Optimal switching and reflected BSDE.

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Outline

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Example: electricity production.

- Electricity price (spot, day-ahead, etc. price) depends on many external factors: consumer demand, weather, oil prices and so on.
  - Barlow (02) (inspired by Föllmer & Schweizer (93)): non-linear Ornstein-Uhlenbeck process

$$P_t = \begin{cases} f_\alpha(S_t), & \text{if } 1 + \alpha S_t > \varepsilon_0, \\ \varepsilon_0^{1/\alpha}, & \text{if } 1 + \alpha S_t \leq \varepsilon_0, \end{cases}$$

    where $dS_t = -\lambda(S_t - a)dt + \sigma dW_t$ and $f_\alpha(x) = (1 + \alpha x)^{1/\alpha}$.

- Electricity can not be stored and is produced only when its profitability is satisfactory. Otherwise the power station is *closed* up to time when the profitability is coming back.

Example: electricity production.

- A power station has several turbines and for each turbine, there are several operating modes besides running at full capacity and keeping it off-line.
- Gas turbine: to run the plant, you must buy natural gas, convert it into electricity and sell the output on the market. When
  - P the electricity price,
  - G the gas price process,
  - K_j the operating costs of the mode j,
  - HR_j the heat rate,

  the rate of payoff is given for example by

$$P_t - HR_j \times G_t - K_j.$$

Example: electricity production.

- Switching from a regime to another is not free and generated switching costs.

Aim: find a sequence of stopping times where one should make startup/shutdown decisions and a sequence of successive modes, in order to maximize the profitability of the station.

Reversible investment problem.

Problem of real options type: investment decisions

- pioneering article: Brennan & Schwartz (85),
- Dixit & Pindyck: *Investment under uncertainty* (94). In particular chapter 7.

Such problems are met in

- management of oil tankers, oil fields,....
- management of raw material mines like copper, gold, steel,...
- dividend policy,
- investment strategy.

Reversible investment problem.

Problem of real options type: investment decisions

- pioneering article: Brennan & Schwartz (85),
- Dixit & Pindyck: *Investment under uncertainty* (94).
- Hamadène and Jeanblanc (04): only two regimes problem called *starting and stopping*.
- Carmona and Ludkovski (06): several regimes; no optimal strategy.
- Hu and Tang (07): RBSDE and optimal switching.

Setting of the problem.

Throughout this talk:

- T is a positive real number;
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space;
- $B = (B_t)_{t \leq T}$: d-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$;
- $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$: completed filtration of the Brownian motion;

Notations:

- $\mathcal{M}^{2,1}$: $\mathbb{F}$-progressively measurable and $\mathbb{R}$-valued processes which belong to $L^2(\Omega \times [0, T], d\mathbb{P} \otimes dt)$;
- $\mathcal{S}^{2,1}$: $\mathbb{F}$-progressively measurable, continuous, $\mathbb{R}$-valued processes $w = (w_t)_{t \leq T}$ such that $\mathbb{E} \sup_{t \leq T} |w_t|^2 < \infty$;

Setting of the problem.

The number of regimes is equal to $M \geq 2$. Changing an output level is costly, requiring various overhead costs.

Definition

For $(i,j) \in \{1,\ldots,M\}^2$, $C_{i,j} : \Omega \times [0,T] \to \mathbb{R} \in \mathcal{S}^{2,1}$*: the switching costs from state* i *to state* j*.*

Assumptions:

- $\mathbb{P}$-a.s. $C_{i,j}$ is continuous,
- there exists a constant α s.t. for every ω and t, $C_{i,j}(\omega,t) \geq \alpha > 0$.

Setting of the problem.

Definition

Rate of payoff in regime $i \in \{1, \dots, M\}$

$$\psi^i : \Omega \times [0, T] \to \mathbb{R} \in \mathcal{M}^{2,1}.$$

In Carmona & Ludkovski, $dX_t = b(X_t)dt + \sigma(X_t)dB_t$, $\psi^i = \psi^i(t, X_t)$, $|\psi^i(t, x)| \leq C(1 + |x|)$, $C_{i,j} = C_{i,j}(t, X_t)$.

In Hu & Tang, $C_{i,j}$ are constant, $\psi^i = \psi^i(t, X_t)$, but X depends also on the strategy.

We ignore time delay effect of having to gradually "ramp-up" and "ramp-down" the turbines.

Strategies and control problem

Operating strategies: double sequences $u = (\delta, \xi)$:

- $\delta = (\tau_k)_{k \geq 1}$ where $0 \leq \tau_{k-1} \leq \tau_k \leq T$: stopping times (switching times),
- $\xi = (\xi_k)_{k \geq 1}$ where ξ_k taking values in $\{1, \ldots, M\}$: successive modes chosen by the strategy u.

Definition (Admissible)

A strategy is called admissible if $\mathbb{P}$-a.s. $\lim_{k \to +\infty} \tau_k = T$. The set of admissible strategies is denoted by $\mathcal{U}(0)$.

Thus u is an $\mathbb{F}$-adapted rcll piecewise-constant process where u_s denotes the operating mode at time s.

STRATEGIES AND CONTROL PROBLEM

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DEFINITION (GLOBAL PROFIT)

Knowing that $u_0 = i$, the average global profit for such a control u is

$$J(i,u) = \mathbb{E}\left[\int_0^T \psi_s^{u_s} ds - \sum_{\tau_k < T} C_{u_{\tau_k^-}, u_{\tau_k}}(\tau_k)\right].$$

Strategies and control problem

Aim: find an optimal strategy, i.e. find u^* such that for every admissible strategy $u \in \mathcal{U}(0)$:

$$J^*(i) = J(i, u^*) \geq J(i, u).$$

Definition (Finite strategy)

The strategy u is finite if during the time interval $[0, T]$ it allows to the manager to make only a finite number of decisions, i.e. $\mathbb{P}(\tau_n < T, \ \forall n \geq 1) = 0$.

Propostion

The supremum over admissible strategies and finite strategies are the same.

The Snell envelope.

Definition

For any stopping time $\tau \in [0, T]$, $\mathcal{T}_\tau$: set of all stopping times θ s.t. a.s. $\tau \leq \theta \leq T$.

Let $X = (X_t)_{t \leq T}$ be an $\mathbb{F}$-adapted $\mathbb{R}$-valued rcll process which belongs to the class (D), i.e. the set of random variables $\{X_\tau, \ \tau \in \mathcal{T}_0\}$ is uniformly integrable.

Proposition and definition

There exists a unique $\mathbb{F}$-adapted rcll $\mathbb{R}$-valued process $Z = (Z_t)_{t \leq T}$, called the Snell envelope of U, such that Z is the smallest super-martingale which dominates X.

The Snell envelope.

Z can be expressed as: for any $\mathbb{F}$-stopping time γ

$$Z_\gamma = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\gamma} \mathbb{E}\left[X_\tau \middle| \mathcal{F}_\gamma\right], \text{ (and } Z_T = U_T).$$

Some properties of the Snell envelope:

- there exist $M = (M_t)_{t \leq T}$, $A = (A_t)_{t \leq T}$ and $B = (B_t)_{t \leq T}$ with M and A continuous, M martingale, A and B non-decreasing, s.t.

$$\forall t \leq T, \; Z_t = M_t - A_t - B_t.$$

The Snell envelope.

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- $\{\Delta_t B = Z_{t-} - Z_t > 0\} \subset \{\Delta_t X < 0\} \cap \{Z_{t-} = X_{t-}\}$.

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Some properties of the Snell envelope:

- there exist $M = (M_t)_{t \leq T}$, $A = (A_t)_{t \leq T}$ and $B = (B_t)_{t \leq T}$ s.t.
$$\forall t \leq T, \; Z_t = M_t - A_t - B_t.$$
- $\{\Delta_t B = Z_{t-} - Z_t > 0\} \subset \{\Delta_t X < 0\} \cap \{Z_{t-} = X_{t-}\}$.
- If $B \equiv 0$, then for any $\mathbb{F}$-stopping time γ, $\tau^*_\gamma = \inf\{\gamma \leq t \leq T, \; Z_t = X_t\} \wedge T$ is optimal after γ, i.e.
$$Z_\gamma = \mathbb{E}\left[X_{\tau^*} | \mathcal{F}_\gamma\right] = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_\gamma} \mathbb{E}\left[X_\tau | \mathcal{F}_\gamma\right].$$

Property of the Snell envelope.

Let X^n be a sequence of rcll processes s.t. $\mathbb{P}$-a.s.: $\forall t \in [0, T],\ X^n_t \leq X^{n+1}_t$. Define X by $X_t = \lim_{n \to +\infty} X^n_t$. We assume that X belongs to the class (D).

- Z^n the Snell envelope of X^n:

$$\mathbb{P}\text{-a.s.},\ \forall t \in [0, T],\ Z^n_t \leq Z^{n+1}_t;$$

- if Z is the increasing limit of Z^n, then Z is the Snell envelope of X, i.e.

$$\forall t \in [0, T],\ Z_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[X_\tau \middle| \mathcal{F}_t\right].$$

Outline

Optimal profit.

Theorem

There exist M continuous processes $Y^i = (Y^i_t)_{t \leq T}$, $i = 1, \ldots, M$, such that for all $i = 1, \ldots, M$ and for every $t \leq T$:

$$Y^i_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi^i_s ds + \max_{k \neq i} \left(Y^k_\tau - C_{i,k} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

Y^i_t will be the optimal profit, knowing that at time t the power station is in regime i. In particular, $Y^i_0 = J^*(i)$. Moreover Y^i is a continuous Snell envelope, therefore we will be able to find an optimal strategy.

Definition of a sequence of processes $(Y^{i,n})_{n\in\mathbb{N}}$.

First consider a restricted situation: fixed upper bound on the total number of switches allowed.

$$\mathcal{U}_n(t) = \{(\xi,\delta) \in \mathcal{U}(t),\ \tau_l = T \text{ for } l \geq n+1\}$$

the set of all admissible strategies on $[t, T]$, with at most n switches.

Definition

$$Y_t^{i,n} = \underset{u\in\mathcal{U}_n(t),\ u_t=i}{\text{ess sup}}\ \mathbb{E}\left[\int_t^T \psi_s^{u_s} ds - \sum_{l\geq 1,\ t\leq\tau_l\leq T} C_{u_{\tau_l^-},u_{\tau_l}} \middle| \mathcal{F}_t\right].$$

$Y^{i,n}$ is the value function when we optimize over $\mathcal{U}_n(t)$.

Properties of the processes $Y^{i,n}$.

Proposition

The processes $Y^{i,n}$ are continuous, $Y^{i,n}_T = 0$, there exists a constant C s.t.

$$\mathbb{E} \sup_{t\in[0,T]} \left| Y^{i,n}_t \right|^2 \leq C \max_{k\in\{1,\dots,M\}} \mathbb{E} \int_0^T \left| \psi^k_s \right|^2 ds.$$

and

$$Y^{i,n+1}_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E} \left[\int_t^\tau \psi^i_s ds + \max_{k\neq i} \left(Y^{k,n}_\tau - C_{i,k} \right) \mathbf{1}_{\tau<T} \middle| \mathcal{F}_t \right].$$

PROOF OF THE *a priori* ESTIMATE.

First remark: $Y_t^{i,0} = \mathbb{E}\left[\int_t^T \psi_s^i ds \middle| \mathcal{F}_t\right], \forall i \in \{1, \dots, M\}$. Hence

$$\mathbb{E} \sup_{t\in[0,T]} \left|Y_t^{i,0}\right|^2 \leq C\mathbb{E} \int_0^T \left|\psi_s^i\right|^2 ds.$$

PROOF OF THE *a priori* ESTIMATE.

First remark: $Y_t^{i,0} = \mathbb{E}\left[\int_t^T \psi_s^i ds \middle| \mathcal{F}_t\right], \forall i \in \{1, \dots, M\}$. For $n \geq 1$,

$$\begin{aligned}
Y_t^{i,n} &= \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left[\int_t^T \psi_s^{u_s} ds - \sum_{l \geq 1,\ t \leq \tau_l \leq T} C(u_{\tau_l^-}, u_{\tau_l}) \middle| \mathcal{F}_t\right] \\
&\geq \mathbb{E}\left[\int_t^T \psi_s^i ds \middle| \mathcal{F}_t\right] \text{ (no switching) and} \\
&\leq \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left[\int_t^T \psi_s^{u_s} ds \middle| \mathcal{F}_t\right] \\
&\leq \max_{k \in \{1,\dots,M\}} \mathbb{E}\left[\int_t^T \left|\psi_s^k ds\right| \middle| \mathcal{F}_t\right]
\end{aligned}$$

Idea of the proof.

For $i \in \{1, \ldots, M\}$, for every $t \in [0, T]$:

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left[\int_t^T \psi_s^{u_s} ds - \sum_{l \geq 1,\ t \leq \tau_l \leq T} C(u_{\tau_l^-}, u_{\tau_l}) \middle| \mathcal{F}_t\right]$$

$$= \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left\{\int_t^{\tau_1} \psi_s^i ds + \left[\left(\int_{\tau_1}^T \psi_s^{u_s} ds - \sum_{l \geq 2,\ \tau_1 \leq \tau_l \leq T} C(u_{\tau_l^-}, u_{\tau_l})\right) - C(i, u_{\tau_1})\right] \mathbf{1}_{\tau_1 < T} \middle| \mathcal{F}_t\right\}$$

$$\leq \operatorname*{ess\,sup}_{\tau_1 \geq t} \mathbb{E}\left[\int_t^{\tau_1} \psi_s^i ds + \max_{k \neq i}\left(Y_{\tau_1}^{k,n-1} - C_{i,k}\right) \mathbf{1}_{\tau_1 < T} \middle| \mathcal{F}_t\right].$$

Idea of the proof.

For $i \in \{1, \dots, M\}$, for every $t \in [0, T]$:

$$
\begin{aligned}
Y_t^{i,n} &= \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left[\int_t^T \psi_s^{u_s} ds - \sum_{l \geq 1,\ t \leq \tau_l \leq T} C(u_{\tau_l^-}, u_{\tau_l}) \middle| \mathcal{F}_t \right] \\
&= \operatorname*{ess\,sup}_{u \in \mathcal{U}_n(t),\ u_t = i} \mathbb{E}\left\{ \int_t^{\tau_1} \psi_s^i ds + \left[\left(\int_{\tau_1}^T \psi_s^{u_s} ds \right.\right.\right. \\
&\quad \left.\left.\left. - \sum_{l \geq 2,\ \tau_1 \leq \tau_l \leq T} C(u_{\tau_l^-}, u_{\tau_l}) \right) - C(i, u_{\tau_1}) \right] \mathbf{1}_{\tau_1 < T} \middle| \mathcal{F}_t \right\} \\
&\leq \operatorname*{ess\,sup}_{\tau_1 \geq t} \mathbb{E}\left[\int_t^{\tau_1} \psi_s^i ds + \max_{k \neq i} \left(Y_{\tau_1}^{k,n-1} - C_{i,k} \right) \mathbf{1}_{\tau_1 < T} \middle| \mathcal{F}_t \right].
\end{aligned}
$$

Idea of the proof.

Now

$$\overline{Y_t^{i,n}} = \operatorname*{ess\,sup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k \neq i} \left(Y_\tau^{k,n-1} - C_{i,k} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

Thus $Y^{i,n} \leq \overline{Y^{i,n}}$ and $\overline{Y^{i,n}}$ is the Snell envelope of the process

$$Z_r^i = \int_t^r \psi_s^i ds + \max_{k \neq i} \left(Y_r^{k,n-1} - C_{i,k} \right) \mathbf{1}_{r < T}.$$

Z^i is continuous, except maybe at time T. But $Z_T^i - Z_{T-}^i = \min_{k \neq i} C_{i,k} > 0$. Since the jump is non negative, $\overline{Y^{i,n}}$ is continuous, and the stopping time $\tau^* = \inf_{s \geq t} \left\{ \overline{Y^{i,n}}_s = Z_s^i \right\}$ is optimal.

Idea of the proof.

Therefore

$$\overline{Y^{i,n}_{\tau^*}} = \mathbb{E}\left[\int_t^{\tau^*} \psi^i_s ds + \max_{k\neq i}\left(Y^{k,n-1}_{\tau^*} - C_{i,k}\right)\mathbf{1}_{\tau^*<T}\middle|\mathcal{F}_t\right].$$

Choosing ξ s.t. $\xi = \underset{k\neq i}{\mathrm{argmax}}\left(Y^{k,n-1}_{\tau^*} - C_{i,k}\right)$, we have

$$\overline{Y^{i,n}_{\tau^*}} = \mathbb{E}\left[\int_t^{\tau^*} \psi^i_s ds + \left(Y^{\xi,n-1}_{\tau^*} - C_{i,\xi}\right)\mathbf{1}_{\tau^*<T}\middle|\mathcal{F}_t\right].$$

Finally recursively $\overline{Y^{i,n}} = Y^{i,n}$.

Hence $Y^{i,n}$ is continuous and $Y^{i,n}_T = 0$.

Construction of Y^i.

Since $\mathcal{U}_n(t) \subset \mathcal{U}_{n+1}(t)$, $\mathbb{P}$-a.s. for every $t \in [0,T]$, $Y_t^{i,n} \leq Y_t^{i,n+1}$. Thus

$$Y_t^i = \lim_{n \to +\infty} Y_t^{i,n}.$$

Recall that for every t

$$Y_t^{i,n+1} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k \neq i} \left(Y_\tau^{k,n} - C_{i,k} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

Control of $Y^{i,n}$ Control: $Y^i \in \mathcal{S}^{2,1}$.

Snell envelope

Passing to the limit:

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k \neq i} \left(Y_\tau^{k} - C_{i,k} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

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Continuity of the optimal profit.

Recall

$$\begin{aligned} Y_t^i &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^i ds + \max_{k \neq i} \left(Y_\tau^k - C_{i,k} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right] \\ &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[Z_\tau^i \middle| \mathcal{F}_t \right] - \int_0^t \psi_s^i ds, \end{aligned}$$

with

$$Z_t^i = \int_0^t \psi_s^i ds + \max_{k \neq i} \left(Y_t^k - C_{i,k} \right) \mathbf{1}_{t < T}.$$

Therefore Y^i is the Snell envelope of Z^i minus a continuous process.

Theorem

The processes Y^i are continuous.

PROOF WHEN $M = 2$.

If $M = 2$, we have for every $t \in [0, T]$

$$Y_t^1 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^1 ds + (Y_\tau^2 - C_{1,2}) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

$$Y_t^2 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^2 ds + \left(Y_\tau^1 - C_{2,1} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

Assume that for $t < T$, $\Delta Y_t^1 \neq 0$. Since Y^1 is a Snell envelope, then Y^2 has a jump at time t. Jump in the Snell Envelope

Proof when $M = 2$.

If $M = 2$, we have for every $t \in [0, T]$

$$Y_t^1 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^1 ds + (Y_\tau^2 - C_{1,2}) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

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Assume that for $t < T$, $\Delta Y_t^1 \neq 0$. Since Y^1 is a Snell envelope, then Y^2 has a jump at time t. Jump in the Snell Envelope

Moreover $Y_{t-}^1 = Y_{t-}^2 - C_{1,2}$ and $Y_{t-}^2 = Y_{t-}^1 - C_{2,1}$. Hence $C_{2,1} + C_{1,2} = 0$, which is impossible. Y^1 and Y^2 have no jump before T.

Proof when $M = 3$.

For every $t \in [0, T]$

$$Y_t^1 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^1 ds + \max(Y_\tau^2 - C_{1,2}, Y_\tau^3 - C_{1,3})\mathbf{1}_{\tau<T} \middle| \mathcal{F}_t\right].$$

$$Y_t^2 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^2 ds + \max(Y_\tau^1 - C_{2,1}, Y_\tau^3 - C_{2,3})\mathbf{1}_{\tau<T} \middle| \mathcal{F}_t\right].$$

$$Y_t^3 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^3 ds + \max(Y_\tau^1 - C_{3,1}, Y_\tau^2 - C_{3,2})\mathbf{1}_{\tau<T} \middle| \mathcal{F}_t\right].$$

OPTIMAL STRATEGY.

If at time 0, the station is in its operating mode i, then an optimal strategy is given by

$$\begin{aligned} \tau_1^* &= \inf\{s \geq 0,\ Y_s^i = \max_{k \neq i}(Y_s^k - C_{i,k})\} \wedge T \\ \xi_1^* &= \operatorname*{argmax}_{k \neq i} (Y_{\tau_1^*}^k - C_{i,\tau_1^*}). \end{aligned}$$

Indeed

$$Y_t^i + \int_0^t \psi_s^i ds = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_0^\tau \psi_s^i ds + \max_{k \neq i}(Y_\tau^k - C_{i,k})\mathbf{1}_{\tau < T} \middle| \mathcal{F}_t\right].$$

◂ Optimal stopping time

Optimal strategy.

If at time 0, the station is in its operating mode i, then an optimal strategy is given by

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Indeed

$$Y_t^i + \int_0^t \psi_s^i ds = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_0^\tau \psi_s^i ds + \max_{k \neq i}(Y_\tau^k - C_{i,k})\mathbf{1}_{\tau < T} \middle| \mathcal{F}_t\right].$$

Optimal stopping time Thus

$$Y_0^i = \mathbb{E}\left[\int_0^{\tau_1^*} \psi_s^i ds + (Y_{\tau_1^*}^{\xi_1^*} - C_{i,\xi_1^*})\mathbf{1}_{\tau_1^* < T}\right].$$

OPTIMAL STRATEGY.

And after this first switch

$$\begin{aligned}\tau_2^* &= \inf\{s \geq \tau_1^*,\ Y_s^{\xi_1^*} = \max_{k\neq\xi_1^*}(Y_s^k - C_{\xi_1^*,k})\} \wedge T\\ \xi_2^* &= \operatorname*{argmax}_{k\neq\xi_1^*}(Y_{\tau_2^*}^k - C_{\xi_1^*,k}).\end{aligned}$$

$$\begin{aligned}Y_{\tau_1^*}^{\xi_1^*} &= \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_{\tau_1^*}} \mathbb{E}\left[\int_{\tau_1^*}^{\tau} \psi_s^{\xi_1^*} ds + \max_{k\neq\xi_1^*}(Y_\tau^k - C_{\xi_1^*,k})\mathbf{1}_{\tau<T}\middle|\mathcal{F}_{\tau_1^*}\right]\\ &= \mathbb{E}\left[\int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds + (Y_{\tau_2^*}^{\xi_2^*} - C_{\xi_1^*,\xi_2^*})\mathbf{1}_{\tau_2^*<T}\middle|\mathcal{F}_{\tau_1^*}\right].\end{aligned}$$

Optimal strategy.

And after this first switch

$$\tau_2^* = \inf\{s \geq \tau_1^*, \ Y_s^{\xi_1^*} = \max_{k \neq \xi_1^*}(Y_s^k - C_{\xi_1^*,k})\} \wedge T$$

$$\xi_2^* = \operatorname*{argmax}_{k \neq \xi_1^*} (Y_{\tau_2^*}^k - C_{\xi_1^*,k}).$$

$$Y_{\tau_1^*}^{\xi_1^*} = \mathbb{E}\left[\int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds + (Y_{\tau_2^*}^{\xi_2^*} - C_{\xi_1^*,\xi_2^*})\mathbf{1}_{\tau_2^* < T} \middle| \mathcal{F}_{\tau_1^*}\right].$$

$$\begin{aligned} Y_0^i &= \mathbb{E}\left[\int_0^{\tau_1^*} \psi_s^i ds + (Y_{\tau_1^*}^{\xi_1^*} - C_{i,\xi_1^*})\mathbf{1}_{\tau_1^* < T}\right] \\ &= \mathbb{E}\left[\int_0^{\tau_1^*} \psi_s^i ds + \int_{\tau_1^*}^{\tau_2^*} \psi_s^{\xi_1^*} ds + (Y_{\tau_2^*}^{\xi_2^*} - C_{\xi_1^*,\xi_2^*} - C_{i,\xi_1^*})\mathbf{1}_{\tau_2^* < T}\right] \end{aligned}$$

OPTIMAL STRATEGY.

Recursively we obtain a sequence of stopping times τ_k^* and of modes ξ_k^*. And

$$Y_0^i = \mathbb{E}\left[\int_0^{\tau_n^*} \psi_s^{u_s} ds - \sum_{1\le k\le n} \left(C_{u_{(\tau_k^*)-},u_{\tau_k^*}} \mathbf{1}_{\tau_k^* < T}\right) + Y_{\tau_n^*}^{\xi_n^*} \mathbf{1}_{\tau_n^* < T}\right]$$

The strategy δ^* is finite: $\mathbb{P}(A) = \mathbb{P}(\{\omega;\ \tau_n^* < T, \forall n\}) = 0$.

OPTIMAL STRATEGY.

Recursively we obtain a sequence of stopping times τ_k^* and of modes ξ_k^*. And

$$Y_0^i = \mathbb{E}\left[\int_0^{\tau_n^*} \psi_s^{u_s} ds - \sum_{1 \le k \le n} \left(C_{u_{(\tau_k^*)-}, u_{\tau_k^*}} \mathbf{1}_{\tau_k^* < T}\right) + Y_{\tau_n^*}^{\xi_n^*} \mathbf{1}_{\tau_n^* < T}\right]$$

The strategy δ^* is finite: $\mathbb{P}(A) = \mathbb{P}(\{\omega;\ \tau_n^* < T, \forall n\}) = 0$.

Y_0^i is the optimal profit. Indeed τ_1^* is optimal, hence

$$\begin{aligned} Y_0^i &\ge \mathbb{E}\left[\int_0^{\tau_1} \psi_s^i ds + \max_{k \neq i}(Y_i^k - C_{i,k}) \mathbf{1}_{\tau_1 < T}\right] \\ &\ge \mathbb{E}\left[\int_0^{\tau_1} \psi_s^i ds + Y_i^{\xi_1} - C_{i,\xi_1} \mathbf{1}_{\tau_1 < T}\right]. \end{aligned}$$

Outline

ONE BARRIER.

Data:

- $\xi \in L^2(\Omega)$, $\mathcal{F}_T$-mesurable and $\mathbb{R}$-valued;
- $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ GENERATOR s.t.
  - for every (y, z), $f(., y, z) \in \mathcal{M}^{2,1}$,
  - there exists K s.t. $|f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + |z - z'|)$;
- $L = (L_t)_{t \in [0,T]}$ a $\mathbb{R}$-valued $\mathbb{F}$-adapted continuous process s.t. $L_T \leq \xi$ a.s. and $L \in \mathcal{S}^{2,1}$.

One barrier.

Definition

A process (Y, Z, K) is a solution of the RBSDE if:

1. *the process $(Y_t, Z_t, K_t)_{0 \le t \le T}$ is $\mathbb{F}$-adapted;*
2. $(Y, Z, K) \in \mathcal{S}^{2,1} \times \mathcal{M}^{2,1} \times \mathcal{S}^{2,1}$;
3. $Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + K_T + K_r - \int_t^T Z_r dB_r,$
   $\forall t \in [0, T]$, $\mathbb{P}$*-a.s;*
4. $\forall t \in [0, T]$, $Y_t \ge L_t$;
5. *K is continuous and non decreasing, $K_0 = 0$ and* $\int_0^T (Y_r - L_r)dK_r.$

Property of a solution.

Proposition

If (Y, Z, K) is a solution,

- *A priori estimate:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2 + K_T^2 + \int_0^T |Z_s|^2 ds\right] \leq C\mathbb{E}\left[\xi^2 + \int_0^T |f(t,0,0)|^2 dt + \sup_{t\in[0,T]} |L_t|^2\right];$$

- *and for every $t \in [0, T]$*

$$Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[\int_t^\tau f(s, Y_s, Z_s)ds + L_\tau \mathbf{1}_{\tau<T} + \xi\mathbf{1}_{\tau=T}\middle|\mathcal{F}_t\right].$$

Existence and uniqueness.

- El Karoui, Kapoudjian, Pardoux, Peng, Quenez (1997):

Theorem

Under the above assumptions, there exists a unique solution (Y, Z, K) of the reflected BSDE.

Idea of the proof: when f does not depend of y and z:

$$
\begin{aligned}
Y_t + \int_0^t f(s)ds &= \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_0^\tau f(s)ds + L_\tau \mathbf{1}_{\tau < T} + \xi \mathbf{1}_{\tau = T} \middle| \mathcal{F}_t \right] \\
& \qquad \text{(Doob-Meyer decomposition)} \\
&= M_t - K_t = \int_0^t Z_s dB_s - K_t.
\end{aligned}
$$

+ Picard iteration.

Application to American options.

An American put option allows to choose the exercise time at any time within the horizon:

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-\tau)}(k - P_\tau)^+ \middle| \mathcal{F}_t \right].$$

The reflected BSDE is described by

$$Y_t = (k - P_T)^+ + \int_t^T (rY_s + (\mu - r)Z_s)ds - \int_t^T \sigma Z_s dB_s + K_T - K_t,$$

subject to the constraints

- $Y_t \geq (k - P_t)^+ = L_t$,
- $\int_0^T (Y_t - L_t)dK_t = 0.$

Two barriers.

- Cvitanic, Karatzas (1996): two barriers: another continuous process $U = (U_t)_{t\in[0,T]}$ s.t. $U \in \mathcal{S}^{2,1}$ with $\xi \leq U_T$ a.s.
  - Mokobodsky's condition: location of the difference of non-negative super-martingales between S and U.
  - Solution $(Y, Z, K^{\pm})$ s.t.

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s;$$

    with
    - $L_t \leq Y_t \leq U_t$, $t \in [0, T]$;
    - K^+ and K^- are non-decreasing continuous processes s.t. $\int_0^T (Y_t - L_t)dK_t^+ = 0$ and $\int_0^T (U_t - Y_t)dK_t^- = 0$.
- Hamadène & Lepeltier (2000).
- Hamadène & Hassani (2005 and 2006).
- ...

Numerical scheme and PDE.

With the penalization method (approximation by a solution (Y^n, Z^n) of a "Lipschitz" BSDE):

- numerical scheme for reflected BSDE:
  - Bally, Pagès (2003) (Markovian case);
  - Bouchard, Touzi (2004);
  - Gobet, Lemor, Warin (2006).

- link with (viscosity) solution of the related obstacle problem for non linear parabolic PDE.
  - El Karoui, Kapoudjian, Pardoux, Peng, Quenez (1997), etc.

Optimal profit for $M = 2$.

There exists $(Y, Z, K^{\pm})$ s.t. for every $s \in [0, T]$:

- $Y_t = \int_t^T (\psi_s^1 - \psi_s^2) ds + (K_T^+ - K_t^+) - (K_T^- - K_t^-) - \int_t^T Z_s dB_s,$
- $L_t = -C_{1,2} \leq Y_t \leq C_{2,1} = U_t,$
  $\int_0^T (Y_s + C_{1,2}) dK_s^+ = \int_0^T (C_{2,1} - Y_s) dK_s^- = 0.$

Recall: if $M = 2$, for every $t \in [0, T]$

$$Y_t^1 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^1 ds + (Y_\tau^2 - C_{1,2}) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

$$Y_t^2 = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\tau \psi_s^2 ds + \left(Y_\tau^1 - C_{2,1} \right) \mathbf{1}_{\tau < T} \middle| \mathcal{F}_t \right].$$

Optimal profit for $M = 2$.

Proposition

For $t \leq T$,

$$Y_t^1 = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \psi_s^1 ds + K_T^+ - K_t^+\right].$$

$$Y_t^2 = \mathbb{E}^{\mathcal{F}_t}\left[\int_t^T \psi_s^2 ds + K_T^- - K_t^-\right].$$

Hence $Y_t = Y_t^1 - Y_t^2$ and the stopping times of the optimal strategy are the ones where the process Y reaches successively the barriers $-C_{1,2}$ and $C_{2,1}$.

Numerical examples.

With the penalization method (approximation by a solution (Y^n, Z^n) of a "Lipschitz" BSDE), numerical schemes for reflected BSDE:

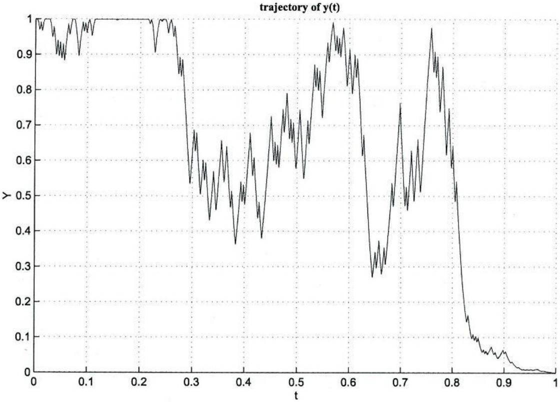
- Bally, Pagès (2003) (Markovian case);
- Bouchard, Touzi (2004);
- Gobet, Lemor, Warin (2006).

Moreover if $C_{1,2}$ and $C_{2,1}$ are constant, the rate of convergence of (Y^n, Z^n) is known (of order $1/n$).

In the following, X is a geometric Brownian motion with parameters μ and σ, and ψ^1 and ψ^2 are functions of x only.

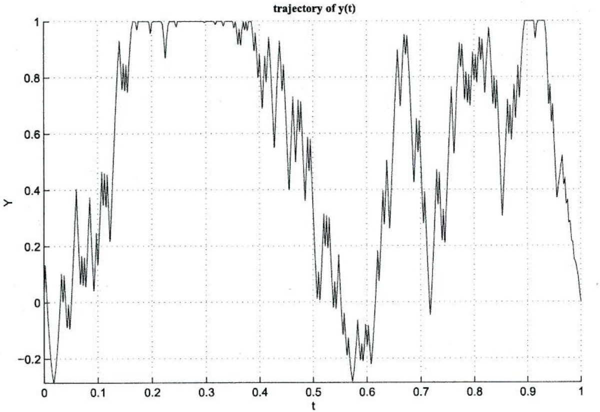
Numerical examples.

$X_0 = 1$, $\mu = 1$, $\sigma = -3$, $C_{2,1} = 1$, $C_{1,2} = 0.5$,
$(\psi^1 - \psi^2)(x) = 0.1x \geq 0$



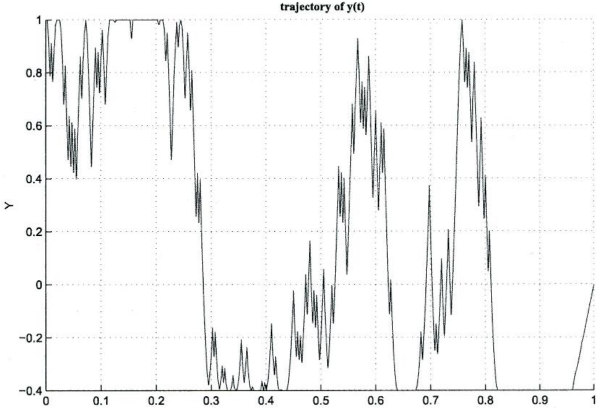
Numerical examples.

$X_0 = 1$, $\mu = 1$, $\sigma = 2$, $C_{2,1} = 1$, $C_{1,2} = 0.3$,
$(\psi^1 - \psi^2)(x) = 0.1x - 6$



Numerical examples.

$X_0 = 1$, $\mu = 0.5$, $\sigma = -3$, $C_{2,1} = 1$, $C_{1,2} = 0.4$,
$(\psi^1 - \psi^2)(x) = 0.7x - 11$



When $M > 2$...

Here for every $i \in \{1, \ldots, M\}$

$$Y_t^i = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}^{\mathcal{F}_t}\left[\int_t^\tau \psi_s^i ds + \max_{k \neq i}(Y_\tau^k - C_{i,k})\mathbf{1}_{\tau < T}\right].$$

We can re-formulated as

- $Y_t^i = \int_t^T \psi_s^i ds + A_T^i - A_t^i - \int_t^T Z_s^i dB_s;$
- $Y_t^i \geq \max_{k \neq i}(Y_t^k - C_{i,k});$
- $\int_t^T (Y_t^i - \max_{k \neq i}(Y_t^k - C_{i,k}))dA_t^i = 0.$

We proved that this system of RBSDE has a unique solution.

Link between our system of RBSDEs and PDE.

For $(t, x) \in [0, T] \times \mathbb{R}^k$, let $X^{t,x}$ be the solution of

$$dX_s^{t,x} = b(s, X_s^{t,x})ds + \sigma(s, X_s^{t,x})dB_s, \ t \leq s \leq T,$$

with $X_s^{t,x} = x$ for $s \leq t$. Its infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma.\sigma^*)_{ij}(t, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial}{\partial x_i}.$$

Assumption [H].

(H1) ψ_i, $i = 1, \ldots, M$, are continuous and

$$|\psi_i(t, x)| \leq C(1 + |x|^\delta), \quad (t, x) \in [0, T] \times \mathbb{R}^k.$$

(H2) For any $1 \leq i, j \leq M$, $C_{i,j}$ are deterministic functions of t.

Link between our system of RBSDEs and PDE.

Let now $(Y_s^{1,t,x}, \dots, Y_s^{M,t,x})_{0 \leq s \leq T}$ be the vector of optimal profits associated with $(\psi_i(s, X_s^{t,x}))_{0 \leq s \leq T}$ and $C_{i,j}(t)$.

Theorem

There exist M deterministic functions $v^1(t,x), \dots, v^M(t,x)$ defined on $[0,T] \times \mathbb{R}^k$ and $\mathbb{R}$-valued s.t.:

(i) $v^1, \dots, v^M$ are continuous in (t,x), are of polynomial growth and satisfy, for each $t \in [0,T]$, and for every $s \in [t,T]$, $Y_s^{i,t,x} = v^i(s, X_s^{t,x})$, for every $1 \leq i \leq M$.

Link between our system of RBSDEs and PDE.

Let now $(Y_s^{1,t,x}, \dots, Y_s^{M,t,x})_{0 \le s \le T}$ be the vector of optimal profits associated with $(\psi_i(s, X_s^{t,x}))_{0 \le s \le T}$ and $C_{i,j}(t)$.

Theorem

There exist M deterministic functions $v^1(t,x), \dots, v^M(t,x)$ defined on $[0,T] \times \mathbb{R}^k$ and $\mathbb{R}$-valued s.t.:

(*i*) $Y_s^{i,t,x} = v^i(s, X_s^{t,x})$, for every $1 \le i \le M$.

(*ii*) The vector of functions $(v^1, \dots, v^M)$ is a viscosity solution for the system of variational inequalities:

$$\begin{cases} \min\limits_{i=1,\dots,M} \{ v^i(t,x) - \max\limits_{j \neq i}(-C_{i,j}(t) + v^j(t,x)), \\ \qquad\qquad\qquad\qquad -\mathcal{L}v^i(t,x) - \psi_i(t,x) \} = 0, \\ v^i(T,x) = 0, \quad 1 \le i \le M, \end{cases}$$

NUMERICAL ASPECTS.

For any $n \in \mathbb{N}$, for all $1 \leq i \leq M$ and $t \in [0, T]$:

$$Y_t^{i,n} = \int_t^T \psi_i(s, X_s)ds + n \int_t^T (L_s^{i,n} - Y_s^{i,n})^+ ds - \int_t^T Z_s^{i,n} dB_s$$

where for every $1 \leq i \leq M$,

$$\forall t \in [0, T], \; L_t^{i,n} = \max_{k \neq i}(-C_{i,k}(t) + Y_t^{k,n}).$$

THEOREM

For every $1 \leq i \leq M$ and all $t \in [0, T]$, the sequence $(Y_t^{i,n})_{n \in \mathbb{N}}$ is non decreasing, $Y_t^{i,n} \leq Y_t^i$ and $Y_t^{i,n}$ converges to Y^i.

Conclusion.

- Positive answer to the question of Carmona & Ludkovski. Extension in the non Markovian case.

- Hu & Tang: existence and uniqueness of the system of RBSDEs "directly" when ψ_i also depends on Y^i and Z^i (but for constant switching costs). Application to optimal switching in the Markovian case, but with X depending on the strategy.

- Some open questions
  - Time delay ?
  - Infinitely many regimes ?
  - Numerical schemes: rate of convergence ?