UNIVERSITÉ DE PROVENCE
U.F.R. de Mathématiques, Informatique et Mécanique

Numéro d'ordre

THÈSE

pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ DE PROVENCE
Discipline : Mathématiques

École Doctorale Mathématiques et Informatique de Marseille
E.D. 184

présentée et soutenue publiquement
par

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le vendredi 10 décembre 2004

Équations différentielles stochastiques rétrogrades avec condition finale singulière

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Remerciements.

Je tiens à remercier tout particulièrement mon directeur de thèse, Etienne Pardoux ; pour ses conseils, sa grande disponibilité et sa générosité avec laquelle il m'a fait partager ses travaux, ses idées et ses intuitions. La pertinence de ses questions et de ses remarques ont toujours su me motiver et me diriger dans mes recherches. Je lui témoigne enfin ma profonde reconnaissance pour m'avoir fait confiance en me proposant de venir travailler avec lui, et pour son soutien constant au cours de ces années.

J'exprime mes plus sincères remerciements à Jean-François Le Gall, qui a accepté de présider le jury de cette thèse. Par sa présence il apporte un point de vue probabiliste différent sur ce sujet. Ses travaux liés aux superdiffusions et aux équations aux dérivées partielles non linéaires ont été une source d'inspiration essentielle dans mon travail.

Je suis très touché que Nicole El Karoui et Ying Hu m'aient fait l'honneur de rapporter cette thèse. Je leur sais gré de l'intérêt qu'ils ont manifesté pour mon travail. Leurs écrits sur la théorie des équations différentielles stochastiques rétrogrades ont été une aide précieuse.

Je tiens à exprimer toute ma gratitude à Elisabeth Rouy, qui a accepté d'être dans le jury de cette thèse. Elle apporte l'indispensable regard de l'analyste sur l'approche probabiliste des équations aux dérivées partielles.

Je tiens à remercier Fabienne Castell, pour avoir bien voulu participer à ce jury, pour sa relecture très minutieuse de ma thèse, et pour sa bonne humeur contagieuse.

Cette thèse doit évidemment beaucoup aux remarques et aux réponses des uns et des autres. Je remercie tout spécialement Franck Boyer et Thierry Gallouët, ainsi que Guy Barles et Laurent Véron de l'Université de Tours.

Je remercie tous les membres de l'équipe de probabilités et statistiques pour leur accueil et leur sympathie. J'adresse un merci tout particulier aux membres successifs du bureau 132 : Marian Ciuca, Karelle Dupoiron, Nicolas Klutchnikoff et Rémi Rhodes.

Je n'oublie pas les membres actifs passés et présents du CyberFoyer, association dont j'ai eu la joie d'être le trésorier pendant trois ans, et parmi eux : Patrick Iglesias et Alexis Monier. Mes pensées vont également aux membres du C.M.I. qui participent à la bonne ambiance du lieu. Je remercie Sylvie Blanc, Nathalie Bonifay, Muriel Gouyache, Gérard Henry, Hervé Masia, Georges Moutouh et Chantal Ravier pour leur gentillesse et leur disponibilité.

Je salue tous ceux avec qui j'ai eu le plaisir de discuter, de partager un café au foyer et de découvrir Marseille (dont ses multiples bars et restaurants). Parmi eux, je pense en particulier à Florence et son petit Paul, Julie, Léa, Maud, Sarah, Sonia, Stéphanie, Sylvaine, Julien, Nicolas, Renaud et Sébastien. Ceux que j'aurais pu oublier sauront, je l'espère, m'excuser.

J'ai une pensée affectueuse pour Jacqueline et Patrice et pour les amis berlinois Mario, Alex, Nadja et Peter. Je remercie chaleureusement toute ma famille, qui m'a soutenu, encouragé et poussé durant toutes mes années d'étude.

Et enfin, je ne saurai l'oublier : un **immense merci** à Muriel !

Sommaire.

Introduction

La théorie des équations différentielles stochastiques rétrogrades (EDSR en abrégé) a connu un formidable développement à partir des années 1990. Ces équations sont apparues en 1973, dans un article de J.M. Bismut [6] qui concernait le contrôle stochastique optimal et la version probabiliste du principe du maximum de Pontryagin. Pourtant le premier résultat général concernant les EDSR ne date que de 1990 et est dû à E. Pardoux et S. Peng [44]. Nous pourrions aborder les équations rétrogrades à travers des problèmes de modélisation : le contrôle stochastique, ou encore le problème du « pricing » et des stratégies de couverture en finance. Mais dans ce travail, la principale motivation provient des équations différentielles ordinaires et des équations aux dérivées partielles.

Considérons le problème de Cauchy suivant:

$$y' = -f(t,y) \quad \text{et} \quad y(T) = x. \tag{0.1}$$

Les données de ce problème sont l'instant terminal $T > 0$, la donnée finale $x \in \mathbb{R}^k$ et la fonction $f : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}^k$. L'inconnue y est en fait une fonction $y : [0,T] \to \mathbb{R}^k$, qui doit satisfaire la condition finale $y(T) = x$ et vérifier l'équation différentielle ordinaire (EDO en abrégé) : $y'(t) = -f(t,y(t))$ pour tout $t \in [0,T]$. On peut réécrire (0.1) sous forme intégrale :

$$\forall t \in [0,T], \quad y(t) = \xi + \int_t^T f(r,y(r))dr.$$

Sous de bonnes hypothèses de régularité sur f par rapport à y (théorème de Cauchy-Lipschitz par exemple), il y a une et une seule solution. De plus, on peut « retourner » le temps pour se ramener au même problème avec une donnée initiale. En effet si y est solution de (0.1), on définit l'application $t \mapsto \tilde{y}(t) = y(T-t)$. La fonction $\tilde{y}$ est solution de :

$$\tilde{y}' = g(t,\tilde{y}) \quad \text{sur } [0,T] \quad \text{et} \quad \tilde{y}(0) = x$$

avec $g(t,.) = f(T-t,.)$. En résumé dans un cadre déterministe, se donner une condition initiale (i.e. en $t = 0$) ou finale (i.e. en $t = T$) revient à résoudre le même problème.

Dans le cas aléatoire, le problème est totalement différent. Sur un espace probabilisé filtré $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbf{P})$ considérons une variable aléatoire ξ supposée mesurable par rapport à la tribu $\mathcal{F}_T$. Cette donnée ξ peut être une fonction du prix d'une action à l'instant T en finance, la filtration représentant les informations disponibles sur le marché à chaque instant, ou la fonction-valeur à optimiser en théorie du contrôle stochastique. On veut résoudre le problème (0.1) suivant :

$$y' = -f(t,y) \text{ et } y(T) = \xi \Longleftrightarrow \forall t \in [0,T], \quad y(t) = \xi + \int_t^T f(r,y(r))dr.$$

C'est l'équation qu'aurait à résoudre un agent cherchant une stratégie de couverture en n'utilisant qu'un actif sans risque (voir l'article [19]). Si ce problème admet une solution, cette solution est elle-même aléatoire car elle dépend de ξ, et à un instant $t < T$, elle est mesurable par rapport à la tribu $\mathcal{F}_T$. Autrement dit à l'instant t elle dépend du futur T. Ceci est inacceptable dans de nombreuses applications ; en finance ceci s'apparente à un délit d'initié. Ainsi on voudrait résoudre la même EDO avec la condition supplémentaire que les solutions n'anticipent pas sur le futur, c'est-à-dire qu'elles soient adaptées à la filtration $(\mathcal{F}_t)_{t\geq 0}$.

C'est là qu'interviennent les EDSR. Ce sont des équations de la forme suivante :

$$Y_t = \xi + \int_t^T f\left(r,Y_r,Z_r\right) dr - \int_t^T Z_r dB_r,\ 0 \leq t \leq T. \tag{0.2}$$

Au début du chapitre 1 nous expliquons sur un exemple une telle forme, via le théorème de représentation des martingales par une intégrale stochastique. Schématiquement disons que la première partie de (0.2) va résoudre l'EDO $y' = -f(t,y)$ avec donnée finale ξ, tandis que simultanément l'intégrale stochastique adapte la solution de l'EDO. Les données de cette équation sont :

1. le temps final $T > 0$ qui peut être déterministe ou aléatoire (temps d'arrêt) ;
2. $(B_t)_{t\geq 0}$ qui est un mouvement brownien standard défini sur un espace de probabilité $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\right)$ et à valeurs dans $\mathbb{R}^d$;
3. la filtration $(\mathcal{F}_t)_{t\geq 0}$ est la filtration augmentée du mouvement brownien, qui vérifie les hypothèses usuelles ;
4. la fonction $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$, mesurable par rapport aux tribus $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^{k\times d})$ et $\mathcal{B}(\mathbb{R}^{k\times d})$;
5. la variable aléatoire ξ à valeurs dans $\mathbb{R}^k$ et supposée $\mathcal{F}_T$-mesurable.

L'application f est appelée le **générateur** et ξ est la **donnée finale**.

Les inconnues sont les processus $\{Y_t\}_{t\in[0,T]}$ et $\{Z_t\}_{t\in[0,T]}$, à qui on impose d'être adaptés à la filtration brownienne : c'est le point crucial de cette théorie et c'est cette condition qui joue le rôle de seconde équation. En 1973, dans l'article [6], ces équations ont été introduites dans le cas où f est linéaire par rapport aux deux variables y et z. En 1990, E. Pardoux et S. Peng [44] ont montré un résultat d'existence et d'unicité des solutions dans un cadre non-linéaire général. Leurs hypothèses étaient : f est lipschitzienne en y et z, et ξ ainsi que le processus $\{f(t,0,0)\}_{t\in[0,T]}$ sont de carré intégrable (voir le théorème 1.1 dans le chapitre suivant). Remarquons que la condition de régularité sur f est très proche de celle du théorème de Cauchy-Lipschitz sur les EDO. Ensuite leur résultat a été étendu par de très nombreux articles et la condition d'intégrabilité sur ξ, ainsi que les hypothèses concernant f ont pu être affaiblies. Il faut noter que dans cette théorie, si ξ et f sont déterministes, alors la solution de l'EDSR est : $Z \equiv 0$ et Y solution de l'équation différentielle $y' = -f(t,y,0)$ avec $Y_T = \xi$. On est ramené alors à la résolution du problème de Cauchy (0.1).

Le cas particulier auquel nous nous sommes intéressés dans cette thèse est celui où le générateur est :

$$\forall y \in \mathbb{R}, \quad f(y) = -y|y|^q, \quad \text{avec } q > 0.$$

Le processus solution (Y,Z) est à valeurs dans $\mathbb{R} \times \mathbb{R}^d$, i.e. $k = 1$. Nous verrons par la suite qu'avec ce générateur f, pour tout ξ appartenant à un certain espace $L^p(\Omega)$, $p \geq 1$, l'EDSR (0.2) admet une unique solution (Y,Z) (dans un certain espace de processus). Or l'EDO associée à cette fonction f est :

$$y' = -f(y) = y|y|^q \quad \text{et} \quad y(T) = x. \tag{0.3}$$

La solution y^x de l'EDO est donnée par la formule :

$$\forall t \in [0,T], \quad y^x(t) = \text{sign}(x) \left(\frac{1}{q(T-t) + \frac{1}{|x|^q}} \right)^{\frac{1}{q}}$$

où $\text{sign}(x) = -1$ si $x < 0$ et $\text{sign}(x) = 1$ si $x > 0$.
Si x est égal à $\pm\infty$, $y^{\pm\infty}$ est encore correctement défini sur $[0,T]$ (par convention $1/\pm\infty = 0$) :

$$y^{\pm\infty}(t) = \pm \left(\frac{1}{q(T-t)} \right)^{\frac{1}{q}}.$$

Et on a une majoration *a priori* fondamentale pour la suite : pour tout $x \in \overline{\mathbb{R}}$ et tout $t \leq T$:

$$|y^x(t)| \leq \left(\frac{1}{q(T-t)} \right)^{\frac{1}{q}}.$$

Ainsi y^x est fini sur $[0,T[$ et borné sur $[0,T-\delta]$ pour $\delta > 0$.

La question que nous nous sommes posés est la suivante : est-ce que ce résultat élémentaire sur l'équation (0.3) est encore vraie pour l'EDSR (2.1) avec ξ éventuellement infini ? Si ξ est une variable aléatoire réelle telle que

$$\mathbf{P}(\xi = +\infty \text{ ou } \xi = -\infty) > 0,$$

l'EDSR

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q dr - \int_t^T Z_r dB_r, \tag{2.1}$$

admet-elle une solution finie sur $[0,T[$? Et si une telle solution existe, est-elle unique ?

Une autre motivation de ce travail provient cette fois de la théorie des équations aux dérivées partielles. Un grand nombre de travaux (parmi lesquels citons [3], [19] ou [42]) ont montré qu'il existe un lien étroit entre les EDSR couplées à une équation différentielle stochastique (ou EDS en abrégé) progressive et les équations aux dérivées partielles du second ordre paraboliques ou elliptiques. Ces travaux peuvent être considérés comme une généralisation non-linéaire de la formule de Feynman-Kac. Ce lien suggère que tout résultat concernant l'existence ou l'unicité de solution obtenu dans une théorie peut avoir sa contre-partie dans l'autre cadre. Ici toujours avec le générateur précédent, l'EDP parabolique associée est :

$$(0.4) \qquad \begin{aligned} \frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - u(t,x)|u(t,x)|^q &= 0,\ (t,x) \in [0,T[\times \mathbb{R}^m, \\ u(T,x) &= h(x),\ x \in \mathbb{R}^m. \end{aligned}$$

L'opérateur $\mathcal{L}$ est un opérateur différentiel du second ordre défini ainsi :

$$\mathcal{L} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} = \frac{1}{2}\mathrm{Trace}(\sigma\sigma^* D^2) + b.\nabla$$

Or ces EDP ont été très largement étudiées, aussi bien par des techniques probabilistes que par des méthodes analytiques.

Dans le cas où l'opérateur $\mathcal{L}$ est uniformément elliptique, ou plus simplement est le laplacien Δ, H. Brézis et A. Friedman [7] et P. Baras et M. Pierre [2] ont montré que cette EDP admet une unique solution si $h \in L^1(\mathbb{R}^m)$ ou si h est une mesure finie ne chargeant pas les ensembles de capacité c_q nulle. Sans entrer dans les détails, cette capacité c_q a la propriété de ne charger les points que si $q < 2/m$ (m est la dimension de l'espace). Ainsi si $q \geq 2/m$, la mesure h ne peut pas par exemple être une mesure de Dirac. Dans leur article [37], M. Marcus et L. Véron démontrent que toute solution positive de l'EDP (0.4) a une *trace finale* :

Théorème (Trace d'une solution) *Si u est une solution positive de l'EDP*

$$\frac{\partial u}{\partial t} + \Delta u - u|u|^q = 0,$$

alors pour tout point $y \in \mathbb{R}^m$ on a l'alternative suivante. Soit
i. pour tout voisinage U de y,

$$\lim_{t \to T} \int_U u(t,x)dx = +\infty,$$

soit
ii. il existe un voisinage ouvert U de y et une fonctionnelle linéaire et positive sur $C_0^\infty(U)$ (ensemble des fonctions de classe C^∞ à support compact inclus dans U) tels que

$$\lim_{t \to T} \int_U u(t,x)\zeta(x)dx = L_U(\zeta) \quad \forall \zeta \in C_0^\infty(U).$$

Ainsi si u est une solution positive de $u_t + \Delta u - u|u|^q = 0$, nous définissons deux ensembles :

$$\mathcal{R} = \left\{ y \in \mathbb{R}^m,\ \exists U \text{ voisinage de } y \text{ t.q. } \limsup_{t \to T} \int_U u(t,x)dx < +\infty \right\},$$

$$\mathcal{S} = \mathbb{R}^m \setminus \mathcal{R} \qquad \text{(ensemble des points singuliers).}$$

Remarquons que l'ensemble $\mathcal{S}$ est fermé et qu'il existe une unique mesure de Radon μ sur $\mathcal{R}$ telle que

$$\lim_{t\to T} \int_{\mathcal{R}} u(t,x)\zeta(x)dx = \int_{\mathcal{R}} \zeta(x)d\mu(x), \quad \forall \zeta \in C_0(U).$$

Le couple $(\mathcal{S},\mu)$ est appelé la *trace finale* de u. A ce couple correspond une unique mesure borélienne ν définie pour tout ensemble borélien A par :

$$\nu(A) = \begin{cases} \infty & \text{si } A \cap \mathcal{S} \neq \emptyset, \\ \mu(A) & \text{si } A \subseteq \mathcal{R}. \end{cases}$$

Dans la suite de leur article [37], les deux auteurs donnent des conditions nécessaires et suffisantes pour qu'à un couple $(\mathcal{S},\mu)$ soit associée une solution u de l'EDP de trace $(\mathcal{S},\mu)$. Sous certaines hypothèses, dont celles de P. Baras et M. Pierre, il existe toujours une solution maximale. Il y a unicité de la solution si $q < 2/m$, mais pas si $q \geq 2/m$. Mieux, si $q \geq 2/m$ et $\mathcal{S} = \mathbb{R}^m$, il est possible de construire une infinité de solutions. Dans les articles [35] et [36], les mêmes auteurs ont obtenu des résultats similaires pour l'EDP elliptique :

$$\begin{cases} \mathcal{L}u(x) - u(x)|u(x)|^q = 0, & \forall x \in D, \\ u(x) = g(x), & \forall x \in \partial D; \end{cases}$$

D étant un ouvert borné de $\mathbb{R}^m$.
Pour toute solution positive u, il existe une *trace au bord*, c'est-à-dire un unique couple $(\mathcal{S},\mu)$, $\mathcal{S}$ étant un sous-ensemble fermé de ∂D sur lequel u explose, et μ étant une mesure de Radon sur $\partial D \setminus \mathcal{S}$. De même, il existe des conditions nécessaires et suffisantes sur $(\mathcal{S},\mu)$ pour avoir existence d'une solution maximale. Et il y a une valeur critique de q égale à $2/(m-1)$ en dessous de laquelle il y a unicité et telle que si $q \geq 2/(m-1)$ il peut y avoir une infinité de solutions.

De leur côté, J.F. Le Gall [28], [29], [30] et E.B. Dynkin et S.E. Kuznetsov [17], [18] ont prouvé des résultats similaires sur ce type d'EDP paraboliques ou elliptiques, grâce à la théorie probabiliste des superprocessus. Une superdiffusion est un modèle mathématique d'un nuage aléatoire. C'est un processus de Markov à valeurs mesures, combinaison d'un processus de branchement avec un mouvement dans l'espace. Il est caractérisé entre autre par le générateur infinitésimal $\mathcal{L}$, qui décrit son mouvement spatial, et le paramètre $1 + q \in]1,2]$, qui décrit le mécanisme de branchement. Si $\mathcal{L} = \Delta$ et $q = 1$, on parle de super-mouvement brownien. Notons qu'on ne sait pas construire ces processus pour q plus grand que 1.

Or la théorie des EDSR donne aussi une interprétation probabiliste de ces EDP. Donc nous nous sommes demandé si nous pouvions obtenir le même genre de résultats que ceux cités précédemment. Pour se faire, il fallait impérativement autoriser la condition finale ξ à prendre des valeurs infinies ; l'ensemble $\{\xi = +\infty\}$ correspondant alors à l'ensemble des points singuliers $\mathcal{S}$.

Nous détaillons un peu notre démarche et nos résultats.

EDSR avec donnée finale singulière (Chapitre 2)

Nous nous sommes donc intéressés à l'EDSR

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r \quad \text{avec } q \in \mathbb{R}_+^*. \tag{2.1}$$

Le mouvement brownien B et le processus Z sont à valeurs dans $\mathbb{R}^d$, tandis que ξ et Y sont à valeurs réelles.
Nous verrons au chapitre suivant que si $\xi \in L^p(\Omega)$ pour un certain $p \geq 1$, l'EDSR admet une unique solution (Y,Z) (avec une condition d'intégrabilité sur (Y,Z)). Que se passe-t-il si maintenant :

$$\mathbf{P}(\xi = +\infty \text{ ou } \xi = -\infty) > 0 \text{ ?}$$

Dans ce cas, ξ est dit « *singulier* ». Tout d'abord la notion de solution (voir la définition 1.1 au chapite 1) doit être modifiée pour tenir compte de l'explosion en T.
Redéfinition d'une solution. *Soit $q > 0$ et ξ une variable aléatoire réelle $\mathcal{F}_T$-mesurable. Le processus (Y,Z) est une solution de l'EDSR*

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r$$

si (Y,Z) vérifie :
(D1) $\mathbf{P}$-p.s. pour tout $0 \leq s \leq t < T$:

$$Y_s = Y_t - \int_s^t Y_r|Y_r|^q dr - \int_s^t Z_r dB_r;$$

(D2) pour tout $t \in [0,T[$,

$$\mathbb{E}\left(\sup_{0\leq s\leq t} |Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) < +\infty;$$

(D3) la limite à gauche en T de Y_t existe $\mathbf{P}$-p.s. et :

$$\lim_{t\to T} Y_t = \xi.$$

La norme $\| \ \|$ qui apparaît dans (D2), est la norme euclidienne usuelle sur $\mathbb{R}^d$.
Si ξ appartient à $L^p(\Omega)$, la solution « classique » associée vérifie bien ces conditions, car on a l'estimation a priori suivante :

$$\mathbf{P} - \text{p.s.} \quad \forall t \in [0,T] \quad Y_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}. \tag{0.5}$$

Dans toute la suite cette inégalité joue un rôle majeur.
Nous supposons ensuite que ξ est positif ou nul et satisfait

$$\mathbf{P}(\xi = +\infty) > 0.$$

Pour les EDSR scalaires, i.e. avec $Y_t \in \mathbb{R}$, une méthode standard consiste à trouver une suite croissante de processus dont la limite est la solution souhaitée, en utilisant un résultat de comparaison des solutions d'une EDSR (voir chapitre 1, théorème 1.2). C'est la technique utilisée par J.P. Lepeltier et J. San Martin [31] pour obtenir l'existence d'une solution avec un générateur seulement supposé continu, ou par M. Kobylanski [24] pour un générateur quadratique en z. Ici l'idée la plus simple est de tronquer ξ par un entier $n \in \mathbb{N}$, puis de faire tendre n vers l'infini. Sans autre hypothèse concernant ξ, on peut construire un processus (Y,Z) qui satisfait les conditions (D1)-(D2) de la définition précédente. Ce couple (Y,Z) est construit comme étant la limite d'une suite croissante de processus (Y^n,Z^n). Les éléments de cette suite sont les solutions (Y^n,Z^n) de l'EDSR 2.1 avec donnée finale $\xi \wedge n$. De plus, le processus Y est le supremum d'une famille de fonctions continues Y^n, donc presque sûrement $t \mapsto Y_t$ est semi-continue inférieurement sur $[0,T]$. En particulier :

$$\mathbf{P}-\text{p.s.} \quad \liminf_{t\to T} Y_t \geq \xi.$$

Ainsi sur l'ensemble $\{\xi = +\infty\}$, Y explose presque sûrement. Mais ce résultat ne spécifie pas si Y vérifie $\mathbf{P}$-p.s. :

$$\lim_{t\to T} Y_t = \xi.$$

Un premier pas dans cette direction a été de prouver l'existence de cette limite. Pour se faire, nous avons remarqué que si ξ est plus grand qu'une constante strictement positive, alors tous les processus Y^n et Y sont minorés eux aussi par une constante positive. L'idée est alors de considérer les processus $(Y^n)^{-q}$ et Y^{-q}, qui sont eux bornés et donc a priori plus aisés à manipuler. C'est ce processus Y^{-q} qui a une limite à gauche en T. Mais nous n'avons pas pu identifier cette limite et montrer qu'elle était égale à $1/\xi^q$.

Ensuite, nous avons trouvé la « vitesse » d'explosion de Y. Sur l'ensemble $\{\xi = +\infty\}$, Y se comporte asymptotiquement exactement comme la solution de l'EDO avec donnée finale $+\infty$, i.e.

$$\lim_{t\to T} (T-t)^{1/q} Y_t = \left(\frac{1}{q}\right)^{1/q} \qquad \text{p.s.}$$

M. Marcus et L. Véron ont montré que les solutions de l'EDP (0.4) se comportent de la même façon sur l'ensemble de points singuliers $\mathcal{S}$ (voir le théorème 3.1 dans [37]). Nous verrons que dans le cas d'un temps d'arrêt, nous ne pouvons pas obtenir de constante aussi explicite.

Restait le problème de la continuité de Y en T. Jusqu'à présent nous avions juste :

$$\lim_{t\to T} Y_t \geq \xi = Y_T.$$

Sans autre condition nous n'avons pas pu prouver l'inégalité opposée. En fait, nous n'avons pas été capables de trouver une majoration, globale et indépendante de n, sur notre suite Y^n, autre que

$$\forall t \in [0,T] \ Y_t^n \leq \left(\frac{1}{q(T-t)}\right)^{1/q}.$$

Evidemment, cette majoration est trop grossière pour obtenir le comportement de Y sur l'ensemble $\{\xi < \infty\}$. Pour obtenir la continuité, nous avons du en quelque sorte

« localiser » et pour cela nous avons du nous restreindre au cadre markovien. La première hypothèse sur ξ est donc la suivante :

$$\text{(H1)} \qquad \xi = g(X_T),$$

où g est une fonction définie sur $\mathbb{R}^m$ à valeurs dans $\overline{\mathbb{R}^+}$ telle que l'ensemble $F = \{g = +\infty\}$ soit fermé ; et où X_T est la valeur à l'instant $t = T$ de la solution de l'équation différentielle stochastique (en abrégé EDS) :

$$X_t = x + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dB_r, \text{ pour } t \in [0,T].$$

Nous supposerons toujours que la dérive $b : [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ et le coefficient de diffusion $\sigma : [0,T] \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ satisfont les hypothèses classiques qui permettent d'avoir une unique solution forte de l'EDS :

1. condition de régularité :

$$\text{(L)} \qquad |b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K|x - y|;$$

2. condition de croissance :

$$\text{(G)} \qquad |b(t,x)| + \|\sigma(t,x)\| \leq K(1 + |x|).$$

Nous imposons également à ξ une hypothèse d'intégrabilité locale : pour tout ensemble compact $\mathcal{K} \subset \mathbb{R}^m \setminus F = \mathbb{R}^m \setminus \{g = +\infty\}$

$$\text{(H2)} \qquad g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in \ L^1\left(\Omega,\mathcal{F}_T,\mathbf{P};\mathbb{R}\right).$$

La preuve de la continuité consiste à multiplier Y par une fonction-test $\varphi(X)$ où φ est à support compact inclus dans l'ensemble $\{g < \infty\}$. Ainsi à l'instant T, on obtient $Y_T^n\varphi(X_T) = (g \wedge n)(X_T)\varphi(X_T)$, qui est une variable aléatoire bornée dans $L^1(\Omega)$, car majorée par $g(X_T)\varphi(X_T)$. L'hypothèse d'intégrabilité locale de g sert ici. On en déduit donc que $\left(Y_T^n\varphi(X_T)\right)_n$ est une suite bornée dans $L^1(\Omega)$. Ensuite nous montrons que $Y\varphi(X)$ est bornée dans $L^1([0,T] \times \Omega)$ et enfin nous trouvons que

$$\mathbb{E}(\varphi(X_T) \liminf_{t\to T} Y_t) \leq \mathbb{E}(\varphi(X_T)g(X_T));$$

ceci étant vrai pour toute fonction φ à support dans $\{g < +\infty\}$. Comme on sait déjà que la limite à gauche en T de $t \mapsto Y_t$ existe, et que cette limite est presque sûrement plus grande que $g(X_T)$, la continuité de Y est prouvée.

Nous verrons que la difficulté majeure provient du terme contenant le processus Z. Quand $q > 2$, nous pouvons assez simplement contrôler ce terme, car $t \mapsto (T - t)^{-2/q}$ est intégrable. Mais si $q \leq 2$, pour se débarasser du terme en Z et n'avoir qu'à contrôler $Y\varphi(X)$, nous avons utilisé le calcul de Malliavin. Le processus Z est en effet la dérivée de Malliavin de Y. Par «intégration par parties», le terme contenant Z ne s'exprime plus qu'en fonction de Y. Pour mener à bien les calculs, il nous faut alors supposer que les fonctions b et σ sont bornés (hypothèse (B)) et que $\sigma\sigma^*$ est uniformément elliptique (hypothèse (E)) ; i.e. il existe $\lambda > 0$ tel que pour tout $(t,x) \in [0,T] \times \mathbb{R}^m$ et tout $y \in \mathbb{R}^m$:

$$\sigma\sigma^*(t,x)y.y \geq \lambda |y|^2.$$

Rappelons que cette hypothèse impose $d \geq m$. Nous avons alors réussi à prouver que, sous les hypothèses (H1)-(H2), et avec soit $q > 2$ soit (B)-(E), Y est continu en T :

$$\lim_{t\to T} Y_t = \xi, \ \mathbf{P} - \text{p.s.}$$

Ainsi nous avons donc pu construire une solution de l'EDSR (2.1) pour un ξ singulier. Malheureusement nous n'avons pas réussi à établir l'unicité de cette solution. Néanmoins la solution (Y,Z) obtenue précédemment est minimale : si $(\bar{Y},\bar{Z})$ est une autre solution positive, alors $\mathbf{P}$-p.s. pour tout $t \in [0,T]$: $\bar{Y}_t \geq Y_t$. La démonstration de ce résultat permet également de voir que si $(\bar{Y},\bar{Z})$ est obtenue par approximation par des solutions bornées, alors $(\bar{Y},\bar{Z}) = (Y,Z)$. De plus toute solution $(\bar{Y},\bar{Z})$ vérifie encore l'estimation (0.5).

Une solution de l'EDSR ayant été construite, nous espérons qu'elle fournit une solution de l'EDP semi-linéaire associée (0.4). Dans cette perspective, la condition (H1) sur ξ est raisonnable. Pour cela nous changeons l'EDS et pour tout $(t,x) \in [0,T] \times \mathbb{R}^m$, nous notons $X^{t,x}$ la solution de l'équation :

$$X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dB_r, \text{ pour } s \in [t,T],$$

et $X_s^{t,x} = x$ pour $s \in [0,t]$.
Les fonctions b et σ vérifient toujours les conditions précédemment énoncées. Comme donnée finale de l'EDSR, nous choisissons $g(X_T^{t,x})$, g étant une fonction de $\mathbb{R}^m$ dans $\overline{\mathbb{R}^+}$ telle que l'ensemble $F = \{g = +\infty\}$ soit fermé et la condition (H2) satisfaite. De plus g est supposée continue de $\mathbb{R}^m$ dans $\overline{\mathbb{R}^+}$. Nous supposons aussi que la solution $(Y^{t,x},Z^{t,x})$ minimale de l'EDSR (2.1), avec $\xi = g(X_T^{t,x})$, existe. Alors $Y_t^{t,x}$ est déterministe et si nous posons $u(t,x) = Y_t^{t,x}$, alors u est semi-continue inférieurement de $[0,T] \times \mathbb{R}^m$ dans $\overline{\mathbb{R}^+}$ et est solution de viscosité de l'EDP (0.4) avec :

$$\forall x \in \mathbb{R}^m \quad \lim_{t\to T} u(t,x) = g(x) = u(T,x). \tag{0.6}$$

De plus cette solution est minimale parmi toutes les solutions de viscosité. Si la condition d'uniforme ellipticité sur σ est satisfaite, cette solution de viscosité appartient à $C^0\left([0,T] \times \mathbb{R}^m; \overline{\mathbb{R}}\right) \cap C^0\left([0,T[\times \mathbb{R}^m; \mathbb{R}\right)$. Elle est même localement höldérienne sur l'ensemble $[0,T[\times\mathbb{R}^m$.

Il faut souligner que ce résultat semble être en contradiction avec ceux de Marcus et Véron ou de Le Gall, Dynkin et Kuznetsov. En effet, il n'y a pas de cas sous-critique ou sur-critique suivant la valeur de q par rapport à la dimension m, ce qui peut facilement s'expliquer par la continuité de la fonction g. Nous ne traitons pas ici le cas des mesures. Mais surtout, nous obtenons une solution minimale de l'EDP, alors que d'après les travaux de J.-F. Le Gall, M. Marcus et L. Véron, il existe toujours une solution maximale et si q est trop grand par rapport à la dimension m, il n'y a pas de solution minimale.

Ceci provient de la différence entre les conditions de bord. La notion de *trace finale* n'est pas équivalente à notre condition (0.6). Dans notre cas, si $\mathcal{S} = \mathbb{R}^m$, on n'a qu'une seule solution, là où il y en a une infinité avec la condition de trace. Finalement, cette différence rend, nous semble-t-il, les résultats difficilement comparables. Notons également que dans le cas $q > 2$, nous pouvons avoir un générateur $\mathcal{L}$ avec un σ dégénéré, ce qui n'est pas le cas de Marcus et Véron.

EDSR avec temps d'arrêt et donnée singulière (Chapitre 3)

Si nous voulons pouvoir donner une interprétation probabiliste des EDP elliptiques

$$\begin{cases} \mathcal{L}u(x) - u(x)|u(x)|^q = 0, & \forall x \in D \subset \mathbb{R}^d, \\ u(x) = g(x), & \forall x \in \partial D, \end{cases}$$

il nous faut traiter le cas où le temps final T est en fait un temps d'arrêt τ par rapport à la filtration brownienne (voir chapitre 1, section 4). Dans ce chapitre nous considérons donc l'EDSR :

$$Y_t = \xi - \int_t^\tau Y_r |Y_r|^q dr - \int_t^\tau Z_r dB_r. \tag{3.3}$$

Le premier problème auquel nous sommes confrontés, est de remplacer l'estimation *a priori* (0.5) valable pour un temps final déterministe par une autre qui tienne compte du temps d'arrêt. L'idée provient d'un résultat concernant les EDP et appelé *inégalité de Keller-Osserman*. Nous allons nous restreindre au cas où τ est le premier temps de sortie d'une diffusion X en dehors d'un ouvert D. Plus précisément, nous supposerons que D est un ouvert borné de $\mathbb{R}^d$ avec un bord ∂D de classe C^2, et que X est la solution d'une EDS :

$$X_t^x = x + \int_0^t b(X_r^x) dr + \int_0^t \sigma(X_r^x) dB_r, \text{ pour } t \geq 0,$$

avec $x \in \mathbb{R}^d$ et

i. condition de Lipschitz sur σ : il existe $K \geq 0$ tel que pour tout (x,y):

$$\|\sigma(x) - \sigma(y)\| \leq K|x - y|; \tag{L}$$

ii. les coefficients sont bornés :

$$|b(x)| + \|\sigma(x)\| \leq K; \tag{B}$$

iii. uniforme ellipticité : il existe une constante $\alpha > 0$ telle que pour tout $x \in \mathbb{R}^d$:

$$\sigma\sigma^*(x) \geq \alpha \mathrm{Id}. \tag{E}$$

Sous ces hypothèses il existe une unique solution forte de l'EDS (voir [53]). Le temps d'arrêt τ est défini, pour tout $x \in \overline{D}$, ainsi :

$$\tau = \tau_x = \inf \left\{ t \geq 0,\ X_t^x \notin \overline{D} \right\}.$$

Alors tous les points du bord de D sont réguliers : si $x \in \partial D$ et si

$$\theta_D^x = \inf \{t > 0,\, X_t^x \in D^c\},$$

alors $\mathbf{P}(\theta_D^x = 0) = 1$ (D^c est le complémentaire de D). Autrement dit, la diffusion X^x qui démarre au point $x \in \partial D$, ne retourne pas immédiatement dans D en y restant un intervalle de temps non vide. De plus, il existe $\beta > 0$ tel que

$$\sup_{x \in \overline{D}} \mathbb{E}\left(e^{\beta \tau_x}\right) < \infty.$$

Sous ces hypothèses, il existe une constante C (qui ne dépend que du domaine D, de q et des bornes sur b et σ) telle que pour tout $x \in \overline{D}$ et toute solution (Y,Z) de l'EDSR (3.3) avec temps final τ_x :

$$\mathbf{P} - \text{p.s.} \quad \forall t \geq 0, \quad |Y_t| \leq \frac{C}{\left(\rho(X_{t\wedge\tau_x}^x)\right)^{\frac{2}{q}}}.$$

La fonction ρ est la distance au bord de D. Cette majoration ne dépend pas de ξ et va servir dans la suite d'estimation *a priori*.

Avec cette majoration, nous pouvons faire le même travail que dans le second chapitre. Pour $x \in \overline{D}$, et tout $\eta > 0$, nous définissons le temps d'arrêt τ_η^x :

$$\tau_\eta^x = \inf \{t \geq 0,\, \rho(X_t^x) \leq \eta\}.$$

Remarquons que pour $x \in \overline{D}$, $\tau_\eta^x \leq \tau_x$ p.s. et quand η tend vers 0, τ_η^x converge vers τ_x p.s. A partir de maintenant, pour alléger les notations, nous fixons $x \in D$ et nous ne précisons plus la dépendance en x.

Pour toute variable aléatoire ξ positive et $\mathcal{F}_\tau$-mesurable, telle que $\mathbf{P}(\xi = \infty) > 0$, il existe un processus (Y,Z) tel que :

1. il existe une constante K telle que pour tout $\eta > 0$ et tout $T \geq 0$:

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_{t\wedge\tau_\eta}|^2 + \int_0^{T\wedge\tau_\eta} |Z_r|^2 dB_r\right) \leq \frac{K}{\eta^{2/q}};$$

2. $\mathbf{P}$-p.s. pour tout $0 \leq t \leq T$ et tout $\eta > 0$:

$$Y_{t\wedge\tau_\eta} = Y_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} Y_r\,|Y_r|^q\,dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} Z_r dB_r;$$

3. sur l'ensemble $\{t \geq \tau\}$, $Y_t = \xi$ et $Z_t = 0$ $\mathbf{P}$-p.s.
4. et de plus, la limite quand $t \to +\infty$ de $Y_{t\wedge\tau}$ existe p.s. et vérifie :

$$\lim_{t\to+\infty} Y_{t\wedge\tau} \geq \xi.$$

Nous pouvons également préciser le comportement de Y sur l'ensemble $\{\xi = +\infty\}$: il existe une constante C telle que

$$\mathbf{P} - \text{p.s. sur l'ensemble } \{\xi = +\infty\} \quad \liminf_{t\to+\infty} \rho(X_{t\wedge\tau})^{2/q} Y_t \geq C.$$

En revanche, nous n'obtenons pas d'expression explicite pour C comme dans le cas déterministe.

Comme précédemment, le problème majeur est d'obtenir la continuité du processus Y, i.e. d'obtenir l'égalité :

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \xi.$$

Là aussi nous avons dû nous limiter au cas où ξ est une fonction de la valeur de la diffusion X à l'instant τ, c'est-à-dire :

$$\text{(A1)} \qquad \xi = g(X_\tau)$$

$g : \mathbb{R}^d \to \overline{\mathbb{R}_+}$ étant une fonction telle que $F = \{g = +\infty\} \cap \partial D$ est un ensemble fermé et telle que pour tout compact $\mathcal{K}$ inclus dans $\partial D \setminus F$, g est bornée sur $\mathcal{K}$ (hypothèse (A2)). Dans le cas T déterministe, nous avions juste besoin de g localement intégrable là où g est finie. De plus pour de raisons techniques, nous supposons aussi que le bord de D est de classe C^3 (cette hypothèse est certainement superflue).
Avec ses conditions, nous démontrons que Y est continu, i.e.

$$\mathbf{P} - \text{p.s.} \quad \lim_{t\to+\infty} Y_{t\wedge\tau} = \xi.$$

La démonstration est légèrement différente de celle du chapitre 2. La difficulté supplémentaire est que nous n'avons plus de bonne majoration sur Z et que nous ne savons pas appliquer le calcul de Malliavin avec un temps d'arrêt pour obtenir $Z_t = D_t Y_t$. La preuve consiste donc ici à trouver une borne sur Y^n qui soit indépendante de n, mais qui dépende de la fonction g. Pour tout ouvert U inclus dans $\partial D \setminus F$, la fonction g est bornée sur U. On majore Y^n par $\rho_U^{-2/q}(X_\tau)$, où ρ_U est une fonction continue strictement positive sur $D \cup U$ et qui prend la valeur 0 sur F. Ainsi si h est une fonction continue à support compact inclus dans U (fonction-test), $h(X)Y^n$ est p.s. bornée indépendamment de n. Le reste des calculs ressemble ensuite à ceux du chapitre 2.

Comme pour un temps déterministe, cette solution est minimale parmi les solutions positives, mais nous ne savons pas montrer l'unicité. A partir de là, nous pouvons construire une solution de viscosité semi-continue inférieurement de l'EDP elliptique :

$$\begin{cases} -\mathcal{L}u(x) + u(x)|u(x)|^q = 0, & \forall x \in D; \\ u(x) = g(x), & \forall x \in \partial D. \end{cases}$$

Cette solution u satisfait :

$$\lim_{\substack{x'\to x \\ x'\in D, x\in\partial D}} u(x') = g(x) = u(x).$$

Nous verrons qu'avec l'hypothèse d'uniforme ellipticité sur σ cette solution est continue sur $\overline{D}$.

Il est important de noter que tous ces résultats peuvent s'obtenir avec des hypothèses différentes sur b et σ. En particulier nous pouvons supprimer l'uniforme ellipticité de σ, pourvu que l'EDS admette une solution forte X, que les points du bord restent réguliers et que le temps d'arrêt ait un moment exponentiel. Le seul point où l'uniforme ellipticité semble indispensable est pour montrer la continuité de la solution de viscosité minimale u. En utilisant un résultat de M. Royer [50], on peut même supposer simplement que τ est intégrable. En effet, pour tout $n \in \mathbb{N}$, $\xi \wedge n$ est dans $L^\infty(\Omega)$ et notre générateur f vérifie toutes les hypothèses de M. Royer pour avoir existence et unicité d'une solution de l'EDSR avec temps d'arrêt.

Quelques pistes de recherche.

Suite aux travaux menés dans cette thèse, nous proposons quelques pistes de recherche susceptibles d'en constituer un prolongement naturel.

Dans les deux chapitres, il manque bien entendu un résultat sur l'unicité des solutions. Nous ne savons pas comparer deux solutions avec ξ singulier. Nous pouvons même construire d'autres solutions, par exemple en grossissant l'ensemble singulier $F = \{g = +\infty\}$. En effet on peut définir la solution minimale (Y_n,Z_n) de l'EDSR (2.1) avec comme donnée finale $\xi_n = g_n(X_T)$, où $g_n = +\infty$ sur l'ensemble F_n défini comme

$$F_n = \{y \in \mathbb{R}^m;\ \mathrm{dist}(Y,F) \leq 1/n\}$$

et avec $g_n = g$ ailleurs. On montre que $\{Y_n\}_{n\in\mathbb{N}}$ est une suite décroisante de processus qui converge vers $\tilde{Y}$. On obtient également une limite pour Z_n. Alors $(\tilde{Y},\tilde{Z})$ est une solution de l'EDSR (2.1) avec condition finale $\xi = g(X_T)$, telle que $\tilde{Y} \geq Y$, sans pouvoir dire s'il y a égalité ou non.

Dans le chapitre 3, nous avons rencontré une difficulté pour dériver, au sens du calcul de Malliavin, la solution d'une EDSR avec temps d'arrêt. A notre connaissance, il n'y a pas de résultat du type $Z_t = D_t Y_t$ quand le temps final est aléatoire. Si nous régularisons les coefficients de la diffusion et si l'hypothèse d'uniforme ellipticité est satisfaite, alors l'EDP elliptique associée à l'EDS progressive-rétrograde admet une solution régulière u et nous pouvons identifier (Y_t,Z_t) avec $u(X_{t\wedge\tau})$ et avec $\nabla u(X_t)\sigma(X_t)\mathbf{1}_{t<\tau}$. Alors on retrouve que Y est la dérivée de Malliavin de Z, ce qui laisse supposer que ceci doit être vrai en général. Nous faisons juste remarquer que même avec toute la régularité nécessaire, la méthode du chapitre 2 reste difficilement applicable, car elle mettrait en jeu la densité du processus tué au bord (voir [5]ou [48]). Et il nous faut connaître le comportement au bord du domaine de cette densité, en particulier la « vitesse » à laquelle la densité s'annule au bord. Celle-ci est bien contrôlée dans le seul cas où l'opérateur $\mathcal{L}$ est symétrique dans $L^2(\mathbb{R}^d)$ (sous forme divergence sans terme d'ordre1). Ces résultats se trouvent par exemple dans le livre de R. Bass [5], celui de E.B. Davies [12] et dans l'article de E.B. Davies et B. Simon [13].

Chapitre 1

Rappel de quelques résultats sur les EDSR

Les équations différentielles stochastiques rétrogrades (EDSR en abrégé dans la suite) sont des équations du type suivant :

$$Y_t = \xi + \int_t^T f\left(r, Y_r, Z_r\right) dr - \int_t^T Z_r dB_r, \ 0 \leq t \leq T. \tag{0.2}$$

$(B_t)_{0 \leq t \leq T}$ est un mouvement brownien standard à valeurs dans $\mathbb{R}^d$, défini sur un espace de probabilité $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbf{P}\right)$, et $(\mathcal{F}_t)_{0 \leq t \leq T}$ est la filtration browienne telle que $\mathcal{F}_0$ contienne tous les éléments $\mathbf{P}$-négligeables de $\mathcal{F}$. Cette filtration est continue à droite et satisfait donc les conditions usuelles (voir [22] p. 90).

La fonction $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ est appelée le **générateur**, T est le **temps final**, et ξ est une variable aléatoire à valeurs dans $\mathbb{R}^k$. La variable aléatoire ξ est appelée **donnée finale** car elle est supposée mesurable par rapport à $\mathcal{F}_T$. Ces notations seront conservées dans toute la suite.

Les inconnues sont les processus $\{Y_t\}_{t \in [0,T]}$ et $\{Z_t\}_{t \in [0,T]}$, à valeurs respectivement dans $\mathbb{R}^k$ et $\mathbb{R}^{k \times d}$, processus qui doivent être adaptés par rapport à la filtration brownienne. C'est le point crucial dans cette théorie, car c'est cette hypothèse qui « fait office » de seconde équation.

Pour illustrer notre propos, considérons l'exemple où f est identiquement nul. Si l'on n'impose pas la condition d'être progressivement mesurable, une solution possible de l'EDSR (0.2) est $y_t = \xi$ et $z_t = 0$ pour tout t. Dans ce cas, le processus y est anticipatif. Si ξ est de carré intégrable, le processus adapté le plus proche de la solution précédente y au sens des moindres carrés, est la martingale $Y_t = \mathbb{E}(\xi | \mathcal{F}_t)$. Alors en utilisant le théorème de représentation des martingales browniennes, il existe un processus Z tel que

$$Y_t = \mathbb{E}(\xi | \mathcal{F}_t) = \mathbb{E}^{\mathcal{F}_t}(\xi) = \mathbb{E}(\xi) + \int_0^t Z_s dB_s = \xi - \int_t^T Z_s dB_s.$$

Ainsi apparaissent la seconde inconnue Z et la «forme» d'une EDSR.

1.1 Notations et définition d'une solution

Fixons d'abord quelques notations. Pour $x,x' \in \mathbb{R}^k$, $|x|$ désigne la norme euclidienne et $\langle x,x' \rangle$ le produit scalaire. Une matrice de taille $k \times d$ est considérée comme un élément de $\mathbb{R}^{k\times d}$; sa norme euclidienne est donnée par $\|y\| = \sqrt{\mathrm{Trace}(yy^*)}$.

Pour tout réel $p > 0$, nous allons définir deux espaces fonctionnels $\mathcal{S}^p(\mathbb{R}^n)$ et $\mathcal{M}^p(\mathbb{R}^n)$. $\mathcal{S}^p(\mathbb{R}^n)$ est l'ensemble des processus $\{X_t\}_{t\in[0,T]}$, continus, à valeurs dans $\mathbb{R}^n$, adaptés, tels que

$$\|X\|_{\mathcal{S}^p} = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right]^{1\wedge 1/p} < +\infty.$$

Si $p \geq 1$, $\|.\|_{\mathcal{S}^p}$ est une norme sur $\mathcal{S}^p(\mathbb{R}^n)$ qui en fait un espace de Banach. Si $p \in]0,1[$, $(X,X') \mapsto \|X - X'\|_{\mathcal{S}^p}$ est une distance sur $\mathcal{S}^p(\mathbb{R}^n)$. Avec cette métrique, $\mathcal{S}^p$ est un espace complet.

$\mathcal{M}^p(\mathbb{R}^n)$ désigne l'ensemble des processus $\{X_t\}_{t\in[0,T]}$ prévisibles à valeurs dans $\mathbb{R}^n$, tels que

$$\|X\|_{\mathcal{M}^p} = \mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right]^{1\wedge 1/p} < +\infty.$$

Si $p \geq 1$, $\mathcal{M}^p(\mathbb{R}^n)$ est un espace de Banach avec la norme $\|.\|_{\mathcal{M}^p}$ et si $p \in]0,1[$, $(X,X') \mapsto \|X - X'\|_{\mathcal{M}^p}$ est une distance sur $\mathcal{M}^p(\mathbb{R}^n)$ qui le rend complet.

Le générateur f est défini sur $\Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ et sera toujours supposé mesurable par rapport à $\mathcal{F} \times \mathcal{B}([0,T]) \times \mathcal{B}(\mathbb{R}^k) \times \mathcal{B}(\mathbb{R}^{k\times d})$. De plus pour tout $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, le processus $\{f(t,y,z)\}_{t\in[0,T]}$ est progressivement mesurable.

Nous définissons maintenant ce qu'est une solution.

Définition 1.1 *Une solution de l'EDSR (0.2) est un couple*

$$(Y,Z) = \{(Y_t,Z_t)\}_{t\in[0,T]}$$

de processus progressivement mesurables, à valeurs dans $\mathbb{R}^k \times \mathbb{R}^{k\times d}$*, tel que,* $\mathbf{P}$*-p.s. :*

- $t \mapsto Z_t$ *est dans* $L^2([0,T])$ *;*
- $t \mapsto f(t,Y_t,Z_t)$ *est dans* $L^1([0,T])$ *;*
- *et*

$$Y_t = \xi + \int_t^T f(r,Y_r,Z_r)\, dr - \int_t^T Z_r dB_r, \quad 0 \leq t \leq T.$$

Remarque 1.1 *Comme les intégrales de l'équation précédente sont bien définies,* Y *est une semi-martingale continue ; ensuite comme* Y *est un processus progressivement mesurable, il est adapté, et en particulier* Y_0 *est déterministe.*

1.2 Existence et unicité des solutions

En 1990, E. Pardoux et S. Peng ([43]) ont démontré l'existence et l'unicité des solutions de l'EDSR (0.2) dans le cas où le générateur f est lipschitzien par rapport aux deux variables y et z.

A partir de ce travail, il y a eu un grand nombre d'articles affinant les hypothèses du théorème de Pardoux et Peng. Ce chapitre n'a pas pour ambition de fournir une liste exhaustive de tous ces résultats. Nous allons nous limiter à ceux qui reviendront dans la suite de cette thèse. Pour plus de références, on pourra consulter la bibliographie des articles [19] ou [42].

1.2.1 Le théorème de E. Pardoux et S. Peng

Nous donnons maintenant un énoncé légèrement différent (que l'on peut trouver dans [42]) du théorème 3.1 de E. Pardoux et S. Peng ([43], 1990) :

Théorème 1.1 *Sous les hypothèses suivantes : il existe une constante $K > 0$ telle que*

1. $\xi \in L^2(\Omega,\mathcal{F}_T)$ *;*
2. $\{f(t,0,0)\}_{0\le t\le T} \in L^2(\Omega \times [0,T])$ *;*
3. f *est uniformément lipschitzienne en* y *et en* z*:*

$$\forall (t,y,y',z,z') \in [0,T]\times\left(\mathbb{R}^k\right)^2\times\left(\mathbb{R}^{k\times d}\right)^2,\ |f(t,y,z)-f(t,y',z')| \le K(|y-y'|+\|z-z'\|);$$

l'EDSR (0.2) admet une unique solution telle que $Z \in \mathcal{M}^2(\mathbb{R}^{k\times d})$.

Un outil-clé dans la démonstration est le théorème de représentation des martingales browniennes (voir [22], théorème III.4.15).

Preuve. Soit $\mathcal{B}^2 = \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k\times d})$, qui est un espace de Banach. On construit une fonction Φ de $\mathcal{B}^2$ dans lui-même, telle que $(Y,Z) \in \mathcal{B}^2$ est une solution de l'EDSR (0.2) si et seulement si (Y,Z) est un point fixe de Φ.

Pour $(U,V) \in \mathcal{B}^2$, on définit $(Y,Z) = \Phi(U,V)$ de la façon suivante. Considérons la variable aléatoire :

$$\zeta = \xi + \int_0^T f(s,U_s,V_s)ds.$$

ζ est de carré intégrable car f est lipschitzienne en y et z, d'où :

$$|\zeta|^2 \le 4\left(|\xi|^2 + \int_0^T |f(r,0,0)|^2dr + K^2 \sup_{0\le t\le T} |U_t|^2 + K^2 \int_0^T |V_r|^2dr\right).$$

Pour Y on choisit :

$$\forall t \in [0,T], \quad Y_t = \mathbb{E}^{\mathcal{F}_t}\left[\xi + \int_t^T f(s,U_s,V_s)ds\right].$$

D'après ce qui précède, cette espérance conditionnelle est bien définie. De plus le théorème de représentation des martingales appliqué à la martingale $\left\{\mathbb{E}^{\mathcal{F}_t}(\zeta)\right\}_{t\in[0,T]}$, permet d'affirmer qu'il existe un processus $Z \in \mathcal{M}^2(\mathbb{R}^{k\times d})$ tel que :

$$\forall t \in [0,T], \quad \mathbb{E}^{\mathcal{F}_t}(\zeta) = \mathbb{E}(\zeta) + \int_0^t Z_r dB_r.$$

Ainsi est construit $(Y,Z) = \Phi(U,V)$. De l'égalité précédente on déduit :

$$Y_t + \int_0^t f(r,U_r,V_r)dr = Y_0 + \int_0^t Z_r dB_r$$

d'où

$$Y_t = \xi + \int_t^T f(r,U_r,V_r)dr - \int_t^T Z_r dB_r.$$

Donc (Y,Z) est solution de (0.2) si et seulement si c'est un point fixe de Φ.

Montrons maintenant que $Y \in \mathcal{S}^2(\mathbb{R}^k)$. Pour tout $0 \leq t \leq T$,

$$Y_t = Y_0 - \int_0^t f(r,U_r,V_r)dr + \int_0^t Z_r dB_r.$$

Comme f est lipschitzienne en y et z,

$$|Y_t| \leq |Y_0| + \int_0^T |f(r,0,0)|dr + K\int_0^T |U_r|dr + K\int_0^T \|V_r\|\, dr + \sup_{0\leq t\leq T}\left|\int_0^t Z_r dB_r\right|.$$

Y_0 est déterministe et comme $(U,V) \in \mathcal{B}^2$, Z étant dans $\mathcal{M}^2$ par construction,

$$\eta = |Y_0| + \int_0^T |f(r,0,0)| + \sup_{0\leq t\leq T}\left|\int_0^t Z_r dB_r\right| + K\int_0^T |U_r|dr + K\int_0^T \|V_r\|\, dr$$

est de carré intégrable. Ainsi

$$\sup_{0\leq t\leq T} |Y_t| \leq \eta,$$

ce qui prouve que Y est dans $\mathcal{S}^2(\mathbb{R})$. Par construction $Z \in \mathcal{M}^2$.

Pour trouver un point fixe, il suffit de prouver que Φ est une contraction stricte de $\mathcal{B}^2$. Pour celà, il faut d'abord remarquer que si $(Y,Z) \in \mathcal{B}^2$, alors $\left\{\int_0^t \langle Y_s, Z_s dB_s\rangle,\ 0 \leq t \leq T\right\}$ est une martingale. En effet les inégalités de Burkholder-Davis-Gundy donnent :

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \langle Y_s, Z_s dB_s\rangle\right|\right] &\leq C\mathbb{E}\left[\left(\int_0^T |Y_r|^2\|Z_r\|^2 dr\right)^{1/2}\right] \\
&\leq C\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_r|\left(\int_0^T \|Z_r\|^2 dr\right)^{1/2}\right] \\
&\leq \frac{C}{2}\left(\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_r|^2\right] + \mathbb{E}\int_0^T \|Z_r\|^2 dr\right) < +\infty.
\end{aligned}$$

Ensuite on prend (U,V) et (U',V') dans $\mathcal{B}^2$, $(Y,Z) = \Phi(U,V)$, $(Y',Z') = \Phi(U',V')$, et on note $(\overline{U},\overline{V}) = (U - U', V - V')$ et $(\overline{Y},\overline{Z}) = (Y - Y', Z - Z')$. La formule d'Itô entraîne que pour tout $\gamma \in \mathbb{R}$ (l'intégrale stochastique est d'espérance nulle) :

$$\begin{aligned}
&e^{\gamma t}\mathbb{E}|\overline{Y}_t|^2 + \mathbb{E}\int_t^T e^{\gamma r}(\gamma|\overline{Y}_r|^2 + \|\overline{Z}_t\|^2)dr \\
&\leq 2K\mathbb{E}\int_t^T e^{\gamma r}|\overline{Y}_r|(|\overline{U}_r| + \|\overline{V}_r\|)dr \\
&\leq 4K^2\mathbb{E}\int_t^T e^{\gamma r}|\overline{Y}_r|^2 dr + \frac{1}{2}\mathbb{E}\int_t^T e^{\gamma r}(|\overline{U}_r|^2 + \|\overline{V}_r\|^2)dr.
\end{aligned}$$

On choisit $\gamma = 1 + 4K^2$, donc

$$\mathbb{E}\int_0^T e^{\gamma r}(|\overline{Y}_r|^2 + \|\overline{Z}_r\|^2)dr \leq \frac{1}{2}\mathbb{E}\int_0^T e^{\gamma r}(|\overline{U}_r|^2 + \|\overline{V}_r\|^2)dr.$$

Ainsi Φ est une contraction stricte de $\mathcal{B}^2$ muni de la norme :

$$|||(Y,Z)||| = \left(\mathbb{E}\int_0^T e^{\gamma t}(\|Y_t\|^2 + \|Z_t\|^2)dt\right)^{1/2}$$

si $\gamma = 1 + 4K^2$. L'espace $\mathcal{B}^2$ muni de cette norme est un espace de Banach. Donc Φ admet un unique point fixe, qui est l'unique solution de l'EDSR (0.2) dans $\mathcal{B}^2$. □

Dans l'énoncé du théorème, on ne précise pas l'espace dans lequel se trouve Y. Mais dans la démonstration on a vu que $Y \in \mathcal{S}^2$. En fait il s'avère que sous une hypothèse assez faible concernant le générateur f, le processus Y est dans $\mathcal{S}^2$ dès que $Z \in \mathcal{M}^2$.

Proposition 1.1 *Supposons qu'il existe un processus $\{f_t\}_{t\in[0,T]}$ positif, dans $\mathcal{M}^2(\mathbb{R})$ et une constante $\lambda > 0$ tels que :*

$$\forall (t,y,z) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}, \quad |f(t,y,z)| \leq \lambda\left(f_t + |y| + \|z\|\right).$$

Si (Y,Z) est solution de l'EDSR (0.2) telle que $Z \in \mathcal{M}^2(\mathbb{R}^{k\times d})$, alors $Y \in \mathcal{S}^2(\mathbb{R}^k)$.

Preuve. Pour tout $0 \leq t \leq T$,

$$Y_t = Y_0 - \int_0^t f(r,Y_r,Z_r)dr + \int_0^t Z_r dB_r$$

ainsi

$$|Y_t| \leq |Y_0| + \lambda\int_0^T \left(|f_r| + \|Z_r\|\right)dr + \sup_{0\leq t\leq T}\left|\int_0^t Z_r dB_r\right| + \lambda\int_0^t |Y_r|\,dr.$$

Y_0 étant déterministe, les hypothèses faites sur les processus Z, $\{f_t\}_{t\in[0,T]}$ impliquent que

$$\zeta = |Y_0| + \lambda\int_0^T \left(|f_r| + \|Z_r\|\right)dr + \sup_{0\leq t\leq T}\left|\int_0^t Z_r dB_r\right|$$

est de carré intégrable. Comme Y est continu, le lemme de Gronwall entraîne que

$$\sup_{0\leq t\leq T}|Y_t| \leq \zeta e^{\lambda T},$$

ce qui prouve que Y est dans $\mathcal{S}^2(\mathbb{R})$. □

Notons que sous les conditions du théorème 1.1, la proposition précédente s'applique et donc $Z \in \mathcal{M}^2$ implique $Y \in \mathcal{S}^2(\mathbb{R}^k)$. Ainsi il n'est pas nécessaire de préciser dans l'énoncé du théorème, l'espace auquel Y appartient.

En revanche cette condition d'intégrabilité sur Z est capitale car c'est elle qui assure l'unicité de la solution. Revenons sur le cas simple en dimension 1 où f est identiquement nul et ξ est de carré intégrable. La solution du théorème 1.1 est donnée pour tout $t \in [0,T]$ par $Y_t = \mathbb{E}^{\mathcal{F}_t}(\xi)$ et Z provient du théorème de représentation des martingales.

Mais d'après l'article de Dudley [15] (voir également [22]), pour tout $T > 0$, il existe un processus progressivement mesurable ψ tel que si pour $0 \leq t \leq T$ $I_t = \int_0^t \psi_r dB_r$, alors :

$$I_0 = 1, \quad I_T = 0 \quad \text{et} \quad \int_0^T \|\psi_r\|^2 dr < +\infty, \quad \mathbf{P} - \text{p.s.}$$

Bien sûr, ψ ne peut pas appartenir à $\mathcal{M}^2$ sinon I serait une martingale. Mais $(Y + \lambda I, Z + \lambda\psi)$ est une autre solution de la même EDSR (au sens de la définition 1.1) et on a ainsi construit une infinité de solutions.

Dans la suite, quand nous écrirons «la solution» sans précision, nous sous-entendrons toujours l'unique solution qui vérifie une condition d'intégrabilité donnée ; par exemple $Z \in \mathcal{M}^2$ (et $Y \in \mathcal{S}^2$).

On a déjà vu sur l'exemple où f est nul que l'intégrale stochastique ou le terme Z, qui apparaissent dans (0.2), sont là pour adapter Y à la filtration du mouvement brownien, ce que confirme encore la proposition suivante.

Proposition 1.2 *Soit $\{(Y_t,Z_t)\}_{t\in[0,T]}$ la solution de l'EDSR (0.2). Supposons que pour un temps d'arrêt $\tau \leq T$ p.s. :*

- *ξ est $\mathcal{F}_\tau$-mesurable ;*
- *$f(t,y,z) = 0$ sur l'intervalle $]\tau,T[$.*

Alors $Y_t = Y_{t\wedge\tau}$ et $Z_t = 0$ sur cet intervalle $]\tau,T[$.

En bref si Z n'est pas nécessaire pour adapter la solution, Z est nul.

Preuve. Comme

$$Y_\tau = \xi - \int_\tau^T Z_r dB_r,$$

$$Y_\tau = \mathbb{E}\left(\xi|\mathcal{F}_\tau\right) = \xi.$$

D'autre part

$$|Y_\tau|^2 + \int_\tau^T \|Z_r\|^2 dr = |\xi|^2 - 2\int_\tau^T \langle Y_r, Z_r dB_r\rangle.$$

Donc

$$|Y_\tau|^2 + \mathbb{E}^{\mathcal{F}_\tau} \int_\tau^T \|Z_r\|^2 dr = |\xi|^2,$$

et ainsi $\int_\tau^T \|Z_r\|^2 dr = 0$ p.s. □

De même si ξ et $f(t,y,z)$ sont déterministes, alors la même démonstration prouve que $Z \equiv 0$ et Y est la solution de l'équation différentielle ordinaire

$$\frac{dY_t}{dt} = -f(t,Y_t,0); \qquad Y_T = \xi.$$

Et dans ce cas, le théorème 1.1 est le théorème de Cauchy-Lipschitz.

1.2.2 Equations linéaires et théorème de comparaison

Les équations linéaires sont celles qui sont apparues les premières en 1973 dans un article de J.M. Bismut ([6]), en théorie du contrôle stochastique (voir par exemple [19], section 3). Comme pour les équations différentielles ordinaires, si f est linéaire, on peut donner une formule explicite de la solution de l'EDSR.

Proposition 1.3 *Soit (α,β) deux processus bornés à valeurs dans $(\mathbb{R},\mathbb{R}^d)$, ϕ un processus dans $\mathcal{M}^2(\mathbb{R})$ et ξ un élément de $L^2(\Omega,\mathcal{F}_T)$ à valeurs réelles.*

L'EDSR linéaire

$$Y_t = \xi + \int_t^T (\alpha_r Y_r + \beta_r Z_r + \phi_r)\, dr - \int_t^T Z_r dB_r, \quad 0 \le t \le T$$

admet une unique solution telle que $Z \in \mathcal{M}^2(\mathbb{R}^d)$, dont Y est donné par la formule suivante

$$Y_t = \Gamma_t^{-1} \mathbb{E}^{\mathcal{F}_t} \left[\xi \Gamma_T + \int_t^T \phi_r \Gamma_r dr \right] \tag{1.1}$$

avec

$$\Gamma_t = \exp\left(\int_0^t \beta_r . dB_r - \frac{1}{2} \int_0^t |\beta_r|^2 dr + \int_0^t \alpha_r dr \right).$$

Preuve. Il faut remarquer d'abord que le processus Γ vérifie :

$$d\Gamma_t = \Gamma_t(\alpha_t dt + \beta_t dB_t), \quad \Gamma_0 = 1$$

et appartient à $\mathcal{S}^2(\mathbb{R})$.

De plus, les hypothèses de cette proposition assurent l'existence et l'unicité d'une solution (Y,Z) à l'EDSR linéaire telle que $Z \in \mathcal{M}^2$. Il suffit de poser $f(t,y,z) = \alpha_t y + \beta_t z + \phi_t$. Par la formule d'Itô on obtient immédiatement que le processus $\Gamma_t Y_t + \int_0^t \phi_r \Gamma_r dr$ est une martingale locale qui est en fait une martingale car $\phi \in \mathcal{M}^2$, et Γ et Y sont dans $\mathcal{S}^2$.

Par suite,

$$\Gamma_t Y_t + \int_0^t \phi_r \Gamma_r dr = \mathbb{E}^{\mathcal{F}_t} \left(\Gamma_T Y_T + \int_0^T \phi_r \Gamma_r dr \right),$$

ce qui donne la formule annoncée. □

Si tous les coefficients de l'EDSR linéaire sont déterministes, ainsi que la donnée finale ξ, alors

$$Z \equiv 0 \quad \text{et} \quad Y_t = \Gamma_t^{-1} \mathbb{E}^{\mathcal{F}_t} \left[\xi \Gamma_T + \int_t^T \phi_r \Gamma_r dr \right] \quad \text{avec } \Gamma_t = \exp\left(\int_0^t \alpha_r dr \right).$$

Pour Y, on retrouve la formule classique de la solution d'une équation différentielle ordinaire linéaire. La première partie correspond à l'équation sans second membre, la seconde partie s'obtenant par la méthode dite de «variation de la constante».

On peut remarquer que si ϕ et ξ sont positifs, alors (1.1) montre que Y est positif. Ceci permet de prouver le théorème de comparaison 1.2, démontré pour la première fois par Peng [47] et qui joue un rôle fondamental dans la résolution d'EDSR en dimension 1 et plus particulièrement dans mon travail.

Théorème 1.2 *Supposons que* $\xi \leq \xi'$ *p.s. et* $f(t,y,z) \leq f'(t,y,z)$ $dt \times d\mathbf{P}$ *p.p. Alors* $Y_t \leq Y'_t$, $0 \leq t \leq T$, *p.s.*

Si de plus $Y_0 = Y'_0$, *alors* $Y_t = Y'_t$, $0 \leq t \leq T$, *p.s. En particulier si* $\mathbf{P}(\xi < \xi') > 0$ *ou* $f(t,y,z) < f'(t,y,z)$, $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$, *sur un ensemble de mesure* $dt \times d\mathbf{P}$ *strictement positive, alors* $Y_0 < Y'_0$.

Preuve. Il suffit de montrer que $(\overline{Y},\overline{Z}) = (Y' - Y, Z' - Z)$ est solution d'une EDSR linéaire avec condition finale $\overline{\xi} = \xi' - \xi$ et $\phi_t = f'(t,Y'_t,Z'_t) - f(t,Y'_t,Z'_t)$. On définit

$$\alpha_t = \begin{cases} (Y'_t - Y_t)^{-1}(f(t,Y'_t,Z'_t) - f(t,Y_t,Z'_t)) & \text{si } Y_t \neq Y'_t \\ 0 & \text{si } Y_t = Y'_t; \end{cases}$$

et le processus β à valeurs dans $\mathbb{R}^d$ comme suit. Pour $1 \leq i \leq d$, $Z_t^{(i)}$ désigne le vecteur de dimension d dont les i premières coordonnées sont égales à celles de Z'_t, et dont les $d-i$ dernières à celles de Z_t. Avec cette notation, on définit pour tout $1 \leq i \leq d$,

$$\beta_t^i = \begin{cases} (Z_t'^i - Z_t^i)^{-1}(f(t,Y_t,Z_t^{(i)}) - f(t,Y_t,Z_t^{(i-1)})) & \text{si } Z_t^i \neq Z_t'^i \\ 0 & \text{si } Z_t^i \neq Z_t'^i. \end{cases}$$

Les processus α et β sont progressivement mesurables et comme f est lipschitzienne en y et z, $|\alpha_t| \leq K$ et $|\beta_t| \leq K$. $(\overline{Y},\overline{Z})$ est solution de l'EDSR :

$$\overline{Y}_t = \overline{\xi} + \int_t^T \left(\alpha_r \overline{Y}_r + \beta_r \overline{Z}_r + \phi_r\right) dr - \int_t^T \overline{Z}_r dB_r.$$

La conclusion vient de la positivité de $\overline{\xi}$ et ϕ. □

Rappelons que cette propriété ne peut être obtenue dans le cas des équations différentielles stochastiques progressives que sous des conditions restrictives sur les coefficients. En particulier les coefficients de diffusion doivent être les mêmes dans les deux équations (voir [22]).

1.2.3 Cadre monotone et théorie L^p

Si nous avons en tête l'énoncé du théorème de Cauchy-Lipschitz, une généralisation du théorème 1.1 consisterait à affaiblir l'hypothèse concernant la régularité de f par rapport à y. Dans le cadre des équations différentielles ordinaires, l'hypothèse f uniformément lipschitzienne n'est pas nécessaire pour obtenir l'existence et l'unicité des solutions.

En dimension 1, on peut tirer parti du théorème de comparaison : par exemple, existence d'une solution (mais perte de l'unicité) avec f simplement continue (cf. [31]), ce qui correspond au théorème de Cauchy-Peano sur les EDO, ou existence et unicité avec f à croissance quadratique en z (cf. [24], [32], [33] ou paragraphe suivant). En dimension quelconque il n'y a pas de comparaison possible. Pour généraliser un peu le cas lipschitzien, on peut utiliser une hypothèse de monotonie sur f.

C'est la différence principale entre le théorème 1.1 cité précédemment et le théorème 2.2 de [42] :

Théorème 1.3 *Sous les hypothèses suivantes : il existe une fonction* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *continue et croissante et deux constantes* $\mu \in \mathbb{R}$ *et* $K > 0$ *telles que*

1. $\xi \in L^2(\Omega,\mathcal{F}_T)$;
2. $|f(t,y,0)| \leq |f(t,0,0)| + \phi(|y|)$, $\forall t,y$, *p.s.* ;
3. $\{f(t,0,0)\}_{0\leq t\leq T} \in L^2(\Omega \times [0,T])$;
4. f *est uniformément lipschitzienne en* z :
$$\forall (t,y,z,z') \in [0,T] \times \mathbb{R}^k \times \left(\mathbb{R}^{k\times d}\right)^2, \quad |f(t,y,z) - f(t,y,z')| \leq K\|z - z'\|;$$
5. f *est "monotone" en* y :
$$\forall (t,y,y',z) \in [0,T] \times \left(\mathbb{R}^k\right)^2 \times \mathbb{R}^{k\times d}, \quad \langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \mu|y - y'|^2;$$
6. $y \mapsto f(t,y,z)$ *est continue pour tout* (t,z), *p.s.* ;

l'EDSR (0.2) admet une unique solution telle que $(Y,Z) \in \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k\times d})$.

Remarque 1.2 *L'hypothèse sur* f *de la proposition 1.1 n'est plus valable, ainsi* $Z \in \mathcal{M}^2(\mathbb{R}^{k\times d})$ *n'entraîne plus* $Y \in \mathcal{S}^2(\mathbb{R}^k)$. *Donc on doit ajouter cette condition dans l'énoncé du théorème.*

Si f *est lipschitzienne en* y *(avec constante* K*) comme dans le théorème 1.1, alors* f *est monotone avec* $\mu = K$ *et on peut prendre* $\phi(x) = Kx$.

Ce résultat a été prouvé également par Ph. Briand et R. Carmona [8] pour une fonction ϕ *polynômiale.*

Preuve du théorème. Nous ne donnons ici que les grandes lignes de la preuve.

L'unicité s'obtient aisément grâce au lemme de Gronwall. S'il existe deux solutions (Y,Z) et (Y',Z'), on applique la formule d'Itô à $|Y - Y'|^2$ (comme $(Y,Z) \in \mathcal{B}^2$, l'intégrale stochastique est de moyenne nulle) :

$$\begin{aligned}\mathbb{E}\,|Y_t - Y_t'|^2 + \int_t^T \|Z_r - Z_r'\|^2 dr &= 2\mathbb{E}\int_t^T \langle Y_r - Y_r', f(r,Y_r,Z_r) - f(r,Y_r',Z_r')\rangle dr \\ &\leq 2\mathbb{E}\int_t^T \left(\mu|Y_r - Y_r'|^2 + K|Y_r - Y_r'|\|Z_r - Z_r'\|\right) dr \\ &\leq (2\mu + K^2)\mathbb{E}\int_t^T |Y_r - Y_r'|^2 dr + \int_t^T \|Z_r - Z_r'\|^2 dr.\end{aligned}$$

Donc

$$\mathbb{E}\,|Y_t - Y_t'|^2 \leq (2\mu + K^2)\mathbb{E}\int_t^T |Y_r - Y_r'|^2 dr$$

et on conclut avec le lemme de Gronwall.

L'existence repose aussi sur un théorème de point fixe, comme pour le théorème 1.1. On munit $\mathcal{B}^2 = \mathcal{S}^2(\mathbb{R}^k) \times \mathcal{M}^2(\mathbb{R}^{k\times d})$ de la norme :

$$\|(Y,Z)\|_\gamma = \left(\mathbb{E}\int_0^T e^{\gamma t}(\|Y_t\|^2 + \|Z_t\|^2) dt\right)^{1/2}$$

avec $\gamma = 1 + 2K^2$. C'est un espace de Banach.

Supposons que l'on sache montrer le résultat suivant : si $V \in \mathcal{M}^2(\mathbb{R}^{k\times d})$, alors il existe une unique solution $(Y,Z) \in \mathcal{B}^2$ de l'EDSR :

$$(1.2) \qquad Y_t = \xi + \int_t^T f(r,Y_r,V_r)dr - \int_t^T Z_r dB_r, \quad 0 \le t \le T.$$

En utilisant le résultat précédent on construit une application $\Phi : \mathcal{B}^2 \to \mathcal{B}^2$ qui à $(U,V) \in \mathcal{B}^2$ associe $(Y,Z) = \Phi(U,V)$, (Y,Z) étant la solution de l'EDSR précédente (1.2). On montre que cette application est une contraction de $(\mathcal{B}^2,\|\|_\gamma)$, donc admet un unique point fixe, solution de l'EDSR (0.2).

La véritable difficulté réside dans la démonstration de l'existence et l'unicité de l'EDSR (1.2). L'unicité se prouve comme précédemment. C'est l'existence qui pose véritablement problème. En résumant énormément, disons qu'il faut approcher f par une suite de fonctions lipschitziennes pour utiliser le théorème 1.1 avec f_n, ce qui donne une suite de processus (Y^n,Z^n), et enfin prouver que cette suite converge vers (Y,Z), solution de l'EDSR (1.2). C'est la convergence de (Y^n,Z^n) qui est difficile à établir. □

La proposition 1.3, avec maintenant l'hypothèse $\alpha_t \le \mu$ pour tout t, et le théorème 1.2 restent valables.

Dans la suite du travail, la dimension sera égale à 1 et le générateur f sera de la forme

$$\forall y \in \mathbb{R},\ f(y) = -y|y|^q \quad \text{avec } q > 0.$$

f n'est pas lipschitzienne, mais satisfait l'hypothèse de monotonie avec $\mu = 0$.

Une autre amélioration possible du théorème 1.1 concerne la donnée finale ξ. Dans l'article [19], N. El Karoui, S. Peng et M.-C. Quenez ont montré qu'il existe une unique solution de l'EDSR (0.2), dans le cas où f est lipschitzienne, mais où ξ et $\{f(t,0,0)\}_{t\in[0,T]}$ sont dans L^p avec $p \in]1,2[$. Récemment Ph. Briand, B. Delyon, Y. Hu, E. Pardoux et L. Stoica dans [9] ont étendu ce résultat pour f monotone et $\xi \in L^p(\Omega)$. Il faut alors distinguer les cas $p > 1$ et $p = 1$. Dans le premier cas :

Théorème 1.4 *Soit $p > 1$. Supposons que les conditions suivantes sont satisfaites :*
(H1) $\xi \in L^p(\Omega,\mathcal{F}_T)$;
(H2)

$$\mathbb{E}\left[\left(\int_0^T |f(t,0,0)|dt\right)^p\right] < +\infty;$$

(H3) il existe $K > 0$ telle que

$$\forall (t,y,z,z') \in [0,T] \times \mathbb{R}^k \times \left(\mathbb{R}^{k\times d}\right)^2, \quad |f(t,y,z) - f(t,y,z')| \le K\|z - z'\|;$$

(H4) il existe $\mu \in \mathbb{R}$ telle que

$$\forall (t,y,y',z) \in [0,T] \times \left(\mathbb{R}^k\right)^2 \times \mathbb{R}^{k\times d}, \quad \langle y - y', f(t,y,z) - f(t,y',z)\rangle \le \mu|y - y'|^2;$$

(H5) $y \mapsto f(t,y,z)$ est continue pour tout (t,z), p.s. ;

(H6) pour tout $r > 0$

$$\psi_r(t) = \sup_{|y|\le r} |f(t,y,0) - f(t,0,0)| \in L^1([0,T] \times \Omega).$$

Alors l'EDSR (0.2) admet une unique solution dans $\mathcal{S}^p(\mathbb{R}^k) \times \mathcal{M}^p(\mathbb{R}^{k\times d})$.

La démonstration repose sur le théorème précédent 1.3 et sur l'estimation a priori suivante

Proposition 1.4 *Sous les hypothèses du théorème 1.4, si (Y,Z) est solution de l'EDSR (0.2) avec $Y \in \mathcal{S}^p$, alors il existe une constante C qui ne dépend que de p, telle que pour $a > \mu + K^2/[1 \wedge (p-1)]$:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} e^{apt}|Y_t|^p + \left(\int_0^T e^{ar}\|Z_r\|^2 dr\right)^{p/2}\right] \le C\mathbb{E}\left[e^{apT}|\xi|^p + \left(\int_0^T e^{ar}|f(r,0,0)|dr\right)^p\right].$$

Les preuves de la proposition et du théorème sont trop longues pour être reproduites ici entièrement. Soulignons simplement qu'une des difficultés nouvelles par rapport au théorème 1.3 est que l'on ne peut plus appliquer directement la formule d'Itô à $|Y|^p$ quand $p < 2$, la fonction $x \to |x|^p$ n'étant pas de classe C^2.

Dans le cas $p = 1$, il y a des difficultés supplémentaires ; la proposition 1.4 par exemple n'est plus valable. Il faut changer un peu les hypothèses. On note Σ_T l'ensemble de tous les temps d'arrêt $\tau \le T$; un processus $(Y_t)_{t\in[0,T]}$ appartient à la classe (D) si la famille $\{Y_\tau\}_{\tau\in\Sigma_T}$ est uniformément intégrable. A un processus Y dans la classe (D), on associe

$$\|Y\|_1 = \sup\{\mathbb{E}(|Y_\tau|);\ \tau \in \Sigma_T\}.$$

L'espace des processus progressivement mesurables et continus appartenant à (D) est complet avec cette norme.

On ajoute aussi sur le générateur f l'hypothèse suivante : il existe deux constantes $\gamma > 0$ et $\alpha \in]0,1[$ et un processus $\{g_t\}_{t\in[0,T]}$ progressivement mesurable et positif tels que :

(H7) $$\forall(t,y,z) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}, \quad |f(t,y,z) - f(t,y,0)| \le \gamma(g_t + |y| + |z|)^\alpha.$$

Cette condition est automatiquement satisfaite si f ne dépend pas de z. Enfin on suppose à présent

(H1') $$\mathbb{E}\left[|\xi| + \int_0^T (|f(r,0,0)| + g_r)\,dr\right] < +\infty.$$

L'unicité est prouvée par :

Théorème 1.5 *Sous les hypothèses (H3)-(H4)-(H5)-(H6) du théorème 1.4 et (H1')-(H7), l'EDSR (0.2) admet au plus une solution telle que Y appartienne à (D) et Z soit dans $\bigcup_{\beta>\alpha} \mathcal{M}^\beta$.*

Pour l'existence on a :

Théorème 1.6 *Toujours avec les conditions (H1')-(H3)-(H4)-(H5)-(H6)-(H7), l'EDSR (0.2) admet une solution telle que* Y *appartienne à (D). De plus pour tout* $\beta \in]0,1[$, $(Y,Z) \in \mathcal{S}^\beta \times \mathcal{M}^\beta$.

Ainsi si on revient sur l'exemple où $f(y) = -y|y|^q$, en combinant les théorèmes 1.4, 1.5 et 1.6, on obtient que pour tout ξ dans $L^p(\Omega)$ pour un certain $p \geq 1$, l'EDSR

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r, \quad 0 \leq t \leq T \tag{2.1}$$

admet une unique solution dans un "bon" espace, que l'on ne précise pas ici.

1.2.4 Le théorème de M. Kobylanski

Revenons sur notre dernier exemple. La dimension k est égale à 1, $q > 0$ est un réel fixé et nous supposons qu'il existe deux constantes $K \geq \alpha > 0$ telle que $\alpha \leq \xi \leq K$ **P**-p.s. Soit (Y,Z) la solution de l'EDSR

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r.$$

Nous verrons dans la section II.3.2, que grâce au théorème de comparaison (1.2), il existe une constante $\beta > 0$ telle que pour tout $0 \leq t \leq T$, $\beta \leq Y_t \leq K$. Donc on peut appliquer la formule d'Itô à Y^{-q}. Il suffit de considérer la fonction qui à $x \in [\beta, +\infty[$ associe x^{-q}, fonction que l'on prolonge sur $\mathbb{R}$ tout entier de manière à obtenir une fonction de classe C^2. On obtient :

$$\frac{1}{(Y_t)^q} = \frac{1}{\xi^q} + q(T-t) - \frac{q(q+1)}{2} \int_t^T \frac{\|Z_s\|^2}{(Y_s)^{q+2}} ds + \int_t^T \frac{qZ_s}{(Y_s)^{q+1}} dB_s$$

Ainsi le processus

$$(U,V) = \left(\frac{1}{Y^q}, \frac{-qZ}{Y^{q+1}} \right)$$

est dans $L^\infty([0,T] \times \Omega) \times L^2([0,T] \times \Omega)$ et satisfait :

$$U_t = \zeta + \int_t^T F(s,U_s,V_s) ds - \int_t^T V_s dB_s, \tag{1.3}$$

avec

$$\zeta = \xi^{-1/q} \quad \text{et} \quad F(t,y,z) = q - \frac{q+1}{2q} \frac{|z|^2}{y \vee 1/K^q}.$$

Le problème est que dans cette EDSR (1.3) le générateur F n'est absolument pas litschitzien en z. Donc aucun des théorèmes précédents ne s'applique. Mais il est à croissance quadratique en z, ce qui est l'objet du théorème de Kobylanski [24].

On considère l'EDSR (0.2) en dimension $k = 1$ et on suppose que le générateur f est continu et vérifie la condition suivante :

$$f(t,y,z) = a_0(t,y,z)y + F_0(t,y,z),$$

avec $\mathbf{P}$-p.s.

$$\beta_0 \leq a_0(t,y,z) \leq \alpha_0 \quad \text{et} \quad |F_0(t,y,z)| \leq b + c(|y|)|z|^2.$$

Le théorème 2.3 de [24] contient le résultat suivant

Théorème 1.7 *Si f vérifie la condition précédente avec $c : \mathbb{R}_+ \to \mathbb{R}_+$ continue et croissante et si $\xi \in L^\infty(\Omega)$, alors l'EDSR (0.2) admet au moins une solution (Y,Z) telle que $Z \in \mathcal{M}^2(\mathbb{R}^d)$ et Y est presque sûrement borné : il existe $K \geq 0$ tel que p.s. $\sup_{t\in[0,T]} |Y_t| \leq K$.*

Le théorème 2.3 de [24] contient aussi le cas où T est un temps d'arrêt τ, τ étant ou borné p.s. ou fini avec $\alpha_0 < 0$; et il précise qu'il existe en fait une solution minimale et une solution maximale. Dans le cas où la croissance de f en y n'est plus linéaire, J.P. Lepeltier et J. San Martin ([32] et [33]) ont étendu le résultat de M. Kobylanski et montré qu'il existe un instant $t^* < T$ et une solution (Y,Z) définie uniquement sur $]t^*,T] \cap [0,T]$, telle que Y est bornée sur $[t,T]$ pour tout $t > t^*$. Si $t^* > 0$, la norme infinie de Y explose quand t tend vers t^* à droite.

Dans notre cas (1.3), il faut prendre $a_0 = 0$, $b = q$ et la fonction c est constante égale à $\frac{q+1}{2q}K^q$ et le théorème précédent s'applique et fournit une solution à l'EDSR (1.3). Se pose maintenant le problème de l'unicité de la solution.

Dans l'article [24], M. Kobylanski prouve un théorème de comparaison, qui entraîne l'unicité de la solution.

Théorème 1.8 *Sont donnés f^1,f^2 deux générateurs, et ξ^1,ξ^2 tels que :*
- *$\xi^1 \leq \xi^2$ p.s. et $f^1 \leq f^2$;*
- *f^1 ou f^2 vérifient : pour tout réel M et tout $\varepsilon > 0$ il existe des constantes b,c telles que pour tout $t \in [0,T]$, $|y| \leq M$ et $z \in \mathbb{R}^d$:*

$$|f(t,y,z)| \leq b + c|z|^2 \quad \text{et} \quad \left|\frac{\partial f}{\partial z}(t,y,z)\right| \leq b + c|z|,$$

$$\frac{\partial f}{\partial y}(t,y,z) \leq b + \varepsilon|z|^2.$$

Si (Y^1,Z^1) et (Y^2,Z^2) sont deux solutions de l'EDSR (0.2) avec paramètres respectivement ξ^1, f^1 et ξ^2, f^2, alors $Y^1_t \leq Y^2_t$ pour tout $t \in [0,T]$, $\mathbf{P}$-p.s.

Dans notre cas particulier, nous ne pouvons pas appliquer directement ce théorème. La dérivée de notre générateur F par rapport à y ne vérifie pas la condition imposée. Néanmoins il est facile de voir que si $(\tilde{U},\tilde{V}) \in \mathcal{S}^\infty \times \mathcal{M}^2$ est solution de l'EDSR (1.3) avec donnée finale ζ, alors $\tilde{U}_t \geq \mathbb{E}(\zeta|\mathcal{F}_t) + q(T-t)$. Donc $\tilde{U}$ est strictement positif sur $[0,T[$. De plus ici $\zeta = \xi^{-q} \geq K^{-q} > 0$. Donc on peut appliquer la formule d'Itô à $\tilde{U}^{-1/q}$ pour retrouver que $\tilde{U}^{-1/q}$ est solution de l'EDSR (2.1) avec donnée finale ξ. Ainsi dans le cas particulier de l'EDSR (1.3), il y a unicité dès qu'il existe une constante $\alpha > 0$ telle que $\zeta \geq \alpha$.

Dans la suite, nous utiliserons ces résultats sur les EDSR à croissance quadratique en z dans les paragraphes 2.2.2 et 3.3.1.

1.3 Application aux EDP, solutions de viscosité

Les applications de la théorie des EDSR sont nombreuses : mathématiques financières, théorie du contrôle... Mais ici nous allons nous intéresser plus particulièrement au lien qui existe avec les équations aux dérivées partielles.

La formule de Feynman-Kac, connue depuis longtemps maintenant, prouve qu'il y a une connection entre les solutions des équations différentielles stochastiques (ou EDS en abrégé) et certaines équations aux dérivées partielles du second ordre linéaires. Considérons $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ une solution régulière de l'équation de la chaleur, à laquelle on ajoute un terme linéaire :

$$\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u - ku + h = 0 \qquad \text{sur } [0,T[\times\mathbb{R}^d,$$
$$u(T,x) = g(x) \qquad x \in \mathbb{R}^d;$$

avec $g : \mathbb{R}^d \to \mathbb{R}$, $k : \mathbb{R}^d \to \mathbb{R}_+$ et $h : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ continues. u régulière veut dire que la dérivée de u en temps et les dérivées premières et secondes de u en espace existent et sont continues sur $[0,T[\times\mathbb{R}^d$. Notons que dans le problème précédent, la donnée g concerne u à l'instant T, ce qui explique le signe devant le laplacien Δ.

Théorème 1.9 (Feynman (1948), Kac (1949)) *Si u et h vérifient en plus :*

$$\max_{0\leq t\leq T} |u(t,x)| + \max_{0\leq t\leq T} |h(t,x)| \leq K e^{a\|x\|^2}; \quad \forall x \in \mathbb{R}^d$$

avec $K > 0$ et $0 < a < 1/2Td$, alors u s'exprime ainsi. Pour $t \in [0,T]$ et $x \in \mathbb{R}^d$:

$$\begin{aligned} u(t,x) &= \mathbb{E}\left[g(B^x_{T-t}) \exp\left\{ -\int_0^{T-t} k(B^x_s) ds \right\} \right. \\ &\quad \left. + \int_0^{T-t} h(t+r,B^x_r) \exp\left\{ -\int_0^r k(B^x_s) ds \right\} dr \right]. \end{aligned}$$

B^x est un mouvement brownien standard de dimension d, tel que $B_0 = x$.

La preuve de ce théorème, ainsi que l'extension de cette formule à d'autres diffusions que le mouvement brownien et donc à d'autres EDP linéaires, peuvent être trouvées dans [22], chapitres 4.4 et 5.7.

Il existe plusieurs généralisations de cette formule pour des EDP non linéaires. Dans la théorie du contrôle (voir par exemple [20]) la fonction-valeur d'un problème de contrôle stochastique optimal peut être identifié avec la solution d'une équation d'Hamilton-Jacobi-Bellman. Une autre approche probabiliste provient de la théorie des super processus : elle permet de relier la loi d'un superprocessus (processus de diffusion et de branchement) avec la solution d'une équation semi-linéaire (voir [16]).

La théorie des EDSR fournit une troisième approche des EDP non linéaires. Mais revenons d'abord sur le cas linéaire et l'équation de la chaleur. Avec un changement de variable, on peut réécrire la formule de Feynman-Kac ainsi :

$$u(t,x) = \mathbb{E}\left[g(B_T^{t,x})\exp\left\{-\int_t^T k(B_s^{t,x})ds\right\} + \int_t^T h(r,B_r^{t,x})\exp\left\{-\int_t^r k(B_s^{t,x})ds\right\}dr\right].$$

avec $B_r^{t,x} = B_r - B_t + x$ pour $r \geq t$.

Soit $(Y^{t,x},Z^{t,x})$ la solution de l'EDSR :

$$(1.4)\quad Y_r^{t,x} = g(B_T^{t,x}) - \int_r^T k(B_s^{t,x})Y_s^{t,x}ds + \int_r^T h(s,B_s^{t,x})ds - \int_r^T Z_r^{t,x}dB_r;\quad t \leq r \leq T.$$

Cette EDSR est linéaire. On applique la proposition 1.3 avec $\alpha_s = -k(B_s^{t,x}) \leq 0$, $\beta_s = 0$ et $\phi_s = h(s,B_s^{t,x})$. La condition imposée à h permet d'affirmer que $\phi \in \mathcal{M}^2$. Alors pour tout $t \leq r \leq T$:

$$Y_r^{t,x} = \mathbb{E}^{\mathcal{F}_r}\left[g(B_T^{t,x})\exp\left\{-\int_r^T k(B_s^{t,x})ds\right\} + \int_r^T h(\theta,B_\theta^{t,x})\exp\left\{-\int_r^\theta k(B_s^{t,x})ds\right\}d\theta\right].$$

On en déduit finalement

$$(1.5)\qquad \forall (t,x) \in [0,T]\times\mathbb{R}^d,\quad u(t,x) = Y_t^{t,x}.$$

Ainsi la solution de l'équation de la chaleur linéaire s'exprime en fonction de la solution d'une EDSR linéaire.

Les articles de Peng ([46], 1991) et de Pardoux et Peng ([44], 1992) ont montré que l'égalité (1.5) se généralise à bien d'autres EDP (ou systèmes d'EDP) paraboliques semi-linéaires, i.e. des équations de la forme suivante : pour tout $(t,x) \in [0,T]\times\mathbb{R}^m$

$$(1.6)\qquad \begin{cases} \partial_t u_i(t,x) + \mathcal{L}u_i(t,x) + f_i(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0, & 1 \leq i \leq k \\ u(T,x) = g(x); \end{cases}$$

l'opérateur $\mathcal{L}$ étant défini ainsi :

$$\mathcal{L} = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i\partial x_j} + \sum_i b_i\frac{\partial}{\partial x_i} = \frac{1}{2}\text{Trace}\left(\sigma\sigma^* D^2\right) + b.\nabla$$

Ici et dans toute la suite ∂_t, ∇ et D^2 désignent respectivement la dérivée première par rapport à la variable de temps t, le gradient et la matrice hessienne par rapport à la variable d'espace $x \in \mathbb{R}^m$.

Nous allons maintenant formuler les hypothèses qui assurent que la formule (1.5) reste valable, si on remplace le mouvement brownien par la solution d'une équation différentielle stochastique de coefficient de diffusion σ et de dérive b, et l'EDSR linéaire (1.4) par une EDSR avec un générateur f quelconque. Les résultats qui suivent se trouvent dans l'article [42]. Commençons par les coefficients de l'EDS.

Soit $b : [0,T]\times\mathbb{R}^m \to \mathbb{R}^m$, $\sigma : [0,T]\times\mathbb{R}^m \to \mathbb{R}^{m\times d}$ des fonctions mesurables qui satisfont les conditions suivantes :

(C1) σ est localement lipschitzienne : pour tout $n \in \mathbb{N}$, il existe une constante $K_n > 0$ telle que pour tout $(t,x,y) \in \mathbb{R}_+\times\mathbb{R}^m\times\mathbb{R}^m$ avec $|x| \leq n$ et $|y| \leq n$:

$$\|\sigma(t,x) - \sigma(t,y)\| \leq K_n|x-y|$$

(C2) b est localement monotone :

$$\langle b(t,x) - b(t,y), x - y\rangle \leq K_n |x - y|^2$$

(C3) $x \to b(t,x)$ est continue sur $\mathbb{R}^m$

(C4) et pour $K > 0$ et tout $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^m$

$$\langle b(t,x),x\rangle \leq K(1 + |x|^2), \qquad \|\sigma(t,x)\| \leq K(1 + |x|).$$

Sous ces hypothèses l'EDS suivante admet une seule solution (forte) $\{X^{t,x},\ t \leq s \leq T\}$:

$$X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dB_r,\ t \leq s \leq T. \tag{1.7}$$

On convient que $X_s^{t,x} = x$ pour $0 \leq s \leq t$. On pourra consulter [22], [49] ou [52] pour plus de détails concernant l'existence et l'unicité de cette solution. L'opérateur différentiel $\mathcal{L}$ est le générateur infinitésimal du processus de Markov $\{X^{t,x},\ t \leq s \leq T\}$.

Nous rappelons ici quelques propriétés du flot de cette EDS (voir [25]).

Proposition 1.5 *Soit $p \geq 1$. Il existe une constante C, qui ne dépend que de T et p, telle que*

$$\forall t \in [0,T],\ \forall x \in \mathbb{R}^m,\ \mathbb{E}\left(\sup_{s\in[0,T]} |X_s^{t,x}|^p\right) \leq C(1 + |x|^p).$$

Soit $2 \leq p < \infty$. Il existe une constante C telle que pour tout (t,x,s), (t',x',s')

$$\mathbb{E}\left(|X_s^{t,x} - X_{s'}^{t',x'}|^p\right) \leq C\left[|x - x'|^p + (1 + |x|^p)(|t - t'|^{p/2} + |s - s'|^{p/2})\right].$$

En particulier, les trajectoires $(t,x,s) \mapsto X_s^{t,x}$ sont höldériennes (localement en x) de paramètre β en t, α en x et β en s, avec $\beta < 1/2$ et $\alpha < 1$.

A l'EDS précédente on ajoute l'EDSR

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dB_r,\ t \leq s \leq T. \tag{1.8}$$

Le couplage des deux équations (1.7)-(1.8) forme une équation différentielle stochastique progressive-rétrograde (ou EDSPR en abrégé).

Les fonctions $g : \mathbb{R}^m \to \mathbb{R}^k$ et $f : [0,T] \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ sont continues et vérifient :

(C5) il existe $K \geq 0$, $p > 0$ et $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ continue et croissante tels que :

$$\begin{aligned} |g(x)| &\leq K(1 + |x|^p), \\ |f(r,x,y,z)| &\leq K(1 + |x|^p + \phi(|y|) + \|z\|), \end{aligned}$$

(C6) et il existe $K \geq 0$, $\mu \in \mathbb{R}$ tels que :

$$\begin{aligned} \langle f(r,x,y,z) - f(r,x,y',z), y - y'\rangle &\leq \mu|y - y'|^2, \\ |f(r,x,y,z) - f(r,x,y,z')| &\leq K\|z - z'\|. \end{aligned}$$

Avec ces conditions, en utilisant par exemple le théorème 1.3, l'EDSR (1.8) admet une unique solution $(Y^{t,x},Z^{t,x})$ telle que $Y^{t,x} \in \mathcal{S}^2(\mathbb{R}^k)$ et $Z^{t,x} \in \mathcal{M}^2(\mathbb{R}^{k\times d})$.

Si u est solution de l'EDP (1.6) et si (X,Y,Z) est solution du système (1.7)-(1.8), pour que la formule (1.5) soit vraie, il faut au moins que $Y_t^{t,x}$ soit déterministe. C'est l'objet de la proposition suivante. Pour $u \leq v$, $\mathcal{F}_v^u$ désigne la tribu engendrée par les accroissements de B, i.e. $\sigma\{B_w - B_u;\ u \leq w \leq v\}$.

Proposition 1.6 *Soit* $(t,x) \in [0,T] \times \mathbb{R}^m$. $\{X_s^{t,x},Y_s^{t,x}\}_{s\in[t,T]}$ *est adapté par rapport à la filtration* $\{\mathcal{F}_s^t\}_{t\leq s\leq T}$. *En particulier* $Y_t^{t,x}$ *est déterministe.*

De plus on peut prendre une version de $\{Z_s^{t,x}\}_{s\in[t,T]}$ *adaptée à la filtration* $\{\mathcal{F}_s^t\}_{t\leq s\leq T}$.

Alors pour tout $(t,x) \in [0,T] \times \mathbb{R}^k$ on peut définir la fonction

$$u(t,x) \triangleq Y_t^{t,x}.$$

Le résultat suivant est la propriété de Markov pour les EDSR et est prouvé dans [19].

Proposition 1.7 *Cette application u est continue et à croissance polynômiale, i.e. il existe $p \geq 0$ tel que*

$$\forall (t,x) \in [0,T] \times \mathbb{R}^k, \quad |u(t,x)| \leq C(1+|x|^p).$$

De plus pour $t \in [0,T]$ et $\theta \in L^{2p}(\Omega,\mathcal{F}_t)$ on a $\mathbf{P}$*-p.s.*

$$Y_t^{t,\theta} = u(t,\theta) \quad et \quad \forall s \in [t,T],\ Y_s^{t,\theta} = u(s,X_s^{t,\theta}).$$

Comme dans le cas linéaire, toute solution régulière de l'EDP fournit une solution de l'EDSR :

Théorème 1.10 *Soit $u \in C^{1,2}([0,T] \times \mathbb{R}^m;\mathbb{R}^k)$ une solution de (1.6) telle que pour un certain c et $q > 0$,*

$$|u(t,x)| + |\nabla u(t,x)| \leq c(1+|x|^q).$$

Alors pour tout $(t,x) \in [0,T] \times \mathbb{R}^m$, $\{u(s,X_s^{t,x}),(\nabla u\sigma)(s,X_s^{t,x});\ t \leq s \leq T\}$ est la solution de l'EDSR (1.8). En particulier $u(t,x) = Y_t^{t,x}$.

$C^{1,2}([0,T] \times \mathbb{R}^m;\mathbb{R}^k)$ désigne l'ensemble des fonctions $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}^k$ dérivable et à dérivée continue respectivement une fois en temps et deux fois en espace.

Preuve. Si u est une solution classique de l'EDP, on définit :

$$Y_r^{t,x} = u(r,X_r^{t,x}), \quad Z_r^{t,x} = \nabla u(r,X_r^{t,x})\sigma(r,X_r^{t,x}), \quad \text{pour } t \leq r \leq T.$$

On a $Y_T^{t,x} = g(X_T^{t,x})$ et on applique la formule d'Itô pour $t \leq r \leq T$:

$$\begin{aligned} u(r,X_r^{t,x}) &= u(t,x) + \int_t^r \frac{\partial u}{\partial t}(s,X_s^{t,x})ds + \int_t^r \mathcal{L}u(s,X_s^{t,x})ds \\ &\quad + \int_t^r \nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x})dB_s \\ &= u(t,x) - \int_t^r f(s,X_s^{t,x},u(s,X_s^{t,x}),\nabla u(s,X_s^{t,x})\sigma(s,X_s^{t,x}))ds + \int_t^r Z_s^{t,x}dB_s. \end{aligned}$$

Ceci montre que $(Y^{t,x},Z^{t,x})$ est une solution de l'EDSR (1.8).

De plus la condition imposée à u et ∇u entraîne

$$\mathbb{E}\left(\sup_{t\leq s\leq T}|u(s,X_s^{t,x})|^2+\int_t^T\|(\nabla u\sigma)(s,X_s^{t,x})\|^2ds\right)<+\infty.$$

□

Ainsi, partant d'une solution classique de l'EDP (1.6), nous construisons une solution à l'EDSR (1.8). Maintenant nous voulons faire le chemin inverse, c'est-à-dire que connaissant une solution de l'EDSR nous allons construire une solution de l'EDP. Les hypothèses que nous avons données jusqu'à présent, impliquent l'existence et l'unicité de la solution de l'EDS et de l'EDSR. Mais elles ne sont pas en général suffisantes pour qu'il y ait existence d'une solution classique de l'EDP. Si par exemple σ est identiquement nulle, avec les hypothèses précédentes, l'équation de transport :

$$\partial_t u+b\nabla u+f=0$$

n'admet pas nécessairement de solution de classe $C^{1,1}$.

Avec des conditions de régularité plus fortes sur les coefficients b, σ, g et f, E. Pardoux et S. Peng ont montré dans [44], que la fonction $(t,x)\mapsto Y_t^{t,x}$ était de classe $C^{1,2}$ en (t,x) (en utilisant le calcul de Malliavin) et ainsi fournissait une solution classique de l'EDP. Néanmoins en gardant les conditions énoncées plus haut, la formule (1.5) est encore valable, sauf que dans ce cas u est une solution faible de l'EDP. Plus particulièrement nous nous intéressons aux solutions faibles dites «de viscosité».

Nous définissons une solution de viscosité du système (1.6). Pour que la définition suivante ait est un sens, nous imposons à f de satisfaire :
pour $1\leq i\leq k$, la i-ème coordonnée de f ne dépend que de la i-ème colonne de la matrice z.
Remarquons que cette condition est immédiatement satisfaite si $k=1$. Ainsi le système (1.6) devient :

$$\begin{cases}\partial_t u_i(t,x)+\mathcal{L}_t u_i(t,x)+f_i(t,x,u(t,x),(\nabla u_i\sigma)(t,x)) &= 0, \qquad 1\leq i\leq k\\ u(T,x) &= g(x)\end{cases}$$

Définition 1.2 (Solution de viscosité continue)

1. *$u\in C([0,T]\times\mathbb{R}^m;\mathbb{R}^k)$ est appelée **sous-solution de viscosité** si $u_i(T,x)\leq g_i(x)$ pour tout $x\in\mathbb{R}^m$ et tout $1\leq i\leq k$, et si de plus pour tout $1\leq i\leq k$ et toute fonction $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^m)$, si $(t_0,x_0)\in[0,T)\times\mathbb{R}^m$ est un maximum local de $u_i-\varphi$, alors :*
$$-\frac{\partial\varphi}{\partial t}(t_0,x_0)-\mathcal{L}\varphi(t_0,x_0)-f_i(t_0,x_0,u(t_0,x_0),(\nabla\varphi\sigma)(t_0,x_0))\leq 0.$$
2. *$u\in C([0,T]\times\mathbb{R}^m;\mathbb{R}^k)$ est appelée **sur-solution de viscosité** si $u_i(T,x)\geq g_i(x)$ pour tout $x\in\mathbb{R}^m$ et tout $1\leq i\leq k$, et si de plus pour tout $1\leq i\leq k$ et toute fonction $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^m)$, si $(t_0,x_0)\in[0,T)\times\mathbb{R}^m$ est un minimum local de $u_i-\varphi$, alors :*
$$-\frac{\partial\varphi}{\partial t}(t_0,x_0)-\mathcal{L}\varphi(t_0,x_0)-f_i(t_0,x_0,u(t_0,x_0),(\nabla\varphi\sigma)(t_0,x_0))\geq 0.$$

3. *$u \in C([0,T] \times \mathbb{R}^m; \mathbb{R}^k)$ est appelée **solution de viscosité** si c'est à la fois une sous- et une sur-solution de viscosité.*

Dans le chapitre suivant nous donnerons une définition légèrement différente de celle-ci, valable pour des solutions discontinues. Il est facile de prouver que toute solution régulière est une solution de viscosité. Le résultat principal de cette partie est le théorème 3.2 de [42] :

Théorème 1.11 *Sous les hypothèses précédentes (C1)-...-(C6), si $(Y^{t,x}, Z^{t,x})$ est la solution de l'EDSR (1.8), et si on définit $u(t,x) \triangleq Y_t^{t,x}$, la fonction u est continue sur $[0,T] \times \mathbb{R}^m$, croît au plus vite de façon polynômiale à l'infini, et est solution de viscosité du système d'EDP (1.6).*

Preuve. La continuité provient de la continuité de $X^{t,x}$ par rapport à t,x et de l'estimation suivante : si (Y,Z) et (Y',Z') sont solution de l'EDSR (0.2) avec donnée finale ξ et ξ' dans $L^2(\Omega, \mathcal{F}_T)$, alors il existe une constante C, qui ne dépend que de T et des constantes de monotonie et de Lipschitz de f, telle que :

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t - Y'_t|^2 + \int_0^T \|Z_t - Z'_t\|^2 dt \right) \leq C\mathbb{E}|\xi - \xi'|^2.$$

C'est un cas particulier du théorème 2.3 de [42].

La croissance polynomiale est due aux estimées données précédemment sur la croissance de $X^{t,x}$ (cf. proposition 1.5), aux hypothèses de croissance imposées à f et g et à l'estimation a priori de la proposition 1.4 avec $p = 2$.

Pour prouver que u est une solution de viscosité, prenons $1 \leq i \leq k$ et $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m)$ et $(t,x) \in [0,T) \times \mathbb{R}^m$ tel que (t,x) soit un maximum local de $u_i - \varphi$. Sans perte de généralité on peut supposer que

$$u_i(t,x) = \varphi(t,x).$$

On raisonne par l'absurde en supposant que

$$-\frac{\partial \varphi}{\partial t}(t,x) - L\varphi(t,x) - f_i(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) > 0.$$

Soit $0 < \alpha \leq T - t$ tel que pour tout $t \leq s \leq t + \alpha$, et tout $|y - x| \leq \alpha$, on ait :

$$u_i(s,y) \leq \varphi(s,y) \quad \text{et} \quad \left(\frac{\partial \varphi}{\partial t} + L\varphi\right)(s,y) + f_i(s,y,u(s,y),(\nabla\varphi\sigma)(s,y)) < 0,$$

et définissons

$$\tau = \inf\left\{s \geq t;\ \left|X_s^{t,x} - x\right| \geq \alpha\right\} \wedge (t + \alpha).$$

Soit maintenant

$$(\overline{Y}_s, \overline{Z}_s) = \left((Y_{s\wedge\tau}^{t,x})^i, \mathbf{1}_{[0,\tau]}(s)(Z_s^{t,x})^i\right).$$

$(\overline{Y}, \overline{Z})$ est solution de l'EDSR unidimensionnelle

$$\overline{Y}_s = u_i(\tau, X_\tau^{t,x}) + \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r) f_i(r, X_r^{t,x}, u(r, X_r^{t,x}), \overline{Z}_r) dr - \int_s^{t+\alpha} \overline{Z}_r dB_r, \quad t \leq s \leq t + \alpha.$$

D'autre part, avec la formule d'Itô, on obtient que

$$(\hat{Y}_s,\hat{Z}_s) = \big(\varphi(s,X^{t,x}_{s\wedge\tau}),\mathbf{1}_{[0,\tau]}(s)(\nabla\varphi\sigma)(s,X^{t,x}_s)\big)$$

est solution de l'EDSR

$$\hat{Y}_s = \varphi(\tau,X^{t,x}_\tau) - \int_s^{t+\alpha} \mathbf{1}_{[0,\tau]}(r)\left(\frac{\partial\varphi}{\partial t} + L\varphi\right)(r,X^{t,x}_r)dr - \int_s^{t+\alpha} \hat{Z}_r dB_r, \quad t \leq s \leq t+\alpha.$$

Comme $u_i \leq \varphi$ et avec le choix de α et τ, on déduit du théorème de comparaison des EDSR 1.2 que $\overline{Y} < \hat{Y}$, d'où $u_i(t,x) < \varphi(t,x)$, ce qui contredit nos hypothèses.

Donc u est une sous-solution de viscosité. Et les mêmes arguments permettent de prouver que u est aussi une sur-solution. □

Pour plus de détails concernant le lien entre les EDSR et les EDP (identification de Z avec $\nabla u\sigma$ ou solution u dans les espaces de Sobolev), on consultera l'article de G. Barles et E. Lesigne ([3]).

Pour clore ce paragraphe, ajoutons qu'il existe des résultats concernant des couplages EDS-EDSR plus forts que ceux décrits précédemment. Jusqu'ici seul le processus X apparaissait dans les deux équations, donc b et σ ne dépendaient ni de u ni de ∇u dans l'EDP (1.6), d'où le nom d'EDP semi-linéaire. Si b et σ dépendent de u ou ∇u on parle d'EDP quasi-linéaires et le système progressif-rétrograde devient, pour tout $t \leq s \leq T$:

$$\begin{aligned} X^{t,x}_s &= x + \int_t^s b(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr + \int_t^s \sigma(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dB_r, \\ Y^{t,x}_s &= g(X^{t,x}_T) + \int_s^T f(r,X^{t,x}_r,Y^{t,x}_r,Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r dB_r. \end{aligned}$$

On pourra consulter les articles [14], [34] et [45] concernant ce type de systèmes.

1.4 Temps final aléatoire

En mathématiques financières (consulter [19] ou [27] pour plus de détails), le temps final T peut par exemple représenter l'échéance d'une option européenne dont on cherche le prix (c'est le «pricing»). Sous certaines hypothèses concernant le prix de l'action sous-jacente, on peut obtenir ainsi la fameuse formule de Black-Scholes. Mais dans certains modèles (type option américaine) cette échéance n'est pas une donnée fixée à l'avance : l'acheteur peut par exemple choisir l'échéance en fonction du prix de l'action. Ce temps devient lui-même aléatoire et il peut être alors intéressant de remplacer le temps T par un temps d'arrêt τ.

Une autre motivation pour introduire ce temps d'arrêt provient de la théorie des EDP elliptiques (voir [43]). En effet si D est un ouvert de $\mathbb{R}^d$ et $f : \partial D \to \mathbb{R}$ est une fonction continue (∂D est le bord de D), le problème de Dirichlet consiste à trouver une fonction $u : \overline{D} \to R$ continue et vérifiant :

$$\begin{cases} \Delta u = 0 & \text{sur } D, \\ u = f & \text{sur } \partial D. \end{cases}$$

Sous réserve que le bord de D soit suffisamment régulier, la solution du problème est donnée par la formule :

$$u(x) \triangleq \mathbb{E} f(B^x_{\tau_D}); \quad x \in \overline{D}.$$

B^x est toujours un mouvement brownien de dimension d, démarrant au point x à l'instant 0, et τ_D est le premier instant de sortie de B^x en dehors de $\overline{D}$. Les conditions nécessaires de régularité de D sont traitées dans [4], [22] ou [48]. Si $f(B^x_{\tau_D})$ est dans $L^2(\Omega)$, on peut alors écrire u de la façon suivante :

$$u(x) = Y^x_0$$

où Y^x est le processus suivant :

$$Y^x_t = \mathbb{E}^{\mathcal{F}_t} f(B^x_{\tau_D}) = f(B^x_{\tau_D}) - \int_{t\wedge\tau_D}^{\tau_D} Z^x_r dB_r, \quad t \geq 0;$$

Z^x étant obtenu via le théorème de représentation des martingales. (Y^x, Z^x) est solution d'une EDSR avec temps final aléatoire τ_D.

Dans la suite, $(B_t)_{t\geq 0}$ est toujours un mouvement brownien standard de dimension d, avec sa filtration $(\mathcal{F}_t)_{t\geq 0}$. τ est un temps d'arrêt par rapport à la filtration brownienne, et ξ est une variable aléatoire mesurable par rapport à $\mathcal{F}_\tau$. On veut résoudre l'EDSR

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(r, Y_r, Z_r) dr - \int_{t\wedge\tau}^{\tau} Z_r dB_r, \; t \geq 0 \tag{1.9}$$

La définition 1.1 d'une solution est légérement modifiée :

Définition 1.3 *Une solution de l'EDSR (1.9) est un couple* $\{(Y_t, Z_t),\, t \geq 0\}$ *de processus progressivement mesurables à valeurs dans* $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ *tels que* $\mathbf{P}$*-p.s.*

- *sur l'ensemble* $\{t > \tau\}$, $Y_t = \xi$ *et* $Z_t = 0$,
- $t \mapsto \mathbf{1}_{t\leq\tau} f(t, Y_t, Z_t)$ *appartient à* $L^1_{loc}(0,\infty)$,
- $t \mapsto Z_t$ *appartient à* $L^2_{loc}(0,\infty)$,
- *et pour tous réels* t *et* T *tels que* $0 \leq t \leq T$

$$Y_{t\wedge\tau} = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(r, Y_r, Z_r) dr - \int_{t\wedge\tau}^{T\wedge\tau} Z_r dB_r.$$

La remarque 1.1, qui suit la définition 1.1, reste valable avec un temps final aléatoire. Nous allons redonner sans démonstration les principaux résultats d'existence et d'unicité des solutions.

Le théorème 3.4 de R. Darling et E. Pardoux, [11] (voir également le théorème 4.1 dans [42]) fournit un résultat d'existence et d'unicité sous des hypothèses très proches de celle du théorème 1.3. De même que dans le cas d'un temps final déterministe, ce résultat a été étendu au cas L^p $(p > 1)$ dans l'article [9]. Nous nous donnons :

(H'1) un temps final aléatoire τ qui est un temps d'arrêt par rapport à la filtration $\{\mathcal{F}_t\}_{t\geq 0}$;

(H'2) un générateur $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ qui satisfait : il existe des nombres réels μ et $K,K' \geq 0$, et une application $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ continue et croissante, tels que :
  - $|f(t,y,z) - f(t,y,z')| \leq K\|z - z'\|$, $\forall t,y,z,z'$, p.s.
  - $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \mu|y - y'|^2$, $\forall t,y,y',z$;
  - $y \mapsto f(t,y,z)$ est continue, $\forall t,z$, p.s. ;

(H'3) une variable aléatoire ξ, $\mathcal{F}_\tau$-mesurable et à valeurs dans $\mathbb{R}^k$ vérifiant pour un certain $p > 1$: pour

$$\lambda > \nu_p = \mu + \frac{K^2}{2(p-1)}$$

(H'3.1) $$\mathbb{E}\left(e^{p\lambda\tau}|\xi|^p + \int_0^\tau e^{p\lambda t}|f(t,0,0)|^p dt\right) < \infty;$$

et

(H'3.2) $$\mathbb{E}\left(\int_0^\tau e^{p\lambda t}|f(t,e^{-\nu_p t}\overline{\xi}_t, e^{-\nu_p t}\overline{\eta}_t)|^p dt\right) < \infty,$$

où $\overline{\xi} = e^{\nu_p\tau}\xi$,

$$\overline{\xi}_t = \mathbb{E}(e^{\nu_p\tau}\xi|\mathcal{F}_t)$$

et enfin $\overline{\eta}$ est prévisible tel que

$$\overline{\xi} = \mathbb{E}(\overline{\xi}) + \int_0^\infty \overline{\eta}_t dB_t \quad \text{et} \quad \mathbb{E}\left[\left(\int_0^{+\infty} |\overline{\eta}_t|^2 dt\right)^{p/2}\right];$$

(H'5) pour tout $r > 0$ et tout $n \in \mathbb{N}$

$$\psi_r(t) = \sup_{|y|\leq r} |f(t,y,0) - f(t,0,0)| \in L^1([0,n]\times\Omega).$$

Théorème 1.12 *Sous toutes les conditions précédentes, l'EDSR (1.9) admet une unique solution satisfaisant*

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{p\lambda t}|Y_t|^2 + \int_0^\tau e^{p\lambda t}|Y_t|^{p-2}\left[|Y_t|^2 + |Z_t|^2\right]dt\right)$$
$$\leq c\mathbb{E}\left(e^{p\lambda\tau}|\xi|^p + \int_0^\tau e^{p\lambda t}|f(t,0,0)|^p dt\right),$$

la constante c ne dépendant que de p, μ et K.

Dans [11] ou [42], ce résultat était prouvé pour $p = 2$. Remarquons que dans le cas où $\mathbf{P}(\tau = +\infty) > 0$, les conditions d'intégrabilité (H'3.1)-(H'3.2) ne sont souvent réalisables que si μ est strictement négatif.

Dans ce travail pour $q > 0$ nous nous intéresserons à l'EDSR suivante :

$$Y_{t\wedge\tau} = \xi - \int_{t\wedge\tau}^\tau Y_r|Y_r|^q dr - \int_{t\wedge\tau}^\tau Z_r dB_r.$$

τ vérifie (H'1), le générateur f satisfait (H'2) et (H'4) avec $\mu = K = 0$ et $f(t,0,0) = 0$. Il reste à vérifier pour quel ξ, (H'3) est vraie.

Terminons ce passage par un théorème important pour nous que l'on trouve dans ([11], corollary 4.4.2)

Théorème 1.13 (Théorème de comparaison) *En dimension $k = 1$, si on considère deux EDSR avec données $\xi \leq \xi'$, $f \leq f'$ et $\tau = \tau'$, alors $Y_t \leq Y'_t$ pour tout $t \geq 0$, $\mathbf{P}$-p.s.*

Revenons maintenant sur le lien entre les EDSR avec temps d'arrêt et les EDP elliptiques dans un ouvert borné de $\mathbb{R}^m$ avec condition de Dirichlet au bord. Là encore ces résultats peuvent être trouvés dans [42].

On note $\{X_t^x;\ t \geq 0\}$ la solution de l'EDS :

$$X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s, \quad t \geq 0. \tag{1.10}$$

b et σ vérifient les mêmes conditions que pour l'EDS (1.7). Soit D un ouvert borné de $\mathbb{R}^m$, dont le bord est de classe C^1 et pour tout $x \in \overline{D}$, on définit le temps d'arrêt :

$$\tau_x = \inf\left\{t \geq 0; X_t^x \notin \overline{D}\right\}.$$

On suppose que

$$\mathbf{P}(\tau_x < +\infty) = 1 \tag{1.11}$$

pour tout $x \in \overline{D}$, que l'ensemble

$$\Gamma = \{x \in \partial D;\ \mathbf{P}(\tau_x > 0) = 0\} \text{ est fermé,} \tag{1.12}$$

et enfin que pour un certain $\lambda > 2\mu + K^2$

$$\sup_{x \in \overline{D}} \mathbb{E}e^{\lambda \tau_x} < \infty. \tag{1.13}$$

Cette dernière condition d'intégrabilité correspond à l'hypothèse (H'3.1) du théorème 1.12 et là aussi elle n'est pas toujours évidente à vérifier, sauf si bien sûr $\mu < 0$ ou si σ est uniformément elliptique (voir par exemple [48]).

Proposition 1.8 *Sous les hypothèses (1.11)-(1.12), l'application $x \mapsto \tau_x$ est p.s. continue sur $\overline{D}$.*

La démonstration se trouve dans [11] ou [42]. On prouve la continuité grâce à la convergence de $(\tau_{x_n})_{n\in\mathbb{N}}$ vers τ_x pour toute suite $(x_n)_{n\in\mathbb{N}}$ d'éléments de $\overline{D}$ qui converge vers x. D'une part on utilise le fait que τ_x est un temps d'arrêt fini pour montrer que la limite supérieure de (τ_{x_n}) est majorée par τ_x. D'autre part, le fait que Γ est fermé permet de montrer que la limite inférieure est minorée par τ_x.

Enfin on se donne une fonction $g : \mathbb{R}^m \to \mathbb{R}^k$ continue et un générateur $f : \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ satisfaisant : il existe $K \geq 0$, $\mu \in \mathbb{R}$, $p > 0$, $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ continue et croissante tels que :

$$\begin{aligned}|f(x,y,z)| &\leq K(1+|x|^p+\phi(|y|)+\|z\|),\\ \langle f(x,y,z)-f(x,y',z),y-y'\rangle &\leq \mu|y-y'|^2,\\ |f(x,y,z)-f(x,y,z')| &\leq K\|z-z'\|.\end{aligned}$$

Soit $\{(Y_t^x,Z_t^x);\ 0\leq t\}$ l'unique solution, au sens du théorème 1.12, de l'EDSR :

$$Y_t^x = g(X_{\tau_x}^x) + \int_{t\wedge\tau_x}^{\tau_x} f(X_s^x,Y_s^x,Z_s^x)ds - \int_{t\wedge\tau_x}^{\tau_x} Z_s^x dB_s, \quad 0\leq t. \tag{1.14}$$

$\mathcal{L}$ désigne toujours le générateur infinitésimal du processus de Markov $\{X_t^x;\ t\geq 0\}$:

$$\mathcal{L} = \frac{1}{2}\mathrm{Trace}(\sigma\sigma^* D^2) + b.\nabla$$

On s'intéresse alors au système d'EDP elliptiques semi-linéaires dans $\mathbb{R}^m$:

$$\begin{aligned}\mathcal{L}u_i(x) + f_i(x,u(x),(\nabla u_i\sigma)(x)) &= 0, \qquad x\in D, \quad 1\leq i\leq k,\\ u_i(x) &= g(x), \qquad x\in\partial D, \ 1\leq i\leq k.\end{aligned} \tag{1.15}$$

Pour un tel système, on définit la notion de solution de viscosité ainsi :

Définition 1.4 (Solution de viscosité dans le cas elliptique)

1. *$u \in C(\overline{D};\mathbb{R}^k)$ est appelée **sous-solution de viscosité** si pour tout $1\leq i\leq k$ et toute fonction $\varphi\in C^2(\mathbb{R}^m)$, si $x_0\in\overline{D}$ est un maximum local de $u_i-\varphi$, alors :*

$$\begin{aligned}-\mathcal{L}\varphi(x_0) - f_i(x_0,u(x_0),(\nabla\varphi\sigma)(x_0)) &\leq 0, \quad si \quad x_0\in D;\\ \min\left(-\mathcal{L}\varphi(x_0) - f_i(x_0,u(x_0),(\nabla\varphi\sigma)(x_0)),\ u_i(x_0)-g_i(x_0)\right) &\leq 0, \quad si \quad x_0\in\partial D.\end{aligned}$$

2. *$u \in C(\overline{D};\mathbb{R}^k)$ est appelée **sur-solution de viscosité** si pour tout $1\leq i\leq k$ et toute fonction $\varphi\in C^2(\mathbb{R}^m)$, si $x_0\in\overline{D}$ est un minimum local de $u_i-\varphi$, alors :*

$$\begin{aligned}-\mathcal{L}\varphi(x_0) - f_i(x_0,u(x_0),(\nabla\varphi\sigma)(x_0)) &\geq 0, \quad si \quad x_0\in D;\\ \min\left(-\mathcal{L}\varphi(x_0) - f_i(x_0,u(x_0),(\nabla\varphi\sigma)(x_0)),\ u_i(x_0)-g_i(x_0)\right) &\geq 0, \quad si \quad x_0\in\partial D.\end{aligned}$$

3. *$u \in C(\overline{D};\mathbb{R}^k)$ est appelée **solution de viscosité** si c'est à la fois une sous- et une sur-solution de viscosité.*

Théorème 1.14 *Sous les conditions précédentes, si on pose $u(x)\triangleq Y_0^x$, u est une fonction continue sur $\overline{D}$ et est une solution de viscosité du système d'équations (1.15).*

Preuve. La continuité de u est prouvée dans [11]. C'est ici que l'on utilise la proposition 1.8.

Nous allons simplement prouver que u est une sous-solution. Soit $1\leq i\leq k$, $\varphi\in C^2(\mathbb{R}^m)$ et soit $x_0\in\overline{D}$ un maximum local de $u_i-\varphi$, tel que $u_i(x_0)=\varphi(x_0)$. Si $x_0\in\Gamma$, alors $\tau_{x_0} = 0$, et donc $u(x_0) = g(x_0)$. On suppose maintenant que $x_0 \notin \Gamma$. Dans ce cas $\tau_x > 0$ **P**-p.s. Raisonnons par l'absurde en supposant

$$(\mathcal{L}\varphi)(x_0) + f_i(x_0,u(x_0),(\nabla\varphi\sigma)(x_0)) < 0.$$

On obtient une contradiction avec les mêmes arguments que dans la preuve du théorème 1.11, en choisissant $\alpha > 0$ tel que, quand $|y - x| \leq \alpha$,

$$\begin{array}{rcl} u_i(y) & \leq & \varphi(y), \\ (\mathcal{L}\varphi)(y) + f_i(y,u(y),(\nabla\varphi\sigma)(y)) & < & 0. \end{array}$$

et

$$\overline{\tau} = \inf\{t > 0, |X_t^x - x| \geq \alpha\} \wedge \tau_x \wedge T$$

T étant un réel positif quelconque. □

Chapitre 2

EDSR avec donnée terminale singulière

2.1 Introduction

In this chapter we are interested by the following backward stochastic differential equation (BSDE in short) with a non-linear generator:

$$Y_t = \xi - \int_t^T Y_r |Y_r|^q dr - \int_t^T Z_r dB_r \quad \text{with } q \in \mathbb{R}_+^*. \tag{2.1}$$

The process $(B_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion on a probability space $\big(\Omega,\mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P}\big)$, with $(\mathcal{F}_t)_{t\geq 0}$ the standard Brownian filtration. The terminal time T is fixed and the $\mathbb{R}$-valued random variable ξ is supposed to be $\mathcal{F}_T$-adapted.

The unknowns are the processes $\{Y_t\}_{t\in[0,T]}$ and $\{Z_t\}_{t\in[0,T]}$ with values respectively in $\mathbb{R}$ and $\mathbb{R}^d$. The unknowns are required to be adapted with respect to the filtration of the Brownian motion.

The generator $f(y) = -y|y|^q$ satisfies all the assumptions of the theorems 4.2 and 6.2-6.3 of [9] (see also the theorems 1.4-1.5-1.6 of the previous chapter): f is continuous on $\mathbb{R}$, does not depend on z, and is monotone: for all $(y,y') \in \mathbb{R}^2$

$$(y - y')(f(y) - f(y')) \leq 0.$$

Therefore there exists a unique solution (Y,Z) for $\xi \in L^p(\Omega)$ for $p \geq 1$ (we do not precise the class of (Y,Z) in which uniqueness holds: the class has been precised in the previous chapter).

We consider this BSDE for the following reason. The solution of the related ordinary differential equation (ODE in short), namely $y' = y|y|^q$, $y_T = x$, is given by the formula:

$$y_t^x = \text{sign}(x) \left(\frac{1}{q(T-t) + \frac{1}{|x|^q}} \right)^{\frac{1}{q}} \qquad t \in [0,T]$$

where $\text{sign}(x) = -1$ if $x < 0$ and $\text{sign}(x) = 1$ if $x > 0$. By convention $1/\pm\infty = 0$ and $\text{sign}(\pm\infty) = \pm 1$. We remark that, even if x is equal to $+\infty$ or $-\infty$, y^x is well-defined on $[0,T]$ and satisfies:

$$|y_t^x| \leq \left(\frac{1}{q(T-t)} \right)^{\frac{1}{q}} \qquad t \in [0,T].$$

Thus y^x is finite on $[0,T[$. A natural question is the following: does this result hold for the BSDE, i.e. can we define a solution of the BSDE (2.1) when $\xi = \pm\infty$ with positive probability?

Moreover these BSDE are connected with the following partial differential equation (PDE in short):

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + Lu(t,x) - u(t,x)|u(t,x)|^q &= 0,\ (t,x) \in [0,T[\times\mathbb{R}^m, \\ u(T,x) &= g(x),\ x \in \mathbb{R}^m. \end{aligned} \tag{2.2}$$

where L is the operator:

$$L = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(t,x)\frac{\partial}{\partial x_i}.$$

Indeed P. Baras and M. Pierre [2], M. Marcus and L. Veron [37] have given existence and uniqueness results for this PDE. In [37] it is shown that every positive solution of (2.2) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^m$ and μ a non negative Radon measure on $\mathcal{R} = \mathbb{R}^m \setminus \mathcal{S}$. The final trace can also be represented by a positive regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R} \end{cases} \quad \forall A \subset \mathbb{R}^m, A \text{ Borel.}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view E.B. Dynkin and S.E. Kuznetsov [17] and J.F. Le Gall [29] have proved similary results for the PDE (2.2) in the case $0 < q \leq 1$: they use the theory of superdiffusions in order to give a probabilistic representation of the PDE (2.2). The aim of this chapter is to prove that the theory of the BSDE gives an other probabilistic representation of the PDE (2.2).

Hence our first problem is to find a solution of the BSDE (2.1) when the terminal data ξ is singular, i.e. when ξ is a real $\mathcal{F}_T$-measurable random variable such that:

$$\mathbf{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0. \tag{2.3}$$

The random variable ξ is not in $L^1(\Omega)$ and the "classical" definition 1.1 of a solution of a BSDE can not be used. Indeed if (Y,Z) is a solution in the sense of the definition 1.1, then $\mathbf{P}$-a.s.

(i) $t \mapsto Z_t$ is in $L^2([0,T])$;

(ii) $t \mapsto -Y_t\,|Y_t|^q$ is in $L^1([0,T])$.

Thus $\mathbf{P}$-a.s. for all $t \in [0,T]$

$$\left| \int_t^T Y_r\,|Y_r|^q\,dr + \int_t^T Z_r dB_r \right| < +\infty.$$

The third condition of the definition implies that for all $t \in [0,T]$

$$Y_t = \xi - \int_t^T Y_r\,|Y_r|^q\,dr - \int_t^T Z_r dB_r = \pm\infty$$

a.s. on the set $\{\xi = +\infty$ ou $\xi = -\infty\}$. This leads to a contradiction with (ii).
We define a solution of the BSDE (2.1) as follows:

Definition 2.1 *Let $q > 0$ and ξ a $\mathcal{F}_T$-measurable random variable. We say that the process (Y,Z) is a solution of the BSDE*

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r$$

if (Y,Z) verifies:
(D1) for all $0 \le s \le t < T$:

$$Y_s = Y_t - \int_s^t Y_r|Y_r|^q dr - \int_s^t Z_r dB_r;$$

(D2) for all $t \in [0,T[$,

$$\mathbb{E}\left(\sup_{0\le s\le t} |Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) < +\infty;$$

(D3) $\mathbf{P}$-a.s.

$$\lim_{t\to T} Y_t = \xi.$$

It is important to remark that the two definitions are compatible. Indeed if $\zeta \in L^1(\Omega)$, if (Y,Z) is a solution in the sense of the definition 1.1) of the BSDE (2.1) with terminal data ζ (theorems 1.5-1.6), then (Y,Z) satisfies (D1) and (D3). The condition (D2) comes from the following lemma:

Lemma 2.1 *If (Y,Z) is a "classical" solution of the BSDE (2.1) with $\zeta \in L^1(\Omega)$, then* $\mathbf{P}$*-a.s.*

$$|Y_t| \le \left(\frac{1}{q(T-t)}\right)^{1/q}, \quad 0 \le t \le T. \tag{2.4}$$

This estimate was already given for the related ODE.

Proof. We apply the Tanaka formula (see [22]) to Y:

$$\begin{aligned} |Y_t| &= |\zeta| - \int_t^T \mathrm{sign}(Y_r)Y_r|Y_r|^q dr - \int_t^T \mathrm{sign}(Y_r)Z_r dB_r + 2(\Lambda_t(0) - \Lambda_T(0)) \\ &= |\zeta| - \int_t^T |Y_r|^{1+q} dr - \int_t^T \mathrm{sign}(Y_r)Z_r dB_r + 2(\Lambda_t(0) - \Lambda_T(0)) \end{aligned}$$

where Λ is a local time of Y. Hence $\Lambda_T(0) \ge \Lambda_t(0)$ a.s. If ζ is in L^∞, there exists $K \ge 0$ such that $|\zeta| \le K$ a.s. With the comparison theorem 1.2 we deduce:

$$|Y_t| \le \left(\frac{1}{q(T-t) + 1/K^q}\right)^{1/q} \le \left(\frac{1}{q(T-t)}\right)^{1/q}.$$

Therefore the estimate (2.4) holds for all $\zeta \in L^\infty$. Finally by a density argument, it holds for all $\zeta \in L^1$. □

We have obtained that every classical solution is a.s. finite on $[0,T[$ and bounded on $[0,T-\delta]$ for all $\delta > 0$. Since f is monotone with $\mu = 0$, the Itô formula leads for all $0 \leq t \leq r < T$ to:

$$\begin{aligned} |Y_t|^2 + \int_t^r \|Z_s\|^2 ds &\leq |Y_r|^2 + 2\int_t^r Y_s Z_s dB_s \\ &\leq \left(\frac{1}{q(T-r)}\right)^{2/q} + 2\int_t^r Y_s Z_s dB_s. \end{aligned}$$

This inequality and the Burkholder-Davis-Gundy inequalities imply the condition (D2).

Main results

The outline of the chapter is as follows. Except in Section 5 ξ is supposed to be non-negative. In the first section without any further assumption on ξ we construct a process (Y,Z) which satisfies all conditions to be a solution in the sense of the previous definition, except the last one. More precisely we establish in Section 1 the

Theorem 2.1
Let $\xi \geq 0$ a.s. There exists a progressively measurable process (Y,Z), with values in $\mathbb{R}^+ \times \mathbb{R}^d$ such that:

1. *for all $t \in [0,T[$, and all $0 \leq s \leq t$:*

(i) $$Y_s = Y_t - \int_s^t (Y_r)^{1+q}\, dr - \int_s^t Z_r dB_r,$$

2. *there exists an universal constant C (independent of q and ξ) s.t. for all $t \in [0,T[$,*

(ii) $$\mathbb{E}\left(\sup_{0\leq s\leq t} |Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) \leq \frac{C}{(q(T-t))^{\frac{2}{q}}},$$

$$0 \leq Y_t \leq \frac{1}{(q(T-t))^{\frac{1}{q}}},$$

3. *Y is continuous on $[0,T[$, the limit of Y_t, when t goes to T, exists and:*

(iii) $$\lim_{t\to T} Y_t \geq \xi,\ \mathbf{P}-a.s.,$$

4. *Z satisfies also:*

(iv) $$\mathbb{E}\left(\int_0^T (T-r)^{2/q}\|Z_r\|^2 dr\right) \leq 8\left(\frac{1}{q}\right)^{\frac{1}{2q}}.$$

Note that this result does not specify whether Y satisfies

$$\lim_{t\to T} Y_t = \xi.$$

In Section 2 we study our process Y in the neighbourhood of T. In a first part we precise the asymptotic behaviour of Y on the "blow-up" set.

Proposition 2.1
On the set $\{\xi = +\infty\}$

$$\lim_{t\to T}(T-t)^{1/q}Y_t = \left(\frac{1}{q}\right)^{1/q} \qquad a.s. \tag{2.5}$$

In the second part we will prove the continuity of Y under stronger conditions on ξ. So far we only have the inequality:

$$\lim_{t\to T} Y_t \geq \xi = Y_T.$$

Without additional assumption, we were unable to prove the converse inequality. The first hypothesis on ξ is the following:

$$\xi = g(X_T), \tag{H1}$$

where g is a function defined on $\mathbb{R}^m$ with values in $\overline{\mathbb{R}^+}$ such that the condition (H1) is satisfied and that the set $F = \{g = +\infty\}$ is closed ; and where X_T is the value at $t = T$ of a diffusion process or more precisely the solution of a stochastic differential equation (in short SDE):

$$X_t = x + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dB_r, \text{ for } t \in [0,T]. \tag{2.6}$$

We will always assume that:

1. Lipschitz condition:

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K|x-y|; \tag{L}$$

2. Growth condition:

$$|b(t,x)| \leq K(1+|x|) \text{ and } \|\sigma(t,x)\| \leq K(1+|x|). \tag{G}$$

Moreover in the case $q \leq 2$ we will add the following conditions:

(B). σ and b are bounded: there exists a constant K s.t.

$$\forall (t,x) \in [0,T]\times\mathbb{R}^m, \quad |b(t,x)| + \|\sigma(t,x)\| \leq K;$$

(E). $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x) \in [0,T]\times\mathbb{R}^m$ and all $y \in \mathbb{R}^m$:

$$\sigma\sigma^*(t,x)y.y \geq \lambda|y|^2.$$

The second hypothesis on ξ is: for all compact set $\mathcal{K} \subset \mathbb{R}^m \setminus F = \mathbb{R}^m \setminus \{g = +\infty\}$

$$g(X_T)\mathbf{1}_{\mathcal{K}}(X_T) \in L^1(\Omega,\mathcal{F}_T,\mathbf{P};\mathbb{R}). \tag{H2}$$

Theorem 2.2 (Continuity of Y at T)
Under the assumptions (H1) and (H2), and with either $q > 2$ or (B)-(E), Y is continuous at time T

$$\lim_{t\to T} Y_t = \xi, \ \mathbf{P}-a.s.$$

In Section 3 we prove that our solution is the minimal solution.

Theorem 2.3 (Minimal solution)
The solution (Y,Z) *obtained in the theorems 2.1 and 2.2 is minimal: if* $(\bar{Y},\bar{Z})$ *is an other non-negative solution in the sense of the definition 2.1, then for all* $t \in [0,T]$, $\mathbf{P}$*-a.s.:* $\bar{Y}_t \geq Y_t$.

Moreover we prove that:

$$\bar{Y}_t \leq \left(\frac{1}{q(T-t)} \right)^{1/q}.$$

The fourth section provides connections between this constructed solution of the BSDE (2.1) and viscosity solutions of related semi-linear PDE (2.2). For all $(t,x) \in [0,T] \times \mathbb{R}^m$, we denote by $X^{t,x}$ the solution of the SDE:

$$X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dB_r, \text{ for } s \in [t,T], \tag{2.7}$$

and $X_s^{t,x} = x$ for $s \in [0,t]$. b and σ verify always the assumptions of the second section. As a final condition of the BSDE, we take $g(X_T^{t,x})$, where g is a function defined from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$ such that the set $F = \{g = +\infty\}$ is closed and such that the condition (H2) is verified. Moreover g is supposed to be continuous from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$.

Theorem 2.4 (Viscosity solution)
The minimal solution of the BSDE (2.1) with $\xi = g(X_T^{t,x})$ *is denoted by* $Y^{t,x}$. *Then* $Y_t^{t,x}$ *is a deterministic number and if we set* $u(t,x) = Y_t^{t,x}$, *then* u *is lower-semicontinuous from* $[0,T] \times \mathbb{R}^m$ *to* $\overline{\mathbb{R}^+}$ *and is a (discontinuous) viscosity solution of the PDE (2.2).*

We prove that the previous solution is minimal among all non-negative viscosity solutions.

Theorem 2.5 (Minimal viscosity solution)
If v *is a non-negative viscosity solution of the PDE (2.2), then for all* $(t,x) \in [0,T] \times \mathbb{R}^m$*:*

$$u(t,x) \leq v(t,x).$$

In Section 5 we extend our results when there is no sign assumption on ξ. In particular ξ should satisfy:

$$\mathbf{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0.$$

2.2 Approximation and construction of a solution

From now and in the sections 2-3-4, ξ satisfies:

$$\mathbf{P}(\xi \geq 0) = 1 \quad \text{and} \quad \mathbf{P}(\xi = +\infty) > 0.$$

In this section we prove Theorem 2.1. For $q > 0$, let us consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(y) = -y|y|^q$. f is continuous and monotone, i.e. satisfies for all $(y,y') \in \mathbb{R}^2$:

$$(y - y')(f(y) - f(y')) \leq 0. \tag{2.8}$$

By theorem 2.2 and example 3.9 in [42], for $\zeta \in L^2(\mathcal{F}_T)$, the following BSDE:

$$Y_t = \zeta + \int_t^T f(Y_s)ds - \int_t^T Z_s dB_s \tag{2.1}$$

has a unique solution (Y,Z) with values in $\mathbb{R} \times \mathbb{R}^d$ such that Y is continuous on $[0,T]$ and that:

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^2 + \int_0^T \|Z_t\|^2 dt\right) < \infty. \tag{2.9}$$

For every $n \in \mathbb{N}^*$, we introduce $\xi_n = \xi \wedge n$. ξ_n belongs to $L^2(\Omega,\mathcal{F}_T,\mathbf{P};\mathbb{R})$. We apply the previous result with ξ_n as the final data, and we create a sequence of random processes (Y^n,Z^n) which satisfy (2.1) and (2.9).

From the comparison theorem 2.4 in [42], if $n \le m$, $0 \le \xi_n \le \xi_m \le m$, which implies for all t in $[0,T]$, a.s.,

$$0 \le Y_t^n \le Y_t^m \le \left(\frac{1}{q(T-t) + \frac{1}{m^q}}\right)^{\frac{1}{q}} \le \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}};$$

and

$$Y_t^n = \xi_n - \int_t^T (Y_s^n)^{1+q} ds - \int_t^T Z_s^n dB_s. \tag{2.10}$$

We define the progressively measurable $\mathbb{R}$-valued process Y, as the increasing limit of the sequence $(Y_t^n)_{n\ge1}$:

$$\forall t \in [0,T],\ Y_t = \lim_{n\to+\infty} Y_t^n. \tag{2.11}$$

Then we obtain for all $0 \le t \le T$

$$0 \le Y_t \le \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}. \tag{2.12}$$

In particular Y is finite on the interval $[0,T[$ and bounded on $[0,T-\delta]$ for all $\delta > 0$.

2.2.1 Proof of (i) and (ii) from Theorem 2.1

Here we will prove the following properties:

1. for all $t \in [0,T[$, for all $0 \le s \le t$:

$$Y_s = Y_t - \int_s^t (Y_r)^{1+q} dr - \int_s^t Z_r dB_r, \tag{i}$$

2. there exists C s.t. for all $t \in [0,T[$,

$$\mathbb{E}\left(\sup_{0\le s\le t} |Y_s|^2 + \int_0^t \|Z_r\|^2 dr\right) \le \frac{C}{(q(T-t))^{\frac{2}{q}}}, \tag{ii}$$

Let $\delta > 0$ and $s \in [0,T-\delta]$. For all $0 \leq t \leq s$, Itô's formula leads to the equality:

$$\begin{aligned}
|Y_t^n - Y_t^m|^2 + \int_t^s \|Z_r^n - Z_r^m\|^2 dr &= |Y_s^n - Y_s^m|^2 - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dB_r \\
&\quad +2\int_t^s (Y_r^n - Y_r^m)\left(f(Y_r^n) - f(Y_r^m)\right) dr \\
&\leq |Y_s^n - Y_s^m|^2 - 2\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dB_r,
\end{aligned}$$

from the monotonicity of f (inequality (2.8)). The property (2.9) implies:

$$\mathbb{E}\left(\int_t^s (Y_r^n - Y_r^m)(Z_r^n - Z_r^m) dB_r\right) = 0.$$

From the Burkholder-Davis-Gundy inequality, we deduce the existence of a universal constant C with:

$$\mathbb{E}\left(\sup_{0\leq t\leq s} |Y_t^n - Y_t^m|^2 + \int_0^s \|Z_r^n - Z_r^m\|^2 dr\right) \leq C\, \mathbb{E}\left(|Y_s^n - Y_s^m|^2\right). \tag{2.13}$$

From the estimate (2.12), for $s \leq T-\delta$, $Y_s^n \leq \frac{1}{(q\delta)^{1/q}}$ and $Y_s \leq \frac{1}{(q\delta)^{1/q}}$. Since Y_s^n converges to Y_s a.s., the dominated convergence theorem and the previous inequality (2.13) imply:

1. for all $\delta > 0$, $(Z^n)_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega \times [0,T-\delta]; \mathbb{R}^d)$, and converges to $Z \in L^2(\Omega \times [0,T-\delta]; \mathbb{R}^d)$,
2. $(Y^n)_{n\geq 1}$ converges to Y uniformly in mean-square on the interval $[0,T-\delta]$, in particular Y is continuous on $[0,T[$,
3. (Y,Z) satisfies for every $0 \leq s < T$, for all $0 \leq t \leq s$:

$$Y_t = Y_s - \int_t^s (Y_r)^{1+q} dr - \int_t^s Z_r dB_r. \tag{i}$$

The relation (i) is proved. Since Y_t is smaller than $1/(q(T-t))^{1/q}$ by (2.12), and since $Z \in L^2(\Omega \times [0,T-\delta]; \mathbb{R}^d)$, applying the Itô formula to $|Y|^2$, we obtain (ii) that with $s < T$ and $0 \leq t \leq s$:

$$\begin{aligned}
|Y_t|^2 + \int_t^s \|Z_r\|^2 dr &= |Y_s|^2 - 2\int_t^s Y_r Z_r dB_r + 2\int_t^s Y_r f(Y_r) dr \\
&\leq \frac{1}{(q(T-s))^{\frac{2}{q}}} - 2\int_t^s Y_r Z_r dB_r,
\end{aligned}$$

again thanks to the monotonicity of f (inequality (2.8)). Therefore we deduce from the Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\left(\sup_{0\leq t\leq s} |Y_t|^2 + \int_0^s \|Z_r\|^2 dr\right) \leq \frac{C}{(q(T-s))^{\frac{2}{q}}}. \tag{ii}$$

2.2.2 Proof of (iii)

From now the process Y is continuous on $[0,T[$ and we define $Y_T = \xi$. The main difficulty will be to prove the continuity at time T. It is easy to prove that:

$$\xi \leq \liminf_{t\to T} Y_t. \tag{2.14}$$

Indeed for all $n \geq 1$ and all $t \in [0,T]$, $Y^n_t \leq Y_t$, therefore:

$$\xi \wedge n = \liminf_{t\to T} Y^n_t \leq \liminf_{t\to T} Y_t.$$

Thus Y is lower semicontinuous on $[0,T]$ (this is clear since Y is the supremum of continuous functions). Without other assumptions on ξ we are unable to prove the continuity of Y at $t = T_-$. But now we will show that Y has a limit on the left at time T.

Proposition 2.2 *The limit of Y_t as t goes to T exists and is therefore greater than or equal to ξ.*

In the proof we will distinguish the case when ξ is greater than a positive constant from the case ξ non-negative.

The case ξ bounded away from zero.

We can show that Y has a limit on the left at T by using Itô's formula applied to the process $1/(Y^n)^q$. We prove the following result:

Proposition 2.3
Suppose there exists a real $\alpha > 0$ such that $\xi \geq \alpha > 0$, $\mathbf{P}$-a.s. Then

$$Y_t = \left(q(T-t) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right) - \Phi_t\right)^{-\frac{1}{q}}, \ 0 \leq t \leq T, \tag{2.15}$$

where Φ is a non negative supermartingale.

Proof. From the comparison result 2.4 of [42], for every $n \in \mathbb{N}^*$ and every $0 \leq t \leq T$:

$$n \geq Y^n_t \geq \left(\frac{1}{q(T-t)+1/\alpha^q}\right)^{1/q} \geq \left(\frac{1}{qT+1/\alpha^q}\right)^{1/q} > 0.$$

By the Itô formula

$$\begin{aligned}\frac{1}{(Y^n_t)^q} &= \frac{1}{(\xi\wedge n)^q} + q(T-t) - \frac{q(q+1)}{2}\int_t^T \frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds + \int_t^T \frac{qZ^n_s}{(Y^n_s)^{q+1}}dB_s \\ &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) + q(T-t) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds.\end{aligned}$$

From [24], theorems 2.3 and 2.6, the process

$$\left(\frac{1}{(Y_t^n)^q}, \frac{-qZ_s^n}{(Y_s^n)^{q+1}}\right)$$

is the unique solution in $L^\infty([0,T] \times \Omega) \times L^2([0,T] \times \Omega)$ of the BSDE:

$$Y_t = \frac{1}{(\xi \wedge n)^q} + \int_t^T \left(q - \frac{q+1}{2q}\frac{\|Z_s\|^2}{Y_s \vee 1/n^q}\right) ds - \int_t^T Z_s dB_s.$$

Let $n \geq m$. Since $\xi \wedge n \geq \xi \wedge m$, we obtain for all $0 \leq t \leq T$:

$$\begin{aligned} 0 \leq \frac{1}{(Y_t^m)^q} - \frac{1}{(Y_t^n)^q} &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge m)^q} - \frac{1}{(\xi \wedge n)^q}\right) \\ &- \frac{q(q+1)}{2}\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^m\|^2}{(Y_s^m)^{q+2}}ds - \mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds\right). \end{aligned}$$

Now:

$$\begin{aligned} &\frac{q(q+1)}{2}\left|\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^m\|^2}{(Y_s^m)^{q+2}}ds - \mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds\right| \\ &\leq \left[\mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge m)^q} - \frac{1}{(\xi \wedge n)^q}\right)\right] \vee \left[\frac{1}{(Y_t^m)^q} - \frac{1}{(Y_t^n)^q}\right]. \end{aligned}$$

For a fixed $t \in [0,T]$, the sequences $\left(\mathbb{E}^{\mathcal{F}_t}\frac{1}{(\xi\wedge n)^q}\right)_{n\geq 1}$ and $\left(\frac{1}{(Y_t^n)^q}\right)_{n\geq 1}$ converge a.s. and in L^1 (dominated convergence theorem). Then $\left(\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds\right)_{n\geq 1}$ converges a.s. and in L^1 and we denote by Φ_t the limit

$$\Phi_t = \lim_{n\to+\infty} \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds.$$

We can also remark that:

$$\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds = \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge n)^q}\right) + q(T-t) - \frac{1}{(Y_t^n)^q},$$

with $Y_t^n \leq 1/(q(T-t))^{1/q}$, so

$$0 \leq \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds \leq \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi \wedge n)^q}\right) \leq \frac{1}{\alpha^q}.$$

Therefore

$$0 \leq \Phi_t \leq \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right).$$

For $r \leq t$,

$$\begin{aligned} \int_r^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds &\geq \int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds, \\ \Longrightarrow \ \mathbb{E}^{\mathcal{F}_r} \int_r^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds &\geq \mathbb{E}^{\mathcal{F}_r} \mathbb{E}^{\mathcal{F}_t} \int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds, \\ \Longrightarrow \qquad \Phi_r &\geq \mathbb{E}^{\mathcal{F}_r} \Phi_t. \end{aligned}$$

We deduce that $(\Phi_t)_{0 \leq t < T}$ is a non-negative bounded supermartingale. Now for all $n \in \mathbb{N}^*$,

$$\frac{1}{(Y_t^n)^q} = q(T-t) + \mathbb{E}^{\mathcal{F}_t} \left(\frac{1}{(\xi \wedge n)^q} \right) - \frac{q(q+1)}{2} \mathbb{E}^{\mathcal{F}_t} \int_t^T \frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}} ds.$$

Fix $t < T$. Taking the limit as $n \to +\infty$, we deduce:

$$\frac{1}{(Y_t)^q} = q(T-t) + \mathbb{E}^{\mathcal{F}_t} \left(\frac{1}{\xi^q} \right) - \Phi_t.$$

From the above expression $(\Phi_t)_{0 \leq t < T}$ is right-continuous. □

Φ being a right-continuous non-negative supermartingale, the limit of Φ_t as t goes to T exists $\mathbf{P}$-a.s. and this limit Φ_T is finite $\mathbf{P}$-a.s., since it is bounded by $1/\alpha^q$. The L^1-bounded martingale $\mathbb{E}^{\mathcal{F}_t} \left(\frac{1}{\xi^q} \right)$ converges a.s. to $1/\xi^q$ as t goes to T, then the limit of Y_t as $t \to T$ exists and is equal to:

$$\lim_{t \to T} Y_t = \frac{1}{(\frac{1}{\xi^q} - \Phi_T)^{1/q}}.$$

If we were able to prove that Φ_T is zero a.s., we would have shown that $Y_T = \xi$.

The case ξ non negative

Now we just assume that $\xi \geq 0$. We cannot apply the Itô formula to $1/(Y^n)^q$ because we have no positive lower bound for Y^n. We will approach Y^n in the following way. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}.$$

This random variable is in L^2 and is greater or equal to $1/m$ a.s. The BSDE (2.1) with $\xi^{n,m}$ as terminal condition has an unique solution $(\widetilde{Y}^{n,m}, \widetilde{Z}^{n,m})$. It is immediate that if $m \leq m'$ and $n \leq n'$ then:

$$\widetilde{Y}^{n,m'} \leq \widetilde{Y}^{n',m}.$$

As for the sequence Y^n, we can define $\widetilde{Y}^m$ as the limit when n grows to $+\infty$ of $Y^{n,m}$. That limit $\widetilde{Y}^m$ is greater than $Y = \lim_{n\to+\infty} Y^n$. But for $m \leq m'$ for $t \in [0,T]$:

$$\begin{aligned}
\widetilde{Y}_t^{n,m} - \widetilde{Y}_t^{n,m'} &= \xi^{n,m} - \xi^{n,m'} - \int_t^T \left[\left(\widetilde{Y}_r^{n,m}\right)^{q+1} - \left(\widetilde{Y}_r^{n,m'}\right)^{q+1}\right] dr \\
&\quad - \int_t^T \left[\widetilde{Z}_r^{n,m} - \widetilde{Z}_r^{n,m'}\right] dB_r \\
&\leq \xi^{n,m} - \xi^{n,m'} - \int_t^T \left[\widetilde{Z}_r^{n,m} - \widetilde{Z}_r^{n,m'}\right] dB_r
\end{aligned}$$

and taking the conditional expectation given $\mathcal{F}_t$:

$$0 \leq \widetilde{Y}_t^{n,m} - \widetilde{Y}_t^{n,m'} \leq \mathbb{E}^{\mathcal{F}_t}\left(\xi^{n,m} - \xi^{n,m'}\right) \leq \frac{1}{m}.$$

Letting $m' \to +\infty$ in the last estimate leads to:

$$0 \leq \widetilde{Y}_t^{n,m} - Y_t^n \leq \frac{1}{m}.$$

Therefore **P**-a.s.:

$$\sup_{t\in[0,T]} \left|\widetilde{Y}_t^m - Y_t\right| \leq \frac{1}{m}.$$

Since $\widetilde{Y}^m$ has a limit on the left at T, so does Y.

2.2.3 Proof of (iv) in Theorem 2.1

In order to complete the proof of theorem 2.1, we need to establish the statement (iv). We first show a weaker property:

Proposition 2.4 *For every $\varepsilon > 0$,*

$$\mathbb{E}\int_0^T (T-s)^{\frac{2}{q}+\varepsilon} \|Z_s\|^2 ds < +\infty.$$

Proof. We define the process $\overline{Y}^{n,\varepsilon}$ by:

$$\overline{Y}_t^{n,\varepsilon} = (T-t)^{(1/q+\varepsilon/2)} Y_t^n,\ 0 \leq t \leq T.$$

From the upper bound (ii), $\overline{Y}_t^{n,\varepsilon} \leq (1/q)^{1/q}(T-t)^{\varepsilon/2}$ and $0 \leq \overline{Y}_t^{n,\varepsilon}$. Moreover:

$$\begin{aligned}
d(\overline{Y}_t^{n,\varepsilon}) &= -\left(\frac{1}{q}+\frac{\varepsilon}{2}\right)(T-t)^{1/q+\varepsilon/2-1}(Y_t^n)dt + (T-t)^{1/q+\varepsilon/2}(Y_t^n)^{1+q}dt \\
&\quad +(T-t)^{(1/q+\varepsilon/2)} Z_t^n dB_t \\
&= \left(-\left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\frac{\overline{Y}_t^{n,\varepsilon}}{T-t} + \frac{(\overline{Y}_t^{n,\varepsilon})^{1+q}}{(T-t)^{1+\varepsilon q/2}}\right) dt + (T-t)^{(1/q+\varepsilon/2)} Z_t^n dB_t.
\end{aligned}$$

Now

$$(T-t)^{-1}\overline{Y}^{n,\varepsilon}_t \leq \left(\frac{1}{q}\right)^{\frac{1}{q}} (T-t)^{\varepsilon/2-1}, \text{ and } (T-t)^{-1-\varepsilon q/2}(\overline{Y}^{n,\varepsilon}_t)^{1+q} \leq \left(\frac{1}{q}\right)^{1+\frac{1}{q}} (T-t)^{\varepsilon/2-1},$$

and $t \mapsto (1/q)^{1/q}(T-t)^{\varepsilon/2-1}$ is integrable on $[0,T]$. Denote by $\overline{Z}^{n,\varepsilon}$ the following process: for $0 \leq t \leq T$

$$\overline{Z}^{n,\varepsilon}_t = (T-t)^{(1/q+\varepsilon/2)} Z^n_t.$$

$\overline{Z}^{n,\varepsilon}$ is in $L^2(\Omega \times [0,T];\mathbb{R}^d)$. So $(\overline{Y}^{n,\varepsilon},\overline{Z}^{n,\varepsilon})$ satisfies for $t \in [0,T]$:

$$\begin{aligned}
\overline{Y}^{n,\varepsilon}_t &= \int_t^T \left[\left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\frac{\overline{Y}^{n,\varepsilon}_r}{T-r} - \frac{(\overline{Y}^{n,\varepsilon}_r)^{1+q}}{(T-r)^{1+\varepsilon q/2}}\right] dr - \int_t^T \overline{Z}^{n,\varepsilon}_r dB_r \\
&= \int_t^T \Theta(r,\overline{Y}^{n,\varepsilon}_r) dr - \int_t^T \overline{Z}^{n,\varepsilon}_r dB_r \qquad (2.16)
\end{aligned}$$

with for $r \in [0,T]$:

$$\begin{aligned}
\Theta(r,\overline{Y}^{n,\varepsilon}_r) &= \left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\frac{\overline{Y}^{n,\varepsilon}_r}{T-r} - \frac{(\overline{Y}^{n,\varepsilon}_r)^{1+q}}{(T-r)^{1+\varepsilon q/2}} \\
&= \frac{\overline{Y}^{n,\varepsilon}_r}{T-r}\left(\frac{1}{q}+\frac{\varepsilon}{2} - \frac{(\overline{Y}^{n,\varepsilon}_r)^{q}}{(T-r)^{\varepsilon q/2}}\right) \\
&\leq \left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\frac{\overline{Y}^{n,\varepsilon}_r}{T-r},
\end{aligned}$$

and since $\overline{Y}^{n,\varepsilon}_t \leq (1/q)^{1/q}(T-t)^{\varepsilon/2}$ we have:

$$\frac{1}{q}+\frac{\varepsilon}{2} - \left(\frac{\overline{Y}^{n,\varepsilon}_r}{(T-r)^{\varepsilon/2}}\right)^q \geq \frac{\varepsilon}{2}, \text{ hence } \Theta(r,\overline{Y}^{n,\varepsilon}_r) \geq 0.$$

If we take the square of both sides of (2.16), we obtain:

$$\begin{aligned}
&\left(\int_t^T \Theta(r,\overline{Y}^{n,\varepsilon}_r)dr\right)^2 + \left(\int_t^T \overline{Z}^{n,\varepsilon}_r dB_r\right)^2 \\
&\quad = \left(\overline{Y}^{n,\varepsilon}_t\right)^2 + 2\left(\int_t^T \Theta(r,\overline{Y}^{n,\varepsilon}_r)dr\right)\left(\int_t^T \overline{Z}^{n,\varepsilon}_r dB_r\right) \\
&\quad \leq (1/q)^{2/q}(T-t)^{\varepsilon} + 2\left(\int_t^T \Theta(r,\overline{Y}^{n,\varepsilon}_r)dr\right)^2 + \frac{1}{2}\left(\int_t^T \overline{Z}^{n,\varepsilon}_r dB_r\right)^2,
\end{aligned}$$

that is:

$$\begin{aligned}
\left(\int_t^T \overline{Z}_r^{n,\varepsilon} dB_r\right)^2 &\leq 2(1/q)^{2/q}(T-t)^\varepsilon + 2\left(\int_t^T \Theta(r,\overline{Y}_r^{n,\varepsilon})dr\right)^2 \\
&\leq 2(1/q)^{2/q}(T-t)^\varepsilon + 2\left(\int_t^T \left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\frac{\overline{Y}_r^{n,\varepsilon}}{T-r}dr\right)^2 \\
&\leq 2(1/q)^{2/q}(T-t)^\varepsilon + 2\left(\left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\int_t^T (1/q)^{1/q}(T-r)^{\varepsilon/2-1}dr\right)^2 \\
&= 2(1/q)^{2/q}(T-t)^\varepsilon + 2\left(\left(\frac{1}{q}+\frac{\varepsilon}{2}\right)\left(\frac{1}{q}\right)^{\frac{1}{q}}\frac{2(T-t)^{\varepsilon/2}}{\varepsilon}\right)^2,
\end{aligned}$$

which implies the following inequality:

$$\left(\int_0^T \overline{Z}_r^{n,\varepsilon} dB_r\right)^2 \leq 2T^\varepsilon(1/q)^{2/q}\left[1+4\frac{(1/q+\varepsilon/2)^2}{\varepsilon^2}\right] = 4T^\varepsilon\left(\frac{1}{q}\right)^{2/q}\left(1+\frac{2}{q^2\varepsilon^2}\right).$$

Since $\overline{Z}^{n,\varepsilon}$ is in $L^2(\Omega\times[0,T];\mathbb{R}^d)$ we have:

$$\mathbb{E}\left(\int_0^T \overline{Z}_r^{n,\varepsilon} dB_r\right)^2 = \mathbb{E}\left(\int_0^T \|\overline{Z}_r^{n,\varepsilon}\|^2 dr\right).$$

Then for every $n \geq 1$:

$$\mathbb{E}\int_0^T (T-r)^{\frac{2}{q}+\varepsilon}\|Z_r^n\|^2 dr \leq K_\varepsilon < +\infty,$$

where K_ε is a constant depending only on T, q and ε. Finally for all $\varepsilon > 0$

$$\mathbb{E}\int_0^T (T-r)^{\frac{2}{q}+\varepsilon}\|Z_r\|^2 ds < +\infty.$$

□

Proposition 2.5 *Suppose there exists a constant $\alpha > 0$ such that* **P**-*a.s.* $\xi \geq \alpha$. *In this case:*

(iv) $$\mathbb{E}\int_0^T (T-s)^{2/q}\|Z_s\|^2 ds < +\infty.$$

Proof. If $\xi \geq \alpha$ by comparison for all integer n and all $t \in [0,T]$:

$$Y_t^n \geq \left(\frac{1}{qT+1/\alpha^q}\right)^{1/q} > 0.$$

Let $\delta > 0$ and $\theta : \mathbb{R} \to \mathbb{R}$, $\theta_q : \mathbb{R} \to \mathbb{R}$ defined by:

$$\begin{cases} \theta(x) = \sqrt{x} & \text{on } [\delta, +\infty[, \\ \theta(x) = 0 & \text{on }]-\infty, 0], \end{cases}$$

$$\left\{\begin{array}{lcll}\theta_q(x) & = & x^{\frac{1}{2q}} & \text{on } [\delta,+\infty[,\\ \theta_q(x) & = & 0 & \text{on }]-\infty,0],\end{array}\right.$$

and such that θ and θ_q are non-negative, non-decreasing and in respectively $C^2(\mathbb{R})$ and $C^1(\mathbb{R})$. We apply the Itô formula on $[0,T-\delta]$ to the function $\theta_q(T-t)\theta(Y^n_t)$, with $\delta < (qT+1/\alpha^q)^{-1/q}$:

$$\begin{aligned}\theta_q(\delta)\theta(Y^n_{T-\delta})-\theta_q(T)\theta(Y^n_0) &= \frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y^n_s)^{1/2}\left((Y^n_s)^q-\frac{1}{q(T-s)}\right)ds\\ &+ \frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y^n_s)^{-1/2}Z^n_s.dB_s\\ &- \frac{1}{8}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z^n_s\|^2}{(Y^n_s)^{3/2}}ds\end{aligned}$$

so:

$$\begin{aligned}\frac{1}{8}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z^n_s\|^2}{(Y^n_s)^{3/2}}ds &\leq T^{1/2q}\theta(Y^n_0)+\frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y^n_s)^{-1/2}Z^n_s.dB_s\\ &+\frac{1}{2}\int_0^{T-\delta}(T-s)^{1/2q}(Y^n_s)^{1/2}\left((Y^n_s)^q-\frac{1}{q(T-s)}\right)ds\end{aligned}$$

and since $Y^n_s \leq 1/(q(T-s))^{1/q}$ and $T^{1/q}Y^n_0 \leq q^{-1/q}$, taking the expectation we obtain:

$$\frac{1}{8}\mathbb{E}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z^n_s\|^2}{(Y^n_s)^{3/2}}ds \leq \theta_q(T)\theta(Y^n_0) \leq (1/q)^{1/2q},$$

that is for all n and all $\delta > 0$:

$$\mathbb{E}\int_0^{T-\delta}(T-s)^{1/2q}\frac{\|Z^n_s\|^2}{(Y^n_s)^{3/2}}ds \leq 8(1/q)^{1/2q}.$$

Now since $1/Y^n_s \geq (q(T-s))^{1/q}$,

$$\mathbb{E}\int_0^{T-\delta}(T-s)^{2/q}\|Z^n_s\|^2 ds \leq 8(1/q)^{2/q},$$

and we deduce that

$$\mathbb{E}\int_0^{T}(T-s)^{2/q}\|Z_s\|^2 ds \leq 8(1/q)^{2/q}.$$

□

Now we come back to the case $\xi \geq 0$. We can not apply the Itô formula because we do not have any positive lower bound for Y^n. But we can approximate the process (Y^n,Z^n) by a sequence of processes $(\widetilde{Y}^{n,m},\widetilde{Z}^{n,m})$ as in the proof of the existence of a limit for

Y at time T. Let us recall that we solve the BSDE (2.1) with $\xi^{n,m} = (\xi \wedge n) \vee \frac{1}{m}$ as terminal condition. For all $\delta > 0$ the Itô formula leads to the inequality:

$$\begin{aligned} \mathbb{E}\int_0^{T-\delta} (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} - Z_r^n \right\|^2 dr &\leq \frac{2}{q}\mathbb{E}\int_0^{T-\delta} (T-s)^{2/q-1} \left| \widetilde{Y}_s^{n,m} - Y_s^n \right|^2 ds \\ &\quad + (\delta)^{2/q} \left| \widetilde{Y}_{T-\delta}^{n,m} - Y_{T-\delta}^n \right|^2 \end{aligned}$$

Let δ go to 0 in the previous inequality. We can do that because for all $t \in [0,T]$, $0 \leq \widetilde{Y}_t^{n,m} - Y_t^n$, which implies $\left| \widetilde{Y}_t^{n,m} - Y_t^n \right|^2 \leq n^2$, and because $(T-.)^{2/q-1}$ is integrable on the interval $[0,T]$. Finally using the inequality

$$0 \leq \widetilde{Y}_s^{n,m} - Y_s^n \leq \frac{1}{m},$$

$$\mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} - Z_r^n \right\|^2 dr \leq \frac{2}{q}\frac{1}{m^2}\int_0^T (T-s)^{2/q-1} ds = \frac{T^{2/q}}{m^2}.$$

Therefore for all $\varepsilon > 0$

$$\begin{aligned} \mathbb{E}\int_0^T (T-r)^{2/q} \left\| Z_r^n \right\|^2 dr &\leq \mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} \right\|^2 dr \\ &+ \mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} - Z_r^n \right\|^2 dr \\ &+ 2\mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} \right\| \left\| \widetilde{Z}_r^{n,m} - Z_r^n \right\| dr \\ &\leq (1+\varepsilon)\mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} \right\|^2 dr \\ &+ (1+\frac{1}{\varepsilon})\mathbb{E}\int_0^T (T-r)^{2/q} \left\| \widetilde{Z}_r^{n,m} - Z_r^n \right\|^2 dr \\ &\leq 8(1+\varepsilon)(1/q)^{2/q} + \frac{T^{2/q}}{m^2}(1+\frac{1}{\varepsilon}). \end{aligned}$$

We have applied the previous result to $\widetilde{Z}^{n,m}$. Now we let first m go to $+\infty$ and then ε go to 0, we have:

$$\mathbb{E}\int_0^T (T-r)^{2/q} \left\| Z_r^n \right\|^2 dr \leq 8(1/q)^{2/q}.$$

The result follows by letting finally n go to ∞.

2.3 Continuity of Y at T

In this section ξ is still supposed non-negative. We precise the behaviour of Y in a neighbourhood of T (proposition 2.1) and we show the continuity of Y at T under stronger assumptions (theorem 2.2).

2.3.1 Lower bound and asymptotic behaviour in a neighbourhood of T

Now we construct an adapted process which is smaller than Y.

Lemma 2.2
For $0 \leq t < T$, $\mathbf{P}$-a.s.

$$Y_t \geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right].$$

Remark 2.1 *The right hand side is obtained through the following operation: first we solve the ordinary differential equation $y' = y^{1+q}$ with ξ as terminal condition, then we project this solution on the σ-algebra $\mathcal{F}_t$.*

Proof. Let $n \in \mathbb{N}^*$ and consider for $0 \leq t \leq T$:

$$\Gamma_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{(\xi\wedge n)^q}}\right)^{1/q}\right].$$

Γ^n is well defined because the term in the conditional expectation is bounded by n. We have:

$$\left(\frac{1}{q(T-t)+\frac{1}{(\xi\wedge n)^q}}\right)^{1/q} = \xi\wedge n - \int_t^T \left[\frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1}{q}}}\right]^{1+q} dr$$

So Γ^n verifies:

$$\begin{aligned}
\Gamma_t^n &= \mathbb{E}^{\mathcal{F}_t}\left(\xi\wedge n - \int_t^T \frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1+q}{q}}} dr\right) \\
&= \mathbb{E}^{\mathcal{F}_t}\left(\xi\wedge n - \int_t^T (\Gamma_r^n)^{1+q} dr - \int_t^T U_r^n dr\right)
\end{aligned}$$

with U^n the adapted and bounded (by n^{1+q}) process:

$$U_r^n = \mathbb{E}^{\mathcal{F}_r}\left(\frac{1}{\left(q(T-r)+\frac{1}{(\xi\wedge n)^q}\right)^{\frac{1+q}{q}}}\right) - (\Gamma_r^n)^{1+q};$$

the Jensen's inequality ($1+q > 1$) showing that $U_r^n \geq 0$ for $0 \leq r \leq T$. Then the comparison theoren 2.4 in [42] allows us to conclude that for all $t \in [0,T]$, a.s.

$$\Gamma_t^n \leq Y_t^n \leq Y_t.$$

We then deduce from the monotone convergence theorem:

$$\lim_{n\to+\infty} \Gamma_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right] = \Gamma_t.$$

□

We now establish the:

Proposition 1 On the set $\{\xi = +\infty\}$

$$\lim_{t\uparrow T}(T-t)^{1/q}Y_t = \left(\frac{1}{q}\right)^{1/q}, \text{ a.s..} \tag{2.5}$$

Proof. Indeed

$$\begin{aligned}\Gamma_t &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+q(T-t)\xi^q}\right)^{1/q} + \frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right)\\ &\geq \mathbb{E}^{\mathcal{F}_t}\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+qT\xi^q}\right)^{1/q} + \frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).\end{aligned}$$

Then

$$(T-t)^{1/q}\Gamma_t \geq (T-t)^{1/q}\mathbb{E}^{\mathcal{F}_t}\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+qT\xi^q}\right)^{1/q} + \left(\frac{1}{q}\right)^{1/q}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).$$

The first term in the right hand side converges to 0 on the set $\{\xi = +\infty\}$ and the second converges to $(1/q)^{1/q}$. Indeed we have

$$0 \leq \frac{\xi^q}{1+qT\xi^q} \leq \frac{1}{qT}$$

and we can apply the martingale convergence theorem. Since Y is bounded from above by $1/(q(T-t))^{1/q}$, this achieves the proof. □

With the assumption (H1) and:

$$\xi \mathbf{1}_{\{\xi<\infty\}} \in L^1(\Omega,\mathcal{F}_T,\mathbf{P}), \tag{H2'}$$

we can prove directly (see Appendix):

$$\lim_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right] = \xi.$$

2.3.2 Continuity at time T: the first step

We now want to prove Theorem 2.2, i.e. $\xi \geq \limsup_{t\to T} Y_t$. Recall that we already know that the limit of Y_t as $t \to T$ exists a.s. From the inequality (2.14):

$$\xi \leq \liminf_{t\to T} Y_t,$$

we just have to show that on the set $\{\xi < +\infty\}$

$$\xi \geq \lim_{t\to T} Y_t = \liminf_{t\to T} Y_t.$$

The main difficulty here is to find a "good" upper bound of Y_t. We shall use a method widely inspired by the article of M. Marcus and L. Véron [37] and more precisely by the proof of lemma 2.2 page 1450. We try to adapt this method to our case.

We make stronger assumptions on ξ. From now and for the rest of this paper we suppose that the conditions (H1) and (H2) hold. For convenience we recall those assumptions:

$$\xi = g(X_T), \tag{H1}$$

where g is a function defined on $\mathbb{R}^k$ with values in $\overline{\mathbb{R}^+}$ such that the condition (H2) is satisfied and that the set $F = \{g = +\infty\}$ is closed ; and where X_T is the value at $t = T$ of a diffusion process or more precisely of the solution of the SDE:

$$X_t = x + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dB_r, \text{ for } t \in [0,T]. \tag{2.6}$$

The functions b and σ are defined on $[0,T] \times \mathbb{R}^m$, with values respectively in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, and such that there exists a constant $K > 0$ s.t. for all t in $[0,T]$ and for all $(x,y) \in \mathbb{R}^m \times \mathbb{R}^m$:

1. Lipschitz condition:

$$|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K|x - y|; \tag{L}$$

2. Growth condition:

$$|b(t,x)| \leq K(1 + |x|) \text{ and } \|\sigma(t,x)\| \leq K(1 + |x|). \tag{G}$$

In the rest of this paper, (L) and (G) are supposed to be satisfied. With these assumptions (see [49] or [52]), for every $x \in \mathbb{R}^m$, there exists an unique solution X, with values in $\mathbb{R}^m$, of the SDE (2.6). Moreover we add the second condition: for all compact set $K \subset (\mathbb{R}^m \setminus F)$

$$g(X_T)\mathbf{1}_K(X_T) \in \ L^1\left(\Omega,\mathcal{F}_T,\mathbf{P};\mathbb{R}\right). \tag{H2}$$

Recall that X is the solution of the SDE (2.6). Let φ be a function in the class $C^2(\mathbb{R}^m)$ with a compact support. Let (Y,Z) be the solution of the BSDE (2.1) with the final condition $\zeta \in L^2(\Omega)$. For any $t \in [0,T]$:

$$\begin{aligned}
Y_t\varphi(X_t) &= Y_0\varphi(X_0) + \int_0^t \varphi(X_r)\left[Y_r|Y_r|^q dr + Z_r.dB_r\right] + \int_0^t Y_r d(\varphi(X_r)) \\
&\quad + \int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr \\
&= Y_0\varphi(X_0) + \int_0^t \varphi(X_r)Y_r|Y_r|^q dr + \int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr \\
&\quad + \int_0^t Y_r\left(\nabla\varphi(X_r).b(r,X_r) + \frac{1}{2}\text{Trace}(D^2\varphi(X_r)\sigma\sigma^*(r,X_r))\right)dr \\
&\quad + \int_0^t \left(Y_r\nabla\varphi(X_r)\sigma(r,X_r) + \varphi(X_r)Z_r\right).dB_r
\end{aligned}$$

where $D^2\varphi$ is the Hessian matrix of φ. Taking the expectation:

$$\begin{aligned}
\mathbb{E}(Y_t\varphi(X_t)) &= \mathbb{E}(Y_0\varphi(X_0)) + \mathbb{E}\int_0^t \varphi(X_r)Y_r|Y_r|^q dr \\
&\quad +\mathbb{E}\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr \\
&\quad +\mathbb{E}\int_0^t Y_r\left(\nabla\varphi(X_r).b(r,X_r) + \frac{1}{2}\text{Trace}(D^2\varphi(X_r)\sigma\sigma^*(r,X_r))\right)dr \\
&= \mathbb{E}(Y_0\varphi(X_0)) + \mathbb{E}\int_0^t \varphi(X_r)Y_r|Y_r|^q dr \qquad (2.17)\\
&\quad +\mathbb{E}\int_0^t Z_r.\nabla\varphi(X_r)\sigma(r,X_r)dr + \mathbb{E}\int_0^t Y_r L\varphi(X_r)dr
\end{aligned}$$

where L is the infinitesimal generator of the process X, i.e. the following second order partial differential operator:

$$L = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j} + \sum_{i=1}^{d} b_i(t,x)\frac{\partial}{\partial x_i}.$$

The idea is to use the relation (2.17) with a suitable function φ. The set $F^c = \{g < +\infty\}$ is open in $\mathbb{R}^m$. Let U be a bounded open set with a regular boundary and such that the compact set $K = \overline{U}$ is included in F^c. We denote by $\Phi = \Phi_U$ a function which is supposed to belong to $C^2(\mathbb{R}^m;\mathbb{R}_+)$ and such that Φ is equal to zero on U^c, is strictly positive on U. Let α be a real number such that

$$\alpha > 2(1 + 1/q).$$

For $n \in \mathbb{N}$ let (Y^n,Z^n) be the solution of the BSDE (2.1) with the final condition $(g \wedge n)(X_T)$. The equality (2.17) becomes for $0 \leq t \leq T$:

$$\begin{aligned}
\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) &= \mathbb{E}(Y_t^n\Phi^\alpha(X_t)) + \mathbb{E}\int_t^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr \\
&+ \mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n L(\Phi^\alpha)(X_r)dr. \qquad (2.18)
\end{aligned}$$

Using (2.18), we will first prove that for every real $\alpha > 2(1 + 1/q)$ and for every n:

$$\mathbb{E}\int_0^T (Y_r^n)^{1+q}\Phi^\alpha(X_r)dr \leq K < \infty,$$

where the constant K is independent of n. In particular we will have to find a bound for the term containing Z in the right hand side of the equation (2.18). We know how to control this term in the two cases of the theorem 2.2: we suppose that either $q > 2$ or (L)-(B)-(E) are satisfied. Thanks to the Fatou lemma $Y^{1+q}\Phi^\alpha(X)$ will belong to $L^1(\Omega \times [0,T])$. Then we will deduce that the limit as t goes to T of Y_t is less than or equal to ξ a.s. on the set $\{\xi < +\infty\}$.

2.3.3 Continuity when $q > 2$

In this section we will suppose that $q > 2$ and that b and σ satisfy the two assumptions (L) and (G).

In that case we can control easily the term

$$\mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r).$$

Using the Cauchy-Schwarz inequality we obtain:

$$\left|\mathbb{E}\int_0^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr\right| \\ \leq \left(\mathbb{E}\int_0^T \|Z_r^n\|^2(T-r)^{2/q}dr\right)^{1/2}\left(\mathbb{E}\int_0^T \frac{\|\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)\|^2}{(T-r)^{2/q}}dr\right)^{1/2}$$

From a previous result of the first section

$$\mathbb{E}\int_0^T \|Z_r^n\|^2(T-r)^{2/q}dr \leq 8\left(\frac{1}{q}\right)^{\frac{2}{q}}.$$

And the second term

$$\mathbb{E}\int_0^T \frac{\|\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)\|^2}{(T-r)^{2/q}}dr$$

is finite if $q > 2$. Indeed recall that Φ is compactly supported and $\alpha > 2$. Hence the numerator is bounded for all $(t,x) \in [0,T] \times \mathbb{R}^m$:

$$\|\nabla(\Phi^\alpha)(x)\sigma(t,x)\|^2 \leq \alpha^2\Phi^{2(\alpha-1)}(x)\|\nabla\Phi(x)\|^2(K(1+|x|))^2 \leq \widetilde{K}.$$

Therefore there exists a constant $k = k(q,\Phi,\alpha,\sigma)$ such that for all $t \in [0,T]$ and $n \in \mathbb{N}$:

$$\left|\mathbb{E}\int_0^t Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr\right| \leq k. \tag{2.19}$$

The term

$$\mathbb{E}\int_0^T Y_r^n L(\Phi^\alpha)(X_r)dr$$

can be bounded using the Hölder inequality. Let p be such that

$$\frac{1}{p}+\frac{1}{1+q}=1,$$

or $p = 1 + 1/q$ and $p/(1+q) = p-1$.

We will prove that

$$\Phi^{-\alpha(p-1)}\left|L(\Phi^\alpha)\right|^p \in L^\infty([0,T]\times\mathbb{R}^m),$$

from which follows that for $0 \leq t \leq T$:

$$\mathbb{E}\left|Y_t^n L(\Phi^\alpha)(X_t)\right| \leq K\mathbb{E}\left[\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right]^{\frac{1}{1+q}}.$$

Using the growth condition (G) on σ we have:

$$\Phi^{-\alpha(p-1)} \left|\text{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p \leq K\Phi^{-\alpha(p-1)} \left\|D^2(\Phi^\alpha)\right\|^p (1+|x|^2)^p,$$

and

$$\begin{array}{rcl} D^2(\Phi^\alpha) & = & \alpha\Phi^{\alpha-1}D^2\Phi + \alpha(\alpha-1)\Phi^{\alpha-2}\nabla\Phi\otimes\nabla\Phi \\ \Phi^{-\alpha(p-1)} \left\|D^2(\Phi^\alpha)\right\|^p & \leq & 2^{p-1}\left(\alpha^p\Phi^{\alpha-p}\left\|D^2\Phi\right\|^p + (\alpha(\alpha-1))^p\Phi^{\alpha-2p}|\nabla\Phi|^{2p}\right). \end{array}$$

Hence:

$$\Phi^{-\alpha(p-1)} \left|\text{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p \leq K\Phi^{\alpha-2p}\Psi$$

where Ψ is the continuous function on $\mathbb{R}^m$:

$$\Psi = \left[\alpha^p\Phi^p\left\|D^2\Phi\right\|^p + (\alpha(\alpha-1))^p|\nabla\Phi|^{2p}\right]\left[1+|x|^{2p}\right].$$

Since $\alpha > 2(1+1/q) = 2p$ and since Φ has a compact support,

$$\Phi^{-\alpha(p-1)} \left|\text{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p \in L^\infty([0,T]\times\mathbb{R}^m).$$

For the second term:

$$\Phi^{-\alpha(p-1)} \left|\nabla(\Phi^\alpha).b(t,.)\right|^p,$$

we have:

$$\begin{array}{rcl} \nabla(\Phi^\alpha) & = & \alpha\Phi^{\alpha-1}\nabla\Phi, \\ \Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha)\right|^p & = & \alpha^p\Phi^{\alpha-p}\left|\nabla\Phi\right|^p. \end{array}$$

Since $\alpha > 2p$ and $|b(t,x)| \leq K(1+|x|)$,

$$\Phi^{-\alpha(p-1)} \left|\nabla(\Phi^\alpha).b(t,.)\right|^p \leq K^p\Phi^{-\alpha(p-1)}\left|\nabla(\Phi^\alpha)\right|^p(1+|x|^p) = (K\alpha)^p\Phi^{\alpha-p}\left|\nabla\Phi\right|^p(1+|x|^p)$$

Therefore

$$\Phi^{-\alpha(p-1)} \left|\nabla(\Phi^\alpha).b(t,.)\right|^p \in L^\infty([0,T]\times\mathbb{R}^m).$$

Since

$$|L(\Phi^\alpha)|^p \leq 2^{p-1}\left|\nabla(\Phi^\alpha).b(t,.)\right|^p + 2^{p-1}\left|\text{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.))\right|^p,$$

we apply the Hölder inequality:

$$\begin{array}{rcl} \mathbb{E}\left|Y_t^nL(\Phi^\alpha)(X_t)\right| & \leq & \left[\mathbb{E}\left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right)\right]^{\frac{1}{1+q}} \times \left[\mathbb{E}\left(\Phi^{-\alpha(p-1)}(X_t)\left|L(\Phi^\alpha)(X_t)\right|^p\right)\right]^{1/p} \\ & \leq & K\left[\mathbb{E}\left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right)\right]^{\frac{1}{1+q}} \end{array}$$

and the constant K depends only on q, b, σ, Φ and α, not on n, neither on t. Finally we have:

$$\begin{array}{rcl} \mathbb{E}\displaystyle\int_0^T\left|Y_r^nL(\Phi^\alpha)(X_r)\right|dr & \leq & K\displaystyle\int_0^T\left[\mathbb{E}\left(\Phi^\alpha(X_r)(Y_r^n)^{1+q}\right)\right]^{\frac{1}{1+q}}dr \\ & \leq & K\left[\mathbb{E}\displaystyle\int_0^T\Phi^\alpha(X_r)(Y_r^n)^{1+q}dr\right]^{\frac{1}{1+q}} \end{array} \tag{2.20}$$

We come back to the equation (2.18):

$$\begin{aligned}\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) &= \mathbb{E}(Y_t^n\Phi^\alpha(X_t)) + \mathbb{E}\int_t^T Z_r^n.\nabla(\Phi^\alpha)(X_r)\sigma(r,X_r)dr \\ &+ \mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n L(\Phi^\alpha)(X_r)dr\end{aligned}$$

$Y_T^n\Phi^\alpha(X_T) \leq g(X_T)\Phi^\alpha(X_T)$; since Φ is equal to zero outside a compact set included in $F^c = \{g < +\infty\}$, using the condition (H2) the left hand side of the previous equality is bounded by a constant independent of n. If $\Phi^\alpha(X)(Y^n)^{1+q}$ is not a bounded sequence in $L^1(\Omega \times [0,T])$, there exists a subsequence n_k along which $\mathbb{E}\int_0^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr$ tends to infinity. With the previous inequalities (2.19) and (2.20) and since $1/(1+q) < 1$ the right member in (2.18) (extracting a subsequence if necessary) would tend to $+\infty$, which is a contradiction with the fact that the left member is bounded. Hence we have proved:

Lemma 2.3 *For* $\alpha > 2(1+1/q)$*, the sequence* $\Phi^\alpha(X)(Y^n)^{1+q}$ *is a bounded sequence in* $L^1(\Omega \times [0,T])$ *and with the Fatou lemma* $Y^{1+q}\Phi^\alpha(X)$ *belongs to* $L^1(\Omega \times [0,T])$*:*

$$\mathbb{E}\int_0^T Y_r^{1+q}\Phi^\alpha(X_r)dr < +\infty.$$

This inequality allows us to show that

$$\liminf_{t\to T} Y_t \leq \xi.$$

Indeed let θ be a function of class $C^2(\mathbb{R}^m;\mathbb{R}^+)$ with a compact support strictly included in U. Let us recall that α is strictly greater than $2(1+1/q) > 2$. Thanks to a result in the proof of the lemma 2.2 of [37] there exists a constant $k = k(\theta,\alpha)$ such that

$$|\theta| \leq k\Phi^\alpha \ , \ |\nabla\theta| \leq k\Phi^{\alpha-1} \text{ and } \left\|D^2\theta\right\| \leq k\Phi^{\alpha-2}.$$

We write again the equation (2.18) for θ, $n \in \mathbb{N}^*$ and $0 \leq t \leq T$:

$$\begin{aligned}\mathbb{E}(Y_T^n\theta(X_T)) &= \mathbb{E}(Y_t^n\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)dr \\ &+ \mathbb{E}\int_t^T Y_r^n\left(\nabla\theta(X_r).b(r,X_r) + \frac{1}{2}\text{Trace}(D^2\theta(X_r)\sigma\sigma^*(r,X_r))\right)dr\end{aligned}$$

that is

$$\begin{aligned}\mathbb{E}((\xi\wedge n)\theta(X_T)) &= \mathbb{E}(Y_t^n\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n L\theta(X_r)dr \\ &+ \mathbb{E}\int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)dr\end{aligned}$$

In the left hand side we use the assumption (H2) to pass to the limit as n tends to ∞. We just have to control the right hand side as n tends to infinity. For the first term there is no problem: we use the dominated convergence theorem. For the second we apply the monotone convergence theorem. For the third one we can do the same calculations using the previously given estimations on θ, $\nabla\theta$ and $D^2\theta$ in terms of power of Φ^α and Hölder's inequality:

$$\Phi^{-\alpha(p-1)} |L\theta|^p \in L^\infty([0,T] \times \mathbb{R}^m). \tag{2.21}$$

Now we can write:

$$Y_r^n L\theta(X_r) = \left(Y_r^n \Phi^{\alpha/(1+q)}\right) \left(\Phi^{-\alpha/(1+q)} L\theta(X_r)\right) = \left(Y_r^n \Phi^{\alpha/(1+q)}\right) \left(\Phi^{-\alpha(p-1)/p} L\theta(X_r)\right),$$

The sequence $Y^n \Phi^{\alpha/(1+q)} = Y^n \Phi^{\alpha(1-1/p)}$ is a bounded sequence in $L^{1+q}(\Omega \times [0,T])$ (see Lemma 2.3). With (2.21), using a weak convergence result and extracting a subsequence if necessary, we can pass to the limit in the term:

$$\mathbb{E} \int_t^T Y_r^n L\theta(X_r) dr.$$

For the remaining term

$$\mathbb{E} \int_t^T Z_r^n . \nabla\theta(X_r) \sigma(r,X_r) dr$$

recall that there exists a constant $k = k(q)$ for all $n \in \mathbb{N}$:

$$\mathbb{E} \int_0^T \|Z_r^n\|^2 (T-r)^{\frac{2}{q}} dr \leq k.$$

Hence there exists a subsequence, which we still denote $Z^n(T-r)^{1/q}$, and which converges weakly in the space $L^2\left(\Omega \times (0,T), d\mathbf{P} \times dt; \mathbb{R}^d\right)$ to a limit, and the limit is $Z(T-r)^{1/q}$, because we already know that Z^n converges to Z in $L^2(\Omega \times (0,T-\delta))$ for all $\delta > 0$. $\nabla\theta(X)\sigma(.,X)(T- .)^{-1/q}$ is $L^2(\Omega \times (0,T))$, because θ is compactly supported and $q > 2$. Therefore

$$\lim_{n \to +\infty} \mathbb{E} \int_t^T Z_r^n . \nabla\theta(X_r) \sigma(r,X_r) dr = \mathbb{E} \int_t^T Z_r . \nabla\theta(X_r) \sigma(r,X_r) dr.$$

Passing to the limit:

$$\begin{aligned} \mathbb{E}(\xi\theta(X_T)) &= \mathbb{E}(Y_t\theta(X_t)) + \mathbb{E} \int_t^T \theta(X_r)(Y_r)^{1+q} dr + \mathbb{E} \int_t^T Y_r L\theta(X_r) dr \\ &+ \mathbb{E} \int_t^T Z_r . \nabla\theta(X_r) \sigma(r,X_r) dr \end{aligned}$$

We let t go to T and we apply Fatou's lemma:

$$\mathbb{E}(\xi\theta(X_T)) = \lim_{t \to T} \mathbb{E}(Y_t\theta(X_t)) \geq \mathbb{E}(\liminf_{t \to T} Y_t\theta(X_T))$$

and this is true for every non-negative function θ whose support is in U. So

$$\xi \geq \liminf_{t \to T} Y_t \quad \mathbf{P} - \text{a.s.}$$

and we have:

$$\liminf_{t \to T} Y_t = \xi.$$

But we have proved before that the limit of Y_t exists, so we have proved that this limit is equal to ξ and therefore the continuity of Y on $[0,T]$ in the case $q > 2$.

2.3.4 When assumptions (L)-(G)-(E) are satisfied and $q > 0$

If we just assume $q > 0$, our previous control on the term containing Z in (2.18) fails. But with the assumptions (L)-(B)-(E):

(L). for all $(t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m$, $|b(t,x) - b(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leq K|x - y|$;

(B). σ and b are bounded: there exists a constant K s.t.

$$\forall (t,x) \in [0,T] \times \mathbb{R}^m, \quad |b(t,x)| + \|\sigma(t,x)\| \leq K;$$

(E). $\sigma\sigma^*$ is uniformly elliptic, i.e. there exists $\lambda > 0$ s.t. for all $(t,x) \in [0,T] \times \mathbb{R}^m$ and all $y \in \mathbb{R}^m$:

$$\sigma\sigma^*(t,x)y.y \geq \lambda |y|^2;$$

we are able to prove that there exists a function ψ such that for $0 < t \leq T$:

$$\mathbb{E} \int_t^T Z_r^n.\nabla\theta(X_r)\sigma(r,X_r)dr = \mathbb{E} \int_t^T Y_r^n \psi(r,X_r)dr,$$

and then we apply again the Hölder inequality in order to control

$$\mathbb{E} \int_t^T Y_r^n \psi(r,X_r)dr$$

by

$$\mathbb{E} \int_t^T (Y_r^n)^{1+q} \Phi^\alpha(X_r)dr.$$

We need the existence of a regular density for the process X solution of the SDE:

$$X_t = x + \int_0^t b(r,X_r^x)dr + \int_0^t \sigma(r,X_r^x)dB_r, \text{ for } t \in [0,T] \text{ and } x \in \mathbb{R}^m. \tag{2.6}$$

According to the article of D.G. Aronson [1], section 6, p. 651, there exists a density (Green's function) for X, $p(x; .,.) \in L^\alpha \left(0,T; H_0^\beta\right)$ where (α,β) are such that $1/\alpha + 1/\alpha' = 1$, $1/\beta + 1/\beta' = 1$ with $2 < \alpha', \beta' \leq \infty$ and $\frac{m}{2\alpha'} + \frac{1}{\beta'} < \frac{1}{2}$.

Moreover in the article of D.W. Stroock [51], theorem II.3.8, p. 344, it is shown that the density is Hölder continuous in x and satisfies the following inequality for $s > 0$:

$$\frac{\exp\left(-Ms - M\frac{|y-x|^2}{s}\right)}{Ms^{m/2}} \leq p(x; s,y) \leq \frac{M\exp\left(Ms - \frac{|y-x|^2}{Ms}\right)}{s^{m/2}}, \tag{2.22}$$

M depending only on the bounds of σ and b, on the fact that σ is uniformly elliptic, and M is independent of the regularity of these functions.

From now on we omit the variable x in $p(x; .,.)$.

A preliminary result

We now prove the following result for the solution (Y,Z) of the BSDE:

$$Y_t = h(X_T) + \int_t^T f(Y_r)dr - \int_t^T Z_r dB_r,$$

with $h(X_T) \in L^2(\Omega)$.

Proposition 2.6
Under the assumptions (L)-(B)-(E), for each function φ in the class $C^2(\mathbb{R}^m)$ with a compact support, there exists a real function ψ defined on $]0,T] \times \mathbb{R}^m$ s.t. for all $t > 0$, $\mathbb{E}(|Y_t \psi(t,X_t)|) < +\infty$ and

$$\mathbb{E}\left[Z_t.\nabla\varphi(X_t)\sigma(t,X_t)\right] = -\,\mathbb{E}\left[Y_t\psi(t,X_t)\right].$$

The function ψ is given by the following formula:

$$\psi(t,x) = \sum_{i=1}^d (\nabla\varphi\sigma)_i(x)\frac{div(p\sigma^i)(t,x)}{p(t,x)} + Trace(D^2\varphi\sigma\sigma^*)(x) + \sum_{i=1}^d \nabla\varphi.[(\nabla\sigma^i)\sigma^i](x);$$

where σ^i is the i-th column of the matrix σ and p is the density of the process X.

Proof. To find this function ψ we use the following result:

Proposition 2.7
For all $1 \leq i \leq d$, $\{D_s^i Y_s, 0 \leq s \leq T\}$ is a version of $\{(Z_s)_i, 0 \leq s \leq T\}$. D denotes the Malliavin derivative.

$(Z_s)_i$ denotes the i-th component of Z_s. This result is the proposition 2.2 of the article of E. Pardoux and S. Peng [43]. Here $D_s^i Y_s$ has the following sense:

$$D_s^i Y_s = \lim_{\substack{r \to s \\ r < s}} D_r^i Y_s.$$

because for a fixed $r \in [t,T]$, for $1 \leq i \leq d$ and for $r \leq s \leq T$:

$$D_r^i Y_s = (Z_r)_i + (1+q)\int_r^s |Y_u|^q D_r^i Y_u du - \int_r^s D_r^i Z_u dB_u.$$

We must calculate:

$$\mathbb{E}\left[Z_t.\nabla\varphi(X_t)\sigma(t,X_t)\right] = \mathbb{E}\left[D_t Y_t.\nabla\varphi\sigma(X_t)\right] = \sum_{i=1}^d \mathbb{E}\left[D_t^i Y_t.(\nabla\varphi\sigma)_i(X_t)\right],$$

where $\nabla\varphi\sigma(X_t) = (\nabla\varphi)(X_t)\sigma(t,X_t)$ and $(\nabla\varphi\sigma)_i(X_t)$ denotes the i-th component of $\nabla\varphi\sigma(X_t)$.

Let h_m^i, $m \in \mathbb{N}^*$, be the following function:

$$h_m^i(r) = m\mathbf{1}_{[t-1/m,t]}(r)e_i,$$

with $r \in [0,T]$ and with $(e_1, \ldots, e_d)$ the canonical basis of $\mathbb{R}^d$. Here we need $t > 0$. We denote

$$D_{h^i_m} Y_t = \langle DY_t, h^i_m \rangle_H,$$

H being the Hilbert space $L^2([0,T];\mathbb{R}^d)$. The integration by parts formula for the Malliavin derivative is the following:
(2.23)

$$\mathbb{E}\left[D_{h^i_m} Y_t (\nabla\varphi\sigma)_i(X_t)\right] = \mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(X_t) \int_0^T h^i_m(r).dB_r\right] - \mathbb{E}\left[Y_t D_{h^i_m}\left((\nabla\varphi\sigma)_i(X_t)\right)\right]$$

Now we calculate the first term of the right hand side.

$$\begin{aligned}\mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(X_t) \int_0^T h^i_m(r).dB_r\right] &= m\, \mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(X_t)(B^i_t - B^i_{t-1/m})\right] \\ &= -m\, \mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(X_t) \int_{t-1/m}^t \frac{\operatorname{div}(p\sigma^i)(u,X_u)}{p(u,X_u)} du\right]\end{aligned}$$

where p is the density of X and σ^i is the i-th column of the matrix σ.

The last equality is justified by the lemmas 3.1 and 4.1 of the article of E. Pardoux [41]. We must show that the assumptions of these lemmas are satisfied. First Y is a function X: there exists a continuous $u : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ such that $Y_t = u(t,X_t)$. Hence we can write:

$$\mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(X_t)(B^i_t - B^i_{t-1/m})\right] = \mathbb{E}\left[g(t,X_t)(B^i_t - B^i_{t-1/m})\right]$$

with $g(t,x) = u(t,x)(\nabla\varphi\sigma)_i(x)$. Since φ has a compact support, so does g. And g is measurable and bounded. So the conditions of the lemma 4.1 are verified, the time dependence of g playing no role in the proof. (L)-(B)-(E) are not exactly the assumptions of the lemmas 3.1 et 4.1. but with these conditions the conclusions are the same. In fact the main problem is to give a sense to the fraction $\operatorname{div}(p\sigma^i)/p$. The condition

$$p \in L^\alpha\left(0,T; H^\beta_0\right)$$

implies that for all i, a.s., the distributional derivative $\frac{\partial p}{\partial x_i}$ equals the usual derivative (see [38], theorem 2.2, p.61). This derivative is zero at any point of $\{(t,x); p(t,x) = 0\}$ where it does exist, since at such a point $p(t,x)$ attains his minimum.

In the case where X is a Brownian motion B (i.e. $\sigma = Id$ and $b = 0$), we can make a easier calculation:

$$m\mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(B_t)(B^i_t - B^i_{t-1/m})\right] = m\mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(B_t)(B^i_t - \mathbb{E}(B^i_{t-1/m}|B_t))\right]$$

and

$$\mathbb{E}(B^i_{t-1/m}|B_t)) = \frac{t-1/m}{t} B^i_t$$

(we need $t > 0$ in order to divide by t), so

$$m\mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(B_t)(B^i_t - B^i_{t-1/m})\right] = \mathbb{E}\left[Y_t (\nabla\varphi\sigma)_i(B_t) \frac{B^i_t}{t}\right].$$

Let m goes to $+\infty$ in the identity (2.23):

$$\mathbb{E}\left[D_t^i Y_t (\nabla\varphi\sigma)_i(X_t)\right] = -\mathbb{E}\left[Y_t(\nabla\varphi\sigma)_i(X_t)\frac{\mathrm{div}(p\sigma^i)(t,X_t)}{p(t,X_t)}\right] - \mathbb{E}\left[Y_t D_t^i\left((\nabla\varphi\sigma)_i(X_t)\right)\right]$$

To find $D_t^i\left((\nabla\varphi\sigma)_i(X_t)\right)$ recall the following result (see proposition 1.2.3 in [39]): since the process X has a density, if μ is a Lipschitz function, then

$$D_t^i \mu(X_t) = \nabla\mu(X_t) D_t^i(X_t) = \nabla\mu(X_t)\sigma^i(X_t).$$

Applying this result with $\mu = (\nabla\varphi\sigma)_i$ we obtain:

$$D_t^i\left((\nabla\varphi\sigma)_i(X_t)\right) = (\sigma^* D^2\varphi\sigma)_{ii}(X_t) + \nabla\varphi.[\nabla\sigma^i.\sigma^i](X_t)$$

Finally

$$\begin{aligned}
\mathbb{E}\left[Z_t.\nabla\varphi(X_t)\sigma(t,X_t)\right] &= \sum_{i=1}^d \mathbb{E}\left[D_t^i Y_t.(\nabla\varphi\sigma)_i(X_t)\right] = \mathbb{E}\left[Y_t\psi(t,X_t)\right] \\
&= -\mathbb{E}\left[Y_t\left(\sum_{i=1}^d (\nabla\varphi\sigma)_i(X_t)\frac{\mathrm{div}(p\sigma^i)(t,X_t)}{p(t,X_t)}\right)\right] \\
&\quad -\mathbb{E}\left[Y_t\left(\mathrm{Trace}(D^2\varphi\sigma\sigma^*)(X_t) + \sum_{i=1}^d \nabla\varphi.[(\nabla\sigma^i)\sigma^i](X_t)\right)\right]
\end{aligned}$$

□

Coming back to the equation (2.18) for $t > 0$:

$$\begin{aligned}
\mathbb{E}(Y_T^n\varphi(X_T)) &= \mathbb{E}(Y_t^n\varphi(X_t)) + \mathbb{E}\int_t^T \varphi(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n\psi(r,X_r)dr \\
&\quad +\mathbb{E}\int_t^T Y_r^n\left(\nabla\varphi(X_r).b(r,X_r) + \frac{1}{2}\mathrm{Trace}(D^2\varphi(X_r)\sigma\sigma^*(r,X_r))\right)dr \\
&= \mathbb{E}(Y_t^n\varphi(X_t)) + \mathbb{E}\int_t^T \varphi(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n\Psi(r,X_r)dr \qquad (2.24)
\end{aligned}$$

with Ψ the following function: for $t \in]0,T]$ and $x \in \mathbb{R}^m$:

$$\begin{aligned}
\Psi(t,x) &= \nabla\varphi(x).b(t,x) - \sum_{i=1}^d\left((\nabla\varphi(x)\sigma(t,x))_i\,\frac{\mathrm{div}(p(t,x)\sigma^i(t,x))}{p(t,x)}\right) \\
&\quad -\frac{1}{2}\mathrm{Trace}(D^2\varphi(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^d\left(\nabla\varphi(x).[\nabla\sigma^i(t,x)\sigma^i(t,x)]\right)
\end{aligned}$$

In the case where X is a Brownian motion B, this formula becomes:

$$\Psi(t,x) = \nabla\varphi(x).\frac{x}{t} - \frac{1}{2}\Delta\varphi(x).$$

Continuity with (L)-(B)-(E)

Recall that U is a bounded open set such that $K = \overline{U}$ is contained in $F^c = \{g < \infty\}$, that $\Phi = \Phi_U$ is a function which is supposed to belong to $C^2(\mathbb{R}^m;\mathbb{R}_+)$ and such that Φ is equal to zero on U^c, is strictly positive on U. α is a real such that $\alpha > 2(1+1/q)$. For $n \in \mathbb{N}$ let (Y^n,Z^n) be the solution of the BSDE (2.1) with the final condition $(g \wedge n)(X_T)$. For $0 < t \leq T$ the relation (2.24) is:

$$(2.25)\quad \mathbb{E}(Y_T^n \Phi^\alpha(X_T)) = \mathbb{E}(Y_t^n \Phi^\alpha(X_t)) + \mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q} dr + \mathbb{E}\int_t^T Y_r^n \Psi_\alpha(r,X_r) dr$$

with Ψ_α the following function: for $t \in]0,T]$ and $x \in \mathbb{R}^m$

$$\begin{aligned}\Psi_\alpha(t,x) &= \nabla(\Phi^\alpha)(x).b(t,x) - \sum_{i=1}^d \left((\nabla(\Phi^\alpha)(x)\sigma(t,x))_i \,\frac{\operatorname{div}(p(t,x)\sigma^i(t,x))}{p(t,x)} \right) \\ &\quad - \frac{1}{2}\operatorname{Trace}(D^2(\Phi^\alpha)(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^d \left(\nabla(\Phi^\alpha)(x).[\nabla\sigma^i(t,x)\sigma^i(t,x)] \right).\end{aligned}$$

Our goal now is to prove that for a fixed $\varepsilon > 0$ and $p = 1 + 1/q$:

$$\Phi^{-\alpha(p-1)} |\Psi_\alpha|^p \in L^\infty([\varepsilon,T] \times \mathbb{R}^m).$$

If it is true, then

$$\mathbb{E}\int_t^T |Y_r^n \Psi_\alpha(r,X_r)|\, dr \leq K \left(\mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q} dr \right)^{\frac{1}{1+q}}$$

and the end of the proof will be the same as in the case $q > 2$.

From the case $q > 2$ we already know that

$$\Phi^{-\alpha(p-1)} |\nabla(\Phi^\alpha).b(t,.)|^p \text{ and } \Phi^{-\alpha(p-1)} \left| \frac{1}{2}\operatorname{Trace}(D^2(\Phi^\alpha)\sigma\sigma^*(t,.) \right|^p$$

are in $L^\infty([0,T] \times \mathbb{R}^m)$. The next term is:

$$\Phi^{-\alpha(p-1)} \left| \sum_{i=1}^d \nabla(\Phi^\alpha).[\nabla\sigma^i(t,.)\sigma^i(t,.)] \right|^p.$$

σ satisfies the conditions (L) and (B). We use again the calculation done for the gradient of Φ^α to deduce that if $\alpha > 2p$ this term is in $L^\infty([0,T] \times \mathbb{R}^m)$. Now we come to the last term which involves the density of X:

$$\Phi^{-\alpha(p-1)} \left| \sum_{i=1}^d (\nabla(\Phi^\alpha)\sigma(t,.))_i \,\frac{\operatorname{div}(p\sigma^i)(t,.)}{p(t,.)} \right|^p = \alpha^p \Phi^{\alpha-p} \left| \sum_{i=1}^d (\nabla\Phi\sigma(t,.))_i \,\frac{\operatorname{div}(p\sigma^i)(t,.)}{p(t,.)} \right|^p.$$

We split this term in two parts:

$$\Phi^{\alpha-p}\left|\sum_{i=1}^{d}\left((\nabla\Phi\sigma(t,.))_i \ \left((\text{div}\sigma^i)+\sigma^i.\frac{(\text{div } p)}{p}\right)(t,.)\right)\right|^p$$
$$\leq (2d)^{p-1}\sum_{i=1}^{d}\Phi^{\alpha-p}\left|(\nabla\Phi\sigma(t,.))_i \ (\text{div } \sigma^i)(t,.)\right|^p$$
$$+(2d)^{p-1}\sum_{i=1}^{d}\Phi^{\alpha-p}\left|(\nabla\Phi\sigma(t,.))_i\sigma^i(t,.)\right|^p \frac{|(\text{div } p)(t,.)|^p}{|p(t,.)|^p}.$$

Here and in the rest of this section, div p denotes the vector $(\frac{\partial p}{\partial x_j})_{j=1,\ldots,m}$. For the first part there is no problem because $\alpha - p > 0$, so $\Phi^{\alpha-p}$ is continuous and compactly supported and $(\nabla\Phi\sigma)_i$ (div σ^i) is bounded because of conditions (L)-(B). For the second part we use the inequality (2.22) and the fact that the support of $\Phi^{\alpha-p}$ is a compact set $\mathcal{K}$. So the minimum of $p(.,.)$ exists on the set $[\varepsilon,T]\times\mathcal{K}$ and is positive. Therefore we control the demominator. For the numerator we need some additional regularity property on div p. This property is given by the theorem 12.1, section 3, p.223 of [26]. Since $p(.,.)$ is solution of the PDE:

$$\partial_t p = Lp,$$

and since the coefficients of L are bounded and Lipschitz in x, the first derivatives of $\sigma\sigma^*$ belong to $L^\infty([0,T]\times\mathbb{R}^m)$. From (B), b is bounded. From (L)-(B)-(E) L is uniformly elliptic and in divergence form. The conclusions of the theorem III.12.1 are valid: $\partial p/\partial x_i$ satisfies an Hölder condition in x. We can now conclude that the second part is bounded by a constant K independent of n and t.

Finally we have:

$$\Phi^{-\alpha(p-1)}\left|\Psi_\alpha\right|^p \in L^\infty([\varepsilon,T]\times\mathbb{R}^m),$$

and thus for $t \geq \varepsilon$:

$$|\mathbb{E}\left[Y_t^n\Psi_\alpha(t,X_t)\right]| \leq K\mathbb{E}\left[\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right]^{\frac{1}{1+q}}, \tag{2.26}$$

where K is a constant independent of t and n. The equation (2.25) is:

$$\mathbb{E}(Y_T^n\Phi^\alpha(X_T)) - \mathbb{E}(Y_\varepsilon^n\Phi^\alpha(X_\varepsilon)) = \int_\varepsilon^T \mathbb{E}\left[\Phi^\alpha(X_r)(Y_r^n)^{1+q}\right]dr + \int_\varepsilon^T \mathbb{E}\left[Y_r^n\Psi_\alpha(r,X_r)\right]dr$$

$Y_T^n\Phi^\alpha(X_T) \leq g(X_T)\Phi^\alpha(X_T)$; since Φ is equal to zero outside a compact set included in $F^c = \{g < +\infty\}$, using the condition (H2) the left hand side of the previous equality is bounded by a constant independent of n. We deduce that $\Phi^\alpha(X)(Y^n)^{1+q}$ is a bounded sequence in $L^1(\Omega\times[\varepsilon,T])$. Otherwise there exists a subsequence along which $\mathbb{E}\int_t^T \Phi^\alpha(X_r)(Y_r^n)^{1+q}dr$ tends to infinity. And with the inequality (2.26) we have:

$$\left|\int_\varepsilon^T \mathbb{E}\left[Y_t^n\Psi_\alpha(t,X_t)\right]dt\right| \leq K\left[\int_\varepsilon^T \mathbb{E}\left(\Phi^\alpha(X_t)(Y_t^n)^{1+q}\right)dt\right]^{\frac{1}{1+q}}.$$

Since $1/(1+q) < 1$ the right member in (2.25) (extracting a subsequence if necessary) would tend to $+\infty$, which is a contradiction with the fact that the left member is bounded.
With the Fatou lemma, for $0 < \varepsilon \leq T$, $Y^{1+q}\Phi^\alpha(X)$ belongs to $L^1(\Omega \times [\varepsilon,T])$.
As Y is bounded on the interval $[0,\varepsilon]$, $\varepsilon < T$, by $1/(q(T-\varepsilon))^{1/q}$, we have proved:

$$\mathbb{E}\int_0^T Y_r^{1+q}\Phi^\alpha(X_r)dr < +\infty. \tag{2.27}$$

As in the case $q > 2$ this inequality allows us to show that

$$\liminf_{t\to T} Y_t \leq \xi.$$

Indeed let θ be a function of class $C^2(\mathbb{R}^d;\mathbb{R}^+)$ with a compact support included in U. α is strictly greater than $2(1+1/q) > 2$. We write again the equation (2.24) for θ, $n \in \mathbb{N}^*$, $\varepsilon > 0$ and $t \geq \varepsilon$:

$$\begin{aligned} \mathbb{E}((\xi\wedge n)\theta(X_T)) &= \mathbb{E}(Y_T^n\theta(X_T)) \\ &= \mathbb{E}(Y_t^n\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r^n)^{1+q}dr + \mathbb{E}\int_t^T Y_r^n\Theta(r,X_r)dr \end{aligned} \tag{2.28}$$

with

$$\begin{aligned} \Theta(t,x) &= \nabla\theta(x).b(t,x) - \sum_{i=1}^d \left((\nabla\theta(x)\sigma(t,x))_i \, \frac{\mathrm{div}(p(t,x)\sigma(t,x)^i)}{p(t,x)} \right) \\ &\quad -\frac{1}{2}\mathrm{Trace}(D^2\theta(x)\sigma\sigma^*(t,x)) - \sum_{i=1}^d \left(\nabla\theta(x).[\nabla\sigma(t,x)^i.\sigma(t,x)^i]\right). \end{aligned}$$

Using the same arguments as in the case $q > 2$, we can pass to the limit:

$$\mathbb{E}(\xi\theta(X_T)) = \mathbb{E}(Y_t\theta(X_t)) + \mathbb{E}\int_t^T \theta(X_r)(Y_r)^{1+q}dr + \mathbb{E}\int_t^T Y_r\Theta(r,X_r)dr.$$

With the inequality (2.27), we let t going to T and we apply Fatou's lemma:

$$\mathbb{E}(\xi\theta(X_T)) = \lim_{t\to T}\mathbb{E}(Y_t\theta(X_t)) \geq \mathbb{E}(\liminf_{t\to T} Y_t\theta(X_T))$$

and this is true for every non-negative function θ whose support is in U. So

$$\xi \geq \liminf_{t\to T} Y_t \ \mathbf{P}-\text{a.s.}$$

and we have:

$$\liminf_{t\to T} Y_t = \xi.$$

But we have proved before that the limit of Y_t exists, so we have demonstrated that this limit is equal to ξ and therefore the continuity of Y on $[0,T]$.

2.4 Minimal solution

For $q > 0$ recall that we denote by f the following function

$$f(y) = -y|y|^q,$$

that it is a continuous and monotone function, i.e. for all reals y and y':

$$(y - y')(f(y) - f(y')) \leq 0.$$

Theorem 2.6 (Minimal solution)
The solution (Y,Z) previously obtained by approximation is the minimal solution: if $(\bar{Y},\bar{Z})$ is an other non-negative solution in the sense of the definition 2.1:

1. *for all $s \in [0,T[$, for all $0 \leq t \leq s$:*

$$\bar{Y}_t = \bar{Y}_s + \int_t^s f\left(\bar{Y}_r\right) dr - \int_t^s \bar{Z}_r dB_r;$$

2. *for all $t \in [0,T[$,*

$$\mathbb{E}\left(\sup_{0 \leq s \leq t} |\bar{Y}_s|^2 + \int_0^t \|\bar{Z}_r\|^2 dr\right) < +\infty;$$

3.

$$\lim_{t \uparrow T} \bar{Y}_t = \xi, \quad \mathbf{P}-a.s.;$$

then for all $t \in [0,T]$, $\mathbf{P}$-a.s.:

$$\bar{Y}_t \geq Y_t.$$

Proposition 2.8 *With the assumptions of the previous theorem, we prove:*

$$\forall t \in [0,T],\ \bar{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

Proof. For every $h > 0$, we define on $[0,T-h]$

$$\Lambda_h(t) = \left(\frac{1}{q(T-h-t)}\right)^{\frac{1}{q}}.$$

Λ_h is the solution of the ordinary differential equation:

$$\Lambda_h'(t) = (\Lambda_h(t))^{1+q}$$

with final condition $\Lambda_h(T-h) = +\infty$. But on the interval $[0,T-h]$, $(\bar{Y},\bar{Z})$ is a solution of the BSDE (2.1) with final condition $\bar{Y}_{T-h}$. From the assumptions $\bar{Y}_{T-h}$ is in $L^2(\Omega)$, so is finite a.s. Now we take the difference between $\bar{Y}$ and Λ_h for all $0 \leq t \leq s < T-h$:

$$\begin{aligned}\Lambda_h(t) - \bar{Y}_t &= \Lambda_h(s) - \bar{Y}_s - \int_t^s \left(\Lambda_h(r)^{1+q} - \left(\bar{Y}_r\right)^{1+q}\right) dr - \int_t^s \bar{Z}_r dB_r \\ &= \Lambda_h(s) - \bar{Y}_s - \int_t^s \alpha_r \left(\Lambda_h(r) - \bar{Y}_r\right) dr - \int_t^s \bar{Z}_r dB_r\end{aligned}$$

with

$$\alpha_r = \begin{cases} \frac{(\Lambda_h(r))^{1+q} - (\bar{Y}_r)^{1+q}}{\Lambda_h(r) - \bar{Y}_r} & \text{for } \bar{Y}_r \neq \Lambda_h(r) \\ 0 & \text{if } \bar{Y}_r = \Lambda_h(r) \end{cases}$$

So α is a non-negative adapted process. Then we deduce:

$$\Lambda_h(t) - \bar{Y}_t = \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \bar{Y}_s\right)\exp\left(\int_t^s -\alpha_r dr\right)\right].$$

Moreover we know that:

$$\bar{Y}_s \leq \mathbb{E}^{\mathcal{F}_s}\left(\bar{Y}_{T-h}\right)$$

Therefore

$$\begin{aligned} \Lambda_h(t) - \bar{Y}_t &\geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \mathbb{E}^{\mathcal{F}_s}\left(\bar{Y}_{T-h}\right)\right)\exp\left(\int_t^s -\alpha_r dr\right)\right] \\ &= \mathbb{E}^{\mathcal{F}_t}\left[\left(\Lambda_h(s) - \bar{Y}_{T-h}\right)\exp\left(\int_t^s -\alpha_r dr\right)\right]. \end{aligned}$$

Now Fatou's lemma leads, as s goes to $T-h$, to:

$$\Lambda_h(t) - \bar{Y}_t \geq 0.$$

This inequality is true for all $t \in [0,T-h]$ and for all $h > 0$. So it is clear that for every $t \in [0,T]$:

$$\bar{Y}_t \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

This achieves the proof of the proposition. □

In the case where $\xi = +\infty$ a.s. this inequality and the lemma 2.2 give the unicity of the solution. If $\xi = +\infty$, there is a unique solution, namely

$$Y_t = \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \text{ and } Z_t = 0.$$

Proof of the theorem. We will demonstrated that $\bar{Y}$ is greater than Y^n for all $n \in \mathbb{N}$, which implies that Y is the minimal solution.

Let (Y^n,Z^n) be the solution of the BSDE (2.1) with $\xi \wedge n$ as terminal condition. By comparison

$$Y_t^n \leq \left(\frac{1}{q(T-t) + 1/n^q}\right)^{1/q} \leq n.$$

Between the instants $0 \leq t \leq s < T$:

$$\begin{aligned} \bar{Y}_t - Y_t^n &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \left((\bar{Y}_r)^{1+q} - (Y_r^n)^{1+q}\right) dr - \int_t^s \left(\bar{Z}_r - Z_r^n\right) dB_r \\ &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \left(\frac{(\bar{Y}_r)^{1+q} - (Y_r^n)^{1+q}}{\bar{Y}_r - Y_r^n}\right)\left(\bar{Y}_r - Y_r^n\right) dr - \int_t^s \left(\bar{Z}_r - Z_r^n\right) dB_r \\ &= \left(\bar{Y}_s - Y_s^n\right) - \int_t^s \alpha_r^n \left(\bar{Y}_r - Y_r^n\right) dr - \int_t^s \left(\bar{Z}_r - Z_r^n\right) dB_r \qquad (2.29) \end{aligned}$$

with

$$\alpha_r^n = \begin{cases} \frac{(\bar{Y}_r)^{1+q}-(Y_r^n)^{1+q}}{\bar{Y}_r-Y_r^n} & \text{for } \bar{Y}_r \neq Y_r^n \\ 0 & \text{if } \bar{Y}_r = Y_r^n \end{cases}$$

The process α^n is well defined, adapted and verifies:

$$0 \leq \alpha_r^n \leq (1+q)(\bar{Y}_r \vee Y_r^n)^q.$$

We deduce that:

$$\bar{Y}_t - Y_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\bar{Y}_s - Y_s^n\right)\exp\left(-\int_t^s \alpha_r^n dr\right)\right]$$

using the linearity of the BSDE (2.29) and the fact that the generator of this BSDE is monotone. Then with Fatou's lemma:

$$\begin{aligned} \bar{Y}_t - Y_t^n &= \liminf_{s\to T} \mathbb{E}^{\mathcal{F}_t}\left[\left(\bar{Y}_s - Y_s^n\right)\exp\left(-\int_t^s \alpha_r^n dr\right)\right] \\ &\geq \mathbb{E}^{\mathcal{F}_t} \liminf_{s\to T}\left[\left(\bar{Y}_s - Y_s^n\right)\exp\left(-\int_t^s \alpha_r^n dr\right)\right] \end{aligned}$$

It is legal to apply the Fatou's lemma because what is inside the conditional expectation has a lower bound equal to $-n$:

$$\bar{Y}_s \geq 0 \text{ (this belongs to the hypothesis)}, \quad -Y_s^n \geq -n \quad \text{and} \quad 1 \geq \exp\left(-\int_t^s \alpha_r^n dr\right).$$

Finally $\bar{Y}_t - Y_t^n \geq 0$. As it is true for every $n \in \mathbb{N}^*$ and every $t \in [0,T]$, we have

$$\bar{Y}_t \geq Y_t.$$

This achieves the proof. □

2.5 Parabolic PDE, viscosity solutions

In the introduction we have said that there is a connection between BSDE whose terminal data is a function of the value at time T of a solution of a SDE (or forward-backward system), and solutions of a large class of semilinear parabolic PDE. Let us precise this connection in our case.

To begin with we modify the equation (2.6). For all $(t,x) \in [0,T] \times \mathbb{R}^m$ we denote by $X^{t,x}$ the solution of the following SDE:

$$X_s^{t,x} = x + \int_t^s b(r,X_r^{t,x})dr + \int_t^s \sigma(r,X_r^{t,x})dB_r, \text{ for } s \in [t,T], \tag{2.7}$$

et $X_s^{t,x} = x$ for $s \in [0,t]$. b and σ satisfy the assumptions (L)-(G), and we add that b and σ are jointly continuous in (t,x). We consider the following BSDE for $t \leq s \leq T$:

$$Y_s^{t,x} = h(X_T^{t,x}) - \int_s^T Y_r^{t,x}|Y_r^{t,x}|^q dr - \int_s^T Z_r^{t,x} dB_r, \tag{2.30}$$

where h is a function defined on $\mathbb{R}^m$ with values in $\mathbb{R}^+$ such that h is continuous and bounded. The two equations (2.7) and (2.30) are called a forward-backward system. This system is connected with the following PDE:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + Lu(t,x) - u(t,x)|u(t,x)|^q &= 0,\ (t,x) \in [0,T[\times\mathbb{R}^m, \\ u(T,x) &= h(x),\ x \in \mathbb{R}^m. \end{aligned} \tag{2.31}$$

where L is the operator:

$$L = \frac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(t,x)\frac{\partial}{\partial x_i}.$$

This result is proved in the theorem 3.2 of the article [42]:

Theorem 3.2 of [42] *If we solve the equations (2.7) and (2.30) and if we define $u(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}^m$ by $u(t,x) = Y_t^{t,x}$, then u is a continuous function and it is a viscosity solution of the PDE (2.31).*

Let us recall the definition of a viscosity solution. It is the same as the definition 7.4 page 38 of [10], adapted to the parabolic case. For $v : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ we define the upper- and lower semicontinuous envelope of v, namely:

$$v^*(t,x) = \limsup_{(t',x')\to(t,x)} v(t',x') \quad \text{and} \quad v_*(t,x) = \liminf_{(t',x')\to(t,x)} v(t',x').$$

v^* and v_* are respectively upper- and lowersemincontinuous and v is continuous if and only if $v = v^* = v_*$.

Definition 2.2 *In this definition h is continuous and bounded on $\mathbb{R}^m$.*

1. *We say that v is a subsolution of (2.31) on $[0,T] \times \mathbb{R}^m$ if $v^* < +\infty$ and if for every function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m)$, (t_0,x_0) is a local maximum of $v^* - \varphi$, then:*
   (a) if $t_0 < T$,

$$-\frac{\partial\varphi}{\partial t}(t_0,x_0) - L\varphi(t_0,x_0) + v^*(t_0,x_0)|v^*(t_0,x_0)|^q \leq 0$$

   (b) if $t_0 = T$,

$$\min\left(v^*(T,x_0) - h(x_0); -\frac{\partial\varphi}{\partial t}(T,x_0) - L\varphi(T,x_0) + v^*(T,x_0)|v^*(T,x_0)|^q\right) \leq 0.$$

2. *We say that v is a supersolution of (2.31) on $[0,T] \times \mathbb{R}^m$ if $v_* > -\infty$ and if for every function $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m)$, (t_0,x_0) is a local minimum of $v_* - \varphi$, then:*
   (a) if $t_0 < T$,

$$-\frac{\partial\varphi}{\partial t}(t_0,x_0) - L\varphi(t_0,x_0) + v_*(t_0,x_0)|v_*(t_0,x_0)|^q \geq 0$$

(b) if $t_0 = T$*,*

$$\max\left(v_*(T,x_0) - h(x_0); -\frac{\partial \varphi}{\partial t}(T,x_0) - L\varphi(T,x_0) + v_*(T,x_0)|v_*(T,x_0)|^q\right) \geq 0.$$

3. A function v *is a viscosity solution if it is both a viscosity sub- and supersolution.*

Now in our case the function h is replaced by the function $g : \mathbb{R}^m \to \overline{\mathbb{R}_+}$ which is supposed to be continuous from $\mathbb{R}^m$ to $\overline{\mathbb{R}^+}$ and such that the set $F = \{g = +\infty\}$ is closed and non empty. We cannot apply the theorem 3.2 in [42], nor the previous definition, because g is unbounded on $\mathbb{R}^m$.

Definition 2.3 (Viscosity solution with unbounded data) *We say that* v *is a viscosity solution of the PDE (2.2) with terminal data* g *if* v *is a viscosity solution on* $[0,T[\times\mathbb{R}^m$ *and satisfies:*

$$\lim_{(t,x)\to(T,x_0)} v(t,x) = g(x_0).$$

We take the notations of the construction of the minimal solution. For all $n \in \mathbb{N}$ and $(t,x) \in [0,T] \times \mathbb{R}^m$ we obtain a sequence of random variables $(Y^{t,x,n},Z^{t,x,n})$ satisfying (2.1):

$$Y^{t,x,n}_s = \left(g(X^{t,x}_T) \wedge n\right) - \int_s^T Y^{t,x,n}_r|Y^{t,x,n}_r|^q dr - \int_s^T Z^{t,x,n}_r dB_r,$$

and (2.9). We know now that this sequence converges to $(Y^{t,x},Z^{t,x})$ which is the minimal solution verifying the conclusions of the theorems 2.1 and 2.2. In particular it means that either $q > 2$ (L)-(G) or else (L)-(B)-(E) are satisfied by σ and b. We want prove the:

Theorem 4 (Viscosity solution) *For all* $(t,x) \in [0,T] \times \mathbb{R}^m$ $Y^{t,x}_t$ *is a deterministic number and if we set* $u(t,x) = Y^{t,x}_t$*, then* u *is lowersemicontinuous from* $[0,T] \times \mathbb{R}^m$ *to* $\overline{\mathbb{R}^+}$ *with* $u(t,x) < \infty$ *whenever* $t < T$ *and* u *is a viscosity solution of the PDE (2.2). In particular* u *satisfies:*

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0) = u(T,x_0).$$

If we define the function u_n by $u_n(t,x) = Y^{t,x,n}_t$, then from theorem 3.2 in [42], we know that u_n is jointly continuous in (t,x) and is a viscosity solution of the following parabolic PDE:

$$\text{(2.32)}\qquad \begin{cases} \partial_t u(t,x) + Lu(t,x) - u(t,x)|u(t,x)|^q &= 0, \quad (t,x) \in [0,T] \times \mathbb{R}^m, \\ u(T,x) &= g(x) \wedge n, \quad x \in \mathbb{R}^m, \end{cases}$$

The fact that g is supposed to be continuous implies that $g \wedge n$ is bounded and continuous on $\mathbb{R}^m$.

By the comparison theorem for BSDE, $(Y^{t,x,n}_t = u_n(t,x))_{n\in\mathbb{N}}$ is a non-decreasing sequence, hence it converges to $Y^{t,x}_t = u(t,x)$ when n tends to infinity. Some remarks about the function u. It is a non-negative function satisfying the following bound:

$$\text{(2.33)}\qquad \forall (t,x) \in [0,T] \times \mathbb{R}^m,\ 0 \leq u(t,x) \leq \frac{1}{(q(T-t))^{\frac{1}{q}}}.$$

Moreover $u(T,x) = g(x)$ for all $x \in \mathbb{R}^m$. At least u is lowersemicontinuous on $[0,T]\times\mathbb{R}^m$ as the supremum of continuous functions (the sequence (u_n) is a non-decreasing sequence), and for all $x_0 \in \mathbb{R}^m$:

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \geq g(x_0).$$

Proof of the theorem 4. The main tool is the half-relaxed upper- and lower- limit of a sequence of functions $\{v_n\}$, i.e.

$$\overline{v}(t,x) = \limsup_{\substack{n\to+\infty \\ (t',x')\to(t,x)}} v_n(t',x') \quad \text{and} \quad \underline{v}(t,x) = \liminf_{\substack{n\to+\infty \\ (t',x')\to(t,x)}} v_n(t',x').$$

In our case $\underline{u} = u \leq \overline{u} = u^*$ because the sequence $\{u_n\}$ is non-decreasing and u_n is continuous for all $n \in \mathbb{N}^*$.

First u is a supersolution of the PDE (2.2) on $[0,T[\times\mathbb{R}^m$. $u = u_* = \underline{u}$ is lower semi-continuous on $[0,T[\times\mathbb{R}^m$. Let $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^m)$ such that (t_0,x_0) is a local maximum of $u-\varphi$ with $t_0 \leq T-2\delta$ for some $\delta > 0$. From the estimate (2.33), for all $n \in \mathbb{N}^*$ and all $(t,x) \in [0,T-\delta]\times\mathbb{R}^m$,

$$u_n(t,x) \leq u(t,x) \leq \left(\frac{1}{q(T-t)}\right)^{1/q}.$$

There exists a sequence $\{(t_n,x_n)\}$ converging to (t_0,x_0) such that $t_n \leq T-\delta$ and (t_n,x_n) is a local maximum of $u_n - \varphi$. Since u_n is a supersolution of the PDE (2.32), passing to the limit with the lemma 6.1, page 33, of [10], we obtain that u is a supersolution of (2.2) on $[0,T[\times\mathbb{R}^m$.

The same argument shows that u^* is a supersolution on $[0,T[\times\mathbb{R}^m$.

As in the case of the BSDE (2.1) the main difficulty is to show that

$$\limsup_{(t,x)\to(T,x_0)} u(t,x) \leq g(x_0) = u(T,x_0).$$

We will prove that u^* is locally bounded on a neighbourhood of T on the set $\{g < +\infty\}$. Then we deduce u^* is a subsolution and we apply this to demonstrate that $u^*(T,x) \leq g(x)$ if $x \in \{g < +\infty\}$, which shows the wanted inequality on u.

Let U be a bounded open set such that the compact set $\mathcal{K} = \overline{U}$ is included in $\{g < +\infty\}$. We use the same calculation as in the proof of the continuity of Y at T. Let θ be a function of class $C^2(\mathbb{R}^m;\mathbb{R}^+)$ with a compact support included in U. We will prove that u_n is uniformly bounded on $[0,T]\times U$. On $[0,T-\delta]\times U$ the bound (2.33) gives immediately the result. It remains to treat the problem on a neighbourhood of T.

First case: $q > 2$:
We write the equality (2.17) between t and T for $x \in U$.

$$\begin{aligned} u_n(t,x)\theta(x) &= \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})) - \mathbb{E}\int_t^T \left[\theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}\right] dr \\ &- \mathbb{E}\int_t^T Y_r^{t,x,n}\left(\nabla\theta(X_r^{t,x}).b(r,X_r^{t,x}) + \frac{1}{2}\mathrm{Trace}(D^2\theta(X_r^{t,x})\sigma\sigma^*(r,X_r^{t,x}))\right) dr \\ &- \mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})dr \end{aligned}$$

The last term is controlled by:

$$\begin{aligned}\left|\mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})dr\right| &\leq \left(\mathbb{E}\int_t^T \|Z_r^{t,x,n}\|^2(T-r)^{2/q}dr\right)^{1/2}\\ &\quad\times\left(\mathbb{E}\int_t^T \frac{\|\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})\|^2}{(T-r)^{2/q}}dr\right)^{1/2}\\ &\leq \widetilde{K}=\widetilde{K}(q,\theta,\sigma).\end{aligned}$$

Here we use the fact that $q>2$, θ is compactly supported, and the condition (G). Thus we have:

$$\begin{aligned}&\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}dr+\mathbb{E}\int_t^T Y_r^{t,x,n}L\theta(X_r^{t,x})dr\\ &\quad\leq \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x}))-\mathbb{E}\int_t^T Z_r^{t,x,n}.\nabla\theta(X_r^{t,x})\sigma(r,X_r^{t,x})dr\end{aligned}$$

The right hand side is bounded by the supremum of $g\theta$ on U and $\widetilde{K}$. In the left hand side the second term is controlled by the first one raised to a power strictly smaller than 1 using Hölder's inequality. Therefore there exists a constant $\tilde{K}$ independent of n,t,x:

$$\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}dr\leq K.$$

We deduce that:

$$u_n(t,x)\theta(x)\leq K=K(T,g,\theta,q).$$

Second case: the assumptions (A1)-(A2)-(A3) are satisfied: For $n\in\mathbb{N}^*$, $t,x\in[0,T]\times U$, $t>0$, the equation (2.28) becomes:

$$\begin{aligned}u_n(t,x)\theta(x) &= \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x}))-\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}dr\\ &\quad-\mathbb{E}\int_t^T Y_r^{t,x,n}\Theta(r,X_r^{t,x})dr.\end{aligned}\tag{2.34}$$

We recall that Θ is defined by:

$$\begin{aligned}\Theta(t,x) &= \nabla\theta(x).b(t,x)-\sum_{i=1}^d\left((\nabla\theta(x)\sigma(t,x))_i\ \frac{\operatorname{div}(p(t,x)\sigma^i(t,x))}{p(t,x)}\right)\\ &\quad-\frac{1}{2}\operatorname{Trace}(D^2\theta(x)\sigma\sigma^*(t,x))-\sum_{i=1}^d\left(\nabla\theta(x).[\nabla\sigma^i(t,x)\sigma^i(t,x)]\right).\end{aligned}$$

As u_n and θ are two non-negative functions,

$$\mathbb{E}\int_t^T \theta(X_r^{t,x})(Y_r^{t,x,n})^{1+q}dr+\mathbb{E}\int_t^T Y_r^{t,x,n}\Theta(r,X_r^{t,x})dr\leq \mathbb{E}(Y_T^{t,x,n}\theta(X_T^{t,x})).$$

The right hand side in the previous inequality is bounded by the supremum of $g\theta$ on U, this supremum is well defined because $g\theta$ is continuous and bounded. And from the calculations made on the BSDE we know that the absolute value of the second term in the left hand side is controlled by the first term (which is non negative) raised to a power strictly smaller than 1. Thus we deduce:

$$\mathbb{E}\int_0^T \theta(B_r^{t,x})(Y_r^{t,x,n})^{1+q}dr \leq K = K(T,g,\theta,q).$$

It is important to note that this constant is independent of n,t,x. If we come back to the inequality (2.34) we deduce that for all $(t,x) \in [0,T] \times U$

$$u_n(t,x)\theta(x) \leq K = K(T,g,\theta,q).$$

Thus u^* is bounded on $[0,T] \times U$ and we can apply the lemma 6.1, page 33, of [10] to the sequence u_n of subsolutions of the PDE (2.32) restricted to $[0,T] \times U$. Therefore u^* is a subsolution of the PDE (2.2). We now use the notion of subsolution to show that $u^* \leq g$.

Let x_0 be in U, $\varepsilon > 0$ and C_ε be a constant such that $C_\varepsilon - \frac{\kappa}{\varepsilon^2} > 0$. κ is a constant we must find. We consider the function $\Phi_\varepsilon \in C^{1,2}([0,T] \times \mathbb{R}^m)$ defined by:

$$\Phi_\varepsilon(t,x) = \frac{|x-x_0|^2}{\varepsilon^2} + C_\varepsilon(T-t).$$

If $u^* - \Phi_\varepsilon$ has a local maximum at $(t_\varepsilon,x_\varepsilon) \in [0,T] \times U$ and if $t_\varepsilon < T$, then we apply the definition of a viscosity subsolution:

$$C_\varepsilon - \frac{1}{\varepsilon^2}\left[\sum_i (\sigma\sigma^*(t_\varepsilon,x_\varepsilon))_{ii} + 2\sum_i b_i(t_\varepsilon,x_\varepsilon)(x_\varepsilon^i - x_0^i)\right] + (u^*(t_\varepsilon,x_\varepsilon))^{1+q} \leq 0.$$

Since b and σ satisfy either the condition (G) or the condition (A3) and since $x_\varepsilon \in U$, we show that what is inside the brackets is bounded by a constant κ. Thus with our choice of C_ε the previous inequality is impossible. Therefore $t_\varepsilon = T$ and $u^*(T,x_\varepsilon) \leq g(x_\varepsilon)$. As it's true for all ε, we deduce that $u^*(T,x_0) \leq g(x_0)$, because x_ε tends to x_0 when ε goes to 0. □

This achieves the proof of the theorem 2.4. The next proposition precises the behaviour of the solution u on a neighbourhood of T.

Proposition 2.9 *The previously defined solution u satisfies for all x in the interior of $\{g = +\infty\}$:*

$$\lim_{t\to T} [q(T-t)]^{1/q}\, u(t,x) = 1.$$

Proof. We take the notations of the lemma 2.2. For all $(t,x) \in [0,T[\times\mathbb{R}^m$

$$Y_t^{t,x,n} \geq \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{q(T-t) + 1/(\xi \wedge n)^q}\right)^{1/q}.$$

Thus for all integer n

$$\begin{aligned}
[q(T-t)]^{1/q}\, u(t,x) &\geq [q(T-t)]^{1/q}\, u_n(t,x) \\
&\geq \left(\frac{q(T-t)}{q(T-t)+1/n^q}\right)^{1/q} \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right) \\
&\quad + \mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{1/q}.
\end{aligned}$$

Therefore

$$[q(T-t)]^{1/q}\, u(t,x) \geq \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right) + \mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{1/q}.$$

We can bounded the last term by

$$\mathbb{E}^{\mathcal{F}_t}\left[\frac{q(T-t)(\xi\wedge n)^q}{1+q(T-t)(\xi\wedge n)^q}\mathbf{1}_{\{\xi<\infty\}}\right]^{1/q} \leq [q(T-t)]^{1/q}\left[\frac{1}{qT}\right]^{1/q}.$$

Hence:

$$\lim_{t\to T} [q(T-t)]^{1/q}\, u(t,x) \geq \lim_{t\to T} \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{g=\infty\}}(X_T^{t,x})\right);$$

this limit is equal to 1 for x in the interior of $\{g=+\infty\}$. We conclude using the bound (2.33). □

2.5.1 Minimal solution

The goal of this paragraph is to demonstrate that the viscosity solution obtained with the BSDE is minimal among all non-negative viscosity solutions (theorem 2.5). We compare a viscosity solution v (in the sense of the definition 2.3) with u_n, for all integer n: for all $(t,x)\in[0,T]\times\mathbb{R}^m$, $u_n(t,x)\leq v_*(t,x)$. We deduce that $u\leq v_*\leq v$. Remark that the only used assumptions in the proof will be (L) and (G). Recall that $g:\mathbb{R}^m\to\overline{\mathbb{R}^+}$ is continuous, which implies that $g\wedge n:\mathbb{R}^m\to\mathbb{R}^+$ is continuous.

Proposition 2.10 *$u_n\leq v_*$, where v is a non-negative viscosity solution of the PDE (2.2).*

Proof. For convenience we omit the term n in the proof. Thus u is a viscosity solution associated to the terminal conditon $f=g\wedge n$. Recall that u is continuous and bounded by n. Moreover we denote by F the function defined on $[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^m\times\mathcal{S}_m^+(\mathbb{R})$ ($\mathcal{S}_m^+(\mathbb{R})$ is the set of symetric real and positive matrices of size $m\times m$) by:

$$F(t,x,r,p,X) = \frac{1}{2}\text{Trace}\left(\sigma\sigma^*(t,x)X\right) + b(t,x).p - r|r|^q.$$

For $\varepsilon>0$ we denote $u_\varepsilon(t,x)=u(t,x)-\frac{\varepsilon}{t}$. u_ε is bounded by u and is a subsolution of the PDE (2.2):

$$\begin{aligned}
-\frac{\partial u_\varepsilon}{\partial t} - Lu_\varepsilon + u_\varepsilon|u_\varepsilon|^q &= -\frac{\partial u}{\partial t} - Lu + u|u|^q - (u|u|^q - u_\varepsilon|u_\varepsilon|^q) - \frac{\varepsilon}{t^2} \\
&\leq -\frac{\varepsilon}{t^2} \leq -\frac{\varepsilon}{T^2}.
\end{aligned}$$

Moreover at T, $u_\varepsilon(T,x) = u(T,x) - \varepsilon/T \le (g \wedge n)(x)$ and at 0, u_ε tends uniformly to $-\infty$. We will prove that $u_\varepsilon \le v$ for every ε, hence we deduce $u \le v$. From now we omit the suffix ε: u is a bounded (by n) continuous subsolution such that for all $x \in \mathbb{R}^m$, $u(0,x) = -\infty$ and $u(T,x) = (g \wedge n)(x) - \varepsilon/T$.

We suppose that there exists $(s,z) \in [0,T] \times \mathbb{R}^m$ such that $u(s,z) - v_*(s,z) \ge \delta > 0$ and we will find a contradiction. First of all it is clear that s is not equal to 0 or T, because $v_*(T,z) = g(z)$.

u and $-v_*$ are bounded from above on $[0,T] \times \mathbb{R}^m$ respectively by n and 0. Thus for $(\alpha,\beta) \in (\mathbb{R}^*)^2$, if we define

$$m(t,x,y) = u(t,x) - v_*(t,y) - \frac{\alpha}{2}|x-y|^2 - \beta\left(|x|^2 + |y|^2\right),$$

m has a supremum on $[0,T] \times \mathbb{R}^m$. Moreover the penalization terms assure that the supremum is attained at a point $(\hat{t},\hat{x},\hat{y}) = (t_{\alpha,\beta},x_{\alpha,\beta},y_{\alpha,\beta})$. Denote by $\mu_{\alpha,\beta}$ this maximum. Since

$$\delta - 2\beta|z|^2 \le u(s,z) - v_*(s,z) - 2\beta|z|^2 = m(s,z,z) \le \mu_{\alpha,\beta},$$

choosing β sufficiently small in order to have $\delta/2 \le \delta - 2\beta|z|^2$, we obtain

$$\delta/2 \le \mu_{\alpha,\beta}. \tag{2.35}$$

From this inequality and since $u \le n$ and $v \ge 0$, we have:

$$\begin{aligned} 0 &\le \mu_{\alpha,\beta} = m(\hat{t},\hat{x},\hat{y}) = u(\hat{t},\hat{x}) - v(\hat{t},\hat{y}) - \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 - \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right) \\ &\le n - \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 - \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right) \end{aligned}$$

hence

$$|\hat{x}|^2 + |\hat{y}|^2 \le \frac{n}{\beta} \quad \text{and} \quad |\hat{x}-\hat{y}|^2 \le \frac{2n}{\alpha}. \tag{2.36}$$

Now we will prove that $\hat{t}$ cannot be equal to zero, neither to T. Since $u(0,.) = -\infty$, $\hat{t}$ cannot be equal to zero. Assume that $\hat{t}$ is equal to T. We have:

$$\begin{aligned} \mu_{\alpha,\beta} &= (g \wedge n)(\hat{x}) - \frac{\varepsilon}{T} - g(\hat{y}) - \frac{\alpha}{2}|\hat{x}-\hat{y}|^2 - \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right) \\ &\le (g \wedge n)(\hat{x}) - (g \wedge n)(\hat{y}) \end{aligned}$$

Let γ^β be a modulus of continuity of $g \wedge n$ defined by

$$\forall \eta \ge 0,\ \gamma^\beta(\eta) = \sup\left\{|(g \wedge n)(x) - (g \wedge n)(y)|;\ |x-y| \le \eta,\ (x,y) \in (B_\beta)^2\right\},$$

where B_β is the closed ball with radius equal to $\sqrt{n/\beta}$. Therefore from (2.35)-(2.36) we obtain:

$$\delta/2 \le \mu_{\alpha,\beta} \le (g \wedge n)(\hat{x}) - (g \wedge n)(\hat{y}) \le \gamma^\beta\left(|\hat{x}-\hat{y}|\right) \le \gamma^\beta\left(\sqrt{\frac{2n}{\alpha}}\right).$$

Since we have supposed that $g \wedge n$ is continuous, $g \wedge n$ is uniformly continuous on B_β and thereby the limit of $\gamma^\beta(\eta)$ is equal to zero as η goes to 0. Hence the previous inequality is false when α is sufficiently large. We deduce that $\hat{t} < T$.

We now use the theorem 8.3 of [10] with u subsolution and v_* supersolution. For all $\nu > 0$ there exists two symetric matrices X and Y of size $m \times m$ and two reals a and b such that

$$\left(\begin{array}{cc} X & 0 \\ 0 & -Y \end{array} \right) \leq A + \nu A^2, \tag{2.37}$$

with A the following matrix:

$$A = \alpha \left(\begin{array}{cc} I & -I \\ -I & I \end{array} \right) + 2\beta \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array} \right),$$

$a - b = 0$ and satisfying:

$$\begin{array}{rcl} -a - F(\hat{t},\hat{x},u(t,\hat{x}),\alpha(\hat{x}-\hat{y}) + 2\beta\hat{x},X) & \leq & -\dfrac{\varepsilon}{T^2} \\ -b - F(\hat{t},\hat{y},v(t,\hat{y}),\alpha(\hat{x}-\hat{y}) - 2\beta\hat{y},Y) & \geq & 0. \end{array}$$

We substract the two previous inequalities:

$$\begin{array}{rcl} \dfrac{\varepsilon}{T^2} & \leq & F(\hat{t},\hat{x},u(t,\hat{x}),\alpha(\hat{x}-\hat{y}) + 2\beta\hat{x},X) - F(\hat{t},\hat{y},v(t,\hat{y}),\alpha(\hat{x}-\hat{y}) - 2\beta\hat{y},Y) \\ & = & \dfrac{1}{2}\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right) - \dfrac{1}{2}\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right) \\ & + & \left(b(\hat{t},\hat{x}) - b(\hat{t},\hat{y})\right).\alpha(\hat{x}-\hat{y}) + 2\beta\left(b(\hat{t},\hat{x}).\hat{x} + b(\hat{t},\hat{y}).\hat{y}\right) \\ & - & u(\hat{t},\hat{x})^{1+q} + v_*(\hat{t},\hat{y})^{1+q} \end{array}$$

Since b is Lipschitz and grows at most linearly, there exists a constant K such that

$$\left(b(\hat{t},\hat{x}) - b(\hat{t},\hat{y})\right).\alpha(\hat{x}-\hat{y}) + 2\beta\left(b(\hat{t},\hat{x}).\hat{x} + b(\hat{t},\hat{y}).\hat{y}\right) \leq \alpha K|\hat{x}-\hat{y}|^2 + 2\beta K(1 + |\hat{x}|^2 + |\hat{y}|^2).$$

Using again the lower bound (2.35) of $\mu_{\alpha,\beta}$ we have:

$$\delta/2 \leq \mu_{\alpha,\beta} \leq u(\hat{t},\hat{x}) - v_*(\hat{t},\hat{y}) \Longrightarrow 0 \leq v_*(\hat{t},\hat{y}) \leq u(\hat{t},\hat{x})$$

thus

$$-u(\hat{t},\hat{x})^{1+q} + v_*(\hat{t},\hat{y})^{1+q} \leq 0.$$

One term remains to be controlled:

$$\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right) - \text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right).$$

For that we use the upper bound (2.37). Let us precise $A + \nu A^2$:

$$A + \nu A^2 = \alpha(1 + 2\alpha\nu + 2\beta\nu) \left(\begin{array}{cc} I & -I \\ -I & I \end{array} \right) + 2\beta(1 + 2\beta\nu) \left(\begin{array}{cc} I & 0 \\ 0 & I \end{array} \right).$$

Therefore there exists a constant $C = 1 + 2\alpha\nu + 2\beta\nu$ such that

$$\frac{1}{C}\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\beta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

If we have choose $\nu = 1/\alpha$, then the constant C is bounded: $3 \leq C \leq 3 + 2\beta/\alpha$. We multiply this inequality by the following non negative matrix:

$$\begin{pmatrix} \sigma\sigma^*(\hat{t},\hat{x}) & \sigma(\hat{t},\hat{x})\sigma^*(\hat{t},\hat{y}) \\ \sigma(\hat{t},\hat{y})\sigma^*(\hat{t},\hat{x}) & \sigma\sigma^*(\hat{t},\hat{y}) \end{pmatrix},$$

and we take the trace:

$$\begin{aligned} \text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right) - \text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right) &\leq C\alpha\left[\sigma(\hat{t},\hat{x}) - \sigma(\hat{t},\hat{y})\right]\left[\sigma^*(\hat{t},\hat{x}) - \sigma^*(\hat{t},\hat{y})\right] \\ &+ 2C\beta\left[\sigma\sigma^*(\hat{t},\hat{x}) + \sigma\sigma^*(\hat{t},\hat{y})\right]. \end{aligned}$$

Using the fact that σ satisfies (L) and (G), we obtain the existence of a constant K such that

$$\text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{x})X\right) - \text{Trace}\left(\sigma\sigma^*(\hat{t},\hat{y})Y\right) \leq K\alpha|\hat{x}-\hat{y}|^2 + K\beta(1 + |\hat{x}|^2 + |\hat{y}|^2).$$

Finally we have:

$$\frac{\varepsilon}{T^2} \leq K\left(\alpha|\hat{x}-\hat{y}|^2 + \beta(1 + |\hat{x}|^2 + |\hat{y}|^2)\right),$$

where K is a constant indenpendent of α and β.

To conclude we just have to let α tend to $+\infty$ and β tend to 0. Denote by M_β the maximum of $u(t,x) - v_*(t,x) - \beta|x|^2$ on $[0,T] \times \mathbb{R}^m$ (the supremum exists because u and $-v_*$ have an upper bound, the existence of the maximum comes from the penalization term). We have for all $(t,x,y) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m$:

$$u(t,x) - v_*(t,x) - 2\beta|x|^2 = m(t,x,x) \leq \mu_{\alpha,\beta}$$

and

$$m(t,x,y) \leq u(t,y) - v_*(t,y) - \beta|y|^2 + u(t,x) - u(t,y) \leq M_\beta + u(t,x) - u(t,y).$$

Thus

$$M_{2\beta} \leq \mu_{\alpha,\beta} \leq M_\beta + u(\hat{t},\hat{x}) - u(\hat{t},\hat{y}) \leq M_\beta + \gamma_u^\beta(|\hat{x}-\hat{y}|),$$

where γ_u^β is a modulus of continuity of u:

$$\forall\eta \geq 0,\ \gamma_u^\beta(\eta) = \sup\left\{|u(t,x) - u(t,y)|;\ |x-y| \leq \eta,\ (x,y) \in (B_\beta)^2,\ t \in [0,T]\right\}.$$

Since u is jointly continuous in $(t,x) \in [0,T] \times \mathbb{R}^m$, $\gamma_u^\beta(\eta)$ goes to zero as η goes to zero. Moreover if

$$M = \sup_{[0,T]\times\mathbb{R}^m}\{u - v_*\}$$

then

$$M_{2\beta} \leq \mu_{\alpha,\beta} \leq M + \gamma_u^\beta(|\hat{x}-\hat{y}|).$$

Since for all $(t,x) \in [0,T] \times \mathbb{R}^m$

$$u(t,x) - v(t,x) - 2\beta|x|^2 = m(t,x,x) \leq \mu_{\alpha,\beta} \leq u(\hat{t},\hat{x}) - v_*(\hat{t},\hat{y})$$

we also have

$$M_{2\beta} \leq u(\hat{t},\hat{x}) - v(\hat{t},\hat{y})$$

and $u(t,x) - v_*(t,y) \leq M + u(t,x) - u(t,y)$, which implies

$$u(\hat{t},\hat{x}) - v_*(\hat{t},\hat{y}) \leq M + \gamma_u^\beta(|\hat{x} - \hat{y}|).$$

Since

$$\lim_{\beta \to 0} M_\beta = M \quad \text{and} \quad \lim_{\alpha \to +\infty} \gamma_u^\beta(\sqrt{2n}/\sqrt{\alpha}) = 0$$

for any fixed $\nu > 0$, there exists β such that $M - \nu \leq M_{2\beta} \leq M$. With this β, with (2.36), there exists α such that

$$0 \leq \gamma_u^\beta(|\hat{x} - \hat{y}|) \leq \gamma_u^\beta(\sqrt{2n}/\sqrt{\alpha}) \leq \nu.$$

Hence

$$M - \nu \leq \mu_{\alpha,\beta} \leq M + \nu \quad \text{and} \quad M - \nu \leq u(\hat{t},\hat{x}) - v(\hat{t},\hat{y}) \leq M + \nu.$$

Since

$$\mu_{\alpha,\beta} = u(\hat{t},\hat{x}) - v_*(\hat{t},\hat{y}) - \frac{\alpha}{2}|\hat{x} - \hat{y}|^2 - \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right)$$

we deduce that

$$\frac{\alpha}{2}|\hat{x} - \hat{y}|^2 + \beta\left(|\hat{x}|^2 + |\hat{y}|^2\right) \leq 2\nu.$$

Coming back to:

$$\frac{\varepsilon}{T^2} \leq K\left(\alpha|\hat{x} - \hat{y}|^2 + 2\beta(1 + |\hat{x}|^2 + |\hat{y}|^2)\right),$$

this leads to a contradiction taking α sufficiently large and β sufficiently small. Hence $u \leq v$ and it is true for every $\varepsilon > 0$, so the result is proved. □

2.5.2 Regularity of the minimal solution

The function u is the minimal non-negative viscosity solution of the PDE (2.2). We know that u is finite on $[0,T[\times\mathbb{R}^m$ (2.33). For $\delta > 0$, u is bounded on $[0,T-\delta] \times \mathbb{R}^m$ by a constant which depends only on δ.

Lemma 2.4 *Under the following assumptions:*

- *the coefficients of the operator L satisfy (L)-(B)-(E),*
- *the coefficients of L are Hölder-continuous in time,*
- *the first derivatives of $\sigma\sigma^*$ are Hölder-continuous in space,*

then for all $\delta > 0$

(2.38) $$u \in C^{1,2}([0,T-\delta] \times \mathbb{R}^m; \mathbb{R}^+).$$

Proof. The proof of the proposition 2.10 shows that there is a unique bounded and continuous viscosity solution of the Cauchy problem:

$$\left\{ \begin{array}{rcll} \partial_t v + Lv - v|v|^q & = & 0, & \text{on } [0,T-\delta] \times \mathbb{R}^m \\ v(T-\delta,x) & = & \phi(x) & \text{on } \mathbb{R}^m \end{array} \right. \tag{2.39}$$

where ϕ is supposed bounded and continuous on $\mathbb{R}^m$.

Moreover the Cauchy problem (2.39) has a classical solution for every bounded and continuous function ϕ. See theorem 8.1 and remark 8.1, section 5 in [26]. More precisely the solution belongs to $H^{1+\beta/2,2+\beta}([0,T] \times \mathbb{R}^m)$, where $H^{1+\beta/2,2+\beta}([0,T] \times \mathbb{R}^m)$ is the set of functions which are $1+\beta/2$-Hölder-continuous in time and $2+\beta$-Hölder-continuous in space.

Recall that u_n is jointly continuous in (t,x) and on $[0,T-\delta] \times \mathbb{R}^m$ u_n is bounded by:

$$0 \leq u_n(t,x) \leq \left(\frac{1}{q\delta} \right)^{1/q}.$$

Thus the problem (2.39) with condition $\phi = u_n(T-\delta,.)$ has a bounded classical solution. Since every classical solution is a viscosity solution and since u_n is the unique bounded and continuous viscosity solution of (2.39), we deduce that:

$$u_n \in H^{1+\beta/2,2+\beta}([0,T-\delta] \times \mathbb{R}^m; \mathbb{R}^+).$$

From the theorem 3.1, section 5 in [26], we also know that the sequence $\{u_n\}$ is locally bounded in $C^{0,1+\alpha}([0,T] \times \mathbb{R}^m)$. Therefore $u(T-\delta,.)$ is continuous on $\mathbb{R}^m$ and if we consider the problem (2.39) with continuous terminal data $u(T-\delta,.)$, with the same argument as for u_n, we obtain that u is a classical solution. □

Proposition 2.11 *If the coefficients of the operator L satisfy (L)-(B)-(E), then the minimal viscosity solution u is continuous on $[0,T[\times\mathbb{R}^m$.*

Proof. In fact we will prove that for every integer n, u_n belongs to a Hölder space $C^\alpha([0,T-\delta] \times \mathcal{K})$, where $\delta > 0$ and $\mathcal{K}$ is a compact subset of $\mathbb{R}^m$. Then we deduce that u belongs to $C^\alpha([0,T-\delta] \times \mathcal{K})$. Thus u is continuous of $[0,T[\times\mathbb{R}^m$.

We can define two sequences $\{b_m\}_{m\in\mathbb{N}}$ and $\{\sigma_m\}_{m\in\mathbb{N}}$ of functions such that for all $m \in \mathbb{N}$, b_m and σ_m satisfy the assumptions of the previous lemma. We assume that the bounds in (L)-(B)-(E) for b_m and σ_m are the bounds of b and σ, i.e. the bounds do not depend on m, and that b_m (resp. σ_m) converges uniformly on every compact subset of $[0,T] \times \mathbb{R}^m$ to b (resp. σ). Therefore the Cauchy problem

$$\left\{ \begin{array}{rcll} \partial_t v + L_m v - v|v|^q & = & 0, & \text{on } [0,T[\times\mathbb{R}^m \\ v(T,x) & = & (g \wedge n)(x) & \text{on } \mathbb{R}^m \end{array} \right.$$

has a unique regular solution $v_{m,n}$. Moreover if $X^{t,x,m}$ is the solution of the SDE

$$X_s^{t,x,m} = x + \int_t^s b_m(r,X_r^{t,x,m})dr + \int_t^s \sigma_m(r,X_r^{t,x,m})dB_r, \text{ for } s \in [t,T],$$

then $(v_{m,n}(X^{t,x,m}),\nabla v_{m,n}(X^{t,x,m})\sigma_m(X^{t,x,m}))$ is the solution of the BSDE (2.1) with terminal data $(g \wedge n)(X_T^{t,x,m})$. We can prove that $v_{m,n}$ converges when $m \to +\infty$ to u_n using standard estimates on the SDE and the BSDE.

Now for every $\delta > 0$ and every compact subset $\mathcal{K}$, from the theorem 10.1 in [26], $v_{m,n}$ belongs to a Hölder space $C^\alpha([0,T-\delta] \times \mathcal{K})$. And the norm $\|v_{m,n}\|_\alpha$ depends only on the bounds in (L)-(B)-(E) (in particular not on m). Therefore $\{u_n\}_n$ is also bounded in this space C^α. □

2.6 Additional results

In the previous sections we have chosen ξ non negative with the generator $f(y) = -y|y|^q$. But of course we can prove the same results when $\xi \leq 0$. In fact if $\xi \in L^\infty(\Omega)$ non-positive and if (Y,Z) is the solution of the BSDE:

$$Y_t = \xi + \int_t^T f(Y_r)dr - \int_t^T Z_r dB_r,$$

it is obvious that $(U,V) = (-Y,-Z)$ is the solution of

$$U_t = (-\xi) + \int_t^T f(U_r)dr - \int_t^T V_r dB_r.$$

Thus if ξ is non-positive and satisfies all conditions of the previous section:

(H1) $$\xi = g(X_T),$$

where $g : \mathbb{R}^m \to \overline{\mathbb{R}^-}$ is such that $\{g = -\infty\}$ is closed and that the assumption (H2) is verified, and with either $q > 2$ and (L)-(G) or (L)-(A1)-(A2), there exists a maximal solution of

$$Y_t = \xi - \int_t^T Y_r|Y_r|^q dr - \int_t^T Z_r dB_r$$

among all negative solutions.

We can also work without sign assumption.

Theorem 2.7 *Let ξ be a $\mathcal{F}_T$-measurable random variable, possibly negative, such that:*

$$\mathbf{P}(\xi = +\infty \text{ or } \xi = -\infty) > 0.$$

Moreover ξ satisfies

(H1) $$\xi = g(X_T),$$

with $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ such that:

1. *the two sets $\{g = +\infty\}$ and $\{g = -\infty\}$ are closed ;*
2. *the condition (H2) is verified.*

The coefficients of the SDE satisfy (L)-(G) and in the case $0 < q \leq 2$*, they also verify (A1)-(A2).*

There exists a process (Y,Z) *which is a solution of the BSDE*

$$Y_t = \xi + \int_t^T f(Y_r)dr - \int_t^T Z_r dB_r$$

in the sense of the definition 2.1 of the introduction.

Proof. For $(n,m) \in (\mathbb{N}^*)^2$ define ξ_m^n by

$$\xi^{n,m} = \xi \wedge n \vee (-m).$$

As $\xi^{n,m}$ is in $L^\infty(\Omega)$, there exists a unique solution $(Y^{n,m},Z^{n,m})$ satisfying

$$Y_t^{n,m} = \xi^{n,m} - \int_t^T Y_r^{n,m}|Y_r^{n,m}|^q dr - \int_t^T Z_r^{n,m} dB_r$$

and

$$\mathbb{E}\left(\sup_{t\in[0,T]} |Y_t^{n,m}|^2 + \int_0^T \|Z_r^{n,m}\|^2 dr\right) < +\infty.$$

With the comparison theorem 2.4 in [42], we have for $0 \leq n \leq n'$ and $0 \leq m \leq m'$, and for all $t \in [0,T]$:

$$-\left(\frac{1}{q(T-t) + \frac{1}{(m')^q}}\right)^{\frac{1}{q}} \leq Y_t^{n,m'} \leq Y_t^{n,m} \leq Y_t^{n',m} \leq \left(\frac{1}{q(T-t) + \frac{1}{(n')^q}}\right)^{\frac{1}{q}}. \tag{2.40}$$

Recall that if x belongs to $\mathbb{R}$, the solution of the ordinary differential equation $y' = y|y|^q$ with $y_T = x$ is equal to

$$\mathrm{sign}(x)\left(\frac{1}{q(T-t) + \frac{1}{|x|^q}}\right)^{\frac{1}{q}}$$

where $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(x) = 1$ if $x > 0$. We define the following process: for $(n,m) \in (\mathbb{N}^*)^2$ and for all $t \in [0,T]$

$$Y_t^{\uparrow,m} = \lim_{n\to+\infty} Y_t^{n,m} \quad \text{and} \quad Y_t^{n,\downarrow} = \lim_{m\to+\infty} Y_t^{n,m}.$$

Therefore with the inequalities (2.40) we have:

$$-\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \leq Y_t^{n,\downarrow} \leq Y_t^{\uparrow,m} \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}.$$

As in the first section we can easily prove that:

1. the limit $Z^{\uparrow,m}$ of $(Z^{n,m})_{n\in\mathbb{N}}$ as n goes to $+\infty$, exists in $L^2(\Omega\times[0,t])$ for all $t < T$;
2. for all $t < T$

$$\mathbb{E}\left(\sup_{s\in[0,t]} |Y_s^{\uparrow,m}|^2 + \int_s^t \|Z_r^{\uparrow,m}\|^2 dr\right) \leq C\left(\frac{1}{q(T-t)}\right)^{\frac{2}{q}};$$

3. for all $t < T$, $\mathbf{P}$-p.s., for all $0 \leq s \leq t$,

$$Y_s^{\uparrow,m} = Y_t^{\uparrow,m} - \int_s^t Y_r^{\uparrow,m}|Y_r^{\uparrow,m}|^q dr - \int_s^t Z_r^{\uparrow,m} dB_r;$$

4. $\mathbf{P}$-p.s.

$$\xi \vee (-m) \leq \liminf_{t\to T} Y_t^{\uparrow,m}. \tag{2.41}$$

The same results are still true for the sequence $(Y^{n,m},Z^{n,m})_{m\in\mathbb{N}}$:

1. the limit $Z^{n,\downarrow}$ of $(Z^{n,m})_{m\in\mathbb{N}}$ as m goes to $+\infty$, exists in $L^2(\Omega\times[0,t])$ for all $t < T$;
2. for all $t < T$

$$\mathbb{E}\left(\sup_{s\in[0,t]} |Y_s^{n,\downarrow}|^2 + \int_s^t \|Z_r^{n,\downarrow}\|^2 dr\right) \leq C\left(\frac{1}{q(T-t)}\right)^{\frac{2}{q}};$$

3. for all $t < T$, $\mathbf{P}$-p.s., for all $0 \leq s \leq t$,

$$Y_s^{n,\downarrow} = Y_t^{n,\downarrow} - \int_s^t Y_r^{n,\downarrow}|Y_r^{n,\downarrow}|^q dr - \int_s^t Z_r^{n,\downarrow} dB_r;$$

4. $\mathbf{P}$-p.s.

$$\limsup_{t\to T} Y_t^{n,\downarrow} \leq \xi \wedge n.$$

Suppose now we have proved the following proposition:

Proposition 2.12

$$\lim_{t\to T} Y_t^{\uparrow,m} = \xi \vee (-m) \quad \textit{and} \quad \lim_{t\to T} Y_t^{n,\downarrow} = \xi \wedge n \qquad \mathbf{P}-a.s. \tag{2.42}$$

Recall that $(Y^{n,\downarrow})_{n\in\mathbb{N}}$ and $(Y^{\uparrow,m})_{m\in\mathbb{N}}$ are respectively a non-decreasing and a non-increasing sequence. Let us define:

$$Y^{\downarrow} = \lim_{n\to+\infty} Y^{n,\downarrow} \quad \text{and} \quad Y^{\uparrow} = \lim_{m\to+\infty} Y^{\uparrow,m}.$$

We have:

$$-\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}} \leq Y_t^{\downarrow} \leq Y_t^{\uparrow} \leq \left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}$$

and for all $(n,m) \in \mathbb{N}^2$

$$\xi \wedge n \leq \liminf_{t\to T} Y_t^{\downarrow} \quad \text{and} \quad \limsup_{t\to T} Y_t^{\uparrow} \leq \xi \vee (-m).$$

Then it is easy to prove that $(Y,Z) = (Y^{\downarrow},Z^{\downarrow})$ or $(Y,Z) = (Y^{\uparrow},Z^{\uparrow})$ satisfy:

1. for all $t < T$

$$\mathbb{E}\left(\sup_{s\in[0,t]} |Y_s|^2 + \int_s^t \|Z_r\|^2 dr\right) \leq C\left(\frac{1}{q(T-t)}\right)^{\frac{2}{q}};$$

2. for all $t < T$, **P**-p.s., for all $0 \le s \le t$,

$$Y_s = Y_t - \int_s^t Y_r |Y_r|^q dr - \int_s^t Z_r dB_r;$$

3. and **P**-a.s.

$$\lim_{t \to T} Y_t = \xi.$$

This achieves the proof of the theorem. □

Proof of the proposition 2.12. Remark that it is enough to show the first limit in (2.42). Indeed $(-Y^{n,\downarrow}, -Z^{n,\downarrow})$ is equal to $(Y^{\uparrow,n}, Z^{\uparrow,n})$.

Now we fix a integer m. For simplicity we denote by (Y^n, Z^n) the process $(Y^{n,m}, Z^{n,m})$ and by (Y,Z) the process $(Y^{\uparrow,m}, Z^{\uparrow,m})$. Define $x^+ = x \vee 0$, $x^- = -(x \wedge 0)$,

$$(Y^{n,+}, Z^{n,+}) = ((Y^n)^+, \left(\mathbf{1}_{\mathbb{R}_+}(Y^n)\right) Z^n) \text{ and } (Y^{n,-}, Z^{n,-}) = ((Y^n)^-, -\left(\mathbf{1}_{\mathbb{R}_-}(Y^n)\right) Z^n).$$

Lemma 2.5 *If ζ is a real-valued random variable, with $\zeta \in L^2(\Omega)$, if (Y,Z) and $(\widetilde{Y}, \widetilde{Z})$ denote the solution of the BSDE:*

$$Y_t = \zeta - \int_t^T Y_r |Y_r|^q dr - \int_t^T Z_r dB_r \quad \text{and} \quad \widetilde{Y}_t = \zeta^+ - \int_t^T \widetilde{Y}_r |\widetilde{Y}_r|^q dr - \int_t^T \widetilde{Z}_r dB_r,$$

then for all $t \in [0,T]$

$$Y_t^+ \le \widetilde{Y}_t.$$

Proof of the lemma. The process Y is a continuous semimartingale, i.e. we write Y as follows

$$Y_t = Y_0 + A_t + M_t$$

with M a continuous martingale and A a process of bounded variations. Therefore there exists a semimartingale local time for Y (see [22], theorem 7.1, page 218), i.e. a non-negative process

$$\Lambda = \{\Lambda_t(\omega);\ t \in [0,T],\, \omega \in \Omega\}$$

such that

$$\begin{aligned} Y_t^+ &= \zeta^+ - \int_t^T \left(\mathbf{1}_{[0,+\infty[}(Y_r)\right) Y_r |Y_r|^q dr - \int_t^T \left(\mathbf{1}_{\mathbb{R}_+}(Y_r)\right) Z_r dB_r + (\Lambda_t - \Lambda_T) \\ &= \zeta^+ - \int_t^T Y_r^+ |Y_r^+|^q dr - \int_t^T \left(\mathbf{1}_{\mathbb{R}_+}(Y_r)\right) Z_r dB_r + (\Lambda_t - \Lambda_T) \end{aligned}$$

Moreover $t \mapsto \Lambda_t$ is non-decreasing a.s. Thus applying the comparison theorem 2.4 in [42], we obtain the announced result and this achieves the proof of the lemma. □

With the same arguments we prove easily that Y^- satisfies:

$$\begin{aligned} Y_t^- &= \zeta^- + \int_t^T \left(\mathbf{1}_{\mathbb{R}_-}(Y_r)\right) Y_r |Y_r|^q dr + \int_t^T \mathbf{1}_{\mathbb{R}_-}(Y_r) Z_r dB_r + (\Lambda_t - \Lambda_T) \\ &= \zeta^- - \int_t^T |Y_r^-|^{1+q} dr + \int_t^T \mathbf{1}_{\mathbb{R}_-}(Y_r) Z_r dB_r + (\Lambda_t - \Lambda_T). \end{aligned}$$

Now with our previous notations we have for all $n \in \mathbb{N}$, for all $t \in [0,T]$

$$Y_t^{n,+} \leq U_t^n \leq U_t \quad \text{and} \quad (Y_t)^+ \leq U_t$$

with

$$U_t^n = \xi^+ \wedge n - \int_t^T U_r^n |U_r^n|^q dr - \int_t^T V_r^n dB_r,$$

and (U,V) is the minimal solution of:

$$U_t = \xi^+ - \int_t^T U_r |U_r|^q dr - \int_t^T V_r dB_r.$$

Since we already know (section 2) that a.s.

$$\lim_{t \to T} U_t = \xi^+$$

we have

$$\limsup_{t \to T} (Y_t)^+ \leq \xi^+.$$

Moreover from (2.41), we obtain

$$\xi \vee (-m) \leq \liminf_{t \to T} (Y_t)^+ \quad \text{and} \quad 0 \leq \liminf_{t \to T} (Y_t)^+.$$

Therefore

$$\lim_{t \to T} (Y_t)^+ = \xi^+.$$

For the negative part, recall that for all $n \in \mathbb{N}$:

$$0 \leq (Y_t^{n,m})^- \leq \left(\frac{1}{q(T-t) + \frac{1}{m^q}} \right)^{\frac{1}{q}} \leq m, \tag{2.43}$$

and for $t \leq T$:

$$Y_t^{n,-} = Y_T^{n,-} - \int_t^T Y_s^{n,-} |Y_s^{n,-}|^q ds - (\Lambda_T^n - \Lambda_t^n) - \int_t^T Z_s^{n,-} dB_s. \tag{2.44}$$

We want to pass to the limit when n goes to ∞ in this equation. With the estimation (2.43) and the dominated convergence theorem, we obtain for all $t \in [0,T]$:

$$\lim_{n \to \infty} \int_t^T Y_s^{n,-} |Y_s^{n,-}|^q ds = \int_t^T (Y_s)^- |(Y_s)^-|^q ds \leq T m^{1+q}.$$

We apply Itô's formula to the process $Y^{n,-}$ and because of the monotony of f we obtain:

$$\frac{1}{2} \mathbb{E} \int_0^T \left\| Z_r^{n,-} \right\|^2 dr \leq \mathbb{E} \left(Y_T^{n,-} \right)^2 - 2\mathbb{E} \int_0^T Y_r^{n,-} d\Lambda_r^n = \mathbb{E} \left(Y_T^{n,-} \right)^2 \leq m^2.$$

Hence for all $t \in [0,T]$

$$\lim_{n \to \infty} \int_t^T Z_r^{n,-} dB_r = - \int_t^T \mathbf{1}_{]-\infty,0]}(Y_r) Z_r dB_r$$

and

$$\mathbb{E}\int_0^T \left\|\mathbf{1}_{]-\infty,0]}(Y_r)Z_r\right\|^2 dr \leq 2m^2.$$

Now for $(n,n') \in \mathbb{N}^2$ and $t \in [0,T]$ we have:

$$\begin{aligned}
|\Lambda_t^n - \Lambda_t^{n'}| &= \left|(Y_0^{n,-} - Y_0^{n',-}) - (Y_t^{n,-} - Y_t^{n',-}) - \int_0^t \left(f(Y_r^{n,-}) - f(Y_r^{n',-})\right) dr \right. \\
&\quad \left. + \int_0^t \left(Z_r^{n,-} - Z_r^{n',-}\right) dB_r \right| \\
&\leq 2 \sup_{t\in[0,T]} \left|Y_t^{n,-} - Y_t^{n',-}\right| + \int_0^T \left|f(Y_r^{n,-}) - f(Y_r^{n',-})\right| dr \\
&\quad + \sup_{t\in[0,T]} \left|\int_0^t \left(Z_r^{n,-} - Z_r^{n',-}\right) dB_r\right|.
\end{aligned}$$

Then we deduce that (Λ^n) converges to a process Λ which is continuous, non decreasing on $[0,T]$. We pass the limit in (2.44):

$$Y_t^- = (\xi \vee (-m))^- - \int_t^T Y_s^-|Y_s^-|^q ds - (\Lambda_T - \Lambda_t) + \int_t^T \mathbf{1}_{]-\infty,0]}(Y_s)Z_s dB_s.$$

Finally we have:

$$\lim_{t\to T} Y_t^- = (\xi \vee (-m))^-$$

with $0 \leq (\xi \vee (-m))^- \leq m$ and

$$\lim_{t\to T} Y_t = \lim_{t\to T} Y_t^+ - \lim_{t\to T} Y_t^- = \xi \vee (-m).$$

□

If ξ is either non negative or non positive, it is obvious that $(Y^\downarrow,Z^\downarrow) = (Y^\uparrow,Z^\uparrow)$ is respectively either the minimal non negative solution or the maximal non positive solution. But in general we are unable to prove the uniqueness of the solution.

For the related PDE recall that for $(t,x) \in [0,T] \times \mathbb{R}^m$ $X^{t,x}$ is the solution of the SDE (2.7). For every $(n,m) \in \mathbb{N}^2$ $Y^{n,m,t,x}$ is the solution of the BSDE (2.1) with $g(X^{t,x}) \wedge n \vee (-m)$ as terminal condition. As in the section 4, if the function $u_{n,m}$ is defined by $u_{n,m}(t,x) = Y_t^{n,m,t,x}$, $u_{n,m}$ is a viscosity solution of the PDE (2.2) with final condition $g \wedge n \vee (-m)$. We can do the same demonstration of in the section 4 to show that

$$u^\uparrow(t,x) = \lim_{m\to+\infty} \lim_{n\to+\infty} u_{n,m}(t,x) \quad \text{and} \quad u^\downarrow(t,x) = \lim_{n\to+\infty} \lim_{m\to+\infty} u_{n,m}(t,x)$$

are two viscosity solutions of the PDE (2.2) with g as terminal condition. In fact we just have to bound $u_{n,m}$ on the complementary of $\{g = +\infty \text{ or} g = -\infty\}$. We use the following lemma:

Lemma 2.6 *Suppose that v is a viscosity solution of the PDE (2.2) with $v(T,x) = h(x)$. h is bounded and continuous on $\mathbb{R}^m$. Then $|v|$ is a subsolution of the same PDE with $|v|(T,x) = |h(x)|$.*

Proof. The definition of a subsolution was given in the section 4, definition 2.2. Since v is continuous, $|v|$ is also continuous. Let $\phi : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ be a $C^{1,2}$ function such that $|v| - \phi$ has a local maximum at (t,x). If $t = T$ we already know that $|v(T,x)| = |h(x)|$. If $t < T$ then there exists a neighbourhood $\mathcal{U} =]t-\varepsilon, t+\varepsilon[\times U$ of (t,x) such that for $(s,y) \in \mathcal{U}$, $0 \leq |v(s,y)| \leq \phi(s,y)$. Moreover we can always suppose that $|v(t,x)| = \phi(t,x)$. If $v(t,x) = 0$, ϕ is a non-negative function on $\mathcal{U}$ and attains his minimum at (t,x). Therefore $\nabla\phi(t,x) = 0$, $\partial_t\phi(t,x) = 0$ and $D^2\phi(t,x) \geq 0$. We deduce:

$$-\partial_t\phi(t,x) - L\phi(t,x) + |v(t,x)|^{1+q} = -\frac{1}{2}\text{Trace}(\sigma\sigma^*(t,x)D^2v(t,x)) \leq 0.$$

If $v(t,x) > 0$, then we can suppose that v is positive on $\mathcal{U}$:

$$\forall (s,y) \in \mathcal{U},\ 0 < v(s,y) = |v(s,y)| \leq \phi(s,y) \quad \text{and} \quad v(t,x) = \phi(t,x).$$

We apply the definition of a subsolution to v and deduce:

$$-\partial_t\phi(t,x) - L\phi(t,x) + v(t,x)|v(t,x)|^q = -\partial_t\phi(t,x) - L\phi(t,x) + |v(t,x)|^{1+q} \leq 0.$$

In the last case $v(t,x) < 0$, we suppose that on $\mathcal{U}$ v is negative. Thus $v - (-\phi)$ has a local minimum at (t,x) and we apply the definition of a supersolution:

$$-\partial_t(-\phi)(t,x) - L(-\phi)(t,x) + v(t,x)|v(t,x)|^q \geq 0;$$

$$\partial_t\phi(t,x) + L\phi(t,x) + v(t,x)|v(t,x)|^q \geq 0;$$

$$-\partial_t\phi(t,x) - L\phi(t,x) + (-v(t,x))|v(t,x)|^q = -\partial_t\phi(t,x) - L\phi(t,x) + |v(t,x)|^{1+q} \leq 0.$$

Finally $|v|$ is a subsolution and this achieves the proof of the lemma. □

By a standard comparison argument (see [10], theorem 8.2 page 48), the solution $u_{n,m}$ is bounded by the viscosity solution U of the PDE (2.2) with terminal argument $|g| \wedge n \wedge m$ and we can use our previous results on U (section 4).

2.7 Appendix

Recall that X is the solution of the SDE:

$$X_t = x + \int_0^t b(r,X_r)dr + \int_0^t \sigma(r,X_r)dB_r, \text{ for } t \in [0,T]; \tag{2.6}$$

with the conditions (L)-(B) on b and σ, and that

$$\xi = g(X_T), \tag{H1}$$

where $g : \mathbb{R}^m \to \overline{\mathbb{R}_+}$ is such that the set $F = \{g = \infty\}$ is a non empty closed set in $\mathbb{R}^m$. We assume:

$$\xi \mathbf{1}_{\{\xi < \infty\}} \in \ L^1(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R}). \tag{H2'}$$

Under the assumptions (H1) and (H2') on ξ, we have **P**-a.s.

$$\lim_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right]=\xi.$$

We have:
(2.45)

$$\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{\frac{1}{q}}\right]=\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=+\infty}\right)+\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{q(T-t)+\frac{1}{\xi^q}}\right)^{\frac{1}{q}}\right]$$

We have already proved in the second section in the proof of the proposition 1, that on the set $\{\xi=+\infty\}$, **P**-a.s.

$$\lim_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right]=+\infty.$$

We recall it for convenience. From (2.45):

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(T-t)+\frac{1}{\xi^q}}\right)^{\frac{1}{q}}\right] &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{\xi^q\mathbf{1}_{\xi<\infty}}{1+q(T-t)\xi^q}\right)^{\frac{1}{q}}+\frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right)\\ &\geq \mathbb{E}^{\mathcal{F}_t}\left(\frac{\xi^q\mathbf{1}_{\xi<\infty}}{1+qT\xi^q}\right)^{\frac{1}{q}}+\frac{1}{(q(T-t))^{1/q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).\end{aligned}$$

The first term in the right hand side converges to 0 on the set $\{\xi=+\infty\}$ and the second converges to $(1/q)^{1/q}$. Indeed we have

$$0\leq\frac{\xi^q}{1+qT\xi^q}\leq\frac{1}{qT}$$

and we can apply the result on convergence of martingales.

We just have to prove the result a.s. on the set $\{\xi<\infty\}$. We claim that under the assumption (H1) on the set $\{\xi<\infty\}$:

$$\lim_{t\to T}\left(\frac{1}{q(T-t)}\right)^{\frac{1}{q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=+\infty}\right)=0. \tag{2.46}$$

The conditional expectation can be calculated:

$$\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{\xi=+\infty\}}\right)=\Lambda_t\left(X_t\right),$$

with for $y\in\mathbb{R}^m$,

$$\begin{aligned}\Lambda_t(y) &= \mathbb{E}\left(\mathbf{1}_{\{g=+\infty\}}(X_T^{t,y})\right)\\ &= \mathbf{P}\left(X_T^{t,y}\in F\right)\end{aligned}$$

with for $s \in [t,T]$,

$$X_s^{t,y} = y + \int_t^s b(r,X_r^{t,y})dr + \int_t^s \sigma(r,X_r^{t,y})dB_r,$$

and $X_s^{t,y} = y$ si $0 \leq s \leq t$.

The set $\{g = +\infty\}$ being denoted by F, we denote by $d(y,F)$ the distance from $y \in \mathbb{R}^m$ to F. Since F is closed, if y does not belong to F, we have $d(y,F) > 0$.

Lemma 2.7 *Supppose y does not belong to F. There exists $t_0 \in [0,T[$ which depends on $d(y,F)$ such that for all $t_0 \leq t < T$:*

$$\Lambda_t(y) = \mathbf{P}\left(X_T^{t,y} \in F\right) \leq (2m)\exp\left(-\frac{d(y,F)^2}{8Am(T-t)}\right),$$

with $A = \|\sigma\sigma^*\|_\infty$.

Proof. If y does not belong to F, since F is closed, the distance $d = d(y,F) > 0$. The intersection of F and the open ball centered at y and with radius d is empty. As b is bounded, there exists $t_0 \in [0,T[$ such that for all $t_0 \leq t \leq T$ we have:

$$\| \int_t^T b(r,X_r^{t,y})dr \| \leq \frac{d}{2}.$$

Thus if $B(x,R)$ is the open ball centered at x and with radius R:

$$B\left(y + \int_t^T b(r,X_r^{t,y})dr\ ,\ \frac{d}{2}\right) \cap F = \emptyset.$$

Recall that $X_T^{t,y}$ satisfies the equation:

$$X_T^{t,y} = y + \int_t^T b(r,X_r^{t,y})dr + \int_t^T \sigma(r,X_r^{t,y})dB_r.$$

Hence if we want $X_T^{t,y}$ to be in F, it is necessary that the norm of $\int_t^T \sigma(r,X_r^{t,y})dB_r$ is greater than $d/2$:

$$\mathbf{P}\left(X_T^{t,y} \in F\right) \leq \mathbf{P}\left(\| \int_t^T \sigma(r,X_r^{t,y})dB_r\| \geq \frac{d}{2}\right).$$

Since σ is bounded $\sigma\sigma^*$ is also bounded. Therefore we can apply the theorem 4.2.1 (p. 87) of [52] to obtain:

$$\begin{aligned}
\mathbf{P}\left(\| \int_t^T \sigma(r,X_r^{t,y})dB_r\| \geq \frac{d}{2}\right) &= \mathbf{P}\left(\|X_T^{t,y} - X_t^{t,y} - \int_t^T b(r,X_r^{t,y})dr\| \geq \frac{d}{2}\right) \\
&\leq \mathbf{P}\left(\sup_{t\leq u\leq T} \|X_u^{t,y} - X_t^{t,y} - \int_t^u b(r,X_r^{t,y})dr\| \geq \frac{d}{2}\right) \\
&\leq (2m)\exp\left(-\frac{\frac{d^2}{4}}{2Am(T-t)}\right).
\end{aligned}$$

This achieves the proof of the lemma. □

Let $\omega \in \Omega$ be in the set $\{\xi < \infty\}$: this means that $X_T(\omega)$ is not in F. There exists $c = c(\omega) = d(X_T(\omega),F) > 0$ such that $B\left(X_T(\omega),2c\right) \cap F$ is empty. By a continuity argument there exists $t_1 = t_1(\omega) \in [0,T[$ such that for $t_1 \leq t \leq T$, $d(X_t(\omega),F) > c$. From the previous lemma there exists $t_0 = t_0(\omega)$ such that for $t_0 \vee t_1 \leq t < T$:

$$\begin{aligned}\left(\frac{1}{q(T-t)}\right)^{1/q} \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{\xi=+\infty\}}\right)(\omega) &= \left(\frac{1}{q(T-t)}\right)^{1/q} \Lambda_t\left(X_t(\omega)\right) \\ &\leq 2m\left(\frac{1}{q(T-t)}\right)^{1/q} \exp\left(-\frac{c^2}{8Am(T-t)}\right).\end{aligned}$$

Finally a.s. on the set $\{\xi < \infty\}$:

$$\lim_{t\to T}\left(\frac{1}{q(T-t)}\right)^{1/q} \mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\{\xi=+\infty\}}\right) = 0.$$

For the remaining term in (2.45)

$$\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{q(T-t)+\frac{1}{\xi^q}}\right)^{1/q}\right] = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\xi^q\mathbf{1}_{\xi<\infty}}{1+q(T-t)\xi^q}\right)^{1/q}\right],$$

we make the following calculation: for all $s \leq t$

$$\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-s)}\right)^{1/q}\right] \leq \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-t)}\right)^{1/q}\right]$$

therefore taking first the limit as t goes to T and then the limit $s \to T$, we obtain:

$$\xi\mathbf{1}_{\xi<\infty} = \lim_{s\to T}\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-s)}\right)^{\frac{1}{q}} \leq \liminf_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-t)}\right)^{\frac{1}{q}}\right].$$

We also have:

$$\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-t)}\right)^{1/q}\right] \leq \mathbb{E}^{\mathcal{F}_t}\left(\xi\mathbf{1}_{\xi<\infty}\right).$$

The hypothesis (H2') implies that the martingale $\left(E^{\mathcal{F}_t}\left(\xi\mathbf{1}_{\xi<\infty}\right)\right)_{0\leq t\leq T}$ converges a.s. to $\xi\mathbf{1}_{\xi<\infty}$ when t converges to T; thus:

$$\limsup_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\mathbf{1}_{\xi<\infty}}{\frac{1}{\xi^q}+q(T-t)}\right)^{\frac{1}{q}}\right] \leq \xi\mathbf{1}_{\xi<\infty}.$$

Finally with the relations (2.45) and (2.46), on the set $\{\xi < \infty\}$

$$\lim_{t\to T}\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{\frac{1}{\xi^q}+q(T-t)}\right)^{\frac{1}{q}}\right] = \xi.$$

Chapitre 3

EDSR avec temps final aléatoire

Introduction

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space, $B = (B_t)_{t\geq 0}$ a Brownian motion defined on this space, with values in $\mathbb{R}^d$. $(\mathcal{F}_t)_{t\geq 0}$ is the standard filtration of the Brownian motion. τ is a $\{\mathcal{F}_t\}$ stopping time, ξ an $\mathcal{F}_\tau$-mesurable random variable in $\mathbb{R}^k$, and $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ a function such that $f(.,y,z)$ is a progressively mesurable process in $\mathbb{R}^k$ for each $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$.

We wish to find a progressively measurable solution (Y,Z) with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ of the equation

$$Y_t = \xi + \int_{t\wedge\tau}^{\tau} f(r,Y_r,Z_r)dr - \int_{t\wedge\tau}^{\tau} Z_r dB_r,\ t \geq 0 \tag{3.1}$$

Let us recall the definition of a solution which can be found in [9].

Definition 3.1 *A solution of the BSDE (3.1) is a pair $\{(Y_t,Z_t),\ t \geq 0\}$ of progressively measurable processes with values in $\mathbb{R}^k \times \mathbb{R}^{k\times d}$ such that, $\mathbf{P}$-a.s.*

- *on the set $\{t > \tau\}$, $Y_t = \xi$ and $Z_t = 0$,*
- *$t \mapsto \mathbf{1}_{t\leq\tau} f(t,Y_t,Z_t)$ belongs to $L^1_{loc}(0,\infty)$,*
- *$t \mapsto Z_t$ belongs to $L^2_{loc}(0,\infty)$,*
- *and for all $0 \leq t \leq T$*

$$Y_{t\wedge\tau} = Y_{T\wedge\tau} + \int_{t\wedge\tau}^{T\wedge\tau} f(r,Y_r,Z_r)dr - \int_{t\wedge\tau}^{T\wedge\tau} Z_r dB_r.$$

A solution is said to be a L^p-solution for some $p > 1$ if moreover for some $\lambda \in \mathbb{R}$

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{p\lambda t}|Y_t|^p + \int_0^\tau e^{p\lambda t}|Y_t|^p dt + \int_0^\tau e^{p\lambda t}|Y_t|^{p-2}\|Z_t\|^2 dt\right) < +\infty.$$

We are given a final time τ which is an $\mathcal{F}_t$-stopping time ; a coefficient $f : \Omega \times \mathbb{R}^+ \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$ which is such that for some real number μ and $K \geq 0$,

(H0) $f(.,y,z)$ is progressively measurable, for all y,z ;

(H1) $|f(t,y,z) - f(t,y,z')| \leq K\|z - z'\|$, $\forall t,y,z,z'$, a.s.
(H2) $\langle y - y', f(t,y,z) - f(t,y',z)\rangle \leq \mu|y - y'|^2$, $\forall t,y,y',z$;
(H3) $y \mapsto f(t,y,z)$ is continuous, $\forall t,z$, a.s. ;
and a final condition ξ which is an $\mathcal{F}_\tau$-measurable and k-dimensional random variable.
We suppose that for all $r > 0$

$$\text{(H4)} \qquad \psi_r(t) = \sup_{|y|\leq r} |f(t,y,0) - f(t,0,0)| \in L^1((0,n)\times\Omega) \quad \forall n \in \mathbb{N}^*.$$

Now for some $p > 1$ we assume that there exists:

$$\lambda > \nu_p = \mu + \frac{K^2}{2(p-1)},$$

such that

$$\text{(H5)} \qquad \mathbb{E}\left[\int_0^\tau e^{p\lambda t}|f(t,0,0)|^p dt\right] < +\infty$$

and

$$\text{(H6)} \qquad \mathbb{E}\left[e^{p\lambda\tau}|\xi|^p + \int_0^\tau e^{p\lambda t}|f(t,e^{-\nu_p t}\overline{\xi}_t, e^{-\nu_p t}\overline{\eta}_t)|^p dt\right] < +\infty,$$

where $\overline{\xi} = e^{\nu_p\tau}\xi$, $\overline{\xi}_t = \mathbb{E}(\overline{\xi}|\mathcal{F}_t)$ and $\overline{\eta}$ is predictable and such that

$$\overline{\xi} = \mathbb{E}\left(\,\overline{\xi}\,\right) + \int_0^{+\infty} \overline{\eta}_t dB_t, \quad \mathbb{E}\left[\left(\int_0^\infty |\overline{\eta}_t|^2 dt\right)^{p/2}\right] < \infty.$$

Let us recall the theorem 5.2 of Ph. Briand, B. Delyon, Y. Hu, E. Pardoux and L. Stoica [9].

Theorem 3.1 *Under the conditions (H0)-...-(H6), there exists a unique solution (Y,Z) of the BSDE (3.1), which moreover satisfies, for $\lambda > \nu_p$ such that (H5) and (H6) hold,*

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{p\lambda t}|Y_t|^p + \int_0^\tau e^{p\lambda r}|Y_r|^{p-2}\left(|Y_r|^2 + \|Z_r\|^2\right) dr\right)$$
$$\leq c\;\mathbb{E}\left(e^{p\lambda\tau}|\xi|^p + \int_0^\tau e^{p\lambda r}|f(t,0,0)|^p dr\right).$$

for some constant $c = c(p,\lambda,K,\mu)$.

Remark 3.1 *The previous theorem is a generalization of the result of R. Darling and E. Pardoux (theorem 3.4 in [11]) or of E. Pardoux (theorem 4.1 in [42]). In [11] or [42], the result is given in the case $p = 2$.*

From now and in the rest of the chapter, we will work under the following assumptions:

1. q is a positive real ;

2. the function f is deterministic and equal to:

$$f(t,y,z) = -y|y|^q.$$

f satisfies all conditions (H0)-...-(H3) of the theorem 3.1, with $K = \mu = 0$, which implies $\nu_p = 0$ for all $p > 1$. Moreover $f(t,0,0) \equiv 0$, so (H5) is always satisfied.

3. D denotes an open bounded subset of $\mathbb{R}^d$, whose boundary is at least of class C^2.

The stopping time τ is defined as follows. For all $x \in \mathbb{R}^d$ let X^x denote the solution of the SDE:

$$X_t^x = x + \int_0^t b(X_r^x)dr + \int_0^t \sigma(X_r^x)dB_r, \text{ for } t \geq 0. \tag{3.2}$$

The functions b and σ are defined on $\mathbb{R}^d$, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, are mesurable and such that:

i. Lipschitz condition: there exists $K \geq 0$ such that for all (x,y):

$$\|\sigma(x) - \sigma(y)\| \leq K|x-y|; \tag{L}$$

ii. Boundedness condition:

$$|b(x)| + \|\sigma(x)\| \leq K; \tag{B}$$

iii. Uniform ellipticity: there exists a constant $\alpha > 0$ such that for all $x \in \mathbb{R}^d$:

$$\sigma\sigma^*(x) \geq \alpha \mathrm{Id}. \tag{E}$$

In the rest of this chapter, (L), (B) and (E) are supposed to be satisfied. Under these assumptions, from a result of A. Yu Veretennikov [53] and [54], the equation (3.2) has a unique strong solution X^x.

For each $x \in \overline{D}$, we define the stopping time:

$$\tau = \tau_x = \inf\left\{t \geq 0,\ X_t^x \notin \overline{D}\right\}.$$

Since D is bounded and since the conditions (B)-(E) hold:

$$\text{every point } x \in \partial D \text{ is regular.} \tag{C0}$$

In particular if $x \in \partial D$ $\tau_x = 0$ a.s. (see [4], corollary 3.2). Moreover we have the following result (see [48], theorem 2.1 and [42], remark 5.6): for all $x \in \overline{D}$, $\tau_x < +\infty$ a.s. and there exists $\beta > 0$ such that:

$$\sup_{x \in \overline{D}} \mathbb{E}\left(e^{\beta\tau_x}\right) < \infty. \tag{C1}$$

From now we are concerned by the following BSDE

$$Y_t = \xi - \int_{t\wedge\tau}^{\tau} Y_r|Y_r|^q dr - \int_{t\wedge\tau}^{\tau} Z_r dB_r. \tag{3.3}$$

From the papers [11], [42] or [46], we know that the BSDE (3.3) with terminal time equal to τ_x and final data equal to $\xi = h(X^x_{\tau_x})$ ($h \in C(\partial D) \Rightarrow \xi \in L^\infty$) is associated with the following PDE:

$$(3.4) \qquad \left\{ \begin{array}{rl} -\mathcal{L}u + u|u|^q = 0 & \text{on } D, \\ u = h & \text{on } \partial D; \end{array} \right.$$

where $\mathcal{L}$ is the second order partial differential operator:

$$(3.5) \qquad \forall \varphi \in C^2_0(\mathbb{R}^d),\ \forall x \in \mathbb{R}^d, \quad \mathcal{L}\varphi(x) = \frac{1}{2}\text{Trace}\left(\sigma\sigma^*(x)D^2\varphi(x)\right) + b(x)\nabla\varphi(x).$$

If (Y^x,Z^x) denotes the solution of the BSDE (see Remark 3.3 in Section 2), the connection is given by the formula:

$$u(x) = Y^x_0.$$

J.F. Le Gall [28] succeeded in describing all solutions of the equation $\Delta u = u^2$ in the unit disk D in $\mathbb{R}^2$ by a purely probabilistic method. He established a 1-1 correspondence between all solutions and all pairs (Γ,ν), where Γ is a closed subset of ∂D and ν is a Radon measure on $\partial D \setminus \Gamma$. The set Γ is the set of points of ∂D where the solution explodes badly. The measure ν can be interpreted as the "boundary value" of u on $\partial D \setminus \Gamma$. The solution corresponding to (Γ,ν) is expressed in terms of the Brownian snake (a path-valued Markov process). In [30], the results announced in [28] are proved in detail and are extended to a general smooth domain in $\mathbb{R}^2$.

The pair (Γ,ν) is called the *boundary trace* for positive solution of the PDE (3.4). The definition of boundary trace in general case was provided by M. Marcus and L. Véron [35] who showed by analitic methods that every positive solution of (3.4) posseses a unique trace. The trace can be described by a (possibly unbounded) positive regular Borel measure $\tilde{\nu}$ on ∂D. The correspondence between (Γ,ν) and $\tilde{\nu}$ is given by:

$$\tilde{\nu}(A) = \left\{ \begin{array}{ll} \nu(A) & \text{if } A \subseteq (\partial D \setminus \Gamma) \\ \infty & \text{if } A \cap \Gamma \neq \emptyset \end{array} \right.$$

for every Borel subset A of ∂D.

The corresponding boundary value problem is presented in [35] in the subcritical case $0 < q < 2/(d-1)$ and in [36] in the supercritical case $q \geq 2/(d-1)$. In the subcritical case, for every pair (Γ,ν) the problem has a unique solution. Remark that in [28] and [30], $q = 1 < 2/(2-1) = 2/(d-1)$. In the supercritical case, Marcus and Véron derive necessary and sufficient conditions for the existence of a maximal solution. Similar conditions were obtained by E.B. Dynkin and S.E. Kuznetsov [18] for $q \leq 1$. Their method relies on probabilistic techniques and is not extendable to $q > 1$, because the main tool is the q-superdiffusion which is not defined for $q > 1$.

The object of the present chapter is to give a probabilistic representation of the solution (3.4) in terms of the solution of the related BSDE (3.3). In general a solution of the PDE has a "blow-up" set Γ. Therefore the final data ξ of the BSDE must be allowed to be infinite with positive probability and the set $\{\xi = +\infty\}$ corresponds to Γ. Hence our first problem is to find a solution of (3.3) when ξ is infinite with positive probability, which implies in particular that (H6) is not satisfied.

Note that there are some differences between our work and the results of J.-F. Le Gall or E.B. Dynkin and S.E. Kuznetsov. With the superprocesses (see [30] or [18]), it should be assumed that $q \leq 1$. In our case there is no restriction on $q > 0$.

Moreover the Dirichlet boundary condition for the PDE (3.4) is not taken in the same sense in the two approaches. With the notion of the *boundary trace* (see [18], [30], [35] and [36]), there always exists a maximal positive solution, and if $q \geq 2(d-1)$, the problem (3.4) may possess more than one positive solution. More precisely assume that D is the unit ball in $\mathbb{R}^d$, that $q \geq 2(d-1)$ and denote by μ_∞ the Borel measure on ∂D which assigns the value $+\infty$ to every non-empty set. Then for every $\varepsilon > 0$ there exists a positive solution of (3.4) such that $u(0) < \varepsilon$ and the trace of u is μ_∞ (see Proposition 5.1 of [36]).

In our case the Dirichlet condition in (3.4) is taken in the viscosity sense (see Definition 3.4 in Section 5). We prove that there exists a minimal positive viscosity solution. But the uniqueness is still open.

Main results

In the first section, we will prove the following estimate:

Theorem 3.2 (a priori estimate) *There exists a constant C (depending on the open set D, on q and on the bound in (B) of b and σ) such that for every $x \in \overline{D}$ and every solution (Y,Z) of the BSDE (3.3) with terminal time τ_x and terminal data $\xi \in L^1(\Omega)$ such that the hypothesis (H6) holds:*

$$\forall t \geq 0, \quad |Y_t| \leq \frac{C}{\left(\rho(X^x_{t\wedge\tau_x})\right)^{\frac{2}{q}}}. \tag{3.6}$$

In the inequality (3.6), the right hand side does not depend on ξ. From now, we fix $x \in D$ and for convenience we omit the variable x.

In Section 2 and in the rest of the chapter, we assume

$$\xi \geq 0 \quad \text{a.s.}$$

and we allow ξ to be infinite with positive probability:

$$\mathbf{P}(\xi = +\infty) > 0.$$

We must modify the definition 3.1 of a BSDE when ξ does not satisfy the condition (H6). ρ will denote the distance from the boundary of D. For $x \in \overline{D}$, for all positive η, let us define the stopping time τ^x_η

$$\tau^x_\eta = \inf\left\{t \geq 0,\, \rho(X^x_t) \leq \eta\right\}.$$

Remark that for $x \in \overline{D}$, $\tau^x_\eta \leq \tau_x$ a.s. and if $x \in D$, when η goes to 0, τ^x_η converges to τ_x a.s. When $x \in \partial D$, for all $\eta > 0$, $\tau^x_\eta = \tau_x = 0$ a.s., because every point $x \in \partial D$ is regular (condition (C0). Therefore we suppose x to be in D and for convenience we omit the variable x.

Definition 3.2 *For a $\mathcal{F}_\tau$-measurable ξ such that $\mathbf{P}(\xi \geq 0) = 1$ and $\mathbf{P}(\xi = \infty) > 0$, the process (Y,Z) is a solution of the BSDE (3.3) if*
(D1) for all $\eta > 0$ and all $T \geq 0$

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |Y_{t\wedge\tau_\eta}|^2 + \int_0^{T\wedge\tau_\eta} |Z_r|^2 dB_r \right) < +\infty;$$

(D2) $\mathbf{P}$-a.s. for all $0 \leq t \leq T$ and all $\eta > 0$:

$$Y_{t\wedge\tau_\eta} = Y_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} Y_r\,|Y_r|^q\,dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} Z_r dB_r;$$

(D3) on the set $\{t \geq \tau\}$, $Y_t = \xi$ and $Z_t = 0$, and $\mathbf{P}$-a.s.

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \xi.$$

We construct a process $\{(Y_t,Z_t);\ t \geq 0\}$ satisfying the conditions (D1)-(D2) of the previous definition. (Y,Z) is the limit of the sequence of processes (Y^n,Z^n), solution (in the sense of the definition 3.1) of the BSDE (3.3) with terminal condition $\xi \wedge n$. Moreover we prove the following results:

Proposition 3.1 *For all $\varepsilon > 1$ there exists K such that*

$$\mathbb{E}\int_0^\tau \|Z_r\|^2 \rho(X_r)^{\frac{4}{q}+\varepsilon} dr \leq K.$$

Proposition 3.2 *On $\{\xi = +\infty\}$ the explosion rate of Y is $\rho^{-2/q}(X_{t\wedge\tau})$: there exists a positive constant C depending on D, q and the bound on b and σ in (B) such that:*

$$\liminf_{t\to+\infty} \rho^{\frac{2}{q}}(X_{t\wedge\tau}) Y_{t\wedge\tau} \geq C \quad \text{a.s. on } \{\xi = +\infty\}.$$

Without other assumption of ξ we cannot prove that (Y,Z) satisfies the condition (D3) of the definition 3.2.

In the Section 3, we first prove that $\mathbf{P}-$a.s. the limit of $Y_{t\wedge\tau}$ as t goes to $+\infty$ exists and:

$$\lim_{t\to+\infty} Y_{t\wedge\tau} \geq \xi.$$

Then we add some assumptions on ξ and on the diffusion X to insure that the condition (D3) holds. We prove:

Theorem 3.3 *Under the assumptions:*

– *the terminal data ξ satisfies:*

$$\text{(A1)} \qquad \xi = g(X_\tau)$$

where $g : \mathbb{R}^d \to \overline{\mathbb{R}_+}$ is a function such that $F = \{g = +\infty\} \cap \partial D$ is a closed set and such that g is locally bounded on $\partial D \setminus F$: for all compact set $\mathcal{K} \subset \partial D \setminus F$ there exists a constant K such that:

$$\text{(A2)} \qquad \forall x \in \mathcal{K}, \quad g(x) \leq K.$$

– *the boundary* ∂D *belongs to* C^3 *;*
the process Y *is continuous, i.e.*

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \xi \quad \mathbf{P} - a.s.$$

If g is a continuous function, the assumption (A2) is satisfied.

In the next section, we prove if there exists a solution $(\overline{Y},\overline{Z})$ of the BSDE (3.3) in the sense of the definition 3.2, then $\overline{Y} \geq Y$. Therefore if the process (Y,Z) is a solution (for example if the assumptions of the theorem 3.3 hold), it is the minimal solution.

In the last section, we show the connection between the BSDE (3.3) with terminal condition $g(X^x_{\tau_x})$ (X^x is the solution of the SDE (3.2) and τ_x is the first exit time of X^x from $\overline{D}$) and the PDE:

$$\left\{ \begin{array}{rl} -\mathcal{L}v + v|v|^q = 0 & \text{on } D; \\ v = g & \text{on } \partial D. \end{array} \right. \tag{3.4}$$

The assumptions of Theorem 3.3 hold. In the previous sections, we have defined a process $\{(Y^x_t,Z^x_t); t \geq\}$ which is a solution of the BSDE (3.3) with terminal data $g(X^x_{\tau_x})$ (in the sense of the definition 3.2). Next we define

$$u(x) = Y^x_0.$$

The conditions (L)-(B)-(E) hold. The main result is:

Theorem 3.4 *Under the assumptions of the theorem 3.3, u is a viscosity solution of the PDE (3.4).*

Here we do not suppose that a viscosity solution is continuous. But under some stronger assumptions on the operator $\mathcal{L}$, we also give some regularity properties of the solution u. Finally we prove that u is the minimal solution.

Important remark

The condition (E) can be relaxed. In the rest of the chapter we can also work with the following assumptions on the diffusion X:

i. b is continuous and satisfies the monotony condtion: there exists $\mu \in \mathbb{R}$ such that for all (x,y):

$$\text{(M)} \qquad \langle x-y|b(x)-b(y)\rangle \leq \mu|x-y|^2;$$

ii. Lipschitz condition: there exists $K \geq 0$ such that for all (x,y):

$$\text{(L)} \qquad \|\sigma(x)-\sigma(y)\| \leq K|x-y|;$$

iii. Boundedness condition:

$$\text{(B)} \qquad |b(x)| + \|\sigma(x)\| \leq K;$$

Remark that the condition (B) implies that $\mu \geq 0$. Under these assumptions, the equation (3.2) has a unique strong solution X^x. For each $x \in \overline{D}$, we define the stopping time:

$$\tau = \tau_x = \inf\left\{t \geq 0,\, X_t^x \notin \overline{D}\right\}.$$

We add the assumptions:

$$\text{(C0)} \qquad \text{every point } x \in \partial D \text{ is regular,}$$

and there exists $\beta > 0$ such that (C1) is true, i.e.

$$\text{(C1)} \qquad \sup_{x \in \overline{D}} \mathbb{E}\left(e^{\beta \tau_x}\right) < \infty.$$

Under the assumptions (M)-(L)-(B) and (C0)-(C1), the results which may be false are: in Section 2, the proposition 3.2 and in Section 5, the propositions 3.11 and 3.12. We are unable to prove that the viscosity solution u is continuous on D without the ellipticity condition.

In the rest of the chapter, all results can be proved without the condition (E). Indeed we use this assumption only in the proofs of the propositions 3.1 and 3.8, in order to control the Green function associated to the process X^x killed at τ. But it seems to be unnecessary.

3.1 An a priori estimate

Let (Y,Z) be the solution of the BSDE (3.3) with terminal data ξ such that the hypothesis (H6) holds. We assume also that $\xi \in L^1(\Omega)$. We will need an a priori inequality which will replace

$$|Y_t| \leq \left(\frac{1}{q(T-t)}\right)^{1/q},$$

used in the case of a deterministic final time T.

The idea comes from the Keller-Osserman inequality which is true for any open set D ([23] and [40]). Denote by ρ the distance from the boundary of $D \subset \mathbb{R}^d$.

Theorem (Keller-Osserman) *There exists a positive constant $C = C(q,d)$ such that if u is any $C^2(D)$ solution of*

$$-\Delta u + u|u|^q = 0 \text{ in } D,$$

then for all $x \in D$

$$|u(x)| \leq \frac{C}{\rho(x)^{2/q}}.$$

Recall that in our case D is supposed to be bounded and $\partial D \in C^2$. We will prove the following:

Theorem 3.2 (a priori estimate) *There exists a constant C (depending on the open set D, on q and on the bound in (B) of b and σ) such that for every $x \in \overline{D}$ and every* *solution (Y,Z) of the BSDE (3.3) with terminal time τ_x and terminal data $\xi \in L^1(\Omega)$ such that (H6) holds, we have:*

$$\forall t \geq 0, \quad |Y_t| \leq \frac{C}{\left(\rho(X^x_{t\wedge\tau_x})\right)^{\frac{2}{q}}}. \tag{3.6}$$

We define the signed distance d

$$d(x) = \begin{cases} \operatorname{dist}(x,\partial D) = \rho(x) & \text{if } x \in D, \\ -\operatorname{dist}(x,\partial D) & \text{if } x \in \mathbb{R}^d \setminus D. \end{cases}$$

For $\mu > 0$ let

$$\Gamma_\mu \triangleq \left\{x \in \mathbb{R}^d \mid |d(x)| < \mu\right\}.$$

The following lemma (see [21], lemma 14.16 p. 354-355) relates the smoothness of the distance function d in Γ_μ to that of the boundary ∂D.

Lemma 3.1 *Let D be bounded and $\partial D \in C^k$ for $k \geq 2$. Then there exists a positive constant μ depending on D such that $d \in C^k(\Gamma_\mu)$.*

Proof of the theorem. Recall that D is an open bounded subset of $\mathbb{R}^d$ with $\partial D \in C^2$. From the previous lemma we already know that there exists a positive constant μ such that on Γ_μ, the signed distance function d belongs to C^2. $d = \rho$ is continuous on $\overline{D}$. There exists a positive constant R (depending only on D) such that for all $x \in \overline{D}$, $0 \leq d(x) = \rho(x) \leq R$. Let $\varphi \in C^\infty(\mathbb{R}^d; [0,1])$ such that φ is equal to 1 on $\mathbb{R}^d \setminus \Gamma_\mu$ and is equal to 0 on $\Gamma_{\mu/2}$.

For $0 < \varepsilon \leq 1$ and $C > 0$ we define a function $f_\varepsilon \in C^2(\mathbb{R}^d; \mathbb{R}_+)$ such that on $\overline{D}$,

$$f_\varepsilon = \frac{C}{[(1-\varphi)\rho + R\varphi + \varepsilon]^{\frac{2}{q}}}.$$

Such a function exists because $(1-\varphi)\rho + R\varphi + \varepsilon \geq \varepsilon$ on $\overline{D}$. Remark that if $x \in \overline{D}$

$$f_\varepsilon(x) \leq \frac{C}{\rho(x)^{2/q}}.$$

We now fix ε, and we drop the index ε for notational simplicity. We denote by θ the function $(1-\varphi)\rho + R\varphi + \varepsilon$, i.e. on $\overline{D}$, $f = C\theta^{-2/q}$. We apply the Itô formula to $f(X^x_{t\wedge\tau_x})$ where $x \in \overline{D}$. For convenience we omit the index x.

$$\begin{aligned} f(X_{t\wedge\tau}) &= f(X_0) + \int_0^{t\wedge\tau} f(X_r)^{1+q} dr + \int_0^{t\wedge\tau} \nabla f(X_r)\sigma(X_r) dB_r \\ &+ \int_0^{t\wedge\tau} \left[\nabla f(X_r)b(X_r) + \frac{1}{2}\operatorname{Trace}\left(\sigma\sigma^*(X_r)D^2 f(X_r)\right) - f(X_r)^{1+q}\right] dr \end{aligned}$$

Now

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= -\frac{2C}{q}\theta^{-\frac{2}{q}-1}\frac{\partial \theta}{\partial x_i} \\ \frac{\partial^2 f}{\partial x_i \partial x_j} &= \frac{2C}{q}(\frac{2}{q}+1)\theta^{-\frac{2}{q}-2}\frac{\partial \theta}{\partial x_i}\frac{\partial \theta}{\partial x_j} - \frac{2C}{q}\theta^{-\frac{2}{q}-1}\frac{\partial^2 \theta}{\partial x_i \partial x_j} \\ f^{1+q} &= C^{1+q}\theta^{-\frac{2}{q}-2} \end{aligned}$$

Therefore

$$\begin{aligned}(\nabla f)b+\frac{1}{2}\text{Trace}\left(\sigma\sigma^* D^2 f\right)-f^{1+q}\\ =-C\theta^{-\frac{2}{q}-2}\left[C^q+\frac{2\theta}{q}(\nabla\theta)b-\frac{1}{q}(\frac{2}{q}+1)\|\sigma\nabla\theta\|^2+\frac{\theta}{q}\text{Trace}(\sigma\sigma^* D^2\theta)\right]\end{aligned}$$

b, σ, θ, $\nabla\theta$ and $D^2\theta$ are bounded on $\overline{D}$. So we can choose the constant C such that for all $x \in \overline{D}$:

$$C^q+\frac{2\theta(x)}{q}(\nabla\theta(x))b(x)-\frac{1}{q}(\frac{2}{q}+1)\|\sigma(x)\nabla\theta(x)\|^2+\frac{\theta(x)}{q}\text{Trace}(\sigma(x)\sigma^*(x)D^2\theta(x))\geq 0.$$

The constant C depends only on D, on q and on the bound in (B) of b and σ. We have obtained for all $0 \leq t \leq T$:

$$f(X_{t\wedge\tau})=f(X_{T\wedge\tau})-\int_{t\wedge\tau}^{T\wedge\tau}\nabla f(X_r)\sigma(X_r)dB_r-\int_{t\wedge\tau}^{T\wedge\tau}f(X_r)^{1+q}dr+\int_{t\wedge\tau}^{T\wedge\tau}U_r dr;$$

with U a nonnegative adapted process, and on $\{t \geq \tau\}$,

$$f(X_{t\wedge\tau})=\frac{C}{\varepsilon^{2/q}}.$$

If (Y,Z) is the solution of the BSDE (3.3) with a final condition ξ in $L^\infty(\Omega,\mathcal{F}_\tau,\mathbf{P})$ (see remark 3.3), we can find $0<\varepsilon<1$ such that :

$$|\xi|\leq\frac{C}{\varepsilon^{2/q}},\text{ a.s.}$$

Moreover the Tanaka formula (see [22]) leads to for all $0 \leq t \leq T$:

$$\begin{aligned}|Y_{t\wedge\tau}| &= |Y_{T\wedge\tau}|-\int_{t\wedge\tau}^{T\wedge\tau}\text{sign}(Y_r)Y_r|Y_r|^q dr-\int_{t\wedge\tau}^{T\wedge\tau}\text{sign}(Y_r)Z_r dB_r\\ &\quad +2(\Lambda_{t\wedge\tau}(0)-\Lambda_{T\wedge\tau}(0))\\ &= |Y_{T\wedge\tau}|-\int_{t\wedge\tau}^{T\wedge\tau}|Y_r|^{1+q}dr-\int_{t\wedge\tau}^{T\wedge\tau}\text{sign}(Y_r)Z_r dB_r+2(\Lambda_t(0)-\Lambda_T(0))\end{aligned}$$

Λ is a local time of Y. Thus $\Lambda_{T\wedge\tau}(0)\geq\Lambda_{t\wedge\tau}(0)$ a.s.

By a comparison theorem (corollary 4.4.2 in [11]) we have a.s.:

$$\forall t\geq 0,\ |Y_t|\leq f_\varepsilon(X_{t\wedge\tau})\leq\frac{C}{\rho(X_{t\wedge\tau})^{2/q}}.$$

By a density argument it is clear that, if (Y,Z) is the solution of the BSDE (3.3) with a final condition ξ satisfying (H6), then

$$\forall t\geq 0,\ |Y_t|\leq\frac{C}{\rho(X_{t\wedge\tau})^{2/q}}.$$

□

Remark 3.2 *The constant C depends only on the bound in (B), on D and q. Moreover in the special case where D is a ball and where the drift in the SDE is equal to 0, the constant C depends only on q and on the bound of σ and not on the center nor the radius of the ball. For example, if X is the Brownian motion, (3.6) is true for any C such that*

$$C^q \geq \frac{4}{q}\left(\frac{2}{q}+1\right)+\frac{4d}{q}.$$

Proof of the remark. In the case of a ball we can give a slightly different proof because we have an explicit expression of the distance function. We will assume that D is the ball centered at y and with radius R. In this case the function ρ is equal to:

$$\rho(x) = R - |x-y| \leq \frac{R^2 - |x-y|^2}{R} = \theta(x),$$

if x is in the ball. The function θ is not of class C^2 on the whole space. We modify θ in order to have a C^2 function. For all $0 < \varepsilon \leq R^2$, on the ball $\overline{G}$, θ_ε will be equal to

$$\theta_\varepsilon(x) = \frac{R^2 + \varepsilon - |x-y|^2}{R}.$$

and on the whole space $\mathbb{R}^d$, θ_ε is positive and of class C^2. Remark that on D, θ_ε is greater than θ. Now we consider the following function:

$$f(x) = \frac{C}{\theta_\varepsilon(x)^{2/q}}$$

where C is a positive constant. Recall that the drift b is supposed to be zero. With the Itô formula we obtain for all $0 \leq t \leq T$:

$$\begin{aligned}
f(X_{t\wedge\tau}) &= f(X_{T\wedge\tau}) - \int_{t\wedge\tau}^{T\wedge\tau} f^{1+q}(X_r)dr - \int_{t\wedge\tau}^{T\wedge\tau} \nabla f(X_r)\sigma(X_r).dB_r \\
&- \int_{t\wedge\tau}^{T\wedge\tau} \left[\frac{1}{2}\text{Trace}\left(\sigma\sigma^*(X_r)D^2 f(X_r)\right) - f(X_r)^{1+q}\right] dr \\
&= f(X_{T\wedge\tau}) - \int_{t\wedge\tau}^{T\wedge\tau} f^{1+q}(X_r)dr - \int_{t\wedge\tau}^{T\wedge\tau} \nabla f(X_r)\sigma(X_r).dB_r \\
&+ \int_{t\wedge\tau}^{T\wedge\tau} C\theta_\varepsilon(X_r)^{-\frac{2}{q}-2}\left[C^q - \frac{2+q}{q^2}\|\sigma(X_r)\nabla\theta_\varepsilon(X_r)\|^2\right. \\
&\quad \left. + \frac{\theta_\varepsilon(X_r)}{q}\text{Trace}\left(\sigma(X_r)\sigma^*(X_r)D^2\theta_\varepsilon(X_r)\right)\right] dr
\end{aligned}$$

Now for $x \in \overline{D}$

$$\nabla\theta_\varepsilon(x) = -\frac{2}{R}(x-y) \quad \text{and} \quad D^2\theta_\varepsilon(x) = -\frac{2}{R}Id.$$

Therefore for $x \in \overline{D}$

$$\begin{aligned} \|\sigma(x)\nabla\theta_\varepsilon(x)\|^2 &= \frac{4}{R^2}\|\sigma(x)(x-y)\|^2 \leq \frac{4K^2}{R^2}|x-y|^2 \leq 4K^2 \\ \mathrm{Trace}(\sigma(x)\sigma^*(x)D^2\theta_\varepsilon(x)) &= -\frac{2}{R}\mathrm{Trace}(\sigma(x)\sigma^*(x)) \geq -\frac{2K}{R} \end{aligned}$$

It suffices to choose C such that

$$C^q \geq \frac{4}{q}\left(\frac{2}{q}+1\right)K^2 + \frac{4K}{q}$$

in order to have:

$$\frac{C}{\theta_\varepsilon(X_r)^{\frac{2}{q}+2}}\left[C^q - \frac{2+q}{q^2}\|\sigma(X_r)\nabla\theta_\varepsilon(X_r)\|^2 + \frac{\theta_\varepsilon(X_r)}{q}\mathrm{Trace}\left(\sigma\sigma^* D^2\theta_\varepsilon(X_r)\right)\right] \geq 0.$$

If X is the Brownian motion B, i.e. $\sigma = Id$

$$C^q \geq \frac{4}{q}\left(\frac{2}{q}+1\right) + \frac{4d}{q}.$$

Here C depends only on the dimension d and on q, like in the Keller-Osserman theorem.

□

3.2 Approximation

We first prove a technical result which gives a sufficient condition on ξ to insure existence and uniqueness of the solution of the BSDE (3.3).

Proposition 3.3 *Under the conditions (C1) on the first exit time τ of the diffusion X, let ξ be a $\mathcal{F}_\tau$-measurable random variable such that $\xi \in L^r$ with $r > 2(1+q)$. Hence there exists $p > 1$ and $\lambda > 0$ such that the condition (H6) is satisfied.*

Proof. With $\alpha > 1$, $\gamma > 1$ such that $1/\alpha + 1/\gamma = 1$, the Hölder inequality leads to:

$$\mathbb{E}\left(e^{p\lambda\tau}|\xi|^p\right) \leq \left[\mathbb{E}\left(e^{\alpha p\lambda\tau}\right)\right]^{1/\alpha}\left[\mathbb{E}\left(|\xi|^{\gamma p}\right)\right]^{1/\gamma}.$$

If $\xi \in L^r$ with $r > 1$, there exists $\gamma > 1$ and $p > 1$ such that $\gamma p \leq r$. From (C1), we can choose $\lambda > 0$ such that $\lambda\alpha p \leq \beta$.

For the rest of the condition (H6), we have:

$$\begin{aligned} \mathbb{E}\left[\int_0^\tau e^{p\lambda t}|f(\mathbb{E}^{\mathcal{F}_t}\xi)|^p dt\right] &\leq \mathbb{E}\left[\int_0^\tau e^{p\lambda t}\mathbb{E}^{\mathcal{F}_t}\left(|\xi|^{p(1+q)}\right)dt\right] \\ &\leq \mathbb{E}\left[\frac{e^{p\lambda\tau}-1}{\lambda p}\sup_{t\in[0,\tau]}\mathbb{E}^{\mathcal{F}_t}\left(|\xi|^{p(1+q)}\right)\right] \\ &\leq \frac{1}{\lambda p}\left[\mathbb{E}e^{\alpha p\lambda\tau}\right]^{1/\alpha}\left[\mathbb{E}\sup_{t\in[0,\tau]}\mathbb{E}^{\mathcal{F}_t}\left(|\xi|^{\gamma p(1+q)}\right)\right]^{1/\gamma} \end{aligned}$$

From the Burkholder-Davis-Gundy inequality we obtain:

$$\mathbb{E} \sup_{t\in[0,\tau]} \mathbb{E}^{\mathcal{F}_t} \left(|\xi|^{\gamma p(1+q)}\right) \leq C \left[\mathbb{E}|\xi|^{2\gamma p(1+q)}\right]^{1/2}.$$

If $r > 2(1+q)$, we can choose $\gamma > 1$, $p > 1$ and $\lambda > 0$ sufficiently small such that $2\gamma p(1+q) < r$ and $\alpha\lambda p \leq \beta$. □

Remark 3.3 *From the previous proposition, if $\xi \in L^r$ for some $r > 2(1+q)$, there exists $p > 1$ and $\lambda > 0$ such that the condition (H6) is satisfied. In our case for all $p > 1$, $\nu_p = 0$ and $f(t,0,0) = 0$. From Theorem 3.1 we deduce that there exists a unique L^p-solution (Y,Z) of the BSDE (3.3). Moreover we have:*

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau} e^{p\lambda t}|Y_t|^p + \int_0^\tau e^{p\lambda r}|Y_r|^{p-2}\left(|Y_r|^2 + \|Z_r\|^2\right)dr\right) < c\ \mathbb{E}e^{p\lambda\tau}|\xi|^p.$$

For a terminal data ξ, if (Y,Z) is a L^p-solution for some $p > 1$, then (Y,Z) is also a $L^{p'}$-solution for all $1 < p' \leq p$. Therefore if (Y',Z') is a $L^{p'}$-solution for some $p' \leq p$, then we can easily prove that $(Y,Z) = (Y',Z')$.

If $\xi \in L^\infty$, from the proof of the proposition (3.3), (H6) holds for every $p > 1$ and $\lambda = \beta/p$.

We are interested in the case where ξ is a non-negative random variable with this new assumption:

$$\mathbf{P}(\xi = +\infty) > 0.$$

We still assume that the conditions (M)-(L)-(B)-(C0)-(C1) hold, and that $\tau = \tau_x$ is the exit time of the diffusion X^x from the set $\overline{D}$. We suppose that ξ is $\mathcal{F}_\tau$-mesurable.

Now for each $n \in \mathbb{N}^*$, let $\xi_n = \xi \wedge n$ be our final condition. With the remark 3.3 we obtain

Lemma 3.2 *There exists a unique solution (Y^n,Z^n) (in the sense of the definition 3.1) of the BSDE (3.3) with terminal data $\xi \wedge n$, i.e.:*

$$\forall t \geq 0, \quad Y_t^n = \xi \wedge n - \int_{t\wedge\tau}^\tau Y_r^n|Y_r^n|^q dr - \int_{t\wedge\tau}^\tau Z_r^n dB_r.$$

From a comparison theorem (see corollary 4.4.2 in [11]) we have a.s.:

$$\forall t \geq 0,\, n \leq m, \quad 0 \leq Y_t^n \leq Y_t^m.$$

Define the progressively measurable process Y by

$$Y_t = \lim_{n\to\infty} Y_t^n \tag{3.7}$$

for all $t \geq 0$.

For $x \in \overline{D}$, for all positive η, let us define the stopping time τ_η^x

$$\tau_\eta^x = \inf\left\{t \geq 0,\, \rho(X_t^x) \leq \eta\right\} = \inf\left\{t \geq 0,\, X_t^x \in \Gamma_\eta\right\}.$$

Remark that for $x \in \overline{D}$, $\tau_\eta^x \leq \tau_x$ a.s. and if $x \in D$, when η goes to 0, τ_η converges to τ a.s. When $x \in \partial D$, for all $\eta > 0$, $\tau_\eta^x = \tau_x = 0$ a.s., because every point $x \in \partial D$ is regular. Therefore we suppose x to be in D and for convenience we omit the variable x.

Proposition 3.4 *The sequence $(Z^n)_{n\in\mathbb{N}^*}$ converges also to a process Z and (Y,Z) satisfies the assumptions (D1) and (D2) of the definition 3.2.*

Proof. From (3.7), we already know that $(Y^n)_{n\in\mathbb{N}^*}$ converges to Y.

From the Itô formula and the Burkholder-Davis-Gundy inequality, there exists a constant K such that for all $\eta > 0$, $n \geq m$ and $0 \leq s$

$$\mathbb{E}\left(\sup_{t\in[0,s]} |Y^n_{t\wedge\tau_\eta} - Y^m_{t\wedge\tau_\eta}|^2\right) + \mathbb{E}\int_0^{s\wedge\tau_\eta} \|Z^n_r - Z^m_r\|^2 dr \leq K\, \mathbb{E}\left(|Y^n_{s\wedge\tau_\eta} - Y^m_{s\wedge\tau_\eta}|^2\right).$$

But with the inequality (3.6),

$$Y^n_{s\wedge\tau_\eta} \leq \frac{C}{\rho(X_{s\wedge\tau_\eta})^{2/q}} \leq \frac{C}{\eta^{2/q}},$$

and with the Lebesgue theorem, we conclude that the sequence $(Y^n_{.\wedge\tau_\eta},Z^n_{.\wedge\tau_\eta})_n$ converges to $(Y_{.\wedge\tau_\eta},Z_{.\wedge\tau_\eta})$ in $L^2(\Omega;C(\mathbb{R}_+;\mathbb{R}_+)) \times L^2(\Omega\times\mathbb{R}^+)$ and $(Y^n_{.\wedge\tau_\eta})$ converges uniformly to $Y_{.\wedge\tau_\eta}$.

Hence (Y,Z) satisfies the following equation: for all $0 \leq t \leq s$, for all $\eta > 0$

$$Y_{t\wedge\tau_\eta} = Y_{s\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{s\wedge\tau_\eta} (Y_r)^{1+q}\, dr - \int_{t\wedge\tau_\eta}^{s\wedge\tau_\eta} Z_r dB_r.$$

From this equation, with the Itô formula and the estimate (3.6) we deduce that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,s]} |Y_{t\wedge\tau_\eta}|^2\right) + \mathbb{E}\int_0^{s\wedge\tau_\eta} \|Z_r\|^2 dr &\leq K\, \mathbb{E}|Y_{s\wedge\tau_\eta}|^2 \\ &\leq \frac{K}{\rho^{2/q}(X_{s\wedge\tau_\eta})} \leq \frac{K}{\eta^{2/q}}.\end{aligned}$$

Therefore (Y,Z) satisfies the conditions (D1)-(D2) of the definition 3.2. □

From the definition 3.1 we also have that on the set $\{t \geq \tau\}$, $Y_t = \xi$ and $Z_t = 0$. With the monotonicity of the sequence Y^n we can conclude that

$$\liminf_{t\to+\infty} Y_{t\wedge\tau} \geq \xi \text{ a.s.}$$

It remains to show the converse inequality:

$$\limsup_{t\to+\infty} Y_{t\wedge\tau} \leq \xi.$$

to have the last condition (D3).

Without more assumptions on ξ we cannot prove (D3). But we are able to give some other estimates on Y and Z.

Proposition 3.1 For all $\varepsilon > 1$ there exists K such that

$$\mathbb{E}\int_0^\tau \|Z_r\|^2 \rho(X_r)^{\frac{4}{q}+\varepsilon} dr \leq K.$$

Proof. We use again the notations Γ_μ, the function $f = (1-\varphi)\rho + R\varphi$ as in the proof of the theorem 3.2, and τ_η for $\eta < \mu$. Recall that $f \in C^2(\overline{D};\mathbb{R})$ and on $\overline{D}$, $f = (1-\varphi)\rho + R\varphi \geq d \geq 0$. Of course $x \mapsto |f(x)|^{4/q+\varepsilon}$ is not in $C^2(\mathbb{R}^d)$, but this function belongs to $C^2(D \setminus \Gamma_\eta)$ and we can define this function on the rest of $(\mathbb{R}^d \setminus D) \cup \Gamma_\eta$ in order to have the required regularity. The Itô formula leads to:

$$\begin{aligned}
&\left(Y^n_{t\wedge\tau_\eta}\right)^2 f(X_{t\wedge\tau_\eta})^{\frac{4}{q}+\varepsilon} = \left(Y^n_0\right)^2 f(X_0)^{\frac{4}{q}+\varepsilon} + 2\int_0^{t\wedge\tau_\eta} (Y^n_r)^{2+q} f(X_r)^{\frac{4}{q}+\varepsilon} dr \\
&+ 2\int_0^{t\wedge\tau_\eta} Y^n_r f(X_r)^{\frac{4}{q}+\varepsilon} Z^n_r dB_r + \int_0^{t\wedge\tau_\eta} \|Z^n_r\|^2 f(X_r)^{\frac{4}{q}+\varepsilon} dr \\
&+ (\frac{4}{q}+\varepsilon)\int_0^{t\wedge\tau_\eta} (Y^n_r)^2 f(X_r)^{\frac{4}{q}+\varepsilon-1} \nabla f(X_r)\left(b(X_r)dr + \sigma(X_r)dB_r\right) \\
&+ \frac{(\frac{4}{q}+\varepsilon)}{2}\int_0^{t\wedge\tau_\eta} (Y^n_r)^2 \left[(\frac{4}{q}+\varepsilon-1) f(X_r)^{\frac{4}{q}+\varepsilon-2}\|\sigma(X_r)\nabla f(X_r)\|^2\right. \\
&\left. + f(X_r)^{\frac{4}{q}+\varepsilon-1}\mathrm{Trace}\left(\sigma\sigma^*(X_r)D^2 f(X_r)\right)\right] dr \\
&+ 2(\frac{4}{q}+\varepsilon)\int_0^{t\wedge\tau_\eta} Y^n_r f(X_r)^{\frac{4}{q}+\varepsilon-1} Z^n_r \nabla f(X_r)\sigma(X_r) dr.
\end{aligned}$$

Then:

$$\begin{aligned}
(3.8)\quad & \mathbb{E}\int_0^{t\wedge\tau_\eta} \|Z^n_r\|^2 f(X_r)^{\frac{4}{q}+\varepsilon} dr + 2(\frac{4}{q}+\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_\eta} Y^n_r f(X_r)^{\frac{4}{q}+\varepsilon-1} Z^n_r \nabla f(X_r)\sigma(X_r) dr \\
&\leq \mathbb{E}\left(\left(Y^n_{t\wedge\tau_\eta}\right)^2 f(X_{t\wedge\tau_\eta})^{\frac{4}{q}+\varepsilon}\right) - (\frac{4}{q}+\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_\eta} (Y^n_r)^2 f(X_r)^{\frac{4}{q}+\varepsilon-1}\nabla f(X_r) b(X_r) dr \\
&-\frac{1}{2}(\frac{4}{q}+\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_\eta} (Y^n_r)^2 f(X_r)^{\frac{4}{q}+\varepsilon-1}\mathrm{Trace}\left(\sigma\sigma^*(X_r)D^2 f(X_r)\right) dr \\
&-\frac{1}{2}(\frac{4}{q}+\varepsilon)(\frac{4}{q}+\varepsilon-1)\mathbb{E}\int_0^{t\wedge\tau_\eta} (Y^n_r)^2 f(X_r)^{\frac{4}{q}} f(X_r)^{\varepsilon-2}\|\sigma(X_r)\nabla f(X_r)\|^2 dr.
\end{aligned}$$

In the proof of the theorem 3.2 we have obtained that there exists some constant C such that for all $n \in \mathbb{N}^*$ and for all $t \geq 0$:

$$(3.9)\qquad (Y^n_{t\wedge\tau})^2 f(X_{t\wedge\tau})^{\frac{4}{q}} \leq C.$$

Moreover b and σ are bounded (assumption (B)), and ∇f and $D^2 f$ are also bounded on $\overline{D}$. Thus the right hand side of (3.8) is bounded by:

$$K\left(1 + \mathbb{E}\int_0^\tau f^{\varepsilon-1}(X_r)dr + \mathbb{E}\int_0^\tau f^{\varepsilon-2}(X_r)dr\right).$$

We denote by $p(t,x,y)$ the density of $\mathbf{P}^x(X_t \in dy; \tau > t)$. $\mathbf{P}^x$ means that the diffusion process X starts from $x \in D$ at time 0. Then

$$\mathbb{E}\int_0^\tau f^{\varepsilon-1}(X_r)dr = \int_0^\infty \int_D f^{\varepsilon-1}(y)p(r,x,y)dydr = \int_D f^{\varepsilon-1}(y)G(x,y)dy$$

where G is the Green function associated to the process X killed at time τ (see [48], section 4.2, theorem 2.5).

We claim that the last integral is finite. Indeed if $B(x,\nu)$ is the ball centered at x with radius $\nu > 0$, since $x \in D$, we can find $\nu > 0$ such that $B(x,\nu) \subset D$ and $B(x,\nu) \cap \Gamma_\nu = \emptyset$. We denote by U the set $D \setminus (B(x,\nu) \cup \Gamma_\nu)$. On U, $f^{\varepsilon-1}G(x,.)$ is a continuous function and is bounded by K. On Γ_ν (resp. on $B(x,\nu)$), $G(x,.)$ (resp. $f^{\varepsilon-1}$) is continuous and bounded by K. On the boundary of D, $f^{\varepsilon-1}$ is singular if $\varepsilon < 1$. Recall the definition of f: $f = (1-\varphi)\rho + R\varphi$. Hence f is equivalent to ρ at the boundary and if $\varepsilon > 0$, $f^{\varepsilon-1}$ is integrable on D. G has a singularity when $y \to x$, but with theorem 2.8 and exercice 4.16 in [48], this singularity is integrable. Therefore we split the integral in three terms:

$$\begin{aligned}
\mathbb{E}\int_0^\tau f^{\varepsilon-1}(X_r)dr &= \int_D f^{\varepsilon-1}(y)G(x,y)dy \\
&\leq \int_{B(x,\nu)} f^{\varepsilon-1}(y)G(x,y)dy + \int_{\Gamma_\nu} f^{\varepsilon-1}(y)G(x,y)dy + \int_U f^{\varepsilon-1}(y)G(x,y)dy \\
&\leq K\int_{B(x,\nu)} G(x,y)dy + K\int_{\Gamma_\nu} f^{\varepsilon-1}(y)dy + K\ \mathrm{Vol}(D) < +\infty.
\end{aligned}$$

From the second integral the same arguments show that if $\varepsilon > 1$

$$\mathbb{E}\int_0^\tau f^{\varepsilon-2}(X_r)dr < +\infty. \tag{3.10}$$

Therefore the right hand side of (3.8) is bounded by a constant K which does not depend on η, n and t. And using the Cauchy-Schwarz inequality

$$\begin{aligned}
\left|\mathbb{E}\int_0^{t\wedge\tau_\eta} Y_r^n f(X_r)^{\frac{4}{q}+\varepsilon-1} Z_r^n \nabla f(X_r)\sigma(X_r)dr\right| &\leq \left(\mathbb{E}\int_0^{t\wedge\tau_\eta} \|Z_r^n\|^2 f(X_r)^{\frac{4}{q}+\varepsilon}dr\right)^{1/2} \\
\times\left(\mathbb{E}\int_0^{t\wedge\tau_\eta} (Y_r^n)^2 f(X_r)^{\frac{4}{q}+\varepsilon-2}\|\nabla f(X_r)\sigma(X_r)\|^2dr\right)^{1/2}.
\end{aligned}$$

But since ∇f and σ are bounded and since (3.9) and (3.10) hold, if $\varepsilon > 1$:

$$\mathbb{E}\int_0^{t\wedge\tau_\eta} (Y_r^n)^2 f(X_r)^{\frac{4}{q}+\varepsilon-2}\|\nabla f(X_r)\sigma(X_r)\|^2dr \leq K.$$

The inequality (3.8) can be written as follows: $A_n + B_n \leq C_n$ with

$$0 \leq A_n = \mathbb{E}\int_0^{t\wedge\tau_\eta} \|Z_r^n\|^2 f(X_r)^{\frac{4}{q}+\varepsilon}dr,$$

$$|B_n| = 2(\frac{4}{q}+\varepsilon)\left|\mathbb{E}\int_0^{t\wedge\tau_\eta} Y_r^n f(X_r)^{\frac{4}{q}+\varepsilon-1} Z_r^n \nabla f(X_r)\sigma(X_r)dr\right| \leq KA_n^{1/2}$$

and $|C_n| \leq K$. Thus for all $n \in \mathbb{N}^*$ and for all $t \geq 0$:

$$\mathbb{E}\int_0^{t\wedge\tau_\eta} \|Z_r^n\|^2 f(X_r)^{\frac{4}{q}+\varepsilon}dr \leq K$$

which implies by Fatou's lemma

$$\mathbb{E}\int_0^\tau \|Z_r\|^2 f(X_r)^{\frac{4}{q}+\varepsilon}dr \leq K.$$

Since $f \geq \rho$ on $\overline{D}$ we obtain the announced result. □

The condition $\varepsilon > 1$ is required in order to insure that

$$\mathbb{E}\int_0^\tau f^{\varepsilon-2}(X_r)dr < \infty.$$

But this integral is equal to

$$\int_D f^{\varepsilon-2}(y)G(x,y)dy$$

where $G(x,y)$ is the Green function associated with the process X^x killed at τ. And if for example the infinitesimal generator of the diffusion X is self-adjoint in $L^2(\mathbb{R}^d)$, i.e. $\mathcal{L} = (1/2)\mathrm{div}(\sigma\sigma^*\nabla)$, then $G(x,y) \leq K\rho(y)$ (see [13], theorem 9.5) and the previous integral is finite for any $\varepsilon > 0$. Hence we obtain that for any $\varepsilon > 0$:

$$\mathbb{E}\int_0^\tau \|Z_r\|^2 \rho(X_r)^{\frac{4}{q}+\varepsilon}dr < \infty.$$

In the next proposition we find a adapted process smaller than Y. This process will give us a lower bound on the explosion rate of Y on the blow-up set $\{\xi = \infty\}$.

Proposition 3.5 (Lower bound on Y) *We define the following process:*

$$\Xi_t = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau - \tau\wedge t) + \frac{1}{\xi^q}}\right)^{1/q}\right].$$

Then for all $t \geq 0$

$$\Xi_t \leq Y_t.$$

Proof. Denote by α_t the quantity:

$$\alpha_t = \left(\frac{1}{q(\tau - \tau\wedge t) + \frac{1}{\xi^q}}\right)^{1/q},$$

if $t < \tau$ and $\alpha_t = \xi$ on $\{t \geq \tau\}$. α solves the following equation:

$$\alpha_t = \alpha_T - \int_t^T \alpha_r^{1+q}\mathbf{1}_{[0,\tau]}(r)dr.$$

Note that $\Xi_t = \mathbb{E}(\alpha_t|\mathcal{F}_t)$. Thanks to Jensen's inequality:

$$\Xi_t \leq \mathbb{E}^{\mathcal{F}_t}\left(\Xi_T - \int_t^T \Xi_r^{1+q}\mathbf{1}_{[0,\tau]}(r)dr\right)$$

and the comparison theorem (corollary 4.4.2 in [11]) achieves the proof. □

We now want to find a lower bound for $\rho(X_{t\wedge\tau})^{2/q}\Xi_t$ when t goes to ∞ on $\{\xi = \infty\}$.

Lemma 3.3 *Let $\rho(x)$ denote the distance of $x \in D$ to the boundary ∂D and τ be the exit time from $\overline{D}$ of the diffusion X. Let the conditions (L)-(B)-(E) hold. Then there exist two positive constants C_1 and C_2 which depend on D, q, σ and b such that for all $x \in D$:*

$$C_1 \leq \rho(x)^{\frac{2}{q}} \mathbb{E}_x \left[\left(\frac{1}{\tau} \right)^{\frac{1}{q}} \right] \leq C_2.$$

Proof. Recall that if $x \in D$, $\mathbf{P}^x(\tau > 0) = 1$ and

$$\mathbb{E}_x \left(\frac{1}{\tau^{1/q}} \right) = \int_0^{+\infty} \mathbf{P}^x \left(\tau < \frac{1}{y^q} \right) dy.$$

If $\tau < h$ then $\sup_{t\in[0,h]} |X_t - x| > \rho(x)$. Therefore we can apply the theorem 4.2.1 (p. 87) of [52] to obtain:

$$\mathbf{P}^x(\tau < h) \leq \mathbf{P}^x \left(\sup_{t\in[0,h]} |X_t - x| > \rho(x) \right) \leq K_1 e^{K_2 h} e^{-K_2 \rho(x)^2/h}.$$

We apply this inequality with $h = 1/y^q$ and $y \geq 1$:

$$\begin{aligned}
\mathbb{E}_x \left(\frac{1}{\tau^{1/q}} \right) &\leq 1 + \int_1^{+\infty} K_1 e^{K_2/y^q} e^{-K_2\rho(x)^2 y^q} dy \\
&\leq 1 + K_1 e^{K_2} \int_1^{+\infty} e^{-K_2\rho(x)^2 y^q} dy \\
&= 1 + \frac{K_1 e^{K_2}}{q\rho(x)^{2/q}} \int_{\rho(x)^2}^{+\infty} e^{-K_2 u} u^{\frac{1}{q}-1} du \\
&\leq 1 + \frac{K_1 e^{K_2}}{q\rho(x)^{2/q}} \int_0^{+\infty} e^{-K_2 u} u^{\frac{1}{q}-1} du.
\end{aligned}$$

Since $-1 + 1/q > -1$, we deduce

$$\rho(x)^{2/q} \mathbb{E}_x \left(\frac{1}{\tau^{1/q}} \right) \leq C_2.$$

For the other inequality remark that

$$\int_0^{+\infty} \mathbf{P}^x \left(\tau < \frac{1}{y^q} \right) dy \geq \int_0^{1/\rho(x)^{2/q}} \mathbf{P}^x \left(\tau < \frac{1}{y^q} \right) dy$$

and $\mathbf{P}^x(\tau < 1/y^q) \geq \mathbf{P}^x(X_{1/y^q} \notin \overline{D})$. We just have to find a lower bound to

$$\rho(x)^{2/q} \int_0^{1/\rho(x)^{2/q}} \mathbf{P}^x(X_{1/y^q} \notin \overline{D}) dy = \int_1^{+\infty} \mathbf{P}^x(X_{\rho(x)^2 u^q} \notin \overline{D}) \frac{du}{u^2}.$$

Let Δ_x be the set $(\mathbb{R}^d \setminus \overline{D}) \cap B(x, 2\rho(x))$ which is not empty, and $\mathrm{Vol}(\Delta_x)$ denotes the volume of the set Δ_x. Using the Aronson estimates of [51], we have for $u \geq 1$

$$\begin{aligned}
\mathbf{P}^x(X_{\rho(x)^2 u^q} \notin \overline{D}) &\geq K_3 \int_{\mathbb{R}^d \setminus \overline{D}} \left(\frac{1}{2\pi u^q \rho(x)^2}\right)^{d/2} \exp\left(-K_4 u^q \rho(x)^2 - K_4 \frac{|y-x|^2}{2u^q\rho(x)^2}\right) dy \\
&\geq K_3 \int_{\Delta_x} \left(\frac{1}{2\pi u^q \rho(x)^2}\right)^{d/2} \exp\left(-K_4 u^q \rho(x)^2 - K_4 \frac{|y-x|^2}{2u^q\rho(x)^2}\right) dy \\
&\geq K_3 \left(\frac{1}{2\pi u^q \rho(x)^2}\right)^{d/2} \exp\left(-K_4 u^q \rho(x)^2\right) \int_{\Delta_x} \exp(-2K_4 u^{-q}) dy \\
&\geq K_3 e^{-2K_4} \left(\frac{1}{2\pi u^q}\right)^{d/2} \exp\left(-K_4 u^q \rho(x)^2\right) \frac{\mathrm{Vol}\,(\Delta_x)}{\rho(x)^d}
\end{aligned}$$

because $u \geq 1$. Thus

$$\rho(x)^{2/q} \int_0^{1/\rho(x)^{2/q}} \mathbf{P}^x(X_{1/y^q} \notin \overline{D}) dy \geq K_5 \left[\int_1^{+\infty} \exp\left(-K_4 u^q \rho(x)^2\right) \frac{du}{u^{2+dq/2}}\right] \frac{\mathrm{Vol}\,(\Delta_x)}{\rho(x)^d}$$

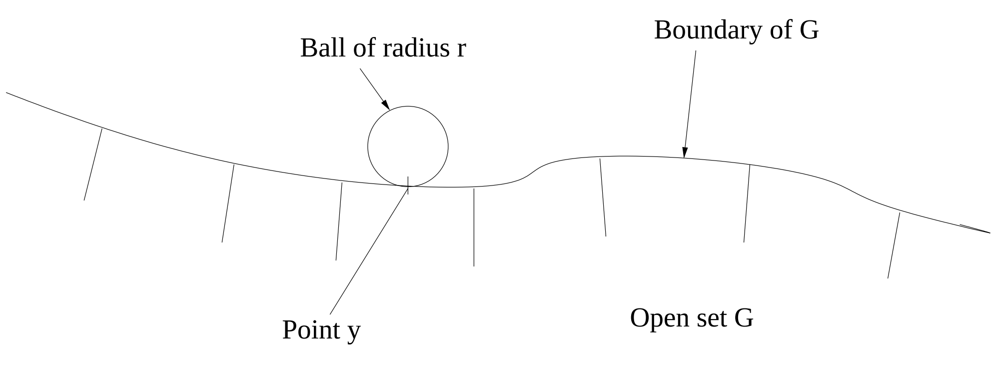
with

$$K_5 = K_3 e^{-2K_4} \left(\frac{1}{2\pi}\right)^{d/2}.$$

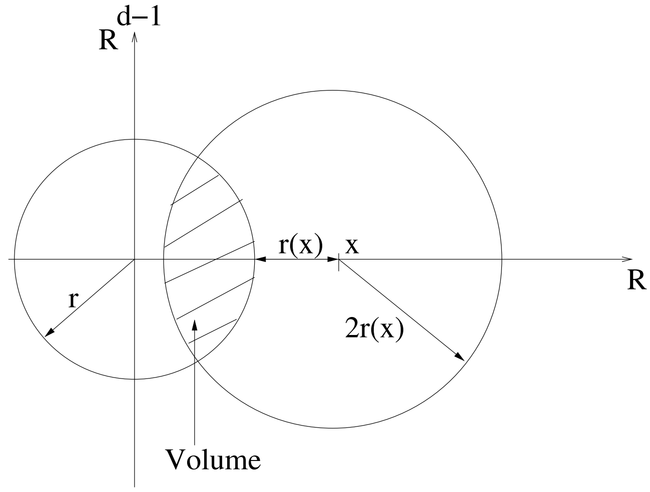
The integral has a lower bound because the open set D is bounded and the dominated convergence theorem shows that:

$$\lim_{\rho(x)\to 0} \int_1^{+\infty} \exp\left(-K_4 u \rho(x)^2\right) \frac{du}{u^{2+d/2}} = \int_1^{+\infty} \frac{du}{u^{2+d/2}} = K_6.$$

We have supposed that $\partial D \in C^2$. Therefore the curvature is continuous on ∂D which is compact ; so the curvature is bounded. There exists $r > 0$ such that each point $y \in \partial D$ lies on the boundary of a ball with radius r and this ball is contained in the complementary of D.



Instead of calculate the volume of $(\mathbb{R}^d \setminus \overline{D}) \cap B(x,2\rho(x))$ our problem is reduced to the following: we find the volume of the intersection of two balls in that case:



If $r < x < 3r$, the volume is equal to

$$V(x) = Cr^d \int_0^{\alpha(x)} (\sin\theta)^d d\theta + C2^d(x-r)^d \int_0^{\beta(x)} (\sin\theta)^d d\theta$$

where C is the volume of the unit ball in $\mathbb{R}^{d-1}$,

$$\alpha(x) = \text{Arccos}\left(\frac{x^2+r^2-4(x-r)^2}{2xr}\right) \text{ and } \beta(x) = \text{Arccos}\left(\frac{5x-3r}{4x}\right).$$

Now we must prove that:

$$\frac{V(x)}{(x-r)^d} \geq K_1.$$

We split $V(x)$ in two parts. For the first part:

$$C2^d(x-r)^d \int_0^{\beta(x)} (\sin\theta)^d d\theta$$

the result is clear because if $r \leq x \leq 2r$:

$$0 < \text{Arccos}(7/8) \leq \beta(x) \leq \frac{\pi}{3}.$$

For the second part:

$$Cr^d \int_0^{\alpha(x)} (\sin\theta)^d d\theta$$

if $r \leq x \leq \frac{4+\sqrt{7}}{3} r$

$$\frac{x^2+r^2-4(x-r)^2}{2xr} \in [0,1] \Longrightarrow \alpha(x) \in [0,\pi/2]$$

and we use the fact that sin is a increasing function on [0,$\pi/2$[; so:

$$\begin{aligned}
\int_0^{\alpha(x)} (\sin\theta)^d d\theta &\le \alpha(x)\left(1-\left(\frac{x^2+r^2-4(x-r)^2}{2xr}\right)^2\right)^{d/2} \\
&\le 2^{d/2}\alpha(x)\left(1-\frac{x^2+r^2-4(x-r)^2}{2xr}\right)^{d/2} \\
&= \left(\frac{3}{xr}\right)^{d/2}\alpha(x)(x-r)^d.
\end{aligned}$$

Therefore if $r < x \le \frac{4+\sqrt{7}}{3}r$,

$$\frac{V(x)}{(x-r)^d} \ge C2^d \int_0^{\mathrm{Arccos}(7/8)} (\sin\theta)^d d\theta > 0.$$

This achieves the proof of:

$$\frac{V(x)}{(x-r)^d} \ge \widetilde{K}$$

and therefore:

$$\rho(x)^2 \mathbb{E}_x\left(\frac{1}{\tau}\right) \ge C_1 = K_5\widetilde{K}.$$

This achieves the proof in the uniformly elliptic case. □

If the diffusion matrix is degenerate, the result on the lower bound may be false. Suppose that $\sigma \equiv 0$ and b is bounded by k. If the exit time τ is smaller than $1/y^q$:

$$\frac{k}{y^q} \ge \int_0^{1/y^q} |b(X_r)|dr \ge \sup_{[0,1/y^q]}\left|\int_0^t b(X_r)dr\right| = \sup_{[0,1/y^q]}|X_t - x| > \rho(x)$$

and thus

$$\rho(x)^{2/q}\mathbb{E}_x\left(\frac{1}{\tau^{1/q}}\right) \le k\rho(x)^{1/q}$$

and the limit, as $\rho(x)$ goes to zero, is zero.

Proposition 3.2 On $\{\xi = +\infty\}$ the explosion rate of Y is $\rho^{-2/q}(X_{t\wedge\tau})$: there exists a positive constant C depending on D, q and the bound on b and σ in (B) such that:

$$\liminf_{t\to+\infty} \rho^{\frac{2}{q}}(X_{t\wedge\tau})Y_{t\wedge\tau} \ge C \quad \text{a.s. on } \{\xi = +\infty\}.$$

Proof. From the proposition 3.5, we work on the process $\Xi = (\Xi_t)_{t\ge 0}$. For all $t \ge 0$

$$\begin{aligned}
\rho^{\frac{2}{q}}(X_{t\wedge\tau})\Xi_t &= \rho^{\frac{2}{q}}(X_{t\wedge\tau})\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau-\tau\wedge t)+\frac{1}{\xi^q}}\right)^{\frac{1}{q}}\right] \\
&= \rho^{\frac{2}{q}}(X_{t\wedge\tau})\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{\xi^q \mathbf{1}_{\xi<\infty}}{1+q\xi^q(\tau-\tau\wedge t)}\right)^{\frac{1}{q}}\right] \\
&+ \rho^{\frac{2}{q}}(X_{t\wedge\tau})\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau-\tau\wedge t)}\right)^{\frac{1}{q}}\mathbf{1}_{\xi=\infty}\right]
\end{aligned}$$

The first term in the right hand side is non-negative. Let $\widetilde{\tau} = \tau - \tau \wedge t$: it is the first exit time of the diffusion starting at $X_{t\wedge\tau}$. Hence

$$\begin{aligned}
&\rho^{\frac{2}{q}}(X_{t\wedge\tau})\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau-\tau\wedge t)}\right)^{\frac{1}{q}}\mathbf{1}_{\xi=\infty}\right] \\
&= \left(\frac{1}{q}\right)^{\frac{1}{q}}\mathbb{E}^{\mathcal{F}_t}\left\{\rho^{\frac{2}{q}}(X_{t\wedge\tau})\mathbb{E}^{X_{t\wedge\tau}}\left[\left(\frac{1}{\widetilde{\tau}}\right)^{\frac{1}{q}}\right]\mathbf{1}_{\xi=\infty}\right\} \\
&\geq C_1\left(\frac{1}{q}\right)^{\frac{1}{q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right).
\end{aligned}$$

where C_1 is the lower bound of the lemma 3.3. Thus we obtain

$$\rho^{\frac{2}{q}}(X_{t\wedge\tau})\Xi_t \geq C_1\left(\frac{1}{q}\right)^{\frac{1}{q}}\mathbb{E}^{\mathcal{F}_t}\left(\mathbf{1}_{\xi=\infty}\right)$$

and we deduce the announced result.

□

3.3 Continuity

Recall that we have constructed a couple of processes (Y,Z) which satisfies for all $\eta > 0$ and all $0 \leq t \leq T$, $Y_t \geq 0$ and:

$$Y_{t\wedge\tau_\eta} = Y_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta}(Y_r)^{1+q}\,dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} Z_r dB_r.$$

Moreover on the set $\{t \geq \tau\}$, $Y_t = \xi$, $Z_t = 0$ and

$$\liminf_{t\to+\infty} Y_{t\wedge\tau} \geq \xi \quad \text{a.s.}$$

We now want to prove the converse inequality, namely:

$$\limsup_{t\to+\infty} Y_{t\wedge\tau} \leq \xi \quad \text{a.s.}$$

Remark that we just have to show this estimate on the set $\{\xi < +\infty\}$.

3.3.1 Existence of the limit

We first prove that the limit of $Y_{t\wedge\tau}$ exists as t goes to $+\infty$ a.s. In the proof we will distinguish the two cases: ξ is greater than a positive constant and ξ is non-negative.

The case where ξ is bounded away from zero.

We can show that $(Y_{t\wedge\tau})_{t\geq 0}$ has a limit as $t \to +\infty$ by using Itô's formula applied to the process $1/(Y^n)^q$. We prove the following result:

Proposition 3.6
Let the conditions (B)-(E) hold. Suppose there exists a real $\alpha > 0$ such that $\xi \geq \alpha > 0$, $\mathbf{P}$-a.s. Then

$$Y_{t\wedge\tau} = \left[\mathbb{E}^{\mathcal{F}_t}\left(q(\tau - t\wedge\tau) + \left(\frac{1}{\xi^q}\right)\right) - \Phi_t\right]^{-\frac{1}{q}}, \; 0 \leq t, \tag{3.11}$$

where Φ is a non-negative supermartingale such that on the set $\{t \geq \tau\}$, $\Phi_t = 0$.

Proof. From the proposition 3.5, for every $n \in \mathbb{N}^*$ and every $0 \leq t$:

$$Y_t^n \geq \Xi_t^n = \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau - \tau\wedge t) + \left(\frac{1}{\xi\wedge n}\right)^q}\right)^{1/q}\right].$$

Since $\xi \geq \alpha$, we have:

$$\begin{aligned}
\Xi_t^n &\geq \mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{q(\tau - \tau\wedge t) + \left(\frac{1}{\alpha}\right)^q}\right)^{1/q}\right] \geq \alpha\mathbb{E}^{\mathcal{F}_t}\left[\left(\frac{1}{1 + q\tau\alpha^q}\right)^{1/q}\right] \\
&\geq \alpha\left(\frac{1}{1 + q\alpha^q\mathbb{E}^{\mathcal{F}_t}(\tau)}\right)^{1/q}.
\end{aligned}$$

Therefore

$$\forall t \geq 0, \quad 0 \leq \frac{1}{(Y_t^n)^q} \leq \frac{1}{\alpha^q}\left(1 + q\alpha^q\mathbb{E}^{\mathcal{F}_t}(\tau)\right) < +\infty. \tag{3.12}$$

because the conditions (B)-(E) hold, which implies in particular that $\tau \in L^1(\Omega)$. Thus for all $t \geq 0$, $(Y_t^n)^{-q}$ belongs to $L^1(\Omega)$. We want to apply the Itô formula to the semi-martingale Y^n with the function x^{-q}. But we just have that a.s. for all $t \geq 0$ $Y_t^n > 0$. For $\varepsilon > 0$ we define a C^2 function $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ such that on $\mathbb{R}_+$

$$f_\varepsilon(x) = \left(\frac{1}{x+\varepsilon}\right)^q.$$

Note that for a fixed $x \in \mathbb{R}_+$ $(f_\varepsilon(x))_{\varepsilon>0}$ is increasing and the limit is equal to $f(x) = x^{-q}$. By the Itô formula for all $0 \leq t \leq T$

$$\begin{aligned}
f_\varepsilon(Y_{t\wedge\tau}^n) &= f_\varepsilon(Y_{T\wedge\tau}^n) - \int_{t\wedge\tau}^{T\wedge\tau} f'_\varepsilon(Y_r^n)Z_r^n dB_r - \int_{t\wedge\tau}^{T\wedge\tau} f'_\varepsilon(Y_r^n)(Y_r^n)^{1+q}dr \\
&- \frac{1}{2}\int_{t\wedge\tau}^{T\wedge\tau} f''_\varepsilon(Y_r^n)\|Z_r^n\|^2 dr \\
&= \mathbb{E}^{\mathcal{F}_t} f_\varepsilon(Y_{T\wedge\tau}^n) - \mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{T\wedge\tau} f'_\varepsilon(Y_r^n)(Y_r^n)^{1+q}dr \qquad (3.13) \\
&- \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{T\wedge\tau} f''_\varepsilon(Y_r^n)\|Z_r^n\|^2 dr
\end{aligned}$$

Now for $x \geq 0$

$$f'_\varepsilon(x)x^{1+q} = -q\left(\frac{x}{x+\varepsilon}\right)^{1+q} \Longrightarrow -q \leq f'_\varepsilon(x)x^{1+q} \leq 0,$$

$$f''_\varepsilon(x) = q(1+q)\left(\frac{1}{x+\varepsilon}\right)^{2+q}.$$

Thereby a.s. and in $L^1(\Omega)$ for all $0 \leq t \leq T$

$$\lim_{\varepsilon \to 0} \mathbb{E}^{\mathcal{F}_t} \int_{t\wedge\tau}^{T\wedge\tau} f'_\varepsilon(Y^n_r)(Y^n_r)^{1+q} dr = -q\mathbb{E}^{\mathcal{F}_t}(T\wedge\tau - t\wedge\tau).$$

From (3.12) we have that a.s. and in $L^1(\Omega)$

$$\lim_{\varepsilon\to 0} \mathbb{E}^{\mathcal{F}_t} f_\varepsilon(Y^n_{T\wedge\tau}) = \mathbb{E}^{\mathcal{F}_t}\frac{1}{(Y^n_{T\wedge\tau})^q} \quad \text{and} \quad \lim_{\varepsilon\to 0} f_\varepsilon(Y^n_{t\wedge\tau}) = \frac{1}{(Y^n_{t\wedge\tau})^q}.$$

For the last term in (3.13) we use the monotone convergence theorem and hence we have proved that for all $0 \leq t \leq T$:

$$\frac{1}{(Y^n_{t\wedge\tau})^q} = \mathbb{E}^{\mathcal{F}_t}\frac{1}{(Y^n_{T\wedge\tau})^q} + q\mathbb{E}^{\mathcal{F}_t}(T\wedge\tau - t\wedge\tau) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{T\wedge\tau} \frac{\|Z^n_r\|^2}{(Y^n_r)^{2+q}} dr$$

Let T go to $+\infty$:

$$\frac{1}{(Y^n_{t\wedge\tau})^q} = \mathbb{E}^{\mathcal{F}_t}\frac{1}{(\xi\wedge n)^q} + q\mathbb{E}^{\mathcal{F}_t}(\tau - t\wedge\tau) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau} \frac{\|Z^n_r\|^2}{(Y^n_r)^{2+q}} dr \tag{3.14}$$

Let $n \geq m$. Since $\xi\wedge n \geq \xi \wedge m$, we obtain for all $0 \leq t$:

$$\begin{aligned} 0 \leq \frac{1}{(Y^m_{t\wedge\tau})^q} - \frac{1}{(Y^n_{t\wedge\tau})^q} &= \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge m)^q} - \frac{1}{(\xi\wedge n)^q}\right) \\ &- \frac{q(q+1)}{2}\left(\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^m_s\|^2}{(Y^m_s)^{q+2}}ds - \mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds\right). \end{aligned}$$

Now:

$$\begin{aligned} &\frac{q(q+1)}{2}\left|\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^m_s\|^2}{(Y^m_s)^{q+2}}ds - \mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds\right| \\ &\quad \leq \left[\mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge m)^q} - \frac{1}{(\xi\wedge n)^q}\right)\right] \vee \left[\frac{1}{(Y^m_{t\wedge\tau})^q} - \frac{1}{(Y^n_{t\wedge\tau})^q}\right]. \end{aligned}$$

For a fixed $t \geq 0$, the sequences $\left(\mathbb{E}^{\mathcal{F}_t}\frac{1}{(\xi\wedge n)^q}\right)_{n\geq 1}$ and $\left(\frac{1}{(Y^n_{t\wedge\tau})^q}\right)_{n\geq 1}$ converge a.s. and in L^1 (dominated convergence theorem). Then $\left(\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds\right)_{n\geq 1}$ converges a.s. and in L^1 and we denote by Φ the limit

$$\Phi_t = \lim_{n\to+\infty}\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z^n_s\|^2}{(Y^n_s)^{q+2}}ds.$$

On the set $\{t \geq \tau\}$, $\Phi_t = 0$ a.s. and we have:

$$\frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds \leq q\mathbb{E}^{\mathcal{F}_t}(\tau - t\wedge\tau) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) \leq q\mathbb{E}^{\mathcal{F}_t}(\tau) + \frac{1}{\alpha^q}.$$

Thus

$$\Phi_t \leq q\mathbb{E}^{\mathcal{F}_t}(\tau) + \frac{1}{\alpha^q}.$$

For $r \leq t$,

$$\begin{aligned}
\int_{r\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds &\geq \int_{t\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds,\\
\Longrightarrow \mathbb{E}^{\mathcal{F}_r}\int_{r\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds &\geq \mathbb{E}^{\mathcal{F}_r}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds,\\
\Longrightarrow \Phi_r &\geq \mathbb{E}^{\mathcal{F}_r}\Phi_t.
\end{aligned}$$

We deduce that $(\Phi_t)_{0\leq t}$ is a non-negative supermartingale. Now for all $n \in \mathbb{N}^*$,

$$\frac{1}{(Y_t^n)^q} = q\mathbb{E}^{\mathcal{F}_t}(\tau - t\wedge\tau) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{(\xi\wedge n)^q}\right) - \frac{q(q+1)}{2}\mathbb{E}^{\mathcal{F}_t}\int_{t\wedge\tau}^{\tau}\frac{\|Z_s^n\|^2}{(Y_s^n)^{q+2}}ds.$$

Fix $t \geq 0$. Taking the limit as $n \to +\infty$, we deduce:

$$\frac{1}{(Y_{t\wedge\tau})^q} = q\mathbb{E}^{\mathcal{F}_t}(\tau - t\wedge\tau) + \mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right) - \Phi_t.$$

This achieves the proof of the proposition. □

Φ being a non-negative supermartingale, the limit of $\Phi_{t\wedge\tau}$ as t goes to $+\infty$ exists **P**-a.s. and this limit Φ_{τ^-} is finite **P**-a.s. The L^1-bounded martingale $\mathbb{E}^{\mathcal{F}_t}\left(\frac{1}{\xi^q}\right)$ converges a.s. to $1/\xi^q$ as t goes to $+\infty$, then the limit of $Y_{t\wedge\tau}$ as $t \to +\infty$ exists and is equal to:

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \frac{1}{(\frac{1}{\xi^q} - \Phi_{\tau^-})^{1/q}}.$$

If we were able to prove that Φ is continuous (or Φ_{τ^-} is zero a.s.), we would have shown that Y is a continuous process.

The case ξ non negative

Now we just assume that $\xi \geq 0$. We cannot apply the same arguments, because Y^n may to equal to zero with positive probability, which implies in particular that $(Y_t^n)^{-q} \notin L^1(\Omega)$. We will approach Y^n in the following way. We define for $n \geq 1$ and $m \geq 1$, $\xi^{n,m}$ by:

$$\xi^{n,m} = (\xi\wedge n) \vee \frac{1}{m}.$$

This random variable is in L^2 and is greater or equal to $1/m$ a.s. The BSDE (3.3) with $\xi^{n,m}$ as terminal condition has an unique solution $(\widetilde{Y}^{n,m},\widetilde{Z}^{n,m})$. It is immediate that if $m \leq m'$ and $n \leq n'$ then:

$$\widetilde{Y}^{n,m'} \leq \widetilde{Y}^{n',m}.$$

As for the sequence Y^n, we can define $\widetilde{Y}^m$ as the limit when n grows to $+\infty$ of $Y^{n,m}$. That limit $\widetilde{Y}^m$ is greater than $Y = \lim_{n\to+\infty} Y^n$. But for $m \leq m'$ for $0 \leq t \leq T$:

$$\begin{aligned}
\widetilde{Y}^{n,m}_{t\wedge\tau} - \widetilde{Y}^{n,m'}_{t\wedge\tau} &= \widetilde{Y}^{n,m}_{T\wedge\tau} - \widetilde{Y}^{n,m'}_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau} \left[\left(\widetilde{Y}^{n,m}_r\right)^{q+1} - \left(\widetilde{Y}^{n,m'}_r\right)^{q+1} \right] dr \\
&\quad - \int_{t\wedge\tau}^{T\wedge\tau} \left[\widetilde{Z}^{n,m}_r - \widetilde{Z}^{n,m'}_r \right] dB_r \\
&\leq \widetilde{Y}^{n,m}_{T\wedge\tau} - \widetilde{Y}^{n,m'}_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau} \left[\widetilde{Z}^{n,m}_r - \widetilde{Z}^{n,m'}_r \right] dB_r
\end{aligned}$$

and taking the conditional expectation given $\mathcal{F}_t$:

$$0 \leq \widetilde{Y}^{n,m}_{t\wedge\tau} - \widetilde{Y}^{n,m'}_{t\wedge\tau} \leq \mathbb{E}^{\mathcal{F}_t}\left(\widetilde{Y}^{n,m}_{T\wedge\tau} - \widetilde{Y}^{n,m'}_{T\wedge\tau}\right) \leq \frac{1}{m}.$$

Letting first $T \to +\infty$ and then $m' \to +\infty$ in the last estimate leads to:

$$0 \leq \widetilde{Y}^{n,m}_{t\wedge\tau} - Y^n_{t\wedge\tau} \leq \frac{1}{m}.$$

Therefore **P**-a.s.:

$$\sup_{t\geq 0} \left| \widetilde{Y}^m_{t\wedge\tau} - Y_{t\wedge\tau} \right| \leq \frac{1}{m}.$$

Since for each $m \geq 0$, $(\widetilde{Y}^m_{t\wedge\tau})_{t\geq 0}$ has a limit on the left at $+\infty$, so does Y.

3.3.2 Continuity of Y

We know now that

(3.15) $$\liminf_{t\to+\infty} Y_{t\wedge\tau} = \lim_{t\to+\infty} Y_{t\wedge\tau} \geq \xi$$

and on the set $\{t \geq \tau\}$, $Y_t = \xi$. In this part we give sufficients conditions which ensure that the process Y is continuous, i.e.

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \xi.$$

It suffices to prove the result on the set $\{\xi < \infty\}$. In the rest of this section we will suppose that $\mathbf{P}(\xi < \infty) > 0$ and:

(A1) $$\xi = g(X_\tau)$$

where $g : \mathbb{R}^d \to \overline{\mathbb{R}_+}$ is a continuous function such that $F = \{g = +\infty\} \cap \partial D$ is a closed set. Moreover: $\mathbf{P}(\xi < \infty) > 0 \Rightarrow F \neq \partial D$. For all compact subset $\mathcal{K}$ of $\mathbb{R}^d$ such that $F \cap \mathcal{K} = \emptyset$, there exists a constant K such that:

(A2) $$\forall x \in \mathcal{K}, \quad g(x) \leq K.$$

A first step

In the first section we have proved the following estimate:

$$\text{(3.6)} \qquad \mathbf{P}-\text{a.s.} \quad \forall t \geq 0, \quad |Y_t| \leq \frac{C}{\left(\rho(X^x_{t\wedge\tau_x})\right)^{\frac{2}{q}}}.$$

where ρ is the distance to the boundary of D. The constant C depends on q, D and the bound on b and σ in (B). Here we want to construct another estimate which depends also on the function g. Our result is the following

Proposition 3.7 *The boundary of D belongs to C^3. If U is a open set such that $\overline{U}\cap F = \emptyset$ and $U \cap \partial D \neq \emptyset$, then there exists a constant $C = C(U,g,q,b,\sigma,D)$ and a open set D_U such that $D \subset D_U$ and if ρ_U denotes the distance to the boundary of D_U we have:*

$$\text{(3.16)} \qquad \mathbf{P}-a.s. \quad \forall n \in \mathbb{N}, \forall t \geq 0,\ Y^n_t \leq \frac{C}{\left(\rho_U(X_{t\wedge\tau})\right)^{2/q}}.$$

Recall that τ is the first exit time from $\overline{D}$.

Proof. We suppose that the set $F = \{g = +\infty\}$ is not equal to ∂D. Hence if we define for all $\varepsilon \geq 0$ the set

$$F_\varepsilon = \{y \in \partial D;\ \text{dist}(y,F) \leq \varepsilon\},$$

there exists $\varepsilon' > 0$ such that $F_{\varepsilon'} \neq \partial D$. Moreover if U is a open subset of $\mathbb{R}^d$ such that $F \cap \overline{U} = \emptyset$ and $U \cap \partial D \neq \emptyset$, there exists $0 < \varepsilon < \varepsilon'$ such that $U \cap \partial D \subset \partial D \setminus F_\varepsilon$.

Recall that D is a bounded open set of $\mathbb{R}^d$ with a boundary $\partial D \in C^2$. Thus there exists $r > 0$ such that on

$$\Gamma_r = \left\{y \in \mathbb{R}^d;\ \text{dist}(y,\partial D) < r\right\},$$

the signed distance d

$$d(x) = \begin{cases} \text{dist}(x,\partial D) = \rho(x) & \text{if } x \in D, \\ -\text{dist}(x,\partial D) & \text{if } x \in \mathbb{R}^d \setminus D. \end{cases}$$

belongs to $C^2(\Gamma_r)$. Moreover for all $y \in \Gamma_r$ there exists a unique $x \in \partial D$ such that $y = x - d(y)\overrightarrow{n}(x)$, where $\overrightarrow{n}(x)$ is the outward normal vector at the point $x \in \partial D$. We have $\|y - x\| = |d(y)| = \text{dist}(y,\partial D)$. The result can be found in [21].

We take a function $\psi_\varepsilon : \mathbb{R}^d \to [0,1]$ such that ψ_ε is of class $C^2(\mathbb{R}^d)$ and $\psi_\varepsilon = 0$ on F_ε and $\psi_\varepsilon = 1$ on $\partial D \setminus F_{2\varepsilon}$. With this function we define the set D_ε as follows:

$$D_\varepsilon = D \cup \left\{y \in \mathbb{R}^d, \exists x \in \partial D \text{ and } \nu \in [0,r/2[\text{ s.t. } y = x + \nu\psi_\varepsilon(x)\overrightarrow{n}(x)\right\},$$

where $\overrightarrow{n}(x)$ always denotes the outward normal vector at the point $x \in \partial D$. We can easily prove that D_ε is included in $D \cup \Gamma_r$, and that

$$F_\varepsilon \subset \partial D_\varepsilon \quad \text{and} \quad \partial D \setminus F_{2\varepsilon} \subset D_\varepsilon.$$

If $\partial D \in C^3$, then the boundary of D_ε is of class C^2, and from our construction the distance to the boundary of D_ε, denoted by ρ_ε, is also a C^2 function on the set Γ_r. Moreover if $y \in \partial D \setminus F_{2\varepsilon}$, then $\rho_\varepsilon(y) = r/2 > 0$.

The proof of (3.16) is similar to the proof of theorem 3.2. Let Φ be a $C^\infty(\mathbb{R}^d)$ function such that $\Phi = 1$ on $D \setminus \Gamma_r$ and $\Phi = 0$ on $\Gamma_{r/2}$. For all $0 < \eta < r/2$ and $C > 0$ we define a function $f = f_\eta \in C^2(\mathbb{R}^d;\mathbb{R}_+)$ such that on $\overline{D}$,

$$f = f_\eta = \frac{C}{[(1-\Phi)\rho_\varepsilon + R_\varepsilon \Phi + \eta]^{\frac{2}{q}}} = \frac{C}{[\Psi_\eta]^{\frac{2}{q}}}.$$

The constant R_ε is the supremum of ρ_ε on $\overline{D_\varepsilon}$ and we can easily see that $R_\varepsilon \leq \sup \rho + r/2$ and that $\Psi_\eta \geq \rho_\varepsilon$ for all $\eta > 0$. Remark also that f is of class C^2 on $\overline{D}$. We apply the Itô formula to $f(X_{t\wedge\tau})$. For all $t \geq 0$:

$$\begin{aligned} f(X_{t\wedge\tau}) &= f(X_0) + \int_0^{t\wedge\tau} f(X_r)^{1+q} dr + \int_0^{t\wedge\tau} \nabla f(X_r)\sigma(X_r) dB_r \\ &+ \int_0^{t\wedge\tau} \left[\nabla f(X_r) b(X_r) + \frac{1}{2} \mathrm{Trace}\left(\sigma\sigma^*(X_r) D^2 f(X_r)\right) - f(X_r)^{1+q} \right] dr \end{aligned}$$

and

$$\begin{aligned} &(\nabla f) b + \frac{1}{2}\mathrm{Trace}\left(\sigma\sigma^* D^2 f\right) - f^{1+q} \\ &\quad = -C\Psi_\eta^{-\frac{2}{q}-2} \left[C^q + \frac{2\Psi_\eta}{q}(\nabla\Psi_\eta) b - \frac{1}{q}(\frac{2}{q}+1)\|\sigma\nabla\Psi_\eta\|^2 + \frac{\Psi_\eta}{q}\mathrm{Trace}(\sigma\sigma^* D^2\Psi_\eta) \right] \end{aligned}$$

b, σ, Ψ_η, $\nabla\Psi_\eta$ and $D^2\Psi_\eta$ are bounded on $\overline{D}$ and the bounds do not depend on η, because $\eta < r/2$. So we can choose the constant C such that on $\overline{D}$:

$$C^q + \frac{2\Psi_\eta}{q}(\nabla\Psi_\eta) b - \frac{1}{q}(\frac{2}{q}+1)\|\sigma\nabla\Psi_\eta\|^2 + \frac{\Psi_\eta}{q}\mathrm{Trace}(\sigma\sigma^* D^2\Psi_\eta) \geq 0. \tag{3.17}$$

The constant C depends only on D, on q and on the bound of b and σ in (B). We have obtained for all $0 \leq t \leq T$:

$$f(X_{t\wedge\tau}) = f(X_{T\wedge\tau}) - \int_{t\wedge\tau}^{T\wedge\tau} \nabla f(X_r)\sigma(X_r) dB_r - \int_{t\wedge\tau}^{T\wedge\tau} f(X_r)^{1+q} dr + \int_{t\wedge\tau}^{T\wedge\tau} U_r dr;$$

with U a nonnegative adapted process. Moreover on $\{t \geq \tau\}$,

$$\begin{aligned} f(X_\tau) &= \frac{C}{\eta^{2/q}} \quad \text{if } X_\tau \in F_\varepsilon, \text{ because on } \partial D,\ \Phi = 0 \text{ and on } F_\varepsilon,\ \rho_\varepsilon = 0; \\ f(X_\tau) &\geq \frac{C}{(\eta + r/2)^{2/q}} \quad \text{if } X_\tau \in \partial D \setminus F_\varepsilon, \text{ because on } \partial D,\ 0 \leq \rho_\varepsilon \leq r/2. \end{aligned}$$

Recall that for all $n \in \mathbb{N}$, (Y^n,Z^n) is the solution of the BSDE (3.3) with terminal time τ and terminal data $g \wedge n$. From the assumption (A2), on the compact set $\overline{\partial D \setminus F_\varepsilon}$, the function g is bounded by a constant $K = K_\varepsilon$. We choose $C > 0$ and $0 < \eta < r/2$ such that

$$\frac{C}{\eta^{2/q}} \geq n \quad \text{and} \quad \frac{C}{(\eta + r/2)^{2/q}} \geq K.$$

We can take $C > Kr^{2/q}$ satisfying (3.17), and $\eta < r/2 \wedge C^{q/2}/n^{q/2}$. Note that C does not depend on η. Therefore if we define for $t \geq 0$:

$$\widetilde{Y}_t = f(X_{t\wedge\tau}) \quad \text{and} \quad \widetilde{Z}_t = \nabla f(X_t)\sigma(X_t)\mathbf{1}_{t<\tau},$$

the process $(\widetilde{Y},\widetilde{Z})$ satisfies $\mathbf{P}$-a.s. for all $0 \leq t \leq T$:

$$\widetilde{Y}_{t\wedge\tau} = \widetilde{Y}_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau} \widetilde{Y}_r^{1+q} dr + \int_{t\wedge\tau}^{T\wedge\tau} U_r dr - \int_{t\wedge\tau}^{T\wedge\tau} \widetilde{Z}_r dB_r,$$

U being a non negative process, and on the set $\{t \geq \tau\}$:

$$\widetilde{Y}_t \geq g(X_\tau) \wedge n.$$

From the comparison theorem (corollary 4.4.2 in [11]) for solutions of a BSDE, we obtain:

$$\mathbf{P} - \text{a.s.} \quad \forall t \geq 0,\ Y_t^n \leq f_\eta(X_{t\wedge\tau}) \leq \frac{C}{(\rho_\varepsilon(X_{t\wedge\tau}))^{2/q}}.$$

Since this inequality holds for all n, we have proved the proposition. □ Using Fatou's lemma we have

Corollary 3.1 *If U is a open set such that $\overline{U} \cap F = \emptyset$ and $U \cap \partial D \neq \emptyset$, then there exists a constant $C = C(U,g,q,b,\sigma,D)$ and a open set D_U such that $D \subset D_U$ and if ρ_U denotes the distance to the boundary of D_U we have:*

$$\mathbf{P} - a.s. \quad \forall t \geq 0,\ Y_t \leq \frac{C}{(\rho_U(X_{t\wedge\tau}))^{2/q}}.$$

The main interest of the proposition 3.7 is that if $h \in C_0(U)$ (h has a compact support included in U) with $\overline{U} \cap F = \emptyset$ and $U \cap \partial D \neq \emptyset$, then $h(X_{.\wedge\tau})Y$ belongs to $L^\infty([0,+\infty[\times\Omega)$: there exists a constant $K = K_U$ such that:

$$\mathbf{P} - \text{a.s.} \quad \forall t \geq 0,\ h(X_{t\wedge\tau})Y_t \leq K.$$

We can also prove the following:

Proposition 3.8 *For all $\nu > 1$ there exists a constant $K = K_{U,\nu} > 0$ such that*

$$\mathbb{E} \int_0^\tau \|Z_t\|^2 \rho_U^{4/q+\nu}(X_t) dt \leq K.$$

Proof. We come back to the proof of the propostion 3.1. Recall that from the previous demonstration there exists a constant C such that $\mathbf{P}$-a.s. we have for all $t \geq 0$:

$$Y_t^n \leq f(X_{t\wedge\tau}) = \frac{C}{[(1-\Phi)(X_{t\wedge\tau})\rho_\varepsilon(X_{t\wedge\tau}) + R_\varepsilon\Phi(X_{t\wedge\tau})]^{\frac{2}{q}}} = \frac{C}{[\Psi(X_{t\wedge\tau})]^{\frac{2}{q}}}.$$

The function ρ_ε is the distance to the boundary of D_ε and R_ε is the maximum of ρ_ε on D_ε. The stopping time τ is the first exit time from $\overline{D}$ of the diffusion X. We can also write the previous inequality as follows:

$$Y_t^n \left[\Psi(X_{t\wedge\tau})\right]^{\frac{2}{q}} \leq C.$$

It is important to note that C does not depend on n. Using Itô's formula we have for $\nu > 0$ and $t \geq 0$:

$$\begin{aligned}
&\mathbb{E}\int_0^{t\wedge\tau} \|Z_r^n\|^2\Psi(X_r)^{\frac{4}{q}+\nu}dr + 2(\frac{4}{q}+\nu)\mathbb{E}\int_0^{t\wedge\tau} Y_r^n\Psi(X_r)^{\frac{4}{q}+\nu-1}Z_r^n\nabla\Psi(X_r)\sigma(X_r)dr \\
&\quad\leq \mathbb{E}\left(\left(Y_{t\wedge\tau}^n\right)^2\Psi(X_{t\wedge\tau})^{\frac{4}{q}+\nu}\right) - (\frac{4}{q}+\nu)\mathbb{E}\int_0^{t\wedge\tau}(Y_r^n)^2\Psi(X_r)^{\frac{4}{q}+\nu-1}\nabla\Psi(X_r)b(X_r)dr \\
&\quad -\frac{1}{2}(\frac{4}{q}+\nu)\mathbb{E}\int_0^{t\wedge\tau}(Y_r^n)^2\Psi(X_r)^{\frac{4}{q}+\varepsilon-1}\mathrm{Trace}\left(\sigma\sigma^*(X_r)D^2\Psi(X_r)\right)dr \\
&\quad -\frac{1}{2}(\frac{4}{q}+\nu)(\frac{4}{q}+\nu-1)\mathbb{E}\int_0^{t\wedge\tau}(Y_r^n)^2\Psi(X_r)^{\frac{4}{q}}\Psi(X_r)^{\nu-2}\|\sigma(X_r)\nabla\Psi(X_r)\|^2dr.
\end{aligned}$$

As for (3.8), the right hand side is bounded by:

$$K\left(1+\mathbb{E}\int_0^\tau \Psi^{\nu-1}(X_r)dr + \mathbb{E}\int_0^\tau \Psi^{\nu-2}(X_r)dr\right)$$

and by the same arguments as in the proof of the proposition 3.1, the two integrals are finite if $\nu > 1$. Thus we can deduce that:

$$\mathbb{E}\int_0^\tau \|Z_r^n\|^2\Psi(X_r)^{\frac{4}{q}+\nu}dr \leq K.$$

The bound K does not depend on t, hence by the Fatou lemma,

$$\mathbb{E}\int_0^\tau \|Z_r\|^2\Psi(X_r)^{\frac{4}{q}+\nu}dr \leq K.$$

Since $\Psi \geq \rho_\varepsilon$ on $\overline{D}$, we obtain the wanted result. □

Continuity: the conclusion

Recall that $F = \{g = +\infty\} \cap \partial D$ is a closed set, that U is an bounded open set such that $\overline{U} \cap F = \emptyset$ and $U \cap \partial D \neq \emptyset$.

Now we take a function $\theta : \mathbb{R}^d \to \mathbb{R}_+$ of class C^2 and with a compact support included in U. For $\beta > 0$ we apply the Itô formula to the process $e^{-\beta t} Y^n_t \theta(X_t)$:

$$\text{(3.18)} \quad \mathbb{E}\left(e^{-\beta\tau}(g \wedge n)(X_\tau)\theta(X_\tau)\right) = \mathbb{E}(e^{-\beta(t\wedge\tau)} Y^n_{t\wedge\tau}\theta(X_{t\wedge\tau})) - \beta\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)Y^n_r dr$$
$$+\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)Y^n_r|Y^n_r|^q dr + \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}Y^n_r \mathcal{L}\theta(X_r)dr$$
$$+\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r} Z^n_r.\nabla\theta(X_r)\sigma(X_r)dr$$

where $\mathcal{L}$ is the second order partial differential operator:

$$\text{(3.5)} \qquad \forall x \in \mathbb{R}^d, \quad \mathcal{L}\varphi(x) = \frac{1}{2}\text{Trace}\left(\sigma\sigma^*(x)D^2\varphi(x)\right) + b(x)\nabla\varphi(x).$$

In (3.18) every term, except maybe the last one, is well-defined because $\beta > 0$, θ is a C^2 function with compact support, and Y^n is bounded by n. Now using the Cauchy-Schwarz inequality we obtain for all $\eta > 1$:

$$\mathbb{E}\int_0^{\tau} e^{-\beta r}\left|Z^n_r.\nabla\theta(X_r)\sigma(X_r)\right| dr \leq \left[\mathbb{E}\int_0^{\tau} \|Z^n_r\|^2 \rho_U^{4/q+\eta}(X_r)dr\right]^{1/2}$$
$$\times\left[\mathbb{E}\int_0^{\tau} e^{-2\beta r}\rho_U^{-4/q-\eta}(X_r)\|\nabla\theta(X_r)\sigma(X_r)\|^2 dr\right]^{1/2}.$$

We already know that σ is bounded on $\overline{D}$. The support of $\nabla\theta$ is included in U. In our previous construction of D_U we have $U \cap \overline{D} \subset D_U$ and on $U \cap \overline{D}$, $\rho_U \geq r/2 > 0$. Therefore $\rho_U^{-4/q-\eta}\nabla\theta$ is a continuous and bounded function. With the proposition 3.8, we deduce:

$$\text{(3.19)} \qquad \mathbb{E}\int_0^{\tau} e^{-\beta r}\left|Z^n_r.\nabla\theta(X_r)\sigma(X_r)\right| dr \leq K.$$

It is important to remark that the constant K does not depend on n.

We want to pass to the limit when $n \to +\infty$ in (3.18). With the monotone convergence theorem we obtain for all $0 \leq t$:

$$\lim_{n\to+\infty} \mathbb{E}\left(e^{-\beta\tau}(g \wedge n)(X_\tau)\theta(X_\tau)\right) = \mathbb{E}\left(e^{-\beta\tau}g(X_\tau)\theta(X_\tau)\right);$$
$$\lim_{n\to+\infty} \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)Y^n_r dr = \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)Y_r dr;$$
$$\lim_{n\to+\infty} \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)\left(Y^n_r\right)^{1+q} dr = \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)\left(Y_r\right)^{1+q} dr.$$

The support of the function $\mathcal{L}\theta$ is included in U. Therefore from the proposition (3.7) $Y^n\mathcal{L}\theta(X)$ is a.s. bounded. From the dominated convergence theorem we deduce that for all $t \geq 0$:

$$\lim_{n\to+\infty} \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}Y^n_r\mathcal{L}\theta(X_r)dr = \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}Y_r\mathcal{L}\theta(X_r)dr.$$

The last term in (3.18) is equal to:

$$\begin{aligned}\mathbb{E}\int_{t\wedge\tau}^{\tau} & e^{-\beta r} Z_r^n.\nabla\theta(X_r)\sigma(X_r)dr \\ & = \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\rho_U^{2/q+\eta/2}(X_r) Z_r^n.\rho_U^{-2/q-\eta/2}(X_r)\nabla\theta(X_r)\sigma(X_r)dr.\end{aligned}$$

From the proposition (3.8), the sequence $\rho_U^{2/q+\eta/2}(X)Z^n\mathbf{1}_{\tau>.}$ is bounded in $L^2([0,+\infty[\times\Omega)$ for all $\eta > 1$. Therefore after extraction of a suitable subsequence, which we omit as an abuse of notation, $\rho_U^{2/q+\eta/2}(X)Z^n\mathbf{1}_{\tau>.}$ converges weakly in $L^2([0,+\infty[\times\Omega)$. The process $e^{-\beta.}\rho_U^{-2/q-\eta/2}(X)\nabla\theta(X)\sigma(X)\mathbf{1}_{\tau>.}$ is in $L^2([0,+\infty[\times\Omega)$, because σ is bounded, and from our construction $\rho_U^{-2/q-\eta/2}(X)\nabla\theta(X)\mathbf{1}_{\tau>.}$ is also bounded. Using the proposition 3.4, we obtain:

$$\lim_{n\to+\infty}\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r} Z_r^n.\nabla\theta(X_r)\sigma(X_r)dr = \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r} Z_r.\nabla\theta(X_r)\sigma(X_r)dr.$$

Finally letting $n\to+\infty$ in (3.18), we have for all $0\leq t$:

$$\begin{aligned}\mathbb{E}\left(e^{-\beta\tau}g(X_\tau)\theta(X_\tau)\right) &= \mathbb{E}(e^{-\beta(t\wedge\tau)}Y_{t\wedge\tau}\theta(X_{t\wedge\tau})) - \beta\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)Y_r dr \\ &+\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}\theta(X_r)\left(Y_r\right)^{1+q}dr + \mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}Y_r\mathcal{L}\theta(X_r)dr \\ &+\mathbb{E}\int_{t\wedge\tau}^{\tau} e^{-\beta r}Z_r.\nabla\theta(X_r)\sigma(X_r)dr.\end{aligned} \tag{3.20}$$

From Proposition 3.7, we know that:

$$\mathbb{E}\int_0^{\tau} e^{-\beta r}\theta(X_r)Y_r dr \leq K,$$

$$\mathbb{E}\int_0^{\tau} e^{-\beta r}\theta(X_r)\left(Y_r\right)^{1+q}dr + \mathbb{E}\int_0^{\tau} e^{-\beta r}Y_r\left|\mathcal{L}\theta(X_r)\right|dr \leq K.$$

For the last term, using the Cauchy-Schwarz inequality we obtain for all $\eta > 1$:

$$\begin{aligned}\mathbb{E}\int_0^{\tau} e^{-\beta r}\left|Z_r.\nabla\theta(X_r)\sigma(X_r)\right|dr &\leq \left[\mathbb{E}\int_0^{\tau}\|Z_r\|^2\rho_U^{4/q+\eta}(X_r)dr\right]^{1/2} \\ &\times\left[\mathbb{E}\int_0^{\tau} e^{-2\beta r}\rho_U^{-4/q-\eta}(X_r)\|\nabla\theta(X_r)\sigma(X_r)\|^2 dr\right]^{1/2}.\end{aligned}$$

From Proposition 3.8, we deduce that:

$$\mathbb{E}\int_0^{\tau} e^{-\beta r}\left|Z_r.\nabla\theta(X_r)\sigma(X_r)\right|dr \leq K.$$

Therefore when t goes to $+\infty$ in the equation (3.20), we obtain, using Fatou's lemma:

$$\begin{aligned}\mathbb{E}\left(e^{-\beta\tau}g(X_\tau)\theta(X_\tau)\right) &= \lim_{t\to+\infty}\mathbb{E}\left(e^{-\beta(t\wedge\tau)}Y_{t\wedge\tau}\theta(X_{t\wedge\tau})\right) \\ &\geq \mathbb{E}\left[e^{-\beta\tau}\theta(X_\tau)\left(\lim_{t\to+\infty}Y_{t\wedge\tau}\right)\right].\end{aligned} \tag{3.21}$$

But recall that we already know that

(3.15) $$\lim_{t\to+\infty} Y_{t\wedge\tau} \geq g(X_\tau).$$

Hence the inequality in (3.21) is in fact a equality, i.e.

$$\mathbb{E}\left(e^{-\beta\tau}g(X_\tau)\theta(X_\tau)\right) = \mathbb{E}\left[e^{-\beta\tau}\theta(X_\tau)\left(\lim_{t\to+\infty} Y_{t\wedge\tau}\right)\right].$$

And using again (3.15), we conclude that

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = g(X_\tau), \quad \mathbf{P}-\text{a.s. on } \{g(X_\tau)<\infty\}.$$

We summarize our result in the

Theorem 3.3 Under the assumptions:

– the terminal data ξ has the form (A1) with the condition (A2) ;

– the boundary ∂D belongs to C^3 ;

the process Y is continuous, i.e.

$$\lim_{t\to+\infty} Y_{t\wedge\tau} = \xi \quad \mathbf{P}-\text{a.s.}$$

3.4 Minimal solution

In the third section we have constructed a process (Y,Z) which satisfies the conditions (D1)-(D2) of the definition 3.2. We will prove now that, if this process is a solution of the BSDE (3.3), i.e. if it satisfies also the condition (D3), then it is the minimal non-negative solution.

Theorem 3.5 *Let the conditions (L)-(B)-(E) hold and let* $\left\{(\overline{Y}_t,\overline{Z}_t);\ t\geq 0\right\}$ *be a non-negative solution of the BSDE (3.3) (solution in the sense of the definition 3.2). Then* $\mathbf{P}$-*a.s. for all* $t\geq 0$

$$\overline{Y}_t \geq Y_t.$$

Proof. Recall that τ_η is the first exit time of $D\setminus\Gamma_\eta$ and we have for all $0\leq t\leq T$:

$$\overline{Y}_{t\wedge\tau_\eta} = \overline{Y}_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} \left(\overline{Y}_r\right)^{1+q} dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} \overline{Z}_r dB_r.$$

For $n\in\mathbb{N}^*$ (Y^n,Z^n) is the solution (in the sense of the definition 3.1) of the BSDE (3.3) with terminal data $\xi\wedge n$. We compare $\overline{Y}$ with Y^n:

$$\begin{aligned}\overline{Y}_{t\wedge\tau_\eta} - Y^n_{t\wedge\tau_\eta} &= \overline{Y}_{T\wedge\tau_\eta} - Y^n_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} \left(\overline{Y}_r\right)^{1+q} - (Y^n_r)^{1+q}\, dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} (\overline{Z}_r - Z^n_r)dB_r \\ &= \overline{Y}_{T\wedge\tau_\eta} - Y^n_{T\wedge\tau_\eta} - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} \alpha^n_r\left(\overline{Y}_r - Y^n_r\right) dr - \int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} (\overline{Z}_r - Z^n_r)dB_r\end{aligned}$$

where the process α_r^n is defined by

$$\begin{aligned} \alpha_r^n &= \frac{\left(\overline{Y}_r\right)^{1+q} - (Y_r^n)^{1+q}}{\overline{Y}_r - Y_r^n} \quad \text{if} \quad \overline{Y}_r \neq Y_r^n \\ \alpha_r^n &= (1+q)\,(Y_r^n)^q \quad \text{if} \quad \overline{Y}_r = Y_r^n. \end{aligned}$$

α^n is a non-negative process and we have a linear BSDE whose solution is:

$$\overline{Y}_{t\wedge\tau_\eta} - Y^n_{t\wedge\tau_\eta} = \mathbb{E}^{\mathcal{F}_t}\left[\left(\overline{Y}_{T\wedge\tau_\eta} - Y^n_{T\wedge\tau_\eta}\right)\exp\left(-\int_{t\wedge\tau_\eta}^{T\wedge\tau_\eta} \alpha_r^n dr\right)\right]. \tag{3.22}$$

From the hypothesis of the theorem, $\overline{Y}$ is non-negative and Y^n is bounded by n. Indeed on the set $\{t \geq \tau\}$, $Y_t^n = \xi \wedge n \leq n$ and for all $0 \leq t \leq T$:

$$\begin{aligned} Y^n_{t\wedge\tau} &= Y^n_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau} (Y_r^n)^{1+q}\,dr - \int_{t\wedge\tau}^{T\wedge\tau} Z_r^n dB_r \\ &\leq Y^n_{T\wedge\tau} - \int_{t\wedge\tau}^{T\wedge\tau} Z_r^n dB_r \end{aligned}$$

thus

$$Y^n_{t\wedge\tau} \leq Y^n_\tau - \int_{t\wedge\tau}^{\tau} Z_r^n dB_r = \xi \wedge n - \int_{t\wedge\tau}^{\tau} Z_r^n dB_r \leq n - \int_{t\wedge\tau}^{\tau} Z_r^n dB_r.$$

Taking the conditional expectation we deduce that for all $t \geq 0$, $Y_t^n \leq n$.

We now pass to the limit in (3.22) first as $\eta \to 0$, then as $T \to +\infty$ and with the Fatou lemma we obtain for all $t \geq 0$:

$$\overline{Y}_{t\wedge\tau} - Y^n_{t\wedge\tau} \geq 0.$$

Therefore $\overline{Y}$ is greater than Y^n for all $n \in \mathbb{N}^*$ and thus than Y. □

Moreover we obtain the following result:

Proposition 3.9 *There exists a constant C which depends only on q, b, σ and D, such that* $\mathbf{P}$*-a.s., for all $t \geq 0$,*

$$\overline{Y}_t \leq \frac{C}{\rho^{2/q}(X_{t\wedge\tau})}.$$

Proof. For all sufficiently small $\eta > 0$ we denote by ρ_η the distance from the boundary of $D_\eta = D \setminus \Gamma_\eta$, i.e.

$$D_\eta = \{x \in D,\, \rho(x) \geq \eta\}.$$

If $x \in D_\eta$, $\rho(x) - \eta \leq \rho_\eta(x) \leq \rho(x)$. We consider the first exit time

$$\tau_\eta = \inf\left\{t \geq 0,\, X_t \notin \overline{D_\eta}\right\}.$$

From the theorem 3.2 we deduce:

$$\forall t \geq 0 \quad \overline{Y}_{t\wedge\tau_\eta} \leq \frac{C}{\rho_\eta^{2/q}(X_{t\wedge\tau_\eta})} \leq \frac{C}{\rho^{2/q}(X_{t\wedge\tau_\eta}) - \eta}.$$

The constant C which appears in the previous inequality may depend on η. In the proof of the theorem 3.2 we use the fact that there exists $\mu > 0$ such that on Γ_μ, the signed distance function is of class C^2. But if $\eta < \mu$, it is also true that ρ_η is of class C^2 on Γ_μ. So in the proof of the theorem we can use the same function φ and the same bound R for ρ_η and ρ. Moreover on Γ_μ, $|\nabla\rho| = 1$ and $D^2\rho$ depends only on the curvature of ∂D. Therefore we can choose a constant C independent of η if $\eta < \mu$.

To conclude let $\eta \to 0$. We obtain the wanted inequality. □

Remark 3.4 *The constant C in the previous proposition, is the same constant as in the theorem 3.6.*

3.5 Viscosity solution of the associated elliptic PDE

Recall that D is a bounded open subset of $\mathbb{R}^d$ with a C^3 boundary. For all $x \in \overline{D}$, $\{X_t^x;\ t \geq 0\}$ is the solution of the SDE (3.2):

$$X_t^x = x + \int_0^t b(X_r^x)dr + \int_0^t \sigma(X_r^x)dB_r, \text{ for } t \geq 0. \tag{3.2}$$

The functions b and σ are defined on $\mathbb{R}^d$, with values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$, and such that:

1. b and σ are continuous on $\mathbb{R}^d$;
2. Monotone condition: there exists $\mu \in \mathbb{R}$ such that for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$:
$$\langle x - y, b(x) - b(y)\rangle \leq \mu|x - y|^2; \tag{M}$$
3. Lipschitz condition: there exists $K \geq 0$ such that for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$:
$$\|\sigma(x) - \sigma(y)\| \leq K|x - y|; \tag{L}$$
4. Boundedness condition:
$$\forall x \in \mathbb{R}^d,\ |b(x)| + \|\sigma(x)\| \leq K. \tag{B}$$

For each $x \in \overline{D}$, we define the stopping time τ_x:

$$\tau_x = \inf\left\{t \geq 0,\ X_t^x \notin \overline{D}\right\}.$$

We assume that:

$$\mathbf{P}(\tau_x < \infty) = 1, \quad \text{for all } x \in \overline{D}, \tag{3.23}$$

that the set of singular points

$$\Gamma = \{x \in \partial D;\ \mathbf{P}(\tau_x > 0) > 0\} \quad \text{is empty}, \tag{C0}$$

and that for some $\lambda > 0$ and all $x \in \overline{D}$,

$$\mathbb{E}e^{\lambda\tau_x} < \infty. \tag{C1}$$

Let us recall the following result (cf. proposition 5.2. in [42]):

Proposition 3.10 *Under the conditions (C0) and (C1), the mapping $x \mapsto \tau_x$ is a.s. continuous on $\overline{D}$.*

Let $g : \partial D \to \overline{\mathbb{R}_+}$ be a continuous function. Therefore the assumption (A2) is satisfied. For all $n \in \mathbb{N}$ we define $g_n = g \wedge n$. Hence g_n is a continuous function. For all $n \in \mathbb{N}$, from the remark 3.3, $\{(Y_t^{x,n}, Z_t^{x,n});\ t \geq 0\}$ is the unique solution (in the sense of the definition 3.1) of the BSDE (3.3):

$$Y_t^{x,n} = g_n(X_{\tau_x}^x) - \int_{t\wedge\tau_x}^{\tau_x} Y_r^{x,n}|Y_r^{x,n}|^q dr - \int_{t\wedge\tau_x}^{\tau_x} Z_r^{x,n} dB_r. \tag{3.24}$$

We denote by u_n the function defined on $\overline{D}$ by

$$u_n(x) \triangleq Y_0^{x,n}$$

and by $\mathcal{L}$ the second order partial differential operator:

$$\forall x \in \mathbb{R}^d, \quad \mathcal{L}\varphi(x) = \frac{1}{2}\text{Trace}\left(\sigma\sigma^*(x)D^2\varphi(x)\right) + b(x)\nabla\varphi(x).$$

For $h \in C(\partial D, \mathbb{R})$ we consider the elliptic PDE:

$$\begin{cases} -\mathcal{L}v + v|v|^q = 0 & \text{on } D; \\ v = h & \text{on } \partial D. \end{cases} \tag{3.4}$$

The following definition can be found in [42] or [10]. If v is a function defined on $\overline{D}$, we denote by v^* (respectively v_*) the upper- (respectively lower-) semicontinuous envelope of v: for all $x \in \overline{D}$

$$v^*(x) = \limsup_{x'\to x,\ x'\in\overline{D}} v(x') \quad \text{and} \quad v_*(x) = \liminf_{x'\to x,\ x'\in\overline{D}} v(x').$$

Definition 3.3 (Viscosity solution)

- *$v : \overline{D} \to \mathbb{R}$ is called a **viscosity subsolution** of (3.4) if $v^* < +\infty$ on $\overline{D}$ and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \overline{D}$ is a point of local maximum of $v^* - \phi$,*

$$\begin{aligned} -\mathcal{L}\phi(x) + v^*(x)|v^*(x)|^q \leq 0 & \quad \textit{if} \quad x \in D; \\ \min\left(-\mathcal{L}\phi(x) + v^*(x)|v^*(x)|^q, v^*(x) - h(x)\right) \leq 0 & \quad \textit{if} \quad x \in \partial D. \end{aligned}$$

- *$v : \overline{D} \to \mathbb{R}$ is called a **viscosity supersolution** of (3.4) if $v_* > -\infty$ on $\overline{D}$ and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \overline{D}$ is a point of local minimum of $v_* - \phi$,*

$$\begin{aligned} -\mathcal{L}\phi(x) + v_*(x)|v_*(x)|^q \geq 0 & \quad \textit{if} \quad x \in D; \\ \max\left(-\mathcal{L}\phi(x) + v_*(x)|v_*(x)|^q, v(x) - h(x)\right) \geq 0 & \quad \textit{if} \quad x \in \partial D. \end{aligned}$$

- *$v : \overline{D} \to \mathbb{R}$ is called a **viscosity solution** of (3.4) if it is both a viscosity sub- and supersolution.*

Let us recall the following result (cf. Theorem 5.3. in [42]):

Theorem 3.6 *Under the assumptions (M)-(L)-(B)-(C0)-(C1), since $g \wedge n$ is continuous on ∂D, u_n is continuous on $\overline{D}$ and it is a viscosity solution of the elliptic PDE (3.4) with boundary data $g \wedge n$.*

Remark 3.5 *Since $g \wedge n$ is continuous on ∂D, from the theorem 3.3 in [10], it follows that u_n is the unique continuous viscosity solution of the PDE (3.4) with terminal data $g \wedge n$.*

From now on we add the uniformly elliptic condition: there exists a constant $\alpha > 0$ such that for all $x \in \mathbb{R}^d$:

$$\sigma\sigma^*(x) \geq \alpha \mathrm{Id}. \tag{E}$$

With this assumption, (C0)-(C1) hold if (B) is true. In the previous sections we have constructed a process $\{(Y_t^x, Z_t^x); t \geq 0\}$ which is a solution of the BSDE (3.3) with terminal data $g(X^x_{\tau_x})$ (in the sense of the definition 3.2). Y^x is the limit of $Y^{x,n}$: for all $t \geq 0$:

$$Y_t^x = \lim_{n \to +\infty} Y_t^{x,n}. \tag{3.7}$$

If we define

$$u(x) \triangleq Y_0^x,$$

then u is the limit of the sequence u_n. Thus u is non negative. Since u is the supremum of continuous functions u_n, u is lower-semicontinuous on $\overline{D}$ and satisfies:

$$\forall x \in \overline{D}, \quad u(x) \leq \frac{C}{\rho^{2/q}(x)}. \tag{3.25}$$

Recall that ρ is the distance from the boundary ∂D and C is a constant which does not depend on g. Moreover $u(x) = g(x)$ on ∂D. Since g is not bounded on ∂D, we cannot apply the theorem 3.6. Moreover the condition $v^* < +\infty$ in the definition 3.3 cannot be satisfied on $\overline{D}$. Therefore we change the definition of a solution.

Definition 3.4 (Unbounded viscosity solution) *We say that v is a viscosity solution of the PDE*

$$\begin{cases} -\mathcal{L}v + v|v|^q = 0 & \text{on } D; \\ v = g & \text{on } \partial D; \end{cases} \tag{3.4}$$

with unbounded terminal data g if v is a viscosity solution on D in the sense of the definition 3.3 and if

$$g(x) \leq \lim_{\substack{x' \to x \\ x' \in D,\ x \in \partial D}} v_*(x') \leq \lim_{\substack{x' \to x \\ x' \in D,\ x \in \partial D}} v^*(x') \leq g(x).$$

Remark that this definition implies that $v^* < +\infty$ and $v_* > -\infty$ on D.

3.5.1 u is a viscosity solution

Lemma 3.4 *The function u is a viscosity solution of the PDE (3.4) on D.*

Proof. We will use the half-relaxed upper- and lower-limit of the sequence of functions u_n:

$$\overline{u}(x) = \limsup_{\substack{n\to+\infty \\ x'\to x}} u_n(x') \quad \text{and} \quad \underline{u}(x) = \liminf_{\substack{n\to+\infty \\ x'\to x}} u_n(x').$$

Since $\{u_n\}$ is a non-decreasing sequence of continuous functions, we have

$$\forall x \in D, \quad u(x) = u_*(x) = \underline{u}(x) \leq u^*(x) = \overline{u}(x).$$

We fix $\eta > 0$ and we prove that on $\widetilde{D} = D \setminus \{x \in D, \, \rho(x) \leq \eta\}$, u is a viscosity solution. We already know that

$$\forall x \in \widetilde{D}, \quad \overline{u}(x) \leq \frac{C}{\eta^{2/q}}.$$

Recall that u_n is a continuous viscosity solution and from the lemma 6.1 of [10], we deduce that u is a viscosity solution of (3.4) on $\widetilde{D}$ and this holds for all $\eta > 0$. Therefore the lemma is proved. □

Since u_n is a non-decreasing sequence of $C^0(\overline{D})$ functions, we have:

$$\forall x \in \partial D, \quad \liminf_{x'\to x, \; x'\in D} u(x') \geq g(x) = u(x). \tag{3.26}$$

Hence u_* is a supersolution of (3.4), because $u_* \geq g$ on ∂D.

Lemma 3.5 *The solution u satisfies the boundary condition, i.e.*

$$\lim_{\substack{x'\to x \\ x'\in D, \; x\in\partial D}} u^*(x') \leq g(x) = u(x).$$

Proof. We already know that

$$\liminf_{\substack{x'\to x \\ x'\in D, x\in\partial D}} u(x') \geq g(x) = u(x).$$

So we just have to prove the converse inequality on the set $\{g < +\infty\}$. If U is a open set such that $\overline{U} \cap F = \emptyset$ and $U \cap \partial D \neq \emptyset$, there exists a open set D_U and a constant C_U such that for all $n \in \mathbb{N}$:

$$\mathbf{P} - \text{a.s.} \quad \forall t \geq 0, \, Y_t^{x,n} \leq \frac{C_U}{\rho_U^{2/q}(X^x_{t\wedge\tau_x})}.$$

Recall that ρ_U is the distance to the boundary of D_U. From the proof of the proposition 3.7, the choice of the set D_U and of the constant C_U does not depend on $x \in \overline{D}$.

We write again the equation (3.18):

$$\begin{aligned} \mathbb{E}\left(e^{-\beta\tau_x}(g\wedge n)(X^x_{\tau_x})\theta(X^x_{\tau_x})\right) &= u_n(x)\theta(x) - \beta\mathbb{E}\int_0^{\tau_x} e^{-\beta r}\theta(X^x_r)Y^{x,n}_r dr \\ &+ \mathbb{E}\int_0^{\tau_x} e^{-\beta r}\theta(X^x_r)Y^{x,n}_r|Y^{x,n}_r|^q dr + \mathbb{E}\int_0^{\tau_x} e^{-\beta r}Y^{x,n}_r\mathcal{L}\theta(X^x_r)dr \\ &+ \mathbb{E}\int_0^{\tau_x} e^{-\beta r}Z^{x,n}_r.\nabla\theta(X^x_r)\sigma(X^x_r)dr \end{aligned} \tag{3.27}$$

The function $\theta : \mathbb{R}^d \to \mathbb{R}_+$ is of class C^2 and has a compact support included in U. The constant β is positive. From Proposition 3.7, there exists a constant K_U such that for all $n \in \mathbb{N}$:

$$\left| \mathbb{E} \int_0^{\tau_x} e^{-\beta r} \left[\theta(X_r^x) Y_r^{x,n} dr + \theta(X_r^x) Y_r^{x,n} |Y_r^{x,n}|^q + Y_r^{x,n} \mathcal{L}\theta(X_r^x) \right] dr \right| \leq K_U \mathbb{E} \int_0^{\tau_x} e^{-\beta r} dr.$$

Since $x \mapsto \tau_x$ is a continuous function on $\overline{D}$ and since $\tau_x = 0$ if $x \in \partial D$, we have:

$$\lim_{\substack{x' \to x \\ x' \in D, x \in \partial D}} \mathbb{E} \int_0^{\tau_{x'}} e^{-\beta r} dr = 0.$$

If $x \in \partial D$ and if $(x_m)_{m \in \mathbb{N}}$ is a sequence of elements of D which converges to x, we replace in (3.27) n by m and x by x_m and we take the limit as $m \to +\infty$. We obtain by Fatou's lemma:

$$\begin{aligned} \limsup_{m \to +\infty} u_m(x_m)\theta(x_m) &= \limsup_{m \to +\infty} \mathbb{E} \left(e^{-\beta \tau_{x_m}} (g \wedge m)(X_{\tau_{x_m}}^{x_m}) \theta(X_{\tau_{x_m}}^{x_m}) \right) \\ &\leq \mathbb{E} \left(\limsup_{m \to +\infty} \left[e^{-\beta \tau_{x_m}} (g \wedge m)(X_{\tau_{x_m}}^{x_m}) \theta(X_{\tau_{x_m}}^{x_m}) \right] \right), \end{aligned}$$

because $g\theta$ is a bounded function. By continuity of $x \mapsto X_{\tau_x}^x$ and of $g\theta$, we have:

$$\limsup_{m \to +\infty} u_m(x_m)\theta(x) = \limsup_{m \to +\infty} u_m(x_m)\theta(x_m) \leq g(x)\theta(x).$$

Finally on $\{g < \infty\}$, we have:

$$\limsup_{\substack{x' \to x \\ x' \in D, \ x \in \partial D}} u^*(x') \leq g(x).$$

With the inequality (3.26), this achieves the proof of the lemma. □

To summarize we have proved:

Theorem 3.4 *Under the assumptions of Theorem 3.3, u is a viscosity solution of the PDE (3.4).*

3.5.2 Some regularity results on u

We want to prove now that u is continuous on $\overline{D}$. Here it seems to be necessary to assume the condition (E).

First we prove that under stronger assumptions on b and σ, u belongs to $C^0(\overline{D}; \overline{\mathbb{R}_+}) \cap C^2(D; \mathbb{R}_+)$.

Proposition 3.11 *Recall that b and σ satisfy always (L)-(B)-(E) and $\partial D \in C^3$. We assume that b and σ belong to $C^1(D)$. Then u is in $C^2(D; \mathbb{R}_+)$.*

In order to prove this result we need the following lemma:

Lemma 3.6 *The assumptions of Proposition 3.11 hold. We consider a continuous function $h : \partial D \to \mathbb{R}$. Suppose that (Y,Z) is the solution (in the sense of the definition 3.1) of the BSDE*

$$Y_t = h(X_\tau) - \int_{t\wedge\tau}^{\tau} Y_r |Y_r|^q \, dr - \int_{t\wedge\tau}^{\tau} Z_r dB_r.$$

Then there exists a function $v : \overline{D} \to \mathbb{R}^+$ of class $C^0(\overline{D}) \cap C^2(D)$ such that

$$\forall t \geq 0, \quad Y_t = v(X_{t\wedge\tau}) \quad \textit{and} \quad Z_t = \nabla v(X_{t\wedge\tau})\sigma(X_{t\wedge\tau})\mathbf{1}_{t<\tau}.$$

Moreover v is solution of the PDE:

$$\left\{ \begin{array}{rl} -\mathcal{L}v + v|v|^q = 0 & \textit{on } D; \\ v = h & \textit{on } \partial D. \end{array} \right. \tag{3.4}$$

Proof. Since h is continuous and since b and σ belongs to $C^1(D)$, from the theorem 15.18 in [21], there exists a unique solution $v \in C^0(\overline{D}) \cap C^2(D)$ of the PDE (3.4) (see also [35]). We prove that for all $t \geq 0$, $Y_t = v(X_{t\wedge\tau})$ and $Z_t = \nabla v(X_{t\wedge\tau})\sigma(X_{t\wedge\tau})\mathbf{1}_{t<\tau}$. We want to apply the Itô formula to the process $v(X)$. But we just have $v \in C^2(D)$ and we do not know if we can define a function $\widetilde{v} \in C^2(\mathbb{R}^d)$ such that $\widetilde{v} = v$ on D.

We will use some arguments of the proof of the theorem 15.18 in [21]. We define a sequence $\{h_m\}$ of functions such that $\{h_m\}$ approximates h uniformly on ∂D and $h_m \in C^{2,\gamma}(\overline{D})$. From the theorem 15.10 in [21], there exists a function v_m such that v_m solves the Dirichlet problem (3.4) with condition h_m on the boundary and $v_m \in C^{2,\gamma}(\overline{D})$. We apply the Itô formula to the process $v_m(X)$: for all $0 \leq t \leq T$

$$\begin{aligned} v_m(X_{t\wedge\tau}) &= v_m(X_{T\wedge\tau}) - \int_{t\wedge\tau}^{T\wedge\tau} (\mathcal{L}v_m)(X_r)dr - \int_{t\wedge\tau}^{T\wedge\tau} \nabla v_m(X_r)\sigma(X_r)dB_r \\ &= v_m(X_{T\wedge\tau}) - \int_{t\wedge\tau}^{T\wedge\tau} v_m(X_r)\,|v_m(X_r)|^q \, dr - \int_{t\wedge\tau}^{T\wedge\tau} \nabla v_m(X_r)\sigma(X_r)\mathbf{1}_{r<\tau}dB_r \end{aligned} \tag{3.28}$$

We denote by (Y^m,Z^m) the solution of the BSDE (3.3) with terminal data $h_m(X_\tau) \in L^\infty(\Omega)$. Uniqueness of solution of this BSDE implies:

$$\forall t \geq 0, \quad Y_t^m = v_m(X_{t\wedge\tau}) \quad \text{and} \quad Z_t^m = \nabla v_m(X_t)\sigma(X_t)\mathbf{1}_{t<\tau}.$$

From (C1) and Remark 3.3, we know that there exists a constant C such that for all $m \in \mathbb{N}$:

$$0 \leq Y_0^m = v_m(x) \leq \mathbb{E}\left[\sup_{t\in[0,\tau]} e^{\beta t}|Y_t^m|^2\right] \leq C\mathbb{E}\left[e^{\beta\tau}|h_m(X_\tau)|^2\right].$$

Since h_m converges uniformly to h on ∂D, h_m is a bounded sequence in $L^\infty(\partial D)$. Therefore the sequence $\{v_m\}$ is uniformly bounded on $\overline{D}$.

From the theorems 6.1, 13.1 and 15.3 in [21], the sequence $\{v_m\}$ converges uniformly on compact subsets of D, together with its first and second derivatives, to the function v.

Since $\{v_m\}$ is uniformly bounded on $\overline{D}$ and converges to v:

$$\forall t \geq 0, \quad \lim_{m\to+\infty} Y_t^m = \lim_{m\to+\infty} v_m(X_{t\wedge\tau}) = v(X_{t\wedge\tau}) \tag{3.29}$$

$$\forall T \geq t \geq 0, \quad \lim_{m\to+\infty} \int_{t\wedge\tau}^{T\wedge\tau} v_m(X_r)\,|v_m(X_r)|^q\,dr = \int_{t\wedge\tau}^{T\wedge\tau} v(X_r)\,|v(X_r)|^q\,dr.$$

Using Itô's formula and the Burkholder-Davis-Gundy inequality we obtain for a constant c independent of m:

$$\mathbb{E}\left[\sup_{0\leq t\leq\tau} |Y_t^m - Y_t|^2 + \int_0^\tau \|Z_t^m - Z_t\|^2 dt\right] \leq c\mathbb{E}\,|h_m(X_\tau) - h(X_\tau)|^2.$$

Therefore with (3.29) we conclude that a.s. $Y_t = v(X_{t\wedge\tau})$ for all $t \geq 0$. Moreover there exists a constant K such that:

$$\mathbb{E}\int_0^\tau \|\nabla v_m(X_r)\sigma(X_r)\mathbf{1}_{r<\tau}\|^2\,dr = \mathbb{E}\int_0^\tau \|Z_r^m\|^2\,dr \leq K < +\infty;$$

and with (3.28) and (3.29) for all $0 \leq t \leq T$:

$$\lim_{m\to+\infty} \int_{t\wedge\tau}^{T\wedge\tau} \nabla v_m(X_r)\sigma(X_r)\mathbf{1}_{r\leq\tau}dB_r = \int_{t\wedge\tau}^{T\wedge\tau} Z_r dB_r.$$

Let $\mathcal{K}$ be a compact subset of D. Since the first derivatives of v_m converge uniformly on $\mathcal{K}$, from the dominated convergence theorem we deduce:

$$\lim_{m\to+\infty} \int_{t\wedge\tau}^{T\wedge\tau} \|\nabla v_m(X_r)\sigma(X_r)\mathbf{1}_{r<\tau}\mathbf{1}_{\mathcal{K}}(X_r) - \nabla v(X_r)\sigma(X_r)\mathbf{1}_{r<\tau}\mathbf{1}_{\mathcal{K}}(X_r)\|^2\,dr = 0.$$

Therefore for all compact subset $\mathcal{K}$ of D, $\mathbf{P}$-a.s.

$$Z_t\mathbf{1}_{\mathcal{K}}(X_t) = \nabla v(X_t)\sigma(X_t)\mathbf{1}_{t<\tau}\mathbf{1}_{\mathcal{K}}(X_t).$$

If $\{\mathcal{K}_m\}$ is an increasing sequence of compact subsets of D such that $\bigcup_m \mathcal{K}_m = D$, for all m:

$$\mathbb{E}\int_0^\tau \|\nabla v(X_t)\sigma(X_t)\mathbf{1}_{t<\tau}\mathbf{1}_{\mathcal{K}_m}(X_t)\|^2\,dt \leq \mathbb{E}\int_0^\tau \|Z_t\|^2\,dt < +\infty$$

and since $\tau > t$ implies $X_t \in D$, with the monotone convergence theorem we deduce:

$$\mathbb{E}\int_0^\tau \|\nabla v(X_t)\sigma(X_t)\mathbf{1}_{t<\tau}\|^2\,dt < \infty.$$

Then

$$\mathbb{E}\int_0^\tau \|\nabla v(X_t)\sigma(X_t)\mathbf{1}_{t<\tau} - Z_t\|^2\,dt = 0.$$

This achieves the proof of the proposition. □

From the lemma 3.6, we can deduce that u_n belongs to $C^0(\overline{D}) \cap C^2(D)$ if σ and b belong to $C^1(D)$.

Proof of the proposition 3.11. We fix $\eta > 0$ and we consider the set $D_\eta = D \setminus \{x \in D,\ \rho(x) \leq \eta\}$ for all $\eta > 0$. From the lemma 3.6, $u_n \in C^2(D)$ and satisfies $-\mathcal{L}u_n + u_n^{1+q} = 0$ on D. And on D_η, u_n is bounded by $C/\eta^{2/q}$. Therefore from the theorem 15.5 in [21], we obtain that ∇u_n is bounded on $D_{2\eta}$. The sequence is bounded in $C^1(D_{2\eta})$, thus the limit u is continuous on $D_{2\eta}$, i.e. u is continuous on D. Moreover we already know that u is continuous on the boundary. Therefore we deduce that u belongs to $C^0(\overline{D},\overline{\mathbb{R}_+})$.

Now if we consider the PDE:

$$\begin{cases} -\mathcal{L}v - v|v|^q = 0 & \text{on } D_\eta; \\ v = u & \text{on } \partial D_\eta. \end{cases}$$

from the theorem 15.18 in [21], the equation has a regular solution $v \in C^0(\overline{D_\eta}) \cap C^2(D_\eta)$. But this solution is also a continuous viscosity solution. Since u is now a continuous viscosity solution of the same PDE, from the comparison result in [10], we deduce that $v = u$, i.e. $u \in C^2(D_\eta)$. Hence u belongs to $C^2(D)$. □

Now we want to prove that u is continuous on D without the regularity conditions on b and σ of the proposition 3.11. We just assume that (M)-(L)-(B)-(E) hold.

Proposition 3.12 *The viscosity solution u is continuous on $\overline{D}$ and is locally Hölder continuous on D.*

Proof. We will show that for all open set $D' \subset D$ such that $\overline{D'} \subset D$, there exists $0 < \alpha < 1$ such that the sequence of functions u_n is bounded in the space $C^\alpha(D')$. $C^\alpha(D')$ is the set of functions v such that

$$\|v\|_\alpha = \sup\left\{\frac{|v(x) - v(y)|}{|x-y|^\alpha},\ (x,y) \in D'\right\} < +\infty.$$

Since u_n converges to u, we deduce that u belongs to $C^\alpha(D')$ and thus is continuous on D.

In order to prove that u_n is a bounded sequence in $C^\alpha(D')$, we will construct a sequence v_m which will belong to $C^\alpha(D')$ and such that there exists a constant K such that for all $m \in \mathbb{N}$, $\|v_m\|_\alpha \leq K$. Let b_m and σ_m be two sequences of functions such that:

1. b_m and σ_m belong to $C^1(D)$ and b_m and σ_m are bounded in $L^\infty(D)$;
2. b_m (resp. σ_m) converges to b (resp. σ), uniformly on D ;
3. σ_m satisfies the condition (E).

Let v_m be the unique solution in $C^0(\overline{D}) \cap C^2(D)$ (see lemma 3.6 or [21]) of the equation:

$$\begin{cases} -\mathcal{L}_m v_m + v_m|v_m|^q = 0 & \text{on } D, \\ v_m = g \wedge n & \text{on } \partial D; \end{cases} \tag{3.4}$$

where $\mathcal{L}_m$ is the operator:

$$\forall x \in \mathbb{R}^d, \quad \mathcal{L}_m\varphi(x) = \frac{1}{2}\text{Trace}\left(\sigma_m\sigma_m^*(x)D^2\varphi(x)\right) + b_m(x)\nabla\varphi(x).$$

For $x \in \overline{D}$, let $X^{x,m}$ be the solution of the SDE

$$\forall t \geq 0, \quad X_t^{x,m} = x + \int_0^t b_m(X_r^{x,m})dr + \int_0^t \sigma_m(X_r^{x,m})dB_r,$$

τ_m is the first exit time from $\overline{D}$ of the diffusion $X^{x,m}$, and $(Y^{x,n,m}, Z^{x,n,m})$ be the solution of the BSDE:

$$Y_t^{x,n,m} = (g \wedge n)(X_{\tau_m}^{x,m}) + \int_t^{\tau_m} Y_r^{x,n,m} |Y_r^{x,n,m}|^q dr - \int_t^{\tau_m} Z^{x,n,m} dB_r.$$

From classical results on SDE, $X^{x,m}$ converges to X^x solution of the SDE (3.2) and the process $(Y^{x,n,m}, Z^{x,n,m})$ converges to $(Y^{x,n}, Z^{x,n})$ solution of the BSDE 3.24 (see proposition 4.4 in [11]).

From the lemma 3.6, we have:

$$v_m(x) = Y_0^{x,n,m} \quad \text{and} \quad u_n(x) = Y_0^{x,n}.$$

Therefore v_m converges to u_n. Moreover we obtain that v_m is a bounded (by n) sequence in $L^\infty(D)$.

Let D' be a open subset of D such that $\overline{D'} \subset D$. We apply the theorem 8.24 in [21]. The function v_m is the solution of:

$$\mathcal{L} v_m = v_m |v_m|^q = g \in L^\infty$$

Therefore there exists a real $0 < \alpha < 1$ and a constant K such that:

$$\|v_m\|_\alpha \leq K \|v_m\|_{L^\infty}.$$

The constants depend on the ellipticity constant of σ_m, on the bound on b_m and σ_m in L^∞ and on the distance between D' and ∂D. We deduce that u_n belongs to $C^\alpha(D')$ and the norm $\|u_n\|_\alpha$ is bounded w.r.t. $n \in \mathbb{N}$.

Finally u belongs to $C^\alpha(D')$. □

3.5.3 Minimal viscosity solution

We prove the

Theorem 3.7 *If v is an other non-negative viscosity solution of the PDE (3.4) (in the sense of the definition 3.4), and if $v_* \geq g$ on ∂D, then $u \leq v$ on $\overline{D}$.*

Proof. We show that for all $n \in \mathbb{N}^*$, $u_n \leq v_*$. The proof is very similar to the proof of the theorem 3.3 in [10].

We fix n, we assume that there exists $z \in \overline{D}$ such that $\delta = u_n(z) - v_*(z) > 0$ and we will find a contradiction. First of all since $u_n = g \wedge n$ and $v_* \geq g$ on ∂D, we can remark that $z \notin \partial D$.

For $\alpha > 0$ we define the function $m_\alpha : \overline{D} \times \overline{D} \to \overline{\mathbb{R}}$ by:

$$m_\alpha(x,y) = u_n(x) - v_*(y) - \frac{\alpha}{2}|x-y|^2.$$

Since $u_n \leq n$ and $v_* \geq 0$, the supremum of m_α is well defined and since $u_n - v_*$ is upper semicontinuous and $\overline{D}$ is compact, the maximum is achieved at some point (x_α, y_α):

$$M_\alpha = \sup_{\overline{D} \times \overline{D}} m_\alpha(x,y) = u_n(x_\alpha) - v_*(y_\alpha) - \frac{\alpha}{2}|x_\alpha - y_\alpha|^2.$$

For all $\alpha > 0$, $M_\alpha \geq u_n(z) - v_*(z) = \delta > 0$. We will use the definition of viscosity solution to contradict this inequality for large α.

From the lemma 3.1 (or the proposition 3.7) in [10], we know that

$$\begin{cases} (i) & \lim_{\alpha\to+\infty} \alpha|x_\alpha - y_\alpha|^2 = 0 \quad \text{and} \\ (ii) & \lim_{\alpha\to+\infty} M_\alpha = u_n(\hat{x}) - v_*(\hat{x}) = \sup_{\overline{D}} ((u_n(x) - v_*(x))) \\ & \text{whenever } \hat{x} \in \overline{D} \text{ is a limit point of } x_\alpha \text{ as } \alpha \to +\infty. \end{cases}$$

It follows from $u_n \leq v_*$ on ∂D that $(x_\alpha,y_\alpha) \in D \times D$ for α large.

Let F defined on $\overline{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}_d^+(\mathbb{R})$ ($\mathcal{S}_d^+(\mathbb{R})$ is the set of symetric real and positive matrices of size $d \times d$) by:

$$F(x,r,p,X) = -\frac{1}{2}\text{Trace}\,(\sigma\sigma^*(x)X) - b(x).p + r|r|^q.$$

We apply the theorem 3.2 in [10]. There exists X,Y in $\mathcal{S}_d^+(\mathbb{R})$ such that

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{3.30}$$

and since u_n is a subsolution and v_* is a supersolution on D:

$$F(x_\alpha,u_n(x_\alpha),\alpha(x_\alpha - y_\alpha),X) \leq 0 \leq F(y_\alpha,v_*(y_\alpha),\alpha(x_\alpha - y_\alpha),Y).$$

Now we have for all $\alpha > 0$, $\delta \leq u_n(x_\alpha) - v_*(y_\alpha)$, i.e. $u_n(x_\alpha) \geq \delta$. Let $\gamma > 0$ be a constant such that $\gamma < (1+q)\delta^q$. If $r \geq s \geq 0$ and $r \geq \delta$, then $\gamma(r-s) \leq r^{1+q} - s^{1+q}$. Thus:

$$\begin{aligned} \gamma\delta &\leq \gamma m_\alpha(x_\alpha,y_\alpha) \leq \gamma\left(u_n(x_\alpha) - v_*(y_\alpha)\right) \leq u_n^{1+q}(x_\alpha) - v_*^{1+q}(y_\alpha) \\ &= F(x_\alpha,u_n(x_\alpha),\alpha(x_\alpha - y_\alpha),X) - F(x_\alpha,v_*(y_\alpha),\alpha(x_\alpha - y_\alpha),X) \\ &\leq F(x_\alpha,u_n(x_\alpha),\alpha(x_\alpha - y_\alpha),X) - F(y_\alpha,v_*(y_\alpha),\alpha(x_\alpha - y_\alpha),Y) \\ &\quad +F(y_\alpha,v_*(y_\alpha),\alpha(x_\alpha - y_\alpha),Y) - F(x_\alpha,v_*(y_\alpha),\alpha(x_\alpha - y_\alpha),X) \\ &\leq \frac{1}{2}\text{Trace}\,(\sigma\sigma^*(x_\alpha)X) - \frac{1}{2}\text{Trace}\,(\sigma\sigma^*(y_\alpha)Y) + \alpha(b(x_\alpha) - b(y_\alpha)).(x_\alpha - y_\alpha) \\ &\qquad \text{hence, using the inequalities (3.30),} \\ &\leq \frac{3}{2}\alpha\text{Trace}\,((\sigma(x_\alpha) - \sigma(y_\alpha))(\sigma^*(x_\alpha) - \sigma^*(y_\alpha))) + \alpha(b(x_\alpha) - b(y_\alpha)).(x_\alpha - y_\alpha) \end{aligned}$$

Since b and σ satisfy the assumptions (L), we obtain:

$$\gamma\delta \leq \frac{3K}{2}\alpha|x_\alpha - y_\alpha|^2.$$

This leads to a contradiction, so $u_n \leq v_*$ on $\overline{D}$. □

Remark that in this proof, we just need that σ is Lipschitz continuous on D and b is monotone in the sense that:

$$(b(x) - b(y)).(x - y) \leq \mu|x - y|^2.$$

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EDSR avec donnée finale singulière.

Dans cette thèse nous nous intéressons à un type particulier d'équations différentielles stochastiques rétrogrades (EDSR en abrégé) non linéaires. Elles possèdent entre autres l'intérêt de fournir une représentation probabiliste des équations de réaction-diffusion associées, dont le terme de réaction est polynômial.

Il existe une autre interprétation probabiliste des solutions de ces EDP, utilisant les superdiffusions. La théorie des superprocessus, ainsi que de récents travaux analytiques, prouvent que ces équations admettent des solutions, qui « explosent » sur un sous-ensemble du bord. Pour interpréter cette singularité grâce aux EDSR, la condition finale doit prendre des valeurs infinies avec probabilité strictement positive. On parle de condition singulière, car une telle donnée n'est pas intégrable, et on sort du cadre connu de résolution des EDSR.

La notion « classique » de solution d'une EDSR est alors inadaptée à une telle donnée et nous commençons par la redéfinir. Nous construisons ensuite un processus qui satisfait toutes les hypothèses pour être une solution, sauf peut-être au voisinage de l'instant final. Le schéma de résolution est le suivant : estimation *a priori*, troncation pour se ramener à une condition finale bornée puis passage à la limite. Pour un temps final déterministe, comme pour un temps d'arrêt, la difficulté majeure provient de la continuité à l'instant terminal. En effet nous n'avons pas de majoration globale qui permette de contrôler précisément le processus près de l'instant final. Par une méthode de « localisation », nous parvenons à prouver que le processus est continu, et donc qu'il est bien solution.

L'existence établie, nous avons prouvé que notre solution est minimale, mais nous ne savons pas en général s'il y a unicité.

Enfin la résolution de ces EDSR nous a permis de construire une solution de viscosité, pour les EDP paraboliques ou elliptiques associées. Elle peut admettre une singularité au bord et est minimale. Notons que l'uniforme ellipticité de l'équation n'est pas une condition nécessaire pour l'existence, mais cette hypothèse permet d'obtenir la continuité de la solution de viscosité.

Mots clés : *équation différentielle stochastique rétrograde ; condition singulière ; temps d'arrêt ; équation aux dérivées partielles ; solution de viscosité.*

BSDE with singular terminal condition.

Works exposed in this thesis deal with some non-linear backward stochastic differential equations (in short BSDEs). Such equations provide among others a probabilistic representation of solutions of the associated reaction-diffusion equations, with a polynomial reaction-term.

For these equations there already exists a probabilistic version of the solutions, using the theory of superdiffusions. The results on superprocesses and other analytic works show that non-negative solutions can have a non empty "blow-up" set on the boundary. Hence in order to use the BSDEs, the final condition must be equal to infinity with positive probability. The data is called singular, because it is not integrable, and we cannot apply some standard results about BSDEs.

The "classical" definition of a solution is not adapted to a non integrable random variable and we must give a weaker definition. Then we build a process which satisfies all assumptions in order to be a solution, except maybe on the neighbourhood of the final time. The scheme of the construction is the following: *a priori* estimate, truncation to have a bounded final condition, and passage to the limit. The main difficulty is the proof of the continuity at the final time. We have no fit global bound on the solution. Therefore we "localize" the problem and we prove that the constructed process is continuous and thus that it is a solution.

We show that our process is the minimal solution of the BSDE, but we do not know in general if uniqueness holds.

Finally this minimal solution provides a viscosity solution of the associated parabolic and elliptic PDE. This solution may have a "blow-up" set and is minimal. Remark that uniform ellipticity of the PDE is not always necessary, but with this assumption we prove that the viscosity solution is continuous.

Keywords: *backward stochastic differential equation ; singular condition ; stopping time ; partial differential equation ; viscosity solution.*