Homogenization of random parabolic operators.

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Nice

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Outline

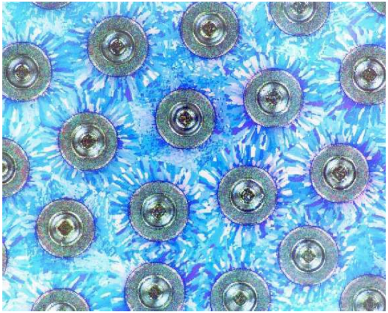
Outline

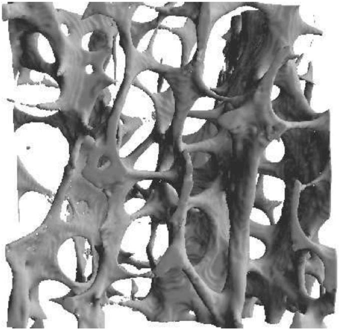
Composite materials.

Physical parameters (conductivity, elasticity, etc.)

- Discontinuous
- Oscillate between different values characterizing each of the components
- Very rapidly oscillations.

Microscopic structure: complicated.

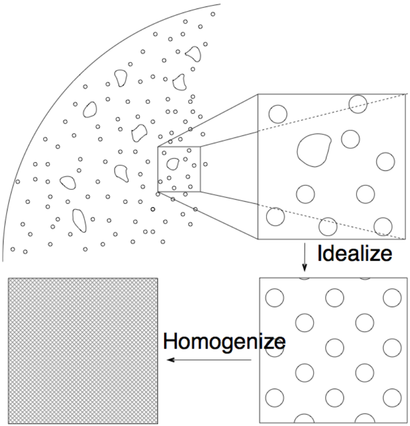




Homogenization process.

Approximation of the macroscopic behaviour

- ε describes the fineness of the microscopic structure.
- The parameter ε tends to zero.

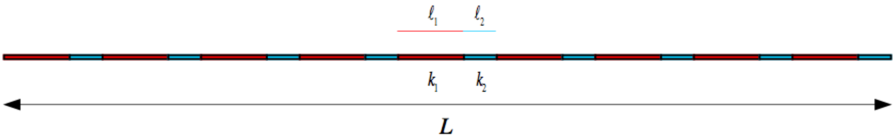


Example in dimension 1.

$$\begin{cases} -\dfrac{d}{dx}\left(a^{\varepsilon}(x)\dfrac{du}{dx}\right) = -\dfrac{d}{dx}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{du}{dx}\right) = f, \quad \text{on } [0, L] \\ u(0) = u(L) = 0 \end{cases}$$

$a : \mathbb{R} \to \mathbb{R}$ is periodic and uniformly elliptic.

Practical example: linear piece submitted to some orthogonal force with density f, a being the rigidity of materials.



Example in dimension 1.

$$\begin{cases} -\dfrac{d}{dx}\left(a^{\varepsilon}(x)\dfrac{du}{dx}\right) = -\dfrac{d}{dx}\left(a\left(\dfrac{x}{\varepsilon}\right)\dfrac{du}{dx}\right) = f, \quad \text{on } [0,L] \\ u(0) = u(L) = 0 \end{cases}$$

If $0 < \alpha \leq a(x) \leq \beta < +\infty$

- Existence of a unique solution u^{ε} (Lax-Milgram theorem).
- Convergence (up to a subsequence) to u^0 (weakly in $H_0^1(0,L)$).
- Convergence of a^{ε} to the mean value $\langle a \rangle$ of a (weakly in L^2).

Belief ? u^0 solution of

$$\begin{cases} -\dfrac{d}{dx}\left(\langle a \rangle \dfrac{du^0}{dx}\right) = -\langle a \rangle \dfrac{d^2 u^0}{dx^2} = f, \quad \text{on } [0,L] \\ u^0(0) = u^0(L) = 0 \end{cases}$$

False in general.

Right answer.

Consider $\xi^\varepsilon = a^\varepsilon(x)\dfrac{du^\varepsilon}{dx}(x)$.

- Convergence in L^2 to ξ^0.
- Implies weak convergence in L^2 of $\dfrac{1}{a^\varepsilon}\xi^\varepsilon$ to $\langle\dfrac{1}{a}\rangle\xi^0$.
- Hence: $\dfrac{du^0}{dx}(x) = \langle\dfrac{1}{a}\rangle\xi^0$.
- Since $-\frac{d\xi^\varepsilon}{dx}(x) = f$:

$$-\frac{d}{dx}\left(\frac{1}{\langle\frac{1}{a}\rangle}\frac{du^0}{dx}(x)\right) = -\frac{1}{\langle\frac{1}{a}\rangle}\frac{d^2u^0}{dx^2}(x) = -A^{\text{eff}}\frac{d^2u^0}{dx^2}(x) = f.$$

Comparison: arithmetic mean $\langle a\rangle \geq A^{\text{eff}} = \frac{1}{\langle\frac{1}{a}\rangle}$ harmonic mean.

Example of the bar:

$$\langle a\rangle = \frac{k_1+k_2}{2}, \quad \text{and} \quad A^{\text{eff}} = \frac{2k_1k_2}{k_1+k_2}.$$

Find the right effective matrix.

In general, a difficult problem. Among other references:

- Bensoussan, Lions, Papanicolaou (1978).
  - Periodic structure.
  - Link with SDE: $u^\varepsilon(x) = \mathbb{E}^x(f(X^\varepsilon_\tau))$

$$X^\varepsilon_t = x + \frac{1}{\varepsilon}\int_0^t b\left(\frac{X^\varepsilon_r}{\varepsilon}\right) dr + \int_0^t \sigma\left(\frac{X^\varepsilon_r}{\varepsilon}\right) dW_r \overset{law}{=} \varepsilon X^1_{t/\varepsilon^2}.$$

- ...

Under some assumptions, compacity argument (G-convergence in the symmetric case): existence of A^{eff} and convergence of subsequences.

A very useful technic: asymptotic expansion using multiple scales.

Asymptotic expansion: back to the example.

Expansion with "slow" and "fast" variables:

$$u^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots$$

where the functions $u_i(x, y)$ are periodic in y.

Plug this into

$$\mathcal{A}^\varepsilon u^\varepsilon = -\frac{d}{dx}\left(a^\varepsilon(x)\frac{du}{dx}\right) = -\frac{d}{dx}\left(a\left(\frac{x}{\varepsilon}\right)\frac{du}{dx}\right) = f$$

and use for $\phi^\varepsilon(x) = \phi(x, x/\varepsilon)$

$$\frac{\partial \phi^\varepsilon}{\partial x}(x) = \left(\frac{\partial \phi}{\partial x} + \frac{1}{\varepsilon}\frac{\partial \phi}{\partial y}\right)(x, \frac{x}{\varepsilon}).$$

Remark: $z \mapsto u_i(z, z/\varepsilon) \approx z \mapsto u_i(x, z/\varepsilon)$ around x.

Asymptotic expansion: terms in ε^{-2}.

Expansion with “slow” and “fast” variables:

$$u^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots$$

For u_0:

$$(\mathcal{A}_0 u_0)(x, x/\varepsilon) = -\frac{\partial}{\partial y}\left(a \frac{\partial u_0}{\partial y}\right)(x, x/\varepsilon) = 0$$

- Problem: fix x as a parameter and find $u_0(x,.)$ periodic s.t. $(\mathcal{A}_0 u_0)(x, y) = 0$.
- Answer: existence of $u_0(x,.)$ and $u_0(x, y) = u^0(x)$.

Asymptotic expansion: terms in ε^{-1}.

Expansion with “slow” and “fast” variables:

$$u^\varepsilon(x) = u^0(x) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots$$

For u_1: find $u_1(x,.)$ periodic s.t.

$$(\mathcal{A}_0 u_1)(x,y) = -\frac{\partial}{\partial y}\left(a\frac{\partial u_1}{\partial y}\right)(x,y) = \frac{\partial a}{\partial y}(y)\frac{\partial u^0}{\partial x}(x).$$

Solution: by separation of variables take $u_1(x,y) = \chi(y)\frac{\partial u^0}{\partial x}(x)$

- with a corrector χ periodic solution of

$$-\frac{d}{dy}\left(a(y)\frac{d\chi}{dy}\right) = \frac{da}{dy}.$$

Asymptotic expansion: terms in ε^0.

Expansion with “slow” and “fast” variables:

$$u^\varepsilon(x) = u^0(x) + \varepsilon\chi(\frac{x}{\varepsilon})\frac{\partial u^0}{\partial x}(x) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots$$

For u_2: find $u_2(x, .)$ periodic s.t.

$$\begin{aligned}(\mathcal{A}_0 u_2)(x,.) &= f(x) - (\mathcal{A}_1 u_1)(x,.) + \frac{\partial}{\partial x}\left(a(.)\frac{\partial u_0}{\partial x}\right)(x,.) \\ &= f(x) + \left[\left(a\frac{\partial \chi}{\partial y}\right) + a + \frac{\partial}{\partial y}(a\chi)\right](y)\frac{\partial^2 u^0}{\partial x^2}(x).\end{aligned}$$

- Necessary (and sufficient) condition: mean of the right-hand side equal zero. Equivalent to:

$$-\langle a(y) + a(y)\frac{d\chi}{dy}\rangle \quad \frac{\partial^2 u^0}{\partial x^2} = f.$$

Conclusion of this expansion.

Expansion with “slow” and “fast” variables:

$$u^\varepsilon(x) = u^0(x) + \varepsilon\chi(x/\varepsilon)\frac{du^0}{dx}(x) + \dots$$

- Short computation (in dimension 1):

$$\langle a(y) + a(y)\frac{d\chi}{dy}\rangle = \frac{1}{\langle 1/a\rangle}.$$

- u^0 solution of

$$-\frac{1}{\langle 1/a\rangle}\frac{d^2u^0}{dx^2} = f$$

- χ periodic solution of

$$-\frac{d}{dy}\left(a(y)\frac{d\chi}{dy}\right) = \frac{da}{dy}.$$

Limit and rate of convergence of u^ε.

▸ Diff case $\alpha = 2$

▸ Stat case $\alpha = 2$

Outline

Random media.

Random field:

$$-\mathrm{div}(a(x/\varepsilon,\omega)\nabla u^\varepsilon) = f.$$

where $a(x,\omega)$ is a random field:

- Uniform ellipticity: $\lambda Id \leq a \leq \frac{1}{\lambda} Id$
- Ergodicity
- ...

Some references:

- S. Olla. Homogenization of diffusion processes in Random Fields.
- G. Chechkin, A. Piatnitski, A. Shamaev. Homogenization (Translations of Mathematical Monographs).
- ...

A parabolic equation.

Parabolic operator:

$$\frac{\partial}{\partial t} - \mathcal{A}^\varepsilon = \frac{\partial}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) \frac{\partial}{\partial x_j}$$

- $\varepsilon > 0$ small parameter and $\alpha > 0$.
- $a_{ij}(z, y)$: matrix supposed to be periodic in z and uniformly elliptic.
- ξ: diffusion process in $\mathbb{R}^d$ with invariant measure with density p, and infinitesimal generator

$$\mathcal{L} = \sum_{i,j} q_{ij}(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} + \sum_i b_i(y) \frac{\partial}{\partial y_i}.$$

Notation: if $\phi(z, y)$ is a periodic function in z, then

- $\langle \phi(., y) \rangle$ is the mean over a period.
- $\overline{\phi(z, .)} = \int_{\mathbb{R}^n} \phi(z, y) p(y) dy$ is the mean w.r.t. p.

Summarize the result

Theorem (Kleptsyna-Piatnitsky, 1997)

Let $u^\varepsilon(x,t)$ be solution of $u^\varepsilon(x,0) = g(x)$ with

$$\frac{\partial u^\varepsilon}{\partial t} - \mathcal{A}^\varepsilon u^\varepsilon = \frac{\partial u^\varepsilon}{\partial t} - \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij}\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right) \frac{\partial u^\varepsilon}{\partial x_j} = 0.$$

Then u^ε converges (in a suitable space) in probability to u^0, solution of

$$\frac{\partial u^0}{\partial t} - \mathrm{Trace}(A^{\mathrm{eff}} D^2 u^0) = 0, \qquad u^0(.,0) = g.$$

1. If $\alpha = 2$, simultaneous homogenization: $A^{\mathrm{eff}} = \overline{\langle a + a\nabla\chi^0\rangle}$, $(\mathcal{A} + \mathcal{L})\chi^0_i = -\mathrm{div}(a_{i.})$.
2. If $\alpha < 2$, periodic homogenization first: $A^{\mathrm{eff}} = \overline{\mathbb{A}}$ where $\mathbb{A}$ homogenized operator (w.r.t. x/ε) for the family $\mathrm{div}(a(x/\varepsilon, y)\nabla)$ (with y as parameter).
3. If $\alpha > 2$, essential random effect: A^{eff} homogenized operator for $\mathrm{div}(\widetilde{a}(x/\varepsilon)\nabla)$ and $\widetilde{a}(z) = \overline{a(z,.)}$.

Assumptions

Denote

$$\mathcal{A} + \mathcal{L} = \frac{\partial}{\partial z_i} a_{ij}(z, y) \frac{\partial}{\partial z_j} + q_{ij}(y) \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} + b_i(y) \frac{\partial}{\partial y_i}$$

Conditions:

- Uniform ellipticity of a and q .
- a and q (and their derivatives) uniformly bounded.
- \+ technical condition on b to have a unique invariant measure with density p.
- \+ technical condition on p and q to obtain required estimates (mixing properties).
- Initial value g: smooth enough (can be weaken).

Some possible extensions.

- Add a drift (Diop et al., 2006):

$$\frac{\partial u^\varepsilon}{\partial t} = \operatorname{div}\left(a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}\right)\nabla u^\varepsilon\right) + \frac{1}{\varepsilon^{1\wedge\alpha/2}} g\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon\right) + h\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^\alpha}, u^\varepsilon\right)$$

- Degenerate case (Pardoux et al., 2009).

- Itô-Lévy case and partial integro-differential equations (Rhodes & Sow, 2009).

- ...

Outline

Case $\alpha = 2$.

Recall that u^ε converges to u^0. Consider

$$U^\varepsilon = \frac{u^\varepsilon - u^0}{\varepsilon} - \chi^0\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\nabla u^0.$$

Theorem

U^ε converges in law in the space $L^2(\mathbb{R}^n \times (0, T))$ to the solution of: 1 2

$$dU^0 = \left[\operatorname{div}\left(A^{\text{eff}}\nabla U^0\right) + \mu\frac{\partial^3 u^0}{\partial x^3}\right] dt + \Lambda^{1/2}\frac{\partial^2 u^0}{\partial x^2}\, dW_t,$$

$$U^0(x, 0) = 0$$

where

- $U^0 : [0, +\infty[\times\mathbb{R}^n \to \mathbb{R}$, W is a standard n^2-dimensional Wiener process,
- and

$$\mu\frac{\partial^3 u^0}{\partial x^3} = \sum_{i,j,k}\mu^{ijk}\frac{\partial^3 u^0}{\partial x_i\partial x_j\partial x_k}, \quad \Lambda^{1/2}\frac{\partial^2 u^0}{\partial x^2}\, dW_t = \sum_{ijkl}(\Lambda^{1/2})^{ijkl}\frac{\partial^2 u^0}{\partial x_i\partial x_j}\, dW_t^{kl}$$

Some remarks

Comments on the assumptions

- Proved for $a(z,s)$ supposed to be periodic in z and stationary ergodic in s with a good mixing property.
- Case $a(z,s) = a(z,\xi_s)$ as particular case.
- Initial condition g: should be sufficiently smooth.
- Diffusive scaling ($\alpha = 2$): crucial.

Comments on the result

- The tensors μ and $\Lambda^{1/2}$ are constant.
- The SPDE is well-posed and defines the limit law of U^ε uniquely.
- $\chi\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)\nabla u^0$ converges weakly in L^2 to zero. Thus the limit in law in the weak topology of $\varepsilon^{-1}(u^\varepsilon - u^0)$ is that of U^ε.

Idea of the proof

Formal asymptotic expansion

$$u^\varepsilon(x,t) = u^0(x,t) + \varepsilon u^1\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \varepsilon^2 u^2\left(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\right) + \dots$$

with functions $u^j(x,t,z,s)$ periodic in z, but not always stationary in s.

Collecting power-like terms

- ε^{-1}: $u^1(x,t,z,s) = \chi(z,s)\nabla_x u^0(x,t)$ with
$$\partial_s \chi - \mathrm{div}_z(a(z,s)\nabla_z \chi) = \mathrm{div}_z(a(z,s)).$$
- ε^0: Convergence of ... in law in $C(0,T;L^2(\mathbb{R}^n))$ to $V^{0,1}$
$$dV^{0,1} = \mathrm{div}(A^{\mathrm{eff}}\nabla V^{0,1})dt + \left(\Lambda^{1/2}\right)^{ijkl}\frac{\partial^2}{\partial x^i \partial x^j}u^0(x,t)dW_{t,kl}$$
- ε^1: Convergence of ... a.s. strongly in $L^2(\mathbb{R}^n \times (0,T))$ to:
$$\frac{\partial \Xi_{0,2}}{\partial t} = \mathrm{div}\left(A^{\mathrm{eff}}\nabla \Xi_{0,2}\right) + \mu\frac{\partial^3 u^0}{\partial x^3}, \quad \Xi_{0,2}(x,0) = 0$$

Summarize

$$\begin{aligned} u^{\varepsilon} &= u^0 + \varepsilon\chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0 + \varepsilon\left[\frac{1}{\varepsilon}V^{\varepsilon,1} + \Xi_{\varepsilon,2}\right] \\ &+ \varepsilon\left[\mathrm{Trace}\left(\varepsilon\chi_{2,2}(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2})D^2u^0\right) + R^{\varepsilon}\right]. \end{aligned}$$

and a.s. strongly in $L^2(\mathbb{R}^n \times (0,T))$

$$\varepsilon\chi_{2,2}, \quad R^{\varepsilon}(x,t) \to 0.$$

Conclusion:

$$\begin{aligned} U^{\varepsilon} &= \frac{u^{\varepsilon} - u^0}{\varepsilon} - \chi\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\nabla u^0 \\ &\approx \frac{1}{\varepsilon}V^{\varepsilon,1} + \Xi_{\varepsilon,2} \longrightarrow V^{0,1} + \Xi_{0,2} \end{aligned}$$

Outline

When $\alpha < 2$.

Recall $u^\varepsilon(x,0) = u^0(x) = g(x)$ and

$$\partial_t u^\varepsilon = \mathrm{div}(a(x/\varepsilon, \xi_{t/\varepsilon^\alpha})\nabla u^\varepsilon), \qquad \partial_t u^0 = A^{\mathrm{eff}} D^2 u^0$$

with

$$A^{\mathrm{eff}} = \overline{\mathbb{A}} = \overline{\langle a + (a\chi^0)_z + a\chi^0_z \rangle}, \quad \mathcal{A}\chi^0 = -a_z.$$

Theorem

Let $\delta = 2 - \alpha$ and $J_0 = \lfloor \frac{1}{\delta} - \frac{1}{2} \rfloor + 1$

$$U^\varepsilon = \varepsilon^{-\alpha/2} \left[u^\varepsilon - u^0 - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k \right]$$

converge in law in $L^2(\mathbb{R}^d \times (0,T))$ to a solution of a linear SPDE, where

$$\partial_t u^k = A^{\mathrm{eff}} u^k_{xx} + \sum_{m=0}^{k-1} \overline{\langle f^{k-m} \rangle} u^m_{xx}, \quad u^k(x,0) = 0.$$

with $\chi^k = \chi^k\left(\frac{x}{\varepsilon}, \xi_{t/\varepsilon^\alpha}\right)$: $\mathcal{A}\chi^k = \mathcal{L}\chi^{k-1}$, and $f^k = (a\chi^k)_z + a\chi^k_z$.

Proof ($\alpha < 2$).

Write

$$V^\varepsilon = \varepsilon^{-\alpha/2} \left[u^\varepsilon - u^0 - \sum_{k=1}^{J_0} \varepsilon^{k\delta} u^k - \varepsilon \left(\sum_{r=0}^{J_0} \sum_{k=0}^{J_0} \varepsilon^{(k+r)\delta} \chi^k \nabla u^r \right) \right]$$

Apply Itô's formula. With an initial condition $V^\varepsilon(.,0)$

$$\begin{aligned} dV^\varepsilon(x,t) - \mathrm{div} \left[a\left(\frac{x}{\varepsilon}, \xi_{\frac{t}{\varepsilon^\alpha}} \right) \nabla V^\varepsilon \right] dt \\ = \varepsilon^{-\alpha/2} H^\varepsilon(x,t) dt + R^\varepsilon(x,t) dt + \varepsilon^{1-\alpha} \Theta^\varepsilon(x,t) dW_t. \end{aligned}$$

Estimates :

- Initial condition and remainder:

$$\mathbb{E}\left(\| V^\varepsilon(.,0) \|^2_{L^2(\mathbb{R}^d)} + \| R^\varepsilon \|^2_{L^2(\mathbb{R}^d \times (0,T))} \right) \leq C \varepsilon^{1-\alpha/2}.$$

- Stochastic integrals (with anticipating stochastic calculus):

$$\mathbb{E}\left(\| V^\varepsilon_W \|^2_{L^2(\mathbb{R}^d \times (0,T))} \right) \leq C \varepsilon^{4-2\alpha} |\log(\varepsilon)|.$$

Case $\alpha > 2$.

Define $\delta = \alpha - 2 > 0$, $J_0 = \left[\frac{\alpha}{2}\right]$, and $J_1 = \lfloor \frac{1}{\delta} + \frac{1}{2} \rfloor$.

Expansion

$$V^\varepsilon = \varepsilon^{-\alpha/2} \left\{ u^\varepsilon - u^0 - \sum_{k=1}^{J_0} \varepsilon^k u^k - \sum_{k=1}^{J_0} \varepsilon^k \sum_{n=1}^{k} \chi^n \left(\frac{x}{\varepsilon}\right) \nabla_x^n u^{k-n} - \sum_{k=1}^{J_1} \varepsilon^{k\delta} v^k \right\}.$$

where u^k and v^k are defined by some recurrence procedure.

Itô's formula:

$$dV^\varepsilon = (\mathcal{A}^\varepsilon V^\varepsilon)dt - \frac{1}{\varepsilon} \left(\kappa_y^1 u_x^0 + \varepsilon^\delta H^\varepsilon + \varepsilon G^\varepsilon \right) q(\xi_{t/\varepsilon^\alpha}) dW_t + R^\varepsilon dt,$$

with $G^\varepsilon, R^\varepsilon = O(\varepsilon^\nu)$, $\nu > 0$ and

$$V^\varepsilon(x,0) = - \sum_{k=1}^{N_0} \varepsilon^{k-\alpha/2} \sum_{n=1}^{k} \chi^n \left(\frac{x}{\varepsilon}\right) \partial_x^n u^{k-n}(x,0) + O(\varepsilon^\nu).$$

Thank you for your attention

Literature.

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Homogenization with randomness

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- G. Chechkin, A. Piatnitski, A. Shamaev. Homogenization, Methods and Applications (Translations of Mathematical Monographs), 2007
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Our articles (with M. Kletpsyna and A. Piatnitski)

- $\alpha = 2$: Homogenization of random parabolic operators. Diffusion approximation. SPA, Volume 125, Issue 5, May 2015.
- $\alpha < 2$: Diffusion approximation for random parabolic operators. II. Almost finished.
- $\alpha > 2$: In progress...