}$ is continuous, which implies that $g\wedge n:\mathbb{R}^m\to\mathbb{R}^+$ is continuous.

Proposition 23. *$u_n\leq v_*$, where v is a non-negative viscosity solution of the PDE (3).*

Proof. This result seems to be a direct consequence of a well-known maximum principle for viscosity solutions (see [21,19] or [20]). But to the best of our knowledge, this principle was not proved for solutions which can take the value $+\infty$. Thus, following the proof of the Theorem 8.2 in [20], we just give here the main points.

Recall that u_n is the bounded (by n) and continuous viscosity solution associated with the terminal condition $f=g\wedge n$. For $\varepsilon>0$, we define $u_{n,\varepsilon}(t,x)=u_n(t,x)-\frac{\varepsilon}{t}$. $u_{n,\varepsilon}$ is bounded by u_n and is a subsolution of the PDE (3) (see [20], proof of Theorem 8.2):

$$-\frac{\partial u_{n,\varepsilon}}{\partial t}-\mathcal{L}u_{n,\varepsilon}+u_{n,\varepsilon}|u_{n,\varepsilon}|^q\leq -\frac{\varepsilon}{T^2}. \tag{46}$$

Moreover, at T, $u_{n,\varepsilon}(T,x)=u_n(T,x)-\varepsilon/T\leq(g\wedge n)(x)$ and at 0, $u_{n,\varepsilon}$ tends uniformly in x to $-\infty$. We will prove that $u_{n,\varepsilon}\leq v_*$ for every ε; hence we deduce $u_n\leq v_*$.

From now on, n and ε are fixed. We suppose that there exists $(s,z)\in[0,T]\times\mathbb{R}^m$ such that $u_{n,\varepsilon}(s,z)-v_*(s,z)\geq\delta>0$ and we will find a contradiction. First of all, it is clear that s is not equal to 0 or T, because $v_*(T,z)=g(z)$.

$u_{n,\varepsilon}$ and $-v_*$ are bounded from above on $[0, T] \times \mathbb{R}^m$ respectively by n and 0. Thus, for $(\alpha, \beta) \in (\mathbb{R}^*)^2$, if we define:

$$m(t, x, y) = u_{n,\varepsilon}(t, x) - v_*(t, y) - \frac{\alpha}{2}|x - y|^2 - \beta \left(|x|^2 + |y|^2\right),$$

m has a supremum on $[0, T] \times \mathbb{R}^m \times \mathbb{R}^m$. Moreover, the penalization terms assure that the supremum is attained at a point $(\hat{t}, \hat{x}, \hat{y}) = (t_{\alpha,\beta}, x_{\alpha,\beta}, y_{\alpha,\beta})$. Denote by $\mu_{\alpha,\beta}$ this maximum. Since

$$\delta - 2\beta|z|^2 \le u_{n,\varepsilon}(s, z) - v_*(s, z) - 2\beta|z|^2 = m(s, z, z) \le \mu_{\alpha,\beta},$$

choosing β sufficiently small in order to have $\delta/2 \le \delta - 2\beta|z|^2$, we obtain

$$\delta/2 \le \mu_{\alpha,\beta}. \tag{47}$$

From this inequality and since $u_{n,\varepsilon} \le n$ and $v_* \ge 0$, we have:

$$\begin{aligned} 0 \le \mu_{\alpha,\beta} &= m(\hat{t}, \hat{x}, \hat{y}) = u_{n,\varepsilon}(\hat{t}, \hat{x}) - v_*(\hat{t}, \hat{y}) - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2 - \beta \left(|\hat{x}|^2 + |\hat{y}|^2\right) \\ &\le n - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2 - \beta \left(|\hat{x}|^2 + |\hat{y}|^2\right) \end{aligned}$$

and hence $|\hat{x}|^2 + |\hat{y}|^2 \le \frac{n}{\beta}$ and $|\hat{x} - \hat{y}|^2 \le \frac{2n}{\alpha}$.

Now, we will prove that $\hat{t}$ cannot be equal to zero, or to T. Recall that $u_{n,\varepsilon}(0, .) = -\infty$, thus $\hat{t}$ cannot be equal to zero. Assume that $\hat{t}$ is equal to T. We have:

$$\begin{aligned} \mu_{\alpha,\beta} &= (g \wedge n)(\hat{x}) - \frac{\varepsilon}{T} - g(\hat{y}) - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2 - \beta \left(|\hat{x}|^2 + |\hat{y}|^2\right) \\ &\le (g \wedge n)(\hat{x}) - (g \wedge n)(\hat{y}). \end{aligned}$$

Let γ^β be a modulus of continuity of $g \wedge n$ defined by:

$$\forall \eta \ge 0, \quad \gamma^\beta(\eta) = \sup\{|(g \wedge n)(x) - (g \wedge n)(y)|;\ |x - y| \le \eta,\ (x, y) \in (B_\beta)^2\},$$

where B_β is the closed ball with radius equal to $\sqrt{n/\beta}$. Therefore, we obtain:

$$\delta/2 \le \mu_{\alpha,\beta} \le (g \wedge n)(\hat{x}) - (g \wedge n)(\hat{y}) \le \gamma^\beta \left(|\hat{x} - \hat{y}|\right) \le \gamma^\beta \left(\sqrt{\frac{2n}{\alpha}}\right).$$

Since we have supposed that $g \wedge n$ is continuous, $g \wedge n$ is uniformly continuous on B_β and thereby the limit of $\gamma^\beta(\eta)$ is equal to zero as η goes to 0. Hence, the previous inequality is false when α is sufficiently large. We deduce that $\hat{t} < T$.

We now use the Theorem 8.3 of [20] with the $u_{n,\varepsilon}$ subsolution and v_* supersolution. For all $\nu > 0$ there exist two symmetric matrices X and Y of size $m \times m$ and two reals a and b such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le A + \nu A^2, \quad \text{with } A = \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\beta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{48}$$

$a - b = 0$ and satisfying:

$$\begin{aligned} &-a - F(\hat{t}, \hat{x}, u_{n,\varepsilon}(t, \hat{x}), \alpha(\hat{x} - \hat{y}) + 2\beta\hat{x}, X) \le -\frac{\varepsilon}{T^2} \\ &-b - F(\hat{t}, \hat{y}, v_*(t, \hat{y}), \alpha(\hat{x} - \hat{y}) - 2\beta\hat{y}, Y) \ge 0. \end{aligned}$$

We subtract the two previous inequalities:

$$\begin{aligned}\frac{\varepsilon}{T^2} &\leq F(\hat{t}, \hat{x}, u_{n,\varepsilon}(t, \hat{x}), \alpha(\hat{x}-\hat{y}) + 2\beta\hat{x}, X) - F(\hat{t}, \hat{y}, v_*(t, \hat{y}), \alpha(\hat{x}-\hat{y}) - 2\beta\hat{y}, Y) \\ &= \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{x})X\right) - \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{y})Y\right) \\ &\quad + \left(b(\hat{t}, \hat{x}) - b(\hat{t}, \hat{y})\right).\alpha(\hat{x}-\hat{y}) + 2\beta\left(b(\hat{t}, \hat{x}).\hat{x} + b(\hat{t}, \hat{y}).\hat{y}\right) \\ &\quad - u_{n,\varepsilon}(\hat{t}, \hat{x})^{1+q} + v_*(\hat{t}, \hat{y})^{1+q}.\end{aligned}$$

Since b is Lipschitz and grows at most linearly, there exists a constant K such that

$$\begin{aligned}&\left(b(\hat{t}, \hat{x}) - b(\hat{t}, \hat{y})\right).\alpha(\hat{x}-\hat{y}) + 2\beta\left(b(\hat{t}, \hat{x}).\hat{x} + b(\hat{t}, \hat{y}).\hat{y}\right) \\ &\quad \leq \alpha K|\hat{x}-\hat{y}|^2 + 2\beta K(1 + |\hat{x}|^2 + |\hat{y}|^2).\end{aligned}$$

Using again the lower bound (47) of $\mu_{\alpha,\beta}$ we have:

$$\delta/2 \leq \mu_{\alpha,\beta} \leq u_{n,\varepsilon}(\hat{t}, \hat{x}) - v_*(\hat{t}, \hat{y}) \Longrightarrow 0 \leq v_*(\hat{t}, \hat{y}) \leq u_{n,\varepsilon}(\hat{t}, \hat{x})$$

and thus $-u_{n,\varepsilon}(\hat{t}, \hat{x})^{1+q} + v_*(\hat{t}, \hat{y})^{1+q} \leq 0$. One term remains to be controlled:

$$\mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{x})X\right) - \mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{y})Y\right).$$

From the upper bound (48), we deduce that there exists a constant $\kappa = 1 + 2\alpha\nu + 2\beta\nu$ such that

$$\frac{1}{\kappa}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\beta\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

If we choose $\nu = 1/\alpha$, then the constant κ is bounded: $3 \leq \kappa \leq 3 + 2\beta/\alpha$. We multiply this inequality by the following non-negative matrix:

$$\begin{pmatrix} \sigma\sigma^*(\hat{t}, \hat{x}) & \sigma(\hat{t}, \hat{x})\sigma^*(\hat{t}, \hat{y}) \\ \sigma(\hat{t}, \hat{y})\sigma^*(\hat{t}, \hat{x}) & \sigma\sigma^*(\hat{t}, \hat{y}) \end{pmatrix},$$

and we take the trace:

$$\begin{aligned}&\mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{x})X\right) - \mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{y})Y\right) \leq 2\kappa\beta\left[\sigma\sigma^*(\hat{t}, \hat{x}) + \sigma\sigma^*(\hat{t}, \hat{y})\right] \\ &\quad + \kappa\alpha\left[\sigma(\hat{t}, \hat{x}) - \sigma(\hat{t}, \hat{y})\right]\left[\sigma^*(\hat{t}, \hat{x}) - \sigma^*(\hat{t}, \hat{y})\right].\end{aligned}$$

Using the fact that σ satisfies (L) and (G), we obtain the existence of a constant K such that

$$\mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{x})X\right) - \mathrm{Trace}\left(\sigma\sigma^*(\hat{t}, \hat{y})Y\right) \leq K\alpha|\hat{x}-\hat{y}|^2 + K\beta(1 + |\hat{x}|^2 + |\hat{y}|^2).$$

Finally we have:

$$\frac{\varepsilon}{T^2} \leq C\left(\alpha|\hat{x}-\hat{y}|^2 + \beta(1 + |\hat{x}|^2 + |\hat{y}|^2)\right), \tag{49}$$

where C is a constant independent of α and β. This inequality is not exactly the same as in the proof of the Theorem 8.2 in [20], because the solutions are defined on $\mathbb{R}^m$ and not on some bounded open set. Thus there is this additional term with β. Since

$$\lim_{\alpha\to+\infty}\lim_{\beta\to 0}\frac{\alpha}{2}|\hat{x}-\hat{y}|^2 + \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right) = 0,$$

the inequality (49) leads to a contradiction taking β sufficiently small and α sufficiently large. Hence $u_{n,\varepsilon} \leq v_*$ and it is true for every $\varepsilon > 0$, so the result is proved. □

4.2. Regularity of the minimal solution

The function u is the minimal non-negative viscosity solution of the PDE (3). We know that u is finite on $[0, T[\times\mathbb{R}^m$ (see (43)). For $\delta > 0$, u is bounded on $[0, T-\delta]\times\mathbb{R}^m$ by a constant which depends only on δ.

Proposition 24. *If the coefficients of the operator $\mathcal{L}$ satisfy (L), (B) and (E), and are Hölder continuous in time, then u is continuous on $[0, T]\times\mathbb{R}^m$, and for all $\delta > 0$:*

$$u \in C^{1,2}([0, T-\delta]\times\mathbb{R}^m; \mathbb{R}^+). \tag{50}$$

Proof. The proof of Proposition 23 shows that there is a unique bounded and continuous viscosity solution of the Cauchy problem:

$$\begin{cases} \partial_t v + \mathcal{L}v - v|v|^q = 0, & \text{on } [0, T-\delta]\times\mathbb{R}^m, \\ v(T-\delta, x) = \phi(x) & \text{on } \mathbb{R}^m \end{cases} \tag{51}$$

where ϕ is supposed bounded and continuous on $\mathbb{R}^m$.

Moreover, the Cauchy problem (51) has a classical solution for every bounded and continuous function ϕ (see Lemma 25 below).

Recall that u_n is jointly continuous in (t, x) and on $[0, T-\delta]\times\mathbb{R}^m$, u_n is bounded by:

$$0 \le u_n(t, x) \le \left(\frac{1}{q\delta}\right)^{1/q}.$$

Thus, the problem (51) with condition $\phi = u_n(T-\delta, .)$ has a bounded classical solution. Since every classical solution is a viscosity solution and since u_n is the unique bounded and continuous viscosity solution of (51), we deduce that:

$$\forall\delta > 0, \quad u_n \in C^{1,2}([0, T-\delta[\times\mathbb{R}^m; \mathbb{R}^+).$$

From the construction of the classical solution u_n, we also know that the sequence $\{u_n\}$ is locally bounded in $C^{\alpha,1+\alpha}([0, T-\delta/2]\times\mathbb{R}^m)$. The bound is given by the L^∞ norm of u_n which is smaller than $(T-\delta/4)^{-1/q}$. Therefore u is continuous on $[T-\delta/2]\times\mathbb{R}^m$ and if we consider the problem (51) with continuous terminal data $u(T-\delta, .)$, with the same argument as for u_n, we obtain that u is a classical solution, i.e. $u \in C^{1,2}([0, T-\delta]\times\mathbb{R}^m; \mathbb{R}^+)$.

In particular, u is continuous on $[0, T[\times\mathbb{R}^m$. And the terminal condition in Theorem 6 shows that u is continuous at time T. $\square$

Lemma 25. *For every bounded and continuous function ϕ, the Cauchy problem (51) has a classical solution v.*

Proof. If $\mathcal{L}$ can be written in divergence form, the conclusion is given by the Theorem 8.1 and Remark 8.1, Section V, in [17]. More precisely, there exists a unique continuous and bounded solution v such that, for all $\delta' > \delta$, v belongs to $H^{1+\beta/2,2+\beta}([0, T-\delta']\times\mathbb{R}^m)$. $H^{1+\beta/2,2+\beta}([0, T-\delta']\times\mathbb{R}^m)$ is the set of functions which are $(1+\beta/2)$-Hölder continuous in time and $(2+\beta)$-Hölder continuous in space. Remark that if the assumption (D) holds, then $\mathcal{L}$ can be written in divergence form.

In general, we use a "bootstrap" method. If v denotes the unique bounded and continuous viscosity solution of (51), if $f = v|v|^q \in L^\infty$, from the theorem 3.1 of [22], the linear Cauchy problem:

$$\begin{cases} \partial_t w + \mathcal{L}w = f, & \text{on } [0, T-\delta] \times \mathbb{R}^m, \\ w(T-\delta, x) = \phi(x) & \text{on } \mathbb{R}^m. \end{cases} \tag{52}$$

has a unique solution w in the class $C([0, T-\delta] \times \mathbb{R}^m; \mathbb{R}^+) \cap \bigcap_{p>1} W^{1,2}_{p,\mathrm{loc}}([0, T-\delta[\times \mathbb{R}^m)$. Now we define two processes:

$$\forall s \geq t, \quad \overline{Y}^{t,x}_s = w(s, X^{t,x}_s), \quad \text{and} \quad \overline{Z}^{t,x}_s = \nabla w(s, X^{t,x}_s)\sigma(s, X^{t,x}_s).$$

We can apply the Itô formula to the function w (see [23], Section 2.10). We have for all $s \geq t$:

$$\begin{aligned} \overline{Y}^{t,x}_s &= \overline{Y}^{t,x}_{T-\delta} + \int_s^{T-\delta} f(r, X^{t,x}_r)\,\mathrm{d}r - \int_s^{T-\delta} \overline{Z}^{t,x}_r\,\mathrm{d}B_r \\ &= \phi(T-\delta, X^{t,x}_{T-\delta}) + \int_s^{T-\delta} (v|v|^q)(r, X^{t,x}_r)\,\mathrm{d}r - \int_s^{T-\delta} \overline{Z}^{t,x}_r\,\mathrm{d}B_r. \end{aligned}$$

Let $(Y^{t,x}, Z^{t,x})$ be the unique solution of the BSDE (1) with final time $T-\delta$ and terminal condition $\phi(X^{t,x}_{T-\delta})$. From the uniqueness of the viscosity solution, we have $Y^{t,x}_t = v(t,x)$. And also for all $s \geq t$, $Y^{t,x}_s = v(s, X^{t,x}_s)$ (see [14], Section 4). Therefore for all $s \geq t$:

$$\begin{aligned} Y^{t,x}_s &= Y^{t,x}_{T-\delta} + \int_s^{T-\delta} Y^{t,x}_s |Y^{t,x}_s|^q\,\mathrm{d}r - \int_s^{T-\delta} Z^{t,x}_r\,\mathrm{d}B_r \\ &= \phi(X^{t,x}_{T-\delta}) + \int_s^{T-\delta} (v|v|^q)(r, X^{t,x}_r)\,\mathrm{d}r - \int_s^{T-\delta} Z^{t,x}_r\,\mathrm{d}B_r. \end{aligned}$$

From the uniqueness of the solution of the BSDE, we obtain that $v = w$. Since $w \in \bigcap_{p>1} W^{1,2}_{p,\mathrm{loc}}([0, T-\delta[\times \mathbb{R}^m)$, v belongs to the space $H^{\beta,1+\beta}$, for all $\beta < 1$, and the Hölder norm of v depends just on the L^∞ bound of v. Thus f is also Hölder continuous, and from the existence result of [17] (see section IV, Theorems 5.1 and 10.1), v is a classical solution. □

5. Sign assumption on the terminal condition ξ

In the previous sections, we have chosen ξ non-negative with the generator $f(y) = -y|y|^q$. But of course, we can prove the same results when $\xi \leq 0$. In fact, if $\xi \in L^\infty(\Omega)$ is non-positive and if (Y, Z) is the solution of the BSDE (1), it is obvious that $(U, V) = (-Y, -Z)$ is the solution of

$$U_t = (-\xi) + \int_t^T f(U_r)\,\mathrm{d}r - \int_t^T V_r\,\mathrm{d}B_r.$$

Thus if ξ is non-positive and satisfies (H1): $\xi = g(X_T)$, where $g : \mathbb{R}^m \to [-\infty, 0]$ is such that $\{g = -\infty\}$ is closed and that the assumptions (H2), (L) and (G) are verified, and with either $q > 2$ or (B), (D), (E) and (H3), there exists a maximal solution of the BSDE (1), among all negative solutions. We can also work without sign assumption and prove Theorem 8.

For $(n, m) \in (\mathbb{N}^*)^2$, we define ξ^n_m by $\xi^n_m = \xi \wedge n \vee (-m)$. As ξ^n_m is in $L^\infty(\Omega)$, there exists a unique solution $(Y^{n,m}, Z^{n,m})$ for the BSDE (1) with terminal condition ξ^n_m. With the comparison Theorem 2.4 in [5], we have for $0 \leq n \leq n'$ and $0 \leq m \leq m'$, and for all $t \in [0, T]$:

$$-\left(\frac{1}{q(T-t) + \frac{1}{(m')^q}}\right)^{\frac{1}{q}} \leq Y^{n,m'}_t \leq Y^{n,m}_t \leq Y^{n',m}_t \leq \left(\frac{1}{q(T-t) + \frac{1}{(n')^q}}\right)^{\frac{1}{q}}. \tag{53}$$

We define the following process: for $(n, m) \in (\mathbb{N}^*)^2$ and for all $t \in [0, T]$

$$Y_t^{\uparrow,m} = \lim_{n\to+\infty} Y_t^{n,m} \quad \text{and} \quad Y_t^{n,\downarrow} = \lim_{m\to+\infty} Y_t^{n,m}.$$

Therefore with the inequalities (53) we have:

$$-\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \le Y_t^{n,\downarrow} \le Y_t^{\uparrow,m} \le \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

As in the first section we can easily prove that:

(1) the limit $Z^{\uparrow,m}$ of $(Z^{n,m})_{n\in\mathbb{N}}$ as n goes to $+\infty$, exists in $L^2(\Omega \times [0, t]; \mathbb{R}^d)$ for all $t < T$;
(2) (D1) and (D2) are satisfied (see Definition 1): for all $t < T$

$$\mathbb{E}\left(\sup_{s\in[0,t]} |Y_s^{\uparrow,m}|^2 + \int_0^t \|Z_r^{\uparrow,m}\|^2 \,\mathrm{d}r\right) \le \left(\frac{1}{q(T-t)}\right)^{\frac{2}{q}}; \tag{54}$$

and $\mathbb{P}$-p.s., for all $0 \le s \le t$,

$$Y_s^{\uparrow,m} = Y_t^{\uparrow,m} - \int_s^t Y_r^{\uparrow,m}|Y_r^{\uparrow,m}|^q \,\mathrm{d}r - \int_s^t Z_r^{\uparrow,m} \,\mathrm{d}B_r; \tag{55}$$

(3) $\mathbb{P}$-p.s.

$$\xi \vee (-m) \le \liminf_{t\to T} Y_t^{\uparrow,m}. \tag{56}$$

The same results are still true for the sequence $(Y^{n,m}, Z^{n,m})_{m\in\mathbb{N}}$, i.e. the limit $Z^{n,\downarrow}$ of $(Z^{n,m})_{m\in\mathbb{N}}$ as m goes to $+\infty$, exists in $L^2(\Omega\times[0, t]; \mathbb{R}^d)$ for all $t < T$, and the limit $(Y^{n,\downarrow}, Z^{n,\downarrow})$ satisfies (54) and (55) and $\mathbb{P}$-p.s.

$$\limsup_{t\to T} Y_t^{n,\downarrow} \le \xi \wedge n.$$

Suppose now we have proved the following proposition:

Proposition 26.

$$\lim_{t\to T} Y_t^{\uparrow,m} = \xi \vee (-m) \quad \textit{and} \quad \lim_{t\to T} Y_t^{n,\downarrow} = \xi \wedge n \quad \mathbb{P}-a.s. \tag{57}$$

Recall that $(Y^{n,\downarrow})_{n\in\mathbb{N}}$ and $(Y^{\uparrow,m})_{m\in\mathbb{N}}$ are respectively a non-decreasing and a non-increasing sequence. Let us define:

$$Y^{\downarrow} = \lim_{n\to+\infty} Y^{n,\downarrow} \quad \text{and} \quad Y^{\uparrow} = \lim_{m\to+\infty} Y^{\uparrow,m}.$$

We have:

$$-\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \le Y_t^{\downarrow} \le Y_t^{\uparrow} \le \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}$$

and for all $(n, m) \in \mathbb{N}^2$, $\xi \wedge n \le \liminf_{t\to T} Y_t^{\downarrow}$, and $\limsup_{t\to T} Y_t^{\uparrow} \le \xi \vee (-m)$. Then it is easy to prove that $(Y, Z) = (Y^{\downarrow}, Z^{\downarrow})$ or $(Y, Z) = (Y^{\uparrow}, Z^{\uparrow})$ satisfy (54) and (55) and

$$\mathbb{P}\text{-a.s.} \quad \lim_{t\to T} Y_t = \xi.$$

This achieves the proof of Theorem 8.

In order to prove Proposition 26, we define $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$. We need the following lemma:

Lemma 27. *If ζ is a real-valued random variable, with $\zeta \in L^2(\Omega)$, if (Y, Z) and $(\widetilde{Y}, \widetilde{Z})$ denote the solution of the BSDE:*

$$Y_t = \zeta - \int_t^T Y_r |Y_r|^q \, \mathrm{d}r - \int_t^T Z_r \, \mathrm{d}B_r \quad \text{and} \quad \widetilde{Y}_t = \zeta^+ - \int_t^T \widetilde{Y}_r |\widetilde{Y}_r|^q \, \mathrm{d}r - \int_t^T \widetilde{Z}_r \, \mathrm{d}B_r,$$

then $\mathbb{P}$-a.s. for all $t \in [0, T]$, $Y_t^+ \leq \widetilde{Y}_t$.

Proof of the lemma. The process Y is a continuous semi-martingale, i.e. we write Y as follows

$$Y_t = Y_0 + A_t + M_t$$

with M a continuous martingale and A a process of bounded variations. Therefore there exists a semi-martingale local time for Y (see [10], Theorem 7.1, page 218), i.e. a non-negative process $\Lambda = \{\Lambda_t(\omega); t \in [0, T], \omega \in \Omega\}$ such that

$$\begin{aligned} Y_t^+ &= \zeta^+ - \int_t^T \left(\mathbf{1}_{\mathbb{R}_+^*}(Y_r)\right) Y_r |Y_r|^q \, \mathrm{d}r - \int_t^T \left(\mathbf{1}_{\mathbb{R}_+^*}(Y_r)\right) Z_r \, \mathrm{d}B_r + (\Lambda_t - \Lambda_T) \\ &= \zeta^+ - \int_t^T Y_r^+ |Y_r^+|^q \, \mathrm{d}r - \int_t^T \left(\mathbf{1}_{\mathbb{R}_+^*}(Y_r)\right) Z_r \, \mathrm{d}B_r + (\Lambda_t - \Lambda_T). \end{aligned}$$

Moreover $t \mapsto \Lambda_t$ is non-decreasing a.s. Thus applying the comparison theorem 2.4 in [5], we obtain the announced result and this achieves the proof of the lemma.

Remark that by the same arguments Y^- satisfies:

$$\begin{aligned} Y_t^- &= \zeta^- + \int_t^T \left(\mathbf{1}_{\mathbb{R}_-^*}(Y_r)\right) Y_r |Y_r|^q \, \mathrm{d}r + \int_t^T \mathbf{1}_{\mathbb{R}_-^*}(Y_r) Z_r \, \mathrm{d}B_r + (\Lambda_t - \Lambda_T) \\ &= \zeta^- - \int_t^T |Y_r^-|^{1+q} \, \mathrm{d}r + \int_t^T \mathbf{1}_{\mathbb{R}_-^*}(Y_r) Z_r \, \mathrm{d}B_r + (\Lambda_t - \Lambda_T). \end{aligned}$$

Proof of Proposition 26. Remark that it is enough to show the first limit in (57). Indeed for all (n, m), $(-Y^{n,m}, -Z^{n,m})$ is the solution of the BSDE (1) with terminal condition $(-\xi) \wedge m \vee (-n)$. Hence, if the result concerning the first limit holds, then

$$\lim_{t \to T} -Y_t^{n,\downarrow} = (-\xi) \vee (-n);$$

which proves the result about the second limit.

Now for the rest of this proof, we fix an integer m. We can give the same proof as in Sections 1 and 2: we show that the limit, when t goes to T, of $Y_t^{\uparrow,m}$ exists and then that this limit is $\xi \vee (-m)$, by multiplying $Y^{n,m}$ with some test process $\varphi(X)$ and passing through the limit when n goes to $+\infty$. Let us explain the main parts.

From the previous lemma, we have for all $n \in \mathbb{N}$, for all $t \in [0, T]$

$$(Y_t^{n,m})^+ \leq U_t^n \leq U_t \quad \text{and} \quad (Y_t^{\uparrow,m})^+ \leq U_t$$

with

$$U_t^n = \xi^+ \wedge n - \int_t^T U_r^n |U_r^n|^q \, \mathrm{d}r - \int_t^T V_r^n \, \mathrm{d}B_r,$$

and (U, V) is the minimal solution of:

$$U_t = \xi^+ - \int_t^T U_r |U_r|^q \, \mathrm{d}r - \int_t^T V_r \, \mathrm{d}B_r.$$

Since we already know (Section 2) that $\lim_{t\to T} U_t = \xi^+$, we have $\limsup_{t\to T} (Y_t^{\uparrow,m})^+ \leq \xi^+$. Moreover from (56), we obtain

$$\xi \vee (-m) \leq \liminf_{t\to T} (Y_t^{\uparrow,m})^+ \quad \text{and} \quad 0 \leq \liminf_{t\to T} (Y_t^{\uparrow,m})^+.$$

Therefore

$$\lim_{t\to T} (Y_t^{\uparrow,m})^+ = \xi^+. \tag{58}$$

For the negative part, recall that for all $n \in \mathbb{N}$ and all $t \in [0, T]$:

$$Y_t^{n,m} \geq -\left(\frac{1}{q(T-t) + \frac{1}{m^q}}\right)^{\frac{1}{q}} \geq -m. \tag{59}$$

Let us prove that the limit, when t goes to T, of $(Y_t^{\uparrow,m})^-$ exists. We have:

$$\begin{aligned} Y_t^{n,m,-} &= (\xi \vee (-m) \wedge n)^- - \int_t^T Y_s^{n,m,-} |Y_s^{n,m,-}|^q \, \mathrm{d}s \\ &\quad - (\Lambda_T^{n,m} - \Lambda_t^{n,m}) + \int_t^T Z_s^{n,m,-} \, \mathrm{d}B_s, \end{aligned}$$

where:

$$Y_t^{n,m,-} = \left(Y_t^{n,m}\right)^-, \quad Z_s^{n,m,-} = \mathbf{1}_{]-\infty,0[}(Y_t^{n,m}) Z_t^{n,m}$$

and $\Lambda^{n,m}$ is the local time of $Y^{n,m}$ (see the proof of the previous lemma). We apply Itô's formula to the process $(Y^{n,m,-})^2$ and because of the monotonicity of f we obtain:

$$\frac{1}{2}\mathbb{E}\int_0^T \left\| Z_r^{n,m,-} \right\|^2 \mathrm{d}r \leq \mathbb{E}\left(Y_T^{n,m,-}\right)^2 - 2\mathbb{E}\int_0^T Y_r^{n,m,-} \, \mathrm{d}\Lambda_r^{n,m} \leq \mathbb{E}\left(Y_T^{n,m,-}\right)^2 \leq m^2.$$

We deduce that the sequence $(Z^{n,m,-})_{n\in\mathbb{N}}$ converges weakly in $L^2(\Omega \times [0, T]; \mathbb{R}^d)$ to $\widetilde{Z}$, and thus that the stochastic integral $\int_\cdot^T Z_r^{n,m,-} \, \mathrm{d}B_r$ also converges weakly to $\int_\cdot^T \widetilde{Z}_r \, \mathrm{d}B_r$ (see [5], Proposition 2.4). Moreover a straightforward calculation proves that for all $\delta > 0$, $(Z^{n,m,-})_{n\in\mathbb{N}}$ converges in $L^2(\Omega \times [0, T-\delta]; \mathbb{R}^d)$ to $Z^{\uparrow,m,-} = \mathbf{1}_{]-\infty,0[}(Y^{\uparrow,m}) Z^{\uparrow,m}$. Hence $\widetilde{Z}$ is a.s. equal to $Z^{\uparrow,m,-}$. Now:

$$\Lambda_t^{n,m} = Y_t^{n,m,-} - Y_0^{n,m,-} - \int_0^t Y_s^{n,m,-} |Y_s^{n,m,-}|^q \, \mathrm{d}s + \int_0^t Z_s^{n,m,-} \, \mathrm{d}B_s. \tag{60}$$

With (59) and the Burkholder–Davis–Gundy inequality, it emerges easily that:

$$\mathbb{E}\int_0^T \left|\Lambda_r^{n,m}\right|^2 \mathrm{d}r \leq C.$$

Therefore the sequence $(\Lambda^{n,m})_{n\in\mathbb{N}}$ converges weakly in $L^2(\Omega \times [0, T]; \mathbb{R}^d)$ to $\widetilde{\Lambda}$. Now taking the weak limit in (60) we obtain:

$$\widetilde{\Lambda}_t = Y_t^{\uparrow,m,-} - Y_0^{\uparrow,m,-} + \int_0^t Y_s^{\uparrow,m,-} |Y_s^{\uparrow,m,-}|^q \, \mathrm{d}s + \int_0^t Z_s^{\uparrow,m,-} \, \mathrm{d}B_s. \tag{61}$$

With (55) and (61), we deduce that on the interval $[0, T-\delta]$ for $\delta > 0$, the process $\widetilde{\Lambda}$ is the local time of the semi-martingale $Y^{\uparrow,m}$. Hence it is a non-decreasing process and the limit as t goes to T of $\widetilde{\Lambda}_t$ exists. We deduce that the limit as t goes to T of $Y_t^{\uparrow,m,-}$ exists, and with (58), we conclude that there exists a limit as t goes to T for $Y_t^{\uparrow,m}$.

Moreover with (59) we can prove that Lemma 14 holds, i.e. if Φ is a non-negative function of class C^2, with compact support included in $\mathbb{R}^m \setminus \{|g| = +\infty\}$, and if $\alpha > 2(1+1/q)$, then there exists a constant C such that

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\int_0^T \left|Y_r^{n,m}\right|^{1+q} \Phi^\alpha(X_r)\,\mathrm{d}r \leq C.$$

Then as in Section 2.3 or 2.4.2, we have $\lim_{t\to T} Y_t^{\uparrow,m} = \xi \vee (-m)$ and thus (57). □

If ξ is either non-negative or non-positive, it is obvious that $(Y^{\downarrow}, Z^{\downarrow}) = (Y^{\uparrow}, Z^{\uparrow})$ is respectively either the minimal non-negative solution or the maximal non-positive solution. But in general we are unable to prove the uniqueness of the solution.

For the related PDE recall that for $(t,x) \in [0,T]\times\mathbb{R}^m$ $X^{t,x}$ is the solution of the SDE (8). For every $(n,m) \in \mathbb{N}^2$ $Y^{n,m,t,x}$ is the solution of the BSDE (1) with $g(X^{t,x}) \wedge n \vee (-m)$ as terminal condition. As in Section 4, if the function $u_{n,m}$ is defined by $u_{n,m}(t,x) = Y_t^{n,m,t,x}$, $u_{n,m}$ is a viscosity solution of the PDE (3) with final condition $g \wedge n \vee (-m)$. We can give the same demonstration as in Section 4 to show that

$$u^{\uparrow}(t,x) = \lim_{m\to+\infty}\lim_{n\to+\infty} u_{n,m}(t,x) \quad \text{and} \quad u^{\downarrow}(t,x) = \lim_{n\to+\infty}\lim_{m\to+\infty} u_{n,m}(t,x)$$

are two viscosity solutions of the PDE (3) with g as terminal condition. In fact we just have to bound $u_{n,m}$ on the complementary of $\{g = +\infty \text{ or } g = -\infty\}$. We use the following lemma:

Lemma 28. *Assume that v is a viscosity solution of the PDE (3) s.t. $v(T,.) = h$. The function h is bounded and continuous on $\mathbb{R}^m$. Then $|v|$ is a subsolution of the same PDE with $|v|(T,x) = |h(x)|$.*

Proof. The definition of a subsolution was given in the Section 4, Definition 20. Since v is continuous, $|v|$ is also continuous. Let $\phi : [0,T]\times\mathbb{R}^m \to \mathbb{R}$ be a $C^{1,2}$ function such that $|v| - \phi$ has a local maximum at (t,x). Moreover we can always suppose that $|v(t,x)| = \phi(t,x)$.

If $t = T$ we already know that $|v(T,x)| = |h(x)|$. If $t < T$ then there exists a neighbourhood $\mathcal{U} =]t-\varepsilon, t+\varepsilon[\times \{y,\ |y-x| \leq \eta\}$ of (t,x), such that for $(s,y) \in \mathcal{U}$, $0 \leq |v(s,y)| \leq \phi(s,y)$. If $v(t,x) = 0$, ϕ is a non-negative function on $\mathcal{U}$ and attains its minimum at (t,x). Therefore $\nabla\phi(t,x) = 0$, $\partial_t\phi(t,x) = 0$ and $D^2\phi(t,x) \geq 0$. We deduce:

$$-\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) + |v(t,x)|^{1+q} = -\frac{1}{2}\mathrm{Trace}(\sigma\sigma^*(t,x)D^2v(t,x)) \leq 0.$$

If $v(t,x) > 0$, then we can suppose that v is positive on $\mathcal{U}$:

$$\forall (s,y) \in \mathcal{U}, \quad 0 < v(s,y) = |v(s,y)| \leq \phi(s,y) \quad \text{and} \quad v(t,x) = \phi(t,x).$$

We apply the definition of a subsolution to v and deduce:

$$-\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) + v(t,x)|v(t,x)|^q = -\partial_t\phi(t,x) - \mathcal{L}\phi(t,x) + |v(t,x)|^{1+q} \leq 0.$$

In the last case $v(t,x) < 0$, we suppose that on $\mathcal{U}$, v is negative. Thus $v - (-\phi)$ has a local minimum at (t,x) and we apply the definition of a supersolution:

$$-\partial_t(-\phi)(t,x) - \mathcal{L}(-\phi)(t,x) + v(t,x)|v(t,x)|^q \geq 0;$$

$$\Longrightarrow -\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) + (-v(t,x))|v(t,x)|^q$$
$$= -\partial_t \phi(t,x) - \mathcal{L}\phi(t,x) + |v(t,x)|^{1+q} \leq 0.$$

Finally $|v|$ is a subsolution and this achieves the proof of the lemma. □

By a standard comparison argument (see [20], theorem 8.2 page 48), the solution $u_{n,m}$ is bounded by the viscosity solution $\widetilde{u}$ of the PDE (3) with terminal argument $|g| \wedge n \wedge m$ and we can use our previous results on $\widetilde{u}$ (Theorem 4, Section 4).

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