Optimal targeting position

and (forward) backward stochastic differential equation.

Based on joint works with

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Outline

1. Introduction: optimal targeting position
   - Motivation: optimal closure
   - Monotone strategy and forward backward SDE
2. Homogeneous case (with T. Kruse)
3. Knightian uncertainty (with C. Zhou)
4. Non homogeneous case (with S. Ankirchner, A. Fromm & T. Kruse)

Outline

Description of our problem.

For $x_0 \in \mathbb{R}$, consider $\mathcal{A} = \{\alpha : \Omega \times [0, T] \to \mathbb{R} \in L^1(0, T),\ a.s.\}$ and

$$X_s^{x_0,\alpha} = x_0 - \int_0^s \alpha_r dr.$$

Problem: minimize over all $\alpha \in \mathcal{A}$

$$J(x_0, \alpha) = \mathbb{E}\left[\int_0^T f\left(s, X_s^{x_0,\alpha}, \alpha_s\right) ds + g\left(X_T^{x_0,\alpha}\right)\right].$$

Two cases:

- Unconstrained problem (UP): no condition on $X_T^{x_0,\alpha}$.
- Constrained problem (CP): $X_T^{x_0,\alpha} = 0$ a.s. $\longrightarrow \alpha \in \mathcal{A}^0$.

Plan

Almgren & Chriss model (2000).

- Continuous-time extension of the Bertsimas & Lo model (1998).
- Execution strategies have absolutely continuous paths, i.e. the remaining position size is determined by trading rate $\alpha_s = \dot{X}_s$

$$X_t = x_0 + \int_0^t \dot{X}_s ds, \quad X_T = 0.$$

- Price impact consists of two components

$$S_t^X = \underbrace{S_t^0}_{\text{Unaffected price}} + \underbrace{\lambda \int_0^t \dot{X}_s ds}_{\text{permanent}} + \underbrace{h(\dot{X}_t)}_{\text{temporary}} .$$

- Gatheral (2010): this choice of the permanent effect rules out price manipulation.

Expected Revenues.

Model:

$$S_t^X = S_t^0 + \lambda(X_t - x) + h(\dot{X}_t).$$

Revenues obtained from following X (with $X_T = 0$)

$$R_T(X) = -\int_0^T S_t^X dX_t.$$

Assume that S^0 is a martingale and integrating by parts:

$$\mathbb{E}\left[R_T(X)\right] = \underbrace{xS_0^0}_{\text{naive book value}} - \underbrace{\lambda\frac{x^2}{2}}_{\text{costs entailed by perm impact}} - \underbrace{\mathbb{E}\left[\int_0^T h(\dot{X}_t)\dot{X}_t dt\right]}_{\text{costs entailed by temp impact}}$$

(Non exhaustive) literature review.

- Mean-variance optimization:

$$\mathbb{E}[R_T(X)] - \delta \mathrm{Var}(R_T(X)) \to \max$$

Almgren & Chriss (1999, 2000), Almgren (2003), and Lorenz & Almgren (2011)

- Expected-Utility maximization:

$$\mathbb{E}[u(R_T(X))] \to \max$$

Schied & Schöneborn (2009), Schied, Schöneborn & Tehranchi (2010), ...

- Time-averaged Risk Measures:

$$\mathbb{E}\left[R_T(X) - \int_0^T f(S_t^0, X_t)dt\right] \to \max$$

Gatheral & Schied (2011), Ankirchner & Kruse (2012), ...

- Models including a dark pool, multi-agent models, transient impact, non-aggresive strategies...

The model: stochastic liquidity.

- Almgren, Hauptmann, Li & Thum (2005): $h(x) \approx \eta sgn(x)|x|^{0.6}$.

- Temporary impact $\eta = (\eta_t,\ t \geq 0)$: depends on time and is random.

$$h_t(\dot{X}_t) = \eta_t \mathrm{sgn}(\dot{X}_t)|\dot{X}_t|^{p-1}$$

with $p > 1$ (shape parameter of the order book (e.g. $p = 1.6$))

Control problem with constraint:

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^\ell \right) ds \right], \quad X_T = 0,$$

over all $\alpha \in \mathcal{A}$ such

$$X_s = x_0 + \int_0^s \alpha_r dr.$$

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with $p > 1$ (shape parameter of the order book (e.g. $p = 1.6$))

Penalized version:

$$v^L(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^\ell \right) ds + L|X_T|^\varrho \right].$$

Questions: when $L \nearrow +\infty$, $v^L(x_0) \nearrow v(x_0)$? Optimal controls ?

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The control problem.

Recall that for $x_0 \in \mathbb{R}$, $\mathcal{A} = \{\alpha : \Omega \times [0,T] \to \mathbb{R} \in L^1(0,T),\ \text{a.s.}\}$ such that

$$X_s^{x_0,\alpha} = x_0 - \int_0^s \alpha_r dr$$

Value function (with or without the constraint on X_T):

$$v(x_0) = \inf_{\alpha \in \mathcal{A} \text{ or } \mathcal{A}^0} J(x_0,\alpha) = \inf_{\alpha \in \mathcal{A} \text{ or } \mathcal{A}^0} \mathbb{E}\left[\int_0^T f\left(s, X_s^{x_0,\alpha}, \alpha_s\right) ds + g(X_T^{x_0,\alpha})\right]$$

Assumptions (uniformly in ω and t):

- $(x,a) \mapsto f(t,x,a)$ and $x \mapsto g(x)$ are convex (f being strictly convex in a).
- $a \mapsto f(t,x,a)$, $x \mapsto f(t,x,0)$ and $x \mapsto g(x)$ attain a minimum at zero with $f(t,0,0) = g(0) = 0$.

Monotone strategies.

Recall that for $x_0 \in \mathbb{R}$, $\mathcal{A} = \{\alpha : \Omega \times [0,T] \to \mathbb{R} \in L^1(0,T), \text{ a.s.}\}$ such that

$$X_s^{x_0,\alpha} = x_0 - \int_0^s \alpha_r dr$$

Value function (with or without the constraint on X_T):

$$v(x_0) \quad = \quad \inf_{\alpha \in \mathcal{A} \text{ or } \mathcal{A}^0} J(x_0, \alpha) = \inf_{\alpha \in \mathcal{A} \text{ or } \mathcal{A}^0} \mathbb{E}\left[\int_0^T f\left(s, X_s^{x_0,\alpha}, \alpha_s\right) ds + g(X_T^{x_0,\alpha})\right]$$

Proposition

Let $x_0 \geq 0$. For any $\alpha \in \mathcal{A}$ there exists $\beta \in \mathcal{A}$ such that $X^{x_0,\beta}$ is non-increasing and non-negative and $J(x_0,\beta) \leq J(x_0,\alpha)$. If $\alpha \in \mathcal{A}$ is optimal, then $(X_s^{x_0,\alpha},\ s \in [0,T])$ is non-increasing and non-negative.

Remark: equivalent properties hold if $x_0 \leq 0$.

- Coherent result with the absence of transaction-triggered price manipulation (Gatheral & Shied (2011), Alfonsi et al. (2012)).

Stochastic maximum principle (Bismut (1973),...).

f is coercive:

$$\forall(\omega, t, x, a), \quad f(t, x, a) \geq b|a|^p.$$

Hamiltonian of our control problem:

$$\mathcal{H}(t, x, a, y) = f(t, x, a) - ay.$$

Convex conjugate of $f(t, x, \cdot)$: $f^*(t, x, \cdot)$

$$\min_{a \in A} \mathcal{H}(t, x, a, y) = -f^*(t, x, y).$$

Optimal closure example

$$f(t, x, a) = \eta_t |a|^p + \gamma_t |x|^\ell,$$

then

$$f^*(t, x, y) = \frac{p-1}{p} \left(\frac{1}{p\eta_t} \right)^{q-1} |y|^q - \gamma_t |x|^\ell, \quad 1/p + 1/q = 1.$$

Stochastic maximum principle (Bismut (1973),...).

Adjoint forward backward SDE: find adapted processes (X, Y, Z) s.t.

$$
\begin{aligned}
X_s &= x_0 - \int_0^s f_y^*(r, X_r, Y_r)dr, \\
Y_s &= g'(X_T) + \int_s^T f_x(r, X_r, f_y^*(r, X_r, Y_r))dr - \int_s^T Z_r dW_r.
\end{aligned}
$$

Verification result

If there exists a solution (X, Y, Z) of the FBSDE (with suitable integrability conditions), then an optimal control is given by

$$\alpha_s = f_y^*(s, X_s, Y_s), \quad s \in [0, T].$$

Remarks:

- Monotone strategy: $X_s \in [0, x_0]$, $\alpha_s \geq 0$.
- A dynamic version of this problem can be easily written.

How to solve a FBSDE ?

Four methods:

1. Fixed-point argument. Works only for small terminal time T.

2. Four-step scheme. Based on PDE arguments and existence of smooth solutions.

3. Continuation method. Based on a monotonicity condition. Suitable for the unconstrained problem.

4. Decoupling field. Lipschitz assumptions on the coefficients.

Constrained case $X_T = 0$:

- How can we include this additional condition in the FBSDE ?

Outline

Homogeneous problem.

Constrained control problem: for some $p > 1$ and for $x_0 \geq 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p \right) ds \right], \quad X_T = 0$$

where

$$X_s = x_0 + \int_0^s \alpha_u du.$$

Penalized problem:

$$v^L(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p \right) ds + L |X_T|^p \right]$$

Heuristics: when $L \nearrow +\infty$, $v^L \nearrow v$.

Relaxing the liquidation constraint.

Possibility for non closure: instead of $X_T = 0$

- Specify a set $\mathcal{S} \subset \mathcal{F}_T$ for closure such that: $X_T \mathbf{1}_{\mathcal{S}} = 0$;
- Penalization on the non closure set $\mathcal{S}^c$.

- Minimize

$$\mathbb{E}(\xi |X_T|^p) = \mathbb{E}(\xi \mathbf{1}_{\mathcal{S}^c} |X_T|^p)$$

with $0 \times \infty = 0$ and a r.v. ξ such that

- $\mathcal{F}_T$-measurable and non negative ;
- $\mathbb{P}(\xi = +\infty) > 0$ and $\mathcal{S} = \{\xi = +\infty\}$;
- $\xi \mathbf{1}_{\mathcal{S}^c} \in L^1(\Omega)$.

Examples:

- binding liquidation: $\xi = +\infty$ a.s. if and only if $X_T = 0$.
- excepted scenarios: $\xi = \infty \mathbf{1}_{\mathcal{S}}$ with e.g.
  - $\mathcal{S} = \{\max_{t \in [0,T]} \eta_t \leq H\}$ for a given threshold H;
  - $\mathcal{S} = \{\int_0^T \eta_t dt \leq H\}$.

Back to the FBSDE.

Control problem: for $x_0 > 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p \right) ds + \xi |X_T|^p \right].$$

Here

$$f(t,x,a) = \eta_s |a|^p + \gamma_s |x|^p, \quad g(x) = \xi |x|^p,$$

$$f^*(t,x,y) = \frac{p-1}{p} \left(\frac{1}{p\eta_t} \right)^{q-1} |y|^q - \gamma_t |x|^p.$$

Adjoint forward backward SDE:

$$X_s = x_0 - \int_0^s \left(\frac{1}{p\eta_r} \right)^{q-1} |Y_r|^{q-1} \text{sign}\,(Y_r) dr,$$

$$Y_s = \xi p |X_T|^{p-1} \text{sign}\,(X_T) + \int_s^T \gamma_r p |X_r|^{p-1} \text{sign}\,(X_r) dr - \int_s^T Z_r dW_r.$$

Back to the FBSDE.

Control problem: for $x_0 > 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T (\eta_s|\alpha_s|^p + \gamma_s|X_s|^p)\, ds + \xi|X_T|^p\right].$$

Here

$$f(t,x,a) = \eta_s|a|^p + \gamma_s|x|^p, \quad g(x) = \xi|x|^p,$$

$$f^*(t,x,y) = \frac{p-1}{p}\left(\frac{1}{p\eta_t}\right)^{q-1}|y|^q - \gamma_t|x|^p.$$

Adjoint forward backward SDE:

$$X_s = x_0 - \int_0^s \left(\frac{1}{p\eta_r}\right)^{q-1} |Y_r|^{q-1}\text{sign}\,(Y_r)dr, \quad X_s \geq 0,$$

$$Y_s = \xi p(X_T)^{p-1} + \int_s^T \gamma_r p(X_r)^{p-1} dr - \int_s^T Z_r dW_r.$$

Back to the FBSDE.

Control problem: for $x_0 > 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T (\eta_s|\alpha_s|^p + \gamma_s|X_s|^p)\, ds + \xi|X_T|^p\right].$$

Here

$$f(t,x,a) = \eta_s|a|^p + \gamma_s|x|^p, \quad g(x) = \xi|x|^p,$$

$$f^*(t,x,y) = \frac{p-1}{p}\left(\frac{1}{p\eta_t}\right)^{q-1}|y|^q - \gamma_t|x|^p.$$

Adjoint forward backward SDE:

$$X_s = x_0 - \int_0^s \left(\frac{1}{p\eta_r}\right)^{q-1}(Y_r)^{q-1}dr, \quad X_s \geq 0,$$

$$Y_s = \xi p(X_T)^{p-1} + \int_s^T \gamma_r p(X_r)^{p-1}dr - \int_s^T Z_r dW_r, \quad Y_s \geq 0.$$

Back to the FBSDE.

Adjoint forward backward SDE:

$$X_s = x_0 - \int_0^s \left(\frac{1}{p\eta_r} \right)^{q-1} (Y_r)^{q-1} dr,$$

$$Y_s = \xi p(X_T)^{p-1} + \int_s^T \gamma_r p(X_r)^{p-1} dr - \int_s^T Z_r dW_r.$$

Variable change (while $X_s > 0$)

$$Y_s = pU_s(X_s)^{p-1} \iff U_s = \frac{Y_s}{p(X_s)^{p-1}} \iff (Y_s)^{q-1} = p^{q-1}(U_s)^{q-1}X_s.$$

Itô's formula:

$$U_s = \xi + \int_s^T \gamma_r dr - \int_s^T (p-1) \left(\frac{1}{\eta_r} \right)^{q-1} \underbrace{\frac{(Y_r)^q}{p^q(X_r)^p}}_{=U_r^q} dr - \int_s^T \frac{Z_r}{p(X_r)^{p-1}} dW_r$$

Back to the FBSDE.

Control problem: for $x_0 > 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left(\eta_s |\alpha_s|^p + \gamma_s |X_s|^p \right) ds + \xi |X_T|^p \right]$$

Decoupled forward backward SDE:

$$\begin{aligned}
X_s &= x_0 - \int_0^s \left(\frac{1}{\eta_r}\right)^{q-1} (U_r)^{q-1} X_r dr, \\
U_s &= \xi + \int_s^T \gamma_r dr - \int_s^T (p-1) \left(\frac{1}{\eta_r}\right)^{q-1} (U_r)^q dr - \int_s^T V_r dW_r \\
Y_s &= u(s, X_s), \quad u(\omega, s, x) = p U_s(\omega) x^{p-1}
\end{aligned}$$

Last equation = decoupling field.

Our aim & related literature.

Value function: for $x_0 \geq 0$

$$v(x_0) = \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T \left(\eta_s|\alpha_s|^p + \gamma_s|X_s|^p\right) ds + \xi|X_T|^p\right]$$

- Related (non exhaustive) literature: $\xi = +\infty$.
  - Ankirchner, Jeanblanc & Kruse (2013). Brownian framework.
  - Schied (2013). Solves a variant of this problem in a Markovian framework using superprocesses.
  - Graewe, Horst & Qiu (2015). Analyze both Markovian and non-Markovian dependence of the coefficients by means of BSPDEs.
  - Bank & Voss (2016). Optimal tracking problems.
- Aims:
  - Relax the constraint at terminal time.
  - No assumption on the filtration (except completeness and right-continuity) $\longrightarrow$ Knightian uncertainty.
  - Extension to random terminal time τ.

Backward stochastic differential equations.

Given

- a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$,
- a terminal time $T > 0$ and a final condition ξ s.t. ξ is a $\mathcal{F}_T$-measurable r.v.

Solve the ODE:

$$\forall t \in [0, T], \quad y_t = \xi + \int_t^T \Psi(s, y_s) ds \Rightarrow y_t \text{ is } \mathcal{F}_T - \text{measurable.}$$

Particular case: $\xi \in L^1$ et $\Psi \equiv 0 \Rightarrow y_t = \xi$. Best adapted approximation:

$$\begin{aligned} Y_t &= \mathbb{E}(y_t|\mathcal{F}_t) = \mathbb{E}(\xi|\mathcal{F}_t) = M_t \quad = \text{ martingale} \\ &= \xi - \int_t^T dM_s, \quad Y \text{ càdlàg process.} \end{aligned}$$

Definition of a BSDE

A BSDE is an equation of the following type:

$$\forall t \in [0,T], \quad Y_t \;=\; \xi + \int_t^T \phi(r, Y_r)dr - \int_t^T dM_s.$$

Data:

- T: (deterministic) terminal time.
- $\Psi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$: generator.
- ξ: terminal condition: an $\mathcal{F}_T$-measurable random variable, with values in $\mathbb{R}$.

Unknowns: $(Y_t, M_t)_{0 \leq t \leq T}$.

In our case:

$$\Psi(t,y) = -(p-1)\frac{|y|^{q-1}}{(\eta_t)^{q-1}} y + \gamma_t \quad \text{and} \quad \mathbb{P}(\xi = +\infty) > 0.$$

Assumptions.

Singular BSDE: for $\xi \geq 0$ with $\mathbb{P}(\xi = +\infty) > 0$

$$U_s = \xi + \int_s^T \left[-(p-1) \left(\frac{1}{\eta_r} \right)^{q-1} |U_r|^{q-1} U_r + \gamma_r \right] dr - \int_s^T dM_r.$$

Assumptions:

- Positivity. $0 < \eta_t < +\infty$, $0 \leq \lambda_t \leq +\infty$, $0 \leq \gamma_t < +\infty$.
- Integrability. For some $\ell > 1$

$$\mathbb{E} \left[\int_0^T (\eta_t + (T-t)^p \gamma_t)^\ell dt \right] < \infty \quad \text{and} \quad \mathbb{E} \left[\int_0^T \frac{1}{\eta_t^{q-1}} dt \right] < \infty.$$

  - First condition: sufficient to obtain a priori estimate.
  - Second condition: necessary to ensure existence of a optimal control.

- Left continuity of the filtration at time T (avoid thin time case).

Existence and verification result (T.K. & A.P., SPA 2016).

Theorem

There exists a minimal (super-)solution of the singular BSDE (U, M), in the sense that (U, M) satisfies the dynamics and some integrability conditions on $[0, T - \varepsilon]$ for any $\varepsilon > 0$ and

$$\mathbb{P} - \text{a.s.} \quad \liminf_{t \to T} U_t \geq \xi.$$

Then the value function is given by:

$$v(x_0) = U_0 |x_0|^p,$$

and an optimal control is given by:

$$X^*_s = x_0 - \int_0^s \left(\frac{U_r}{\eta_r} \right)^{q-1} X^*_r dr = x_0 \exp \left[- \int_0^s \left(\frac{U_r}{\eta_r} \right)^{q-1} dr \right].$$

X^* belongs to $\mathcal{A}(x_0)$, satisfies the terminal state constraint $X^*_T \mathbf{1}_{\xi = +\infty} = 0$.

Extensions.

- Dark-pool trading. BSDE with unknows (U, ϕ, M)

$$U_t = \xi - (p-1)\int_t^T \left[\frac{U_s^q}{\eta_s^{q-1}}\right] ds + \int_t^T \gamma_s ds - \int_t^T \vartheta(s, U_s, \phi_s) ds$$
$$- \int_t^T \int_{\mathcal{E}} \phi_s(e)\widetilde{\pi}(ds, de) - \int_t^T dM_s.$$

- Random terminal time T = exit time of a continuous diffusion.
  - Existence of an a priori estimate $\approx$ Keller-Osserman inequality.
  - Example: $T = \inf\{t \geq 0, S_t^0 \leq H\}$.

- U càdlàg on $[0, T]$. The left limit at time T exists (A.P., ESAIM P&S, '16).

Outline

Knightian uncertainty.

- An old concept : Knight (1921). Distinction between risk vs uncertainty. From a single probability $\mathbb{P}$ to a set of probability $\mathcal{P}$.
  - Quantitative finance: model risk. Given recent market behaviour, this has become a very acute and concrete problematic for practitioners and risk managers.
  - Economics: theory of decision under uncertainty, monetary policy, psychology and behaviour of investors during period of stress.

- Triggered development of new mathematical tools.
  - Quasi-sure stochastic analysis, non-linear expectations, G-Brownian motions, second order BSDE.
  - See among many others Peng (2010-2011), Denis and Martini (2006), Soner et al. (2011), ...

Uncertainty and expected utility.

How to model and represent preferences of agents under uncertainty ?

- von Neumann and Morgenstern (1947):

$$\sup_X \mathbb{E}_{\mathbb{P}_O} U(X).$$

  $\mathbb{P}_O$ is the objective probability (fixed). Allais Paradox.

- Savage (1954): $\sup_X \mathbb{E}_{\mathbb{P}_S} U(X)$. $\mathbb{P}_S$ is a subjective probability. Ellsberg Paradox.
- Gilboa and Schmeidler (1989): $\sup_X \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}} U(X)$. $\mathcal{P}$ range of all possible subjective beliefs : **worst case expected utility**.

How do we model uncertainty ?

A set of probability $\mathcal{P}$ on a measurable space $(\Omega, \mathcal{F})$ which is non-dominated (different models do not agree on null-event).

Warning: several classical tools may be false !

Typical example.

Brownian motion with drift and volatility

$$X_t = x_0 + \sigma W_t + \mu t, \quad \text{law of } X = \mathbb{P}_{\mu,\sigma}.$$

- For a given $\sigma > 0$, Girsanov theorem: $X_t = x_0 + \sigma \widetilde{W}_t$

$$\forall \mu \in \mathbb{R}, \quad \mathbb{P}_{\mu,\sigma} \ll \mathbb{P}_{0,\sigma}.$$

- $\mathbb{P}_{\mu,\sigma}\left[\text{quadratic variation of } (X_t,\ 0 \leq t \leq T) = \sigma^2 T\right] = 1.$

Monotone convergence theorem: counter example

$$\mathcal{P} = \{\mathbb{P}_{0,\sqrt{p}}, \quad p \in \mathbb{N}^*\}, \qquad Y_n = (W_1)^2/n.$$

- $Y_n \downarrow 0$, $\mathbb{P}$-a.s., for any $\mathbb{P} \in \mathcal{P}$ and $\mathbb{E}^{\mathbb{P}_p}(Y_n) = \frac{p}{n}$.
- Hence $\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}}(Y_n) = +\infty$.

Liquidation under uncertainty

Minimize the expected execution costs under $\mathbb{P}$

$$J(t, X, \mathbb{P}) = \mathbb{E}^{\mathbb{P}} \left[\int_t^T (\eta_s |\alpha_s|^p + \gamma_s |X_s|^p) \, ds + \xi |X_T|^p \middle| \mathcal{F}_t \right]$$

Define

$$v(t, x_0) = \operatorname*{essinf}_{X \in \mathcal{A}^0(t,x_0)} \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} J(t, X, \mathbb{P})$$

where $\mathcal{A}^0(t, x_0)$ the set of admissible controls $X \in \mathcal{A}(t, x_0)$ such that

$$X_s = x_0 + \int_t^s \alpha_s ds, \quad X_T \mathbf{1}_{\mathcal{S}} = 0, \quad \mathcal{P} - q.s.$$

Idea: consider

$$\operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} \operatorname*{essinf}_{X \in \mathcal{A}^0_{\mathbb{P}}(t,x_0)} J(t, X, \mathbb{P}) = |x_0|^p \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} U^{\mathbb{P}}_t \leq v(t, x_0)$$

where $U^{\mathbb{P}}$ solution of the related singular BSDE under $\mathbb{P}$.

Second order BSDE: setting

Setting: $\Omega = C([0, T], \mathbb{R}^d)$ and

- long and boring description ! But important to be able to control the negligible sets.

Example: $\mathcal{P} = \{\mathbb{P}^a\}$ with:

$$\mathbb{P}^a = \mathbb{P}_0 \circ (\mathfrak{X}^a)^{-1}, \quad \mathfrak{X}^a_t = \int_0^t a_s^{1/2} d\mathfrak{X}_s$$

for all processes a of the form

$$a_s = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} a_i^n(s) \mathbf{1}_{E_i^n} \mathbf{1}_{[\tau_n, \tau_{n+1})}(s),$$

- $(a_i^n)_{i,n}$ are deterministic mappings such that $0 < \underline{a} \leq a_i^n(t)$ for any $t \geq 0$,
- $(\tau_n)_n$ is a nondecreasing sequence of stopping times with $\tau_0 = 0$
- for each n, $\{E_i^n, \ i \geq 1\} \subset \mathcal{F}_{\tau_n}$ forms a partition of Ω.

Second order BSDE (2BSDE).

We consider the 2BSDE

$$Y_t = \xi + \int_t^T \Psi(u, \mathfrak{X}_{\cdot \wedge u}, Y_u, a_u^{1/2} Z_u, a_u, b_u^{\mathbb{P}}) du - \int_t^T Z_u d\mathfrak{X}_u^{c,\mathbb{P}} - \int_t^T dM_u^{\mathbb{P}} + \int_t^T dK_u^{\mathbb{P}}.$$

Definition

$(Y, Z, M^{\mathbb{P}}, K^{\mathbb{P}})$ is a solution if

- *the 2BSDE is satisfied $\mathcal{P}$-q.s., that is $\mathbb{P}$-a.s. for any $\mathbb{P} \in \mathcal{P}$.*
- *the family $(K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P})$ satisfies some minimality condition.*

Roughly speaking:

$$Y_t = \text{essup}_{\mathbb{P} \in \mathcal{P}_t} Y_t^{\mathbb{P}}.$$

Literature for square integrable ξ:

- Soner, Touzi & Zhang (2011-2013). Lipschitz generator.
- Possamaï (2013). Monotone generator with linear growth.
- Possamaï, Tan & Zhou (2016).

Control problem with uncertainty.

2BSDE: $U_t = \operatorname*{esssup}_{\mathbb{P} \in \mathcal{P}_t} U_t^{\mathbb{P}}$ solves for any $\mathbb{P} \in \mathcal{P}$ and $0 \leq s \leq t < T$:

$$U_s = U_t - \int_s^t (p-1) \frac{U_r^q}{(\eta_r)^{q-1}} du + \int_s^t \gamma_r dr - \int_s^t dM_r^{\mathbb{P}} + \int_s^t dK_r^{\mathbb{P}}$$

where

- $M^{\mathbb{P}}$ is a martingale,
- $K^{\mathbb{P}}$ is non decreasing under $\mathbb{P}$ and with a minimality condition on the family $(K^{\mathbb{P}}, \ \mathbb{P} \in \mathcal{P})$ ($\approx$ Skorokhod condition).

Quasi surely,

$$\liminf_{t \to T} U_t \geq \xi.$$

Optimality: $v(t, x_0) = |x_0|^p U_t$ with an optimal state process:

$$X_s^* = x_0 - \int_t^s \left(\frac{U_r}{\eta_r} \right)^{q-1} X_r^* dr.$$

Outline

Back to the general case.

For $x_0 \geq 0$, $\mathcal{A} = \{\alpha : \Omega \times [0, T] \to \mathbb{R} \in L^1(0, T) \text{ a.s.}\}$ such that

$$X_s^{x_0,\alpha} = x_0 - \int_0^s \alpha_r dr$$

And

$$v(x_0) \quad = \quad \inf_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^T f\left(s, X_s^{x_0,\alpha}, \alpha_s\right) ds + g(X_T^{x_0,\alpha})\right]$$

Adjoint forward backward SDE:

$$\begin{aligned} X_s &= x_0 - \int_0^s f_y^*(r, X_r, Y_r) dr, \quad X_s \in [0, x], \quad \text{non increasing}, \\ Y_s &= g'(X_T) + \int_s^T f_x(r, X_r, f_y^*(r, X_r, Y_r)) dr - \int_s^T Z_r dW_r \end{aligned}$$

Decoupling field: definition.

From A. Fromm & P. Imkeller and J. Ma et al. (2015).

Definition

Let $t \in [0, T]$. $u: [t, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ with $u(T, \cdot) = \xi(\cdot)$ a.e. is a decoupling field for the FBSDE $(\xi, (\mu, \sigma, f))$ on $[t, T]$ if:

- *for all $t_1, t_2 \in [t, T]$ with $t_1 \leq t_2$*
- *and any $\mathcal{F}_{t_1}$-measurable $X_{t_1} : \Omega \to \mathbb{R}$*

there exist progressively measurable processes (X, Y, Z) on $[t_1, t_2]$ such that

$$X_s = X_{t_1} + \int_{t_1}^{s} b(r, X_r, Y_r, Z_r)dr,$$

$$Y_s = Y_{t_2} + \int_{s}^{t_2} F(r, X_r, Y_r, Z_r)dr - \int_{s}^{t_2} Z_r dW_r,$$

and $Y_s = u(s, X_s)$ for all $s \in [t_1, t_2]$.

Existence and uniqueness result of a decoupling field.

Assumption (SLC):

1. (μ, σ, f) are Lipschitz continuous in (x, y, z) with Lipschitz constant L,
2. $\|(|\mu| + |f| + |\sigma|)(\cdot, \cdot, 0, 0, 0)\|_\infty < \infty$,
3. $\xi \colon \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable such that $\|\xi(\cdot, 0)\|_\infty < \infty$ and $L_{\xi,x} < +\infty$.

Theorem (Fromm & Imkeller (2015))

Let $(\xi, (\mu, \sigma, f))$ satisfy SLC. Then there exists a unique strongly regular decoupling field u on some interval $I_{\max} \subset [0, T]$.

Furthermore, either $I_{\max} = [0, T]$ or $I_{\max} = (t_{\min}, T]$, where $0 \leq t_{\min} < T$. In the latter case we have

$$\lim_{t \downarrow t_{\min}} L_{u(t,\cdot),x} = +\infty.$$

For our FBSDE in the unconstrained case.

For $x_0 \geq 0$

$$X_s = x_0 - \int_0^s f_y^*(r, X_r, Y_r)dr, \quad X_s \in [0, x], \quad \text{non increasing,}$$

$$Y_s = g'(X_T) + \int_s^T f_x(r, X_r, f_y^*(r, X_r, Y_r))dr - \int_s^T Z_r dW_r$$

Assumptions:

- $f_y^*(t, x, y)$, $g'(x)$ and $f_x(t, x, f_y^*(t, x, y))$ uniformly Lipschitz continuous in $(x, y) \in [0, \infty) \times [0, \infty)$,
- $g'(0) = f_x(t, 0, 0) = 0$ for all ω, t,
- $\sup_{x \geq 0,\, a \in A_+} |f_{xx}(t, x, a)|$ is bounded uniformly in (ω, t).

Proposition

- *There exists a unique regular decoupling field u on $I_{\max} = [0, T]$ s.t. $u(t, x) = 0$ for all $x \leq 0$ and $t \in [0, T]$.*
- *The solution (X, Y, Z) with $Y_s = u(s, X_s)$ is s.t. X and Y are both bounded and $X_s > 0$ for all $s \in [0, T]$.*

For the constrained case ?

Extension in two cases: let L be a fixed parameter.

1. Quadratic setting:
   - The whole Hessian matrix $D^2 f(s,x,a)$ of f w.r.t. $(x,a) \in [0,\infty) \times A_+$ is uniformly bounded independently of (ω, s, x, a).
   - $g(x) = Lx^2$.

   Examples:

$$f(t,x,a) = \eta_t \frac{|a|^3 + 2|a|^2}{|a|+1} + \gamma_t |x|^2, \qquad f(t,x,a) = \eta_t |a|^2 + \gamma_t |x|^2.$$

2. Additive power setting: for $p \leq 2$ and $\ell \geq 2$

$$f(t,x,a) = \eta_t |a|^p + \gamma_t |x|^\ell, \qquad g(x) = L|x|^p.$$

   Note that g is not Lipschitz continuous on $[0,+\infty[$.

The scheme.

First step: there exists a solution (X^L, Y^L, Z^L) with decoupling field u^L.

Second step: the decoupling field u^L is non decreasing w.r.t. L. Let u^∞ its limit.

Third step: regularity of u^∞:

- Quadratic case: u^∞ is Lipschitz continuous on $[0, T-\varepsilon] \times [0, x_0]$ for any $\varepsilon > 0$.
- Additive case ($p < 2$): $u^\infty(t,x) = |x|^{p-1} v^\infty(t,x)$ and v^∞ is Lipschitz continuous on $[0, T-\varepsilon] \times [0, x_0]$ for any $\varepsilon > 0$.

Fourth step: the sequence X^L is non increasing w.r.t. L and its limit X^∞ satisfies:

$$X_s^\infty = x_0 + \int_0^s f_y^*(r, X_r^\infty, u^\infty(r, X_r^\infty))dr, \qquad X_T^\infty = 0.$$

X^∞ is an optimal state process.

The scheme.

First step: there exists a solution (X^L, Y^L, Z^L) with decoupling field u^L.

Fourth step: the sequence X^L is non increasing w.r.t. L and its limit X^∞ satisfies:

$$X_s^\infty = x_0 + \int_0^s f_y^*(r, X_r^\infty, u^\infty(r, X_r^\infty))dr, \qquad X_T^\infty = 0.$$

X^∞ is an optimal state process.

Fifth step: the sequence (Y^L, Z^L) converges on $[0, T) \times \Omega$ and its limit (Y^∞, Z^∞) satisfies for any $0 \leq s \leq t < T$:

$$\begin{aligned} Y_s^\infty &= u^\infty(s, X_s^\infty) \\ Y_s^\infty &= Y_t^\infty + \int_s^t f_x(r, X_r^\infty, f_y^*(r, X_r^\infty, Y_r^\infty))dr - \int_s^t Z_r^\infty dW_r. \end{aligned}$$

Thank you for your attention !

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