ENSTA, troisième année

Processus de Lévy,

et applications en finance.

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Introduction and definition of a Lévy process

Let us mention some facts concerning financial continuous time models with continuous trajectories. As we want to deal with jumps, here we just discuss the inconvenients of these models. But keep in mind that they are very important and it is not possible to understand models with jumps if the continuous models are not known.

- **Black-Scholes model.** This model is a benchmark, is the most used and is a reference. Of course it has well known advantages but also inconvenients. Among them the scale invariance of the Brownian motion implies that one month trading, one day trading or one second tradong are almost the same.
- **Local volatility models.** They are natural extensions of the Black-Scholes model. For this class of models, a perfect hedging is possible. In practice, we know that there is always a residual risk.
- **Stochastic volatility models.** Here a perfect delta hedging is not possible. But some practical facts are not catch. It is difficult to obtain heavy tails (which can be a large risk management problem) and large sudden moves can not occur. In practice this behaviour can be observed for short maturity options.

All these model defaults can be removed when models with discontinuous trajectories are used. For a more detailled review of these problems, read the introduction of the book of R. Cont and P. Tankov.

But a natural question is: why should jumps be introduce in a continuous time models? Indeed a discrete time model naturally contains jumps ! The answer is quite simple: in a continuous time model we can use the stochastic calculus, which is a very powerful tool to make computations. Moreover from the works of Aït-Sahalia and Jacod (*Testing for jumps in a discrete observed process, 2007*), we have statistical tests to determine the presence of jumps in discrete time data.

To complete this introduction, we give two financial problems where jumps are required.

- **Credit risk.** The main difficulty here is to compute the default probability, i.e. the probability that the default time of a company occurs after a fixed time t: $\mathbb{P}(\tau > t)$. Classically there are two approachs. The reduced models give directly the law of τ whereas in the structural models τ is a first pssage-time:

$$\tau = \inf\{t \geq 0,\ X_t < H\},$$

where X is a stochastic process modelling for example the stock price of the company and H is a barrier. If X has no jump, τ is a *predictable* stopping time. In practice default time are not predictable (Enron, November 2001, for example): the default time can not be anticipated by the actors of the market.

- **High-frequency trading.** Since few years, intra-day trading and also intra-minute trading have been developped. At the time scale, traders have to consider the tick, the minimum movement by which the price of a security, option, or index changes. Hence the price process is completely discontinuous ! At a classical time scale, these very small jumps are not seen and continuous models can be considered.

To finish the introduction, we briefly explain the classical model for the risk process for an insurance company. In 1903 Lundberg proposed to modelize the revenue of an insurance company as a process X defined by:

$$X_t = x + ct - \sum_{i=1}^{N_t} \xi_i \tag{1}$$

where the company collects premiums at a fixed rate c from its customers, x is the initial capital, N is the arrival times of claims of the customers, causing the revenue to jump downwards. The size of claims ξ_j is independent and identically distributed. X is a compound Poisson process with drift c. Fundamental quantities of interest are the distribution of the time to ruin and the deficit at ruin; otherwise

$$\tau_0 = \inf\{t > 0,\ X_t < 0\}, \qquad X_{\tau_0} \text{ on } \{\tau_0 < +\infty\}.$$

This model has be widely studied and some extensions have been proposed where the compound Poisson process is replaced by a general Lévy process with jumps in $]-\infty, 0[$. Such processes are called spectrally negative.

To finish this introduction, note that Lévy processes are used for

- Population size models, genealogy, mutations,
- Phylogenetic analysis,
- Limit of Galton-Watson trees,
- Branching processes,
- Epidemiology models,

and so on.

Definitions and first properties

Definition 0.1 *A stochastic process $(X_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is a* Lévy process *if*

1. *its increments are* independent*: for every increasing sequence $t_0, \ldots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independents;*
2. $X_0 = 0$ *a.s.*
3. *its increments are* stationnary*: the law of $X_{t+h} - X_t$ does not depend on t;*
4. *X satisfies the property called* stochastic continuity*: for any $\varepsilon > 0$,*
$$\lim_{h\to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0;$$
5. *there exists a subset Ω_0 s.t. $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, $t \mapsto X_t(\omega)$* is RCLL.

RCLL means right continuous with left limits. The fourth condition excludes processes with deterministic jump times. For a fixed t, the probability to have a jump is zero.

Some remarks on the hypotheses:

- If we remove Assumption 5, we speak about Lévy process in law.
- If we remove Assumption 3, we obtain an additive process.
- Dropping Assumptions 3 and 5, we have an additive process in law.

If a filtration $(\mathcal{F}_t)_{t\geq 0}$ is already given on $(\Omega, \mathcal{F}, \mathbb{P})$, then the definition becomes:

Definition 0.2 *A stochastic process $(X_t)_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in Re ^{d}, is a* Lévy process *if*

1. $X_0 = 0$ *a.s.*
2. *its increments are* independent*: for any $s \leq t$, the r.v. $X_t - X_s$ is independent of $\mathcal{F}_s$;*
3. *its increments are* stationnary*;*
4. *X satisfies the property called* stochastic continuity*;*
5. *a.s. $t \mapsto X_t(\omega)$* is RCLL .

Let us remark that:

- If $\mathcal{F}_t = \mathcal{F}_t^X$, the two definitions are equivalent.
- If $\{\mathcal{F}_t\}$ is a larger filtration than $(\mathcal{F}_t^X \subset \mathcal{F}_t)$ and if $X_t - X_s$ is independent of $\mathcal{F}_s$, then $\{X_t;\ 0 \leq t < +\infty\}$ is a Lévy process under the large filtration.

For a process $X = \{X_t;\ t \geq 0\}$, we define

- $\mathcal{N}_\infty = \mathcal{N}$ the set of $\mathbb{P}$-negligible events.
- For any $0 \leq t \leq \infty$, the augmented filtration: $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \cup \mathcal{N})$.

Theorem 0.1 *Let $X = \{X_t;\ t \geq 0\}$ be a Lévy process. Then*

- *the augmented filtration $\{\mathcal{F}_t\}$ is right-continuous.*
- *With respect to the enlarged filtration, $\{X_t,\ t \geq 0\}$ is still a Lévy process.*

About the regularity of the paths, let us mention the next result:

Theorem 0.2 *A Lévy process (or an additive process) in law has a RCLL modification.*

Proof. Admitted. □

We can also prove that 2, 3 and 5 imply 4. Remenber that the law of a random variable (r.v. in short) is characterized by its characteristic function.

Proposition 0.1 *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Then there exists a function $\psi : \mathbb{R}^d \to \mathbb{R}$ called* characteristic exponent *of X such that:*

$$\boxed{\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)}.} \tag{2}$$

Proof. Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a right-continuous function such that for every x and y, $f(x+y) = f(x)f(y)$, then there exists $\alpha \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, $f(x) = \exp(\alpha x)$. □

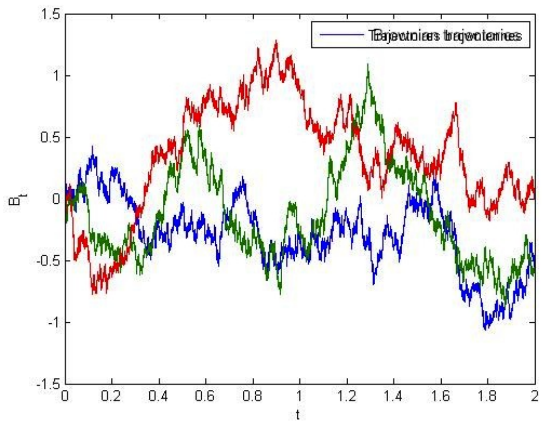
By definition, a Brownian motion is a Lévy process satisfying

1. for every $t > 0$, X_t is Gaussian with mean vector zero and covariance matrix $t\mathrm{Id}$;
2. the process X has continuous sample paths a.s.

Id is the identity matrix of dimension d. And $|z|$ will denote the Euclidean norm on $\mathbb{R}^d$. Its characteristic function is given by :

$$\mathbb{E}(e^{i\langle z, B_t\rangle}) = \exp(-t|z|^2/2).$$

In other words, its characteristic exponent ψ is : $\psi(z) = -|z|^2/2$. A typical trajectory of a Brownian motion is drawn below.


Trajectoires browniennes
1.5
1
0.5
0
-0.5
-1
-1.5
B_t
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
t

Part I

Lévy processes: properties and simulation

Chapter 1

A first class: the jump-diffusion processes

In this first part, we analyze a special class of Lévy processes : the **jump-diffusion processes**. The study of the general Lévy process will be done in the second part. Roughly speaking a jump-diffusion process is the sum of a Brownian motion and a compound Poisson process. For the financial point of view let us recall the main features of a jump-diffusion process.

- The prices are diffusion process, with jumps at randon times.
- The jumps are rare events. Therefore this model can be used for cracks or large losses.

The advantages are the following:

- the price structure is easy to understand, to describe and to simulate.
- Hence efficient Monte Carlo methods can be applied to compute path-depend prices.
- And this model is very performant to interpolate the implicit volatility smiles.

But there are some inconvenients.

- the densities are not known in closed formula,
- and statistical estimation or moments/quantiles computations are difficult to realize.

1.1 Poisson process

The second classical Lévy process is the Poisson process.

Definition 1.1 *A stochastic process $(X_t)_{t\geq 0}$, with values in $\mathbb{R}$, is a* Poisson process *with intensity $\lambda > 0$ if it is a Lévy process s.t. for every $t > 0$, X_t has a Poisson law with parameter λt.*

It means that for each $t > 0$, for every $k \in \mathbb{N}$,

$$\mathbb{P}(X_t = k) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t).$$

So in fact X is a process with values in $\mathbb{N}$.

Proposition 1.1 (Construction) *If $(T_n)_{n\in\mathbb{N}}$ is a random walk on $\mathbb{R}$ s.t. for every $n \geq 1$, $T_n - T_{n-1}$ is exponentially distributed with parameter λ (with $T_0 = 0$), then the process $(X_t)_{t\geq 0}$ defined by*

$$X_t = n \Longleftrightarrow T_n \leq t < T_{n+1}$$

is a Poisson process with intensity λ.

Proof. Let us prove first that X is a Lévy process. By definition $X_0 = 0$ and X is a RCLL process. Moreover for any $\varepsilon > 0$

$$|X_{t+h} - X_t| \geq \varepsilon \Longleftrightarrow \exists n \in \mathbb{N},\ t < T_n \leq t + \varepsilon.$$

But since

$$T_n = \sum_{i=1}^n T_i - T_{i-1}$$

T_n is a sum of n independent exponentially distributed r.v. Therefore T_n is Gamma distributed, i.e. has the density

$$f(x) = \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} \mathbf{1}_{x>0}.$$

Hence

$$\mathbb{P}(t < T_n \leq t + \varepsilon) = \int_t^{t+\varepsilon} f(x) dx \underset{\varepsilon \to 0}{\longrightarrow} 0 \Longrightarrow \lim_{\varepsilon \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0.$$

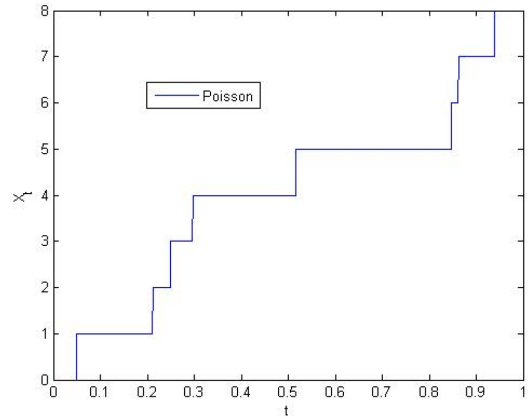
We will prove the stationnary and the independence of the increments below.

Now for $t > 0$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{P}(N_t = n) &= = \mathbb{P}(T_n \leq t < T_{n+1}) = \mathbb{P}(T_n \leq t < (T_{n+1} - T_n) + T_n) \\
&= \mathbb{P}(T_n \leq t,\ T_{n+1} - T_n > t - T_n) \\
&= \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} \left(\int_{t-x}^{+\infty} \lambda e^{-\lambda y} dy \right) dx \\
&= \int_0^t \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} e^{-\lambda(t-x)} dx \\
&= e^{-\lambda t} \frac{\lambda^n t^n}{n!}.
\end{aligned}$$

Finally X is a Poisson process. □

Here is a typical trajectory of a Poisson process. Contrary to the Brownian motion, it is a pure jump process !



The jump distribution of a Poisson process can be completely described. If we denote by T_i the jump times of the process, we have the following result.

Proposition 1.2 *Let $n \geq 1$ and $t > 0$. The conditional law of $T_1, \ldots, T_n$ knowing $X_t = n$ coincides with the law of the order statistics $U_{(1)}, \ldots, U_{(n)}$ of n independent variables, uniformly distributed on $[0,t]$.*

Proof. Let B a Borelian subset of $\mathbb{R}^n$. Then

$$\begin{aligned}
&\mathbb{P}((U_{(1)}, \ldots, U_{(n)}) \in B) \\
&\quad = \sum_{\sigma \in \mathcal{S}_n} \mathbb{P}\left((U_{\sigma(1)}, \ldots, U_{\sigma(n)}) \in B,\ U_{\sigma(1)} < \ldots < U_{\sigma(n)}\right) \\
&\quad = \sum_{\sigma \in \mathcal{S}_n} \int \mathbf{1}_{u_{\sigma(1)} < \ldots < u_{\sigma(n)}} \mathbf{1}_B(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \frac{1}{t^n} du_1 \ldots du_n \\
&\quad = \frac{n!}{t^n} \int \mathbf{1}_{u_1 < \ldots < u_n} \mathbf{1}_B(u_1, \ldots, u_n) du_1 \ldots du_n.
\end{aligned}$$

Now $(T_1, \ldots, T_n, T_{n+1})$ has for density $\lambda^{n+1} e^{-\lambda t_{n+1}} \mathbf{1}_{0<t_1<\ldots<t_n<t_{n+1}}$. Hence for any Borelian B:

$$\begin{aligned}
&\mathbb{P}((T_1, \ldots, T_n) \in B,\ N_t = n) \\
&\quad = \mathbb{P}((T_1, \ldots, T_n) \in B,\ T_n \leq t < T_{n+1}) \\
&\quad = \int \int \lambda^{n+1} e^{-\lambda t_{n+1}} \mathbf{1}_{0<t_1<\ldots<t_n<t_{n+1}} \mathbf{1}_B(t_1, \ldots, t_n) dt_1 \ldots dt_{n+1} \\
&\quad = \lambda^n e^{-\lambda t} \int_{[0,t]^n} \mathbf{1}_B(t_1, \ldots, t_n) \mathbf{1}_{0<t_1<\ldots<t_n} dt_1 \ldots dt_n
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{P}((T_1, \ldots, T_n) \in B | N_t = n) &= \frac{\mathbb{P}((T_1, \ldots, T_n) \in B,\ N_t = n)}{\mathbb{P}(N_t = n)} \\
&= \frac{n!}{t^n} \int \mathbf{1}_{t_1<\ldots<t_n} \mathbf{1}_B(t_1, \ldots, t_n) dt_1 \ldots dt_n,
\end{aligned}$$

which achieves the proof. $\square$

This last proposition will be very useful to simulate Poisson process. For any interval I, we denote by $N(I)$ the number of jumps of X_t, $t \in I$.

Proposition 1.3 *For $0 < s < t$ and $n \geq 1$, the conditional law of X_s knowing $X_t = n$ is binomial with parameters n and s/t.*

For $0 = t_0 < t_1 < \ldots < t_k = t$ and $I_j =]t_{j-1}, t_j]$, the conditional law of $(N(I_1), \ldots, N(I_k))$ knowing $X_t = n$ is multinomial with parameters n, $(t_1 - t_0)/t, \ldots, (t_k - t_{k-1})/t$.

Proof. Let us denote by $J_{n_1,\ldots,n_k}$ a partition of the set $\{0, 1, \ldots, n\}$ in k subsets $J_1, \ldots, J_k$ of length $n_1, \ldots, n_k$ with $n_1 + \ldots + n_k = n$. Denote by $\mathcal{J}$ the set of all such partitions. Remember that the number of elements of $\mathcal{J}$ is equal to $\dfrac{n!}{n_1! \ldots n_k!}$.Then

$$\begin{aligned}
\mathbb{P}(N(I_1) = n_1, \ldots, N(I_k) = n_k | N_t = n) &= \mathbb{P}(\exists J_{n_1,\ldots,n_k} \in \mathcal{J},\ \forall i \in J_l,\ t_{l-1} < U_i \leq t_l) \\
&= \sum_{J_{n_1,\ldots,n_k} \in \mathcal{J}} \mathbb{P}(\forall i \in J_l,\ U_i \in]t_{l-1}, t_l]) \\
&= \sum_{J_{n_1,\ldots,n_k} \in \mathcal{J}} \prod_{l=1}^{k} \left(\frac{t_l - t_{l-1}}{t} \right)^{n_l} \\
&= \frac{n!}{n_1! \ldots n_k!} \prod_{l=1}^{k} \left(\frac{t_l - t_{l-1}}{t} \right)^{n_l}.
\end{aligned}$$

Now take $k = 1$, $t_1 = s$ to obtain that $X_s = N([0, s])$ knowing $N_t = n$ is binomial with parameters n and s/t. □

The last proposition allows us to finish the proof of Proposition 1.1. Indeed let $0 = t_0 < t_1 < \ldots < t_k$ and $n_1, \ldots, n_k$ be non negative integers with $n = \sum n_k$. Then

$$\begin{aligned}
&\mathbb{P}(X_{t_1} = n_1, X_{t_2} - X_{t_1} = n_2, \ldots, X_{t_k} - X_{t_{k-1}} = n_k) \\
&= \mathbb{P}(X_{t_k} = n)\mathbb{P}(X_{t_1} = n_1, X_{t_2} - X_{t_1} = n_2, \ldots, X_{t_k} - X_{t_{k-1}} = n_k | X_{t_k} = n) \\
&= \mathbb{P}(X_{t_k} = n) \\
&\quad \mathbb{P}(T_{n_1} \leq t_1 < T_{n_1+1},\ T_{n_1+n_2} \leq t_2 < T_{n_1+n_2+1}, \ldots, T_{n_1+\ldots+n_k} \leq t_k | X_{t_k} = n) \\
&= \mathbb{P}(X_{t_k} = n)\mathbb{P}(N(I_1) = n_1, \ldots N(I_k) = n_k | X_{t_k} = n) \\
&= e^{-\lambda t_k} \frac{(\lambda t_k)^n}{n!} \frac{n!}{n_1! \ldots n_k!} \prod_{l=1}^{k} \left(\frac{t_l - t_{l-1}}{t} \right)^{n_l} \\
&= \prod_{l=1}^{k} e^{-\lambda(t_l - t_{l-1})} \frac{(\lambda(t_l - t_{l-1}))^{n_l}}{n_l!}.
\end{aligned}$$

Hence we proved that the increments are stationnary and independent.

1.2 Compound Poisson processes

The Poisson process is very simple. The jump sizes are always equal to one. Therefore we will complicate a little bit to obtain the compound Poisson process. We consider a Poisson process $(P_t)_{t \geq 0}$ with intensity λ and jump times T_n, and a sequence $(Y_n)_{n \in \mathbb{N}^*}$ of $\mathbb{R}^d$-valued r.v. such that

1. Y_n are i.i.d. with distribution measure π;

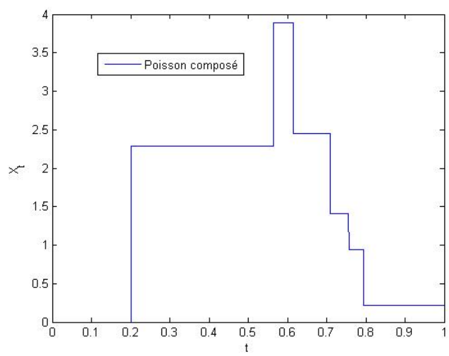
2. $(P_t)_{t\geq 0}$ and $(Y_n)_{n\in\mathbb{N}^*}$ are independent.

We define

$$X_t = \sum_{n=1}^{P_t} Y_n = \sum_{n=1}^{+\infty} Y_n \mathbf{1}_{[0,t]}(T_n). \tag{1.1}$$

Definition 1.2 *The process $(X_t)_{t\geq 0}$ is a* compound Poisson processes *with intensity λ and jump distribution π.*

Let us draw an example. This is a compound Poisson processes with Gaussian jumps used in the Merton model.



Proposition 1.4 *The process $(X_t)_{t\geq 0}$ is a Lévy process, with piecewise constant trajectories and characteristic function:*

$$\begin{aligned} \forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z,X_t\rangle}\right) &= \exp\left(t\lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx)\right) \\ &= \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right). \end{aligned} \tag{1.2}$$

Proof. X satisfies all properties of Definition 0.1. Therefore it is a Lévy process. Between two jumps of the Poisson process P, X is constant. Then the trajectories are piecewise constant. Now let us compute the characteristic function.

$$\begin{aligned} \mathbb{E}\left(e^{i\langle z,X_t\rangle}\right) &= \mathbb{E}\left(\exp\left(i\langle z, \sum_{n=1}^{P_t} Y_n\rangle\right)\right) = \mathbb{E}\left(\exp\left(\sum_{n=1}^{P_t} i\langle z, Y_n\rangle\right)\right) \\ &= \sum_{k=0}^{+\infty} \mathbb{E}\left(\exp\left(\sum_{n=1}^{P_t} i\langle z, Y_n\rangle\right) \Big| P_t = k\right) \mathbb{P}(P_t = k) \\ &= \sum_{k=0}^{+\infty} \mathbb{E}\left(\prod_{n=1}^{k} \exp\left(i\langle z, Y_n\rangle\right)\right) \mathbb{P}(P_t = k) = \sum_{k=0}^{+\infty} \prod_{n=1}^{k} \mathbb{E}\exp\left(i\langle z, Y_n\rangle\right) \mathbb{P}(P_t = k) \\ &= \sum_{k=0}^{+\infty} \left(\mathbb{E}\exp\left(i\langle z, Y_1\rangle\right)\right)^k e^{-\lambda t} \frac{(\lambda t)^k}{k!} = e^{-\lambda t} \exp\left[\lambda t \mathbb{E}\exp\left(i\langle z, Y_1\rangle\right)\right] \\ &= \exp\left[\lambda t \left(\mathbb{E}\exp\left(i\langle z, Y_1\rangle\right) - 1\right)\right]. \end{aligned}$$

But $\mathbb{E}\exp\left(i\langle z, Y_1\rangle\right) - 1 = \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx)$, and we obtain the result. $\square$

The quantity ν is a finite measure defined on $\mathbb{R}^d$ by

$$\boxed{\nu(A) = \lambda\pi(A),\ A \in \mathcal{B}(\mathbb{R}^d).}$$

Finite measure means that $\nu(\mathbb{R}^d) < +\infty$.

Definition 1.3 (Lévy measure) *ν is called the* ***Lévy measure*** *of the compound Poisson process. Moreover*

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1], \quad \Delta X_t \neq 0,\ \Delta X_t \in A\}\right].$$

Definition 1.4 *The law μ of X_1 is called* compound Poisson distribution *and has a characteristic function given by:*

$$\hat{\mu}(z) = \exp\left(\lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx)\right).$$

Proposition 1.5 *Let X be a compound Poisson process and A and B two disjointed subsets of $\mathbb{R}^d$. Then:*

$$Y_t = \sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in A} \quad \text{and} \quad Z_t = \sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in B}$$

are two independent compound Poisson processes.

Proof. The reader will prove that Y and Z are two compound Poisson processes. Moreover the intensity of Y is $\lambda\pi(A)$ and the jump size distribution of Y is given by the probability π^A defined by:

$$\forall C \subset \mathbb{R}^d, \quad \pi^A(C) = \frac{\pi(C \cap A)}{\pi(A)}.$$

Now

$$\begin{aligned}
\mathbb{E}\exp(iuY_t + ivZ_t) &= \mathbb{E}\exp\left(iu\sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in A} + iv\sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in B}\right) \\
&= \mathbb{E}\exp\left(\sum_{n=1}^{P_t} Y_n(iu\mathbf{1}_{Y_n \in A} + iv\mathbf{1}_{Y_n \in B})\right) \\
&= \mathbb{E}\exp(iuY_t)\mathbb{E}\exp(ivZ_t).
\end{aligned}$$

The last equality has to be proved properly, but the trick is the same as the method used to compute the characteristic function of a compound Poisson process. $\square$

1.3 Jump-diffusion process

Definition 1.5 *A* jump-diffusion process X *is the sum of a Brownian motion and of a independent compound Poisson process. Therefore a jump-diffusion process is a Lévy process.*

In other words we have a k-dimensional Brownian motion $(W_t)_{t\geq 0}$, a $d\times k$ matrix A, a d-dimensioinal vector γ, a Poisson process $(P_t)_{t\geq 0}$ with intensity λ and jump times T_n, and a sequence $(Y_n)_{n\in\mathbb{N}^*}$ of $\mathbb{R}^d$-valued r.v. such that

1. Y_n are i.i.d. with distribution measure π;
2. $(W_t)_{t\geq 0}$, $(P_t)_{t\geq 0}$ and $(Y_n)_{n\in\mathbb{N}^*}$ are independent.

And we define

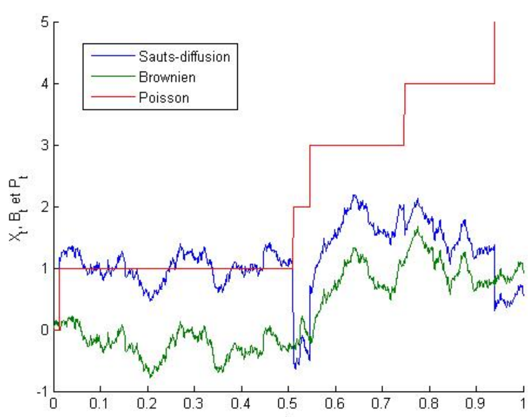
$$\boxed{X_t = AW_t + \gamma t + \sum_{n=1}^{P_t} Y_n = AW_t + \gamma t + \sum_{n=1}^{+\infty} Y_n \mathbf{1}_{[0,t]}(T_n).} \tag{1.3}$$

The characteristic exponent of X is given for any $z\in\mathbb{R}^d$ by:

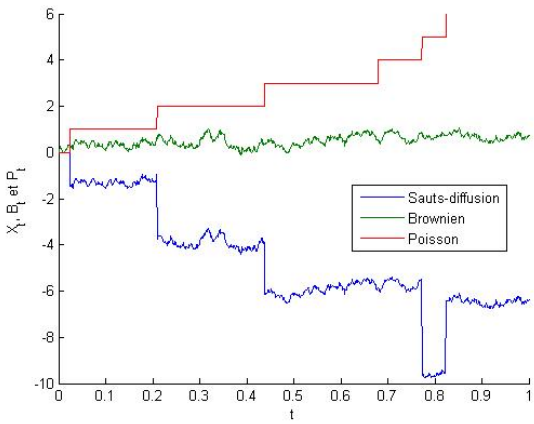
$$\boxed{\begin{aligned} \psi_X(z) &= -\frac{t}{2}\langle z, AA^*z\rangle + it\langle z,\gamma\rangle + t\lambda\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle}-1)\pi(dx) \\ &= -\frac{t}{2}\langle z, AA^*z\rangle + it\langle z,\gamma\rangle + t\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle}-1)\nu(dx). \end{aligned}} \tag{1.4}$$

A^* is the transpose matrix of A.

Let us draw two examples. The first one is a jump-diffusion process with Gaussian jumps (used in the Merton model).



The second one is used in the Kou model where the jump sizes are given by a non symmetric Laplace distribution.



Sauts-diffusion
Brownien
Poisson
t
X_t, B_t et P_t

1.4 Exercises

Exercice 1.1 Soit $N = (N_t)$ un processus de Poisson d'intensité λ. Montrer que $\lim_{t\to+\infty} \frac{N_t}{t} = \lambda$ p.s. et que le processus $\left(\frac{N_t - \lambda t}{\sqrt{\lambda t}}\right)_{t\geq 0}$ converge en loi quand t tend vers $+\infty$ vers une loi gaussienne centrée réduite.

Exercice 1.2 Montrer que si $N = (N_t)$ et $(\tilde{N}_t)_{t\geq 0}$ sont deux processus de Poisson indépendants, d'intensité respective λ et μ, alors la somme est un processus de Poisson d'intensité $\lambda + \mu$.

Exercice 1.3 Soient $(N_t)_{t\geq 0}$ et $(\tilde{N}_t)_{t\geq 0}$ deux processus de Poisson indépendants, d'intensité respective λ et μ.

1. Montrer que le processus $(X_t)_{t\geq 0}$ défini par $X_t = N_t - \tilde{N}_t$ est un processus de Lévy dont on déterminera le triplet caractéristique.

2. On suppose $\lambda \neq \mu$. Montrer que $(X_t/t)_{t\geq 0}$ et $(X_t)_{t\geq 0}$ convergent p.s. dans $\overline{\mathbb{R}}$ quand t tend vers $+\infty$. Préciser la limite de X_t suivant le signe de $\lambda - \mu$.

Exercice 1.4 Une machine possède une durée de vie τ_1 de la loi exponentielle de paramètre λ. Lorsqu'elle tombe en panne, elle est immédiatement remplacée par une machine semblable de durée de vie τ_2, et ainsi de suite. On suppose les durées de vie τ_i indépendantes et identiquement distribuées. La première machine commence à travailler à l'instant 0 et les instants T_k de panne forment un processus de Poisson d'intensité λ.

1. Pour un instant $t > 0$ fixé, soit D_t la durée écoulée depuis la mise en fonctionnement de la machine en marche à l'instant t. Dans quel ensemble la variable aléatoire D_t prend-elle ses valeurs ? Quelle est la loi de D_t ? Montrer que lorsque t tend vers $+\infty$, cette loi admet une loi limite.

2. Soit S_t la variable aléatoire positive telle que $t + S_t$ soit l'instant de défaillance de la machine en fonctionnement à l'instant t. Quelle est la loi de S_t, du couple (D_t, S_t) ? Quelle est la limite de cette dernière lorsque t tend vers $+\infty$?

3. Quelle est la loi de $D_t + S_t$, la durée de vie de la machine en fonctionnement à l'instant t ? Comparer la limite de cette loi quand t tend vers $+\infty$ avec la loi commune des τ_n.

Exercice 1.5 On se donne une mesure ν sur $\mathbb{R}^*$ de densité par rapport à la mesure de Lebesgue :

$$\nu(x) = \frac{c_1}{|x|^{1+\alpha_1}} e^{-\lambda_1|x|}\mathbf{1}_{x<0} + \frac{c_2}{x^{1+\alpha_2}} e^{-\lambda_2 x}\mathbf{1}_{x>0},$$

où c_1 et c_2 sont des constantes positives ou nulles, λ_1 et λ_2 sont strictement positives, tandis que α_1 et α_2 sont des réels strictement négatifs. On considère le processus de Lévy X de triplet caractéristique $(0, \nu, \gamma)$.

1. Montrer que X est un processus de Poisson composé avec dérive.
2. Déterminer l'intensité du processus de Poisson sous-jacent et la distribution des sauts.
3. À quelle condition, ce même processus est-il croissant ?

Exercice 1.6 Soit $X = (X_t)_{t \geq 0}$ un processus de sauts-diffusion de triplet caractéristique (b, σ, ν). Déterminer la fonction caractéristique de X_t pour $t > 0$. En déduire une condition nécessaire et suffisante sur (b, σ, ν) pour que

1. $\mathbb{E}|X_t|^n < +\infty$,
2. $\mathbb{E}e^{uX_t} < +\infty$.

En déduire les valeurs de $\mathbb{E}X_t$, $\operatorname{Var} X_t$, $c_3 = \mu_3 = \mathbb{E}(X_t - \mathbb{E}(X_t))^3$, $\mu_4 = \mathbb{E}(X_t - \mathbb{E}(X_t))^4$, $c_4 = \mu_4 - 3\operatorname{Var}$, et du skewness $s(X_t) = \dfrac{c_3}{\operatorname{Var}^{3/2}}$ et du kurtosis $\kappa = \dfrac{c_4}{\operatorname{Var}^{2}}$.

Chapter 2

Theory of Lévy processes

For jump-diffusion processes, the Lévy measure ν of the process is finite : $\nu(\mathbb{R}^d) < +\infty$. Here we remove this condition and we will just assume a weaker assumption :

$$\int_{\mathbb{R}^d} (1 \wedge |x|^2)\nu(dx) < +\infty,$$

with $u \wedge v = \min(u, v)$. The Lévy measure ν of a general Lévy process will satisfy this condition and in general $\nu(\mathbb{R}^d) = +\infty$.

If the prices are given by a Lévy process with a general measure ν, the prices are processes with a infinite number of jumps during any time period. This case is called **infinite activity model**. Of course the price structure is less intuitive and it is often more complicated to simulate. The advantages are:

- a more realistic description of the prices at different time scales. This property is very important for high-frequency trading for example.
- since the process is often obtained as subordinator of a Brownian motion (time change), there are closed formulas or more tractable than for the jump-diffusion models.

Let us recall some definitions.

Definition 2.1 *A stochastic process $(X_t)_{t \geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is a* Lévy process *if*

1. *its increments are* independent*: for every increasing sequence $t_0, \ldots, t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independents;*
2. *$X_0 = 0$ a.s.*
3. *its increments are* stationnary*: the law of $X_{t+h} - X_t$ does not depend on t;*
4. *X satisfies the property called* stochastic continuity*: for any $\varepsilon > 0$,*

$$\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0;$$

5. *there exists a subset Ω_0 s.t. $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, $t \mapsto X_t(\omega)$* is RCLL.

RCLL means right continuous with left limits. The fourth condition excludes processes with deterministic jump times. For a fixed t, the probability to have a jump is zero.

Remember that the law of a random variable (r.v. in short) is characterized by its characteristic function.

Proposition 2.1 *Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Then there exists a function $\psi : \mathbb{R}^d \to \mathbb{R}$ called* characteristic exponent *of X such that:*

$$\boxed{\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)}.} \tag{2}$$

2.1 Poisson measures

Let us introduce some useful definitions for general Lévy processes.

Definition 2.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ by a probability space, $E \subset \mathbb{R}^k$, and ρ a measure on $(E, \mathcal{E})$. A* Poisson random measure on E with intensity ρ *is a function with values in $\mathbb{N}$:*

$$\begin{array}{rccc} M: & \Omega \times \mathcal{E} & \to & \mathbb{N} \qquad s.t. \\ & (\omega, A) & \mapsto & M(\omega, A), \end{array}$$

1. *$\forall \omega \in \Omega$, $M(\omega, .)$ is a Radon measure on E, that is, $\forall A \in \mathcal{E}$ measurable and bounded, $M(A) < +\infty$ is a r.v. with values in $\mathbb{N}$;*
2. *$\forall A \in \mathcal{E}$, $M(., A) = M(A)$ is a Poisson r.v. with parameter $\rho(A)$;*
3. *for any $A_1, \ldots, A_n$ disjointed sets, the r.v. $M(A_1), \ldots, M(A_n)$ are independents.*

For a compound Poisson process X with intensity λ and jump distribution π, we define a (random) measure on $[0, +\infty[\times \mathbb{R}^d \setminus \{0\}$ by:

$$\boxed{\forall B \subset E = [0, +\infty[\times \mathbb{R}^d \setminus \{0\}, \quad J_X(\omega, B) = \#\{(t, X_t(\omega) - X_{t^-}(\omega)) \in B\}}. \tag{2.1}$$

Proposition 2.2 (Poisson measure) *$J_X : \Omega \times E \to \mathbb{N}$ is a Poisson measure on $E = [0, +\infty[\times \mathbb{R}^d \setminus \{0\}$ with intensity*

$$\rho(dt \times dx) = dt\nu(dx) = \lambda dt\pi(dx).$$

As a consequence, we obtain

$$\boxed{\nu(A) = \mathbb{E}(J_X([0, 1] \times A)) = \mathbb{E}\left[\#\{t \in [0, 1], \quad \Delta X_t \neq 0, \ \Delta X_t \in A\}\right]}. \tag{2.2}$$

Remember that ν is the Lévy measure of X (see Definition 1.3).

Proof. From Equation (2.1), J_X is an integer valued measure. Let us begin to check that $J_X(B)$ is a Poisson r.v. Let P be the Poisson process counting the jumps of X. Conditionally on the trajectory of P, the jump sizes Y_i are i.i.d. and $J_X([t_1, t_2] \times A)$ is a sum of $P_{t_2} - P_{t_1}$ i.i.d. Bernoulli r.v. taking value 1 with probability $\pi(A)$. Then

$$\begin{aligned}
&\mathbb{E} \exp\left(iuJ_X([t_1, t_2] \times A)\right) = \mathbb{E}\left\{\mathbb{E}\left[\exp\left(iuJ_X([t_1, t_2] \times A\right) | P_t,\ t \geq 0\right]\right\} \\
&= \mathbb{E}\left[\left(e^{iu}\pi(A) + 1 - \pi(A)\right)^{P_{t_2} - P_{t_1}}\right] = \exp\left(\lambda(t_2 - t_1)\pi(A)(e^{iu} - 1)\right).
\end{aligned}$$

Thus $J_X([t_1, t_2] \times A)$ is a Poisson r.v. with parameter $\lambda(t_2 - t_1)\pi(A)$, which was to be shown.

Now let us check the independence of measures of disjoints sets. First let us show that if A and B are two disjoint Borel sets in $\mathbb{R}^d$, then $J_X([t_1, t_2] \times A)$ and $J_X([t_1, t_2] \times B)$ are independent. Again conditionally on the trajectory of P, the expression $iuJ_X([t_1, t_2] \times A) + ivJ_X([t_1, t_2] \times B)$ is a sum of $P_{t_2} - P_{t_1}$ i.i.d. r.v. taking values:

- iu with probability $\pi(A)$;
- iv with probability $\pi(B)$;
- 0 with probability $1 - \pi(A) - \pi(B)$.

And now it is easy to check that

$$\begin{aligned}
&\mathbb{E} \exp\left(iuJ_X([t_1, t_2] \times A) + ivJ_X([t_1, t_2] \times B)\right) = \mathbb{E}\left[\left((e^{iu} - 1)\pi(A) + (e^{iv} - 1)\pi(B) + 1\right)^{P_{t_2} - P_{t_1}}\right] \\
&\quad = \exp\left[\lambda(t_2 - t_1)(\pi(A)(e^{iu} - 1) + \pi(B)(e^{iv} - 1)\right] \\
&\quad = \mathbb{E} \exp\left(iuJ_X([t_1, t_2] \times A)\right) \mathbb{E} \exp\left(iuJ_X([t_1, t_2] \times B)\right).
\end{aligned}$$

If $[t_1, t_2]$ and $[s_1, s_2]$ are disjoint sets, the independence of $J_X([t_1, t_2] \times A)$ and $J_X([s_1, s_2] \times A)$ follows directly from the independence of the increments of the process X.

The independence of jump measures of any finite number of disjoint sets of $[0, +\infty[\times \mathbb{R}^d$ follows from the additivity of J_X and from the fact that the method used works for any finite number of sets.

Equation (2.2) comes from the fact that a Poisson r.v. with parameter μ has an exceptation equal to μ. □

Moreover we have

$$\mathbb{E}(J_X([0, t] \times A)) = t\nu(A) = t\lambda \int_A \pi(dx)$$

and we can write X as follows:

$$\boxed{X_t = \sum_{s \in [0,t]} \Delta X_s = \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx).}$$

It is important to note that J_X completely characterize the jumps of X. And since X is a pure jump process, it gives the complete process X.

Proposition 2.3 *If $B_1, \ldots, B_k$ are disjointed sets s.t. $\bigcup_{j=1}^{k} B_j =]0, t] \times \mathbb{R}^d \setminus \{0\}$, then the conditional law of $(J(B_1), \ldots, J(B_k))$ knowing $P_t = n$ is multinomial with parameters n, $(\lambda t)^{-1}\rho(B_1), \ldots, (\lambda t)^{-1}\rho(B_k)$.*

Proof. Sketch the proof of Proposition 1.3. □

Now let M be a Poisson random measure with intensity measure μ and let A be a measurable subset s.t. $0 < \mu(A) < +\infty$. Then the following two random measures on the subsets of A have the same distribution conditionally on $M(A)$:

- $M|_A$, the restriction of M to A.
- $\widehat{M}_A$ defined by $\widehat{M}_A(B) = \sharp\{X_i \in B\}$ for all measurable subsets B of A, where X_i, $i = 1, \ldots, M(A)$ are independent and distributed on A with the law $\frac{\mu(dx)}{\mu(A)}$.

This implies in particular that

$$\boxed{\mathbb{E}\exp\left(\int_A f(x)M(dx)\right) = \exp\left(\int_A (e^{f(x)} - 1)\mu(dx)\right)} \tag{2.3}$$

for any function f such that $\int_A e^{f(x)}\mu(dx) < +\infty$. This can be obtained by conditioning the expectation on $\mu(A)$ and by the previous result on $M|_A$ and $\widehat{M}_A$.

This equation allows to establish a one-to-one correspondence between compound Poisson processes and Poisson random measures with intensity measures of the form $\nu(dx)dt$ with ν finite. Indeed, let ν be a finite measure on $\mathbb{R}^d$ and let M be a Poisson random measure on $\mathbb{R}^d \times [0, +\infty[$ with intensity measure $\nu(dx)dt$. Then one can show that the last equation defines a compound Poisson process with Lévy measure ν.

2.2 Infinitely divisible distributions

Here we study a wide family of probability distributions. This family is related to the law of Lévy process.

Let μ be a probability measure on $\mathbb{R}^d$. Denote by μ^n the convolution product of μ n times μ with herself:

$$\mu^n = \underbrace{\mu * \ldots * \mu}_{n \text{ times}}.$$

Remember that if $X_1, \ldots, X_n$ are i.i.d. r.v. with law μ, the law of $X_1 + \ldots + X_n$ is given by μ^n.

Definition 2.3 *A probability measure on $\mathbb{R}^d$ is* infinitely divisible *if for every $n \in \mathbb{N}^*$, there exists a probability μ_n s.t. $\mu = \mu_n^n$.*

In other words if X is a r.v. with law μ, there exists $Y_1, \ldots, Y_n$ i.i.d. r.v. such that for any positive function f,

$$\mathbb{E}(f(X)) = \mathbb{E}(f(Y_1 + \ldots + Y_n)).$$

The law of Y_1 is denoted μ_n.

Denote by $\hat{\mu}$ the Fourier transform (or the characteristic function) of μ. Then μ is infinitely divisible if $\hat{\mu}$ has a n-th root, which is a characteristic function: for any $z \in \mathbb{R}^d$,

$$\hat{\mu}(z) = (\hat{\mu}_n(z))^n .$$

Gaussian, Cauchy, Poisson, compound Poisson, exponential, gamma, geometric distributions are all infinitely divisible distributions. Moreover an immediate consequence of Definition 2.1 is the following.

Proposition 2.4 *If X is a Lévy processs in law, the law of X_t is infinitely divisible.*

For a jump-diffusion process we have:

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right].$$

Now let us give some technical results concerning this family.

Lemma 2.1 *The convolution product between two infinitely divisible distributions is infinitely divisible.*

Proof. Straightforward if we recall that the Fourier transform of a convolution product is the product of the Fourier transforms. □

Lemma 2.2 (Technical properties)

1. *If μ is infinitely divisible, then $\hat{\mu}$ has no zero on $\mathbb{R}^d$.*
2. *The limit in law of a sequence of infinitely divisible distributions is infinitely divisible.*
3. *If μ is infinitely divisible, then μ^t is well defined and is infinitely divisible for any $t \in [0, +\infty[$ ($\mu^0 = \delta_0$ is the Dirac mass).*

Proof. The proof is rather technical and based on complex analysis. Therefore we admit the result. □

As an exercise, use the first property of this lemma to prove that the uniform law is not infinitely divisible.

The next result is the key point of this paragraph. Denote by $D = \{x \in \mathbb{R}^d,\ |x| \leq 1\}$ the unit ball in $\mathbb{R}^d$.

Theorem 2.1 (Lévy–Khintchine decomposition) *If μ is infinitely divisible on $\mathbb{R}^d$, then $\hat{\mu}$ has the following representation:*

$$\boxed{\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x))\nu(dx)\right]} \tag{2.4}$$

with

- *$A \in \mathcal{S}_d^+(\mathbb{R})$ a symetric positive matrix;*
- *ν a measure on $\mathbb{R}^d$ s.t.*

$$\boxed{\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < +\infty;} \tag{2.5}$$

- *$\gamma \in \mathbb{R}^d$ a vector.*

Moreover this representation of $\hat{\mu}$ is unique.

Conversely if $A \in \mathcal{S}_d^+(\mathbb{R})$, if ν is a measure satisfying (2.5), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible law μ on $\mathbb{R}^d$ with characteristic function given by the formula (2.4).

Definition 2.4 *(A, ν, γ) is called the* characteristic triple *of μ. A is the* Gaussian covariance matrix*, ν the* Lévy measure*.*

Proposition 2.5 *If μ is given by its triple (A, ν, γ), the characteristic triple of μ^t is $(tA, t\nu, t\gamma)$.*

Proof. Left as an exercise. □

We admit the proof of the theorem 2.1. But let us make some remarks on γ. Let $c : \mathbb{R}^d \to \mathbb{R}$ be a bounded function s.t.

$$\begin{cases} c(x) = 1 + o(|x|) & \text{when } |x| \to 0, \\ c(x) = O(1/|x|) & \text{when } |x| \to +\infty. \end{cases} \tag{2.6}$$

Then we can also give the following representation for $\hat{\mu}$:

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_c, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x))\nu(dx)\right]$$

with

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x(c(x) - \mathbf{1}_D(x))\nu(dx).$$

Definition 2.5 *The triple is denoted by $(A, \nu, \gamma_c)_c$ and the previous formula is also a Lévy-Khintchine decomposition of μ.*

It is very important to note that A and ν are intrinsic values of μ. But the vector γ_c depends on the choice of the function c, called the truncation function. We can take $c(x) = \mathbf{1}_{|x|\le\varepsilon}(x)$ with $\varepsilon > 0$, $c(x) = 1/(1 + |x|^2)$, or $c(x) = (\sin x)/x$ in dimension 1. If $c(x) = \mathbf{1}_D(x)$ then we remove the subscript c on γ.

In some particular cases we can relax the assumptions on c.

1. If $\displaystyle\int_{|x|\le 1} |x|\nu(dx) < \infty$, with $c \equiv 0$,

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right].$$

Definition 2.6 *γ_0 is the* drift *of μ.*

2. If $\displaystyle\int_{|x|\geq 1} |x|\nu(dx) < \infty$, with $c \equiv 1$,

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_1, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle)\nu(dx)\right].$$

Definition 2.7 *γ_1 is the* center *of μ. And $\gamma_1 = \int_{\mathbb{R}^d} x\mu(dx)$.*

Now let us give some examples.

Properties 2.1

- *$\nu \equiv 0$ if and only if μ is Gaussian.*
- *If μ is a compound Poisson measure, $A = 0$, $\nu = \lambda\pi$ and $\gamma_0 = 0$.*
- *If $d = 1$ and μ is the Poisson distribution, $A = 0$, $\nu = \lambda\delta_1$, $\gamma_0 = 0$.*
- *If μ is the Γ distribution with parameters c and α*

$$\text{density: } \frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x} \mathbf{1}_{Re_+^*}(x),$$

  then $A = 0$, $\gamma_0 = 0$ and $\nu(dx) = c\dfrac{e^{-\alpha x}}{x}\mathbf{1}_{Re_+^*}(x)$.

- *If μ is the stable distribution with index 1/2,*

$$\text{density: } \frac{c}{\sqrt{2\pi}} e^{-c^2/(2x)} x^{-3/2} \mathbf{1}_{Re_+^*}(x),$$

  then $A = 0$, $\gamma_0 = 0$ and $\nu(dx) = \dfrac{c}{\sqrt{2\pi}} x^{-3/2}\mathbf{1}_{Re_+^*}(x)$.

Proof. See the exercises 2.2 and 2.3. □

Now define the set $C^\sharp = \{f \in C_b(\mathbb{R}^d, \mathbb{R}),\ f \equiv 0 \text{ around } 0\}$ and denote by $c \in C_b(\mathbb{R}^d, \mathbb{R})$ a truncation function satisfying (2.6).

Theorem 2.2 *Let μ_n be a sequence of infinitely divisible laws with triple $(A_n, \nu_n, \gamma_n)_c$. μ_n converges to μ if and only if μ infinitely divisible with triple $(A, \nu, \gamma)_c$ with*

- *if $f \in C^\sharp$, $\displaystyle\lim_{n\to+\infty} \int_{\mathbb{R}^d} f(x)\nu_n(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx)$;*
- *if $A_{n,\varepsilon}$ is defined by: $\langle z, A_{n,\varepsilon} z\rangle = \langle z, A_n z\rangle + \displaystyle\int_{|x|\leq\varepsilon} \langle x, z\rangle^2 \nu_n(dx)$, then:*

$$\forall z \in \mathbb{R}^d, \quad \lim_{\varepsilon\to 0} \limsup_{n\to+\infty} |\langle z, A_{n,\varepsilon} z\rangle - \langle z, Az\rangle| = 0;$$

- $\gamma_n \to \gamma$.

Corollary 2.1 *Any infinitely divisible law (with $A = 0$) is a limit of a sequence of compound Poisson distributions.*

Corollary 2.2 *If $t_n \downarrow 0$, and if ν is the Lévy measure of an infinitely divisible law μ, then for all $f \in C^\sharp$,*

$$\lim_{n\to+\infty} \frac{1}{t_n} \int_{\mathbb{R}^d} f(x)\mu^{t_n}(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx).$$

Let us finish this paragraph with the link with Lévy processes.

Theorem 2.3

1. *If X is a Lévy process (in law), the law of X_t is given by μ^t where μ is the law of X_1.*
2. *If μ is infinitely divisible on $\mathbb{R}^d$, then there exists a Lévy process in law s.t. $\mathbb{P}_{X_1} = \mu$.*

This theorem says that there is a bijection between the family of infinitely divisible laws and the Lévy processes. Since we know the Fourier transform of an infinitely divisible law, we can apply the Lévy-Khintchine decomposition to a Lévy process.

Proposition 2.6 *Let $(X_t)_{t\geq 0}$ be a Lévy process in law. Then there exists $\psi : \mathbb{R}^d \to \mathbb{R}$, the characteristic exponent of X s.t.:*

$$\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)},$$

with characteristic triple (A, ν, γ) *s.t.*

$$\boxed{\psi(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x))\nu(dx).} \tag{2.7}$$

And a triple (A, ν, γ) s.t. ν satisfies the condition (2.5), is the "law" of a Lévy process.

This proposition allows us to classify the Lévy processes.

Definition 2.8 *Let X be a Lévy process with triple (A, ν, γ). X is called of*

- type A *if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$,*
- type B *if $A = 0$, $\nu(\mathbb{R}^d) = \infty$ and $\displaystyle\int_{|x|\leq 1} |x|\nu(dx) < +\infty$,*
- type C *if $A \neq 0$ or $\displaystyle\int_{|x|\leq 1} |x|\nu(dx) = +\infty$.*

Remark that for a type A or B Lévy process, the drift γ_0 is well defined.

2.3 Decomposition of a Lévy process

We have seen that a compound Poisson process Y can be represented by a Poisson measure with intensity $\nu(ds)dt$ where ν is a finite measure defined by:

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1], \quad \Delta Y_t \neq 0,\ \Delta Y_t \in A\}\right], \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

Now if we add a Brownian motion with drift $bt + W_t$, independent of Y, then the sum $X_t = bt + W_t + Y_t$ is a Lévy process which can be written as follows:

$$\begin{aligned} X_t &= bt + W_t + \sum_{s \in [0,t]} \Delta Y_s = bt + W_t + \sum_{s \in [0,t]} \Delta X_s \\ &= bt + W_t + \int_{[0,t]\times\mathbb{R}^d} x J_X(ds \times dx). \end{aligned}$$

The characteristic exponent of this process X is then:

$$\psi(z) = -\frac{1}{2}|z|^2 + i\langle b, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx).$$

Therefore we obtain almost the formula (2.7), except that the condition on ν is more restrictive than (2.5).

Let us come to the general case. Let $E = [0,+\infty[\times\mathbb{R}^d \setminus \{0\}$ and X a Lévy process with characteristic triple (A, ν, γ). We define:

$$\forall B \subset \mathcal{B}(E), \quad J_X(\omega, B) = \#\{(t, X_t(\omega) - X_{t^-}(\omega)) \in B\}.$$

The definition of J_X is the same as for a compound Poisson process (see equation (2.1)). But here since ν satisfies only (2.5), and thus is not necessary finite, we have to remove 0 from $\mathbb{R}^d$. Nevertheless J_X satisfies the same properties.

Proposition 2.7 *The jumps measure J_X is a Poisson measure on E with intensity $dt\nu(dx)$. And ν satisfies* (2.5).

Proof. Indeed since X is RCLL, then the set $\{t,\ |X_t - X_{t^-}| = |\Delta X_t| \geq \varepsilon > 0\}$ is finite and the Poisson random measure (of any closed set not containing 0) can be constructed as Proposition 2.2. The intensity measure of J_X is homogeneous and equal to $\nu(dx)dt$.

We admit the following lemma:

Lemma 2.3 *Let (X_t, Y_t) be a Lévy process. If (Y_t) is compound Poisson and (X_t) and (Y_t) never jump together, then they are independent.*

Now we prove that ν satisfies the condition (2.5). In fact since the Lévy measure of any closed set not containing zero is finite, it is sufficient to prove that for some $\delta > 0$, $\int_{|x|\leq\delta} |x|^2\nu(dx) < +\infty$. Let us define

$$Y_t = \sum_{0\leq s\leq t} \Delta X_s \mathbf{1}_{[1,+\infty[}(|\Delta X_s|) = \int_{1\leq|x|,\ 0\leq s\leq t} x J_X(ds \times dx),$$

and

$$X_t^\varepsilon = \sum_{0\leq s\leq t} \Delta X_s \mathbf{1}_{[\varepsilon,1[}(|\Delta X_s|) = \int_{\varepsilon\leq|x|<1,\ 0\leq s\leq t} x J_X(ds \times dx). \tag{2.8}$$

Y and X^ε are compound Poisson processes. Moreover from Proposition 1.5, Y and X^ε are independent. Now we define $R^\varepsilon = X - Y - X^\varepsilon$. $(X^\varepsilon, R^\varepsilon)$ is a Lévy process and from the previous lemma, X^ε and R^ε are independent. Therefore

$$\mathbb{E} \exp i\langle z, X_t\rangle = (\mathbb{E} \exp i\langle z, Y_t\rangle)\,(\mathbb{E} \exp i\langle z, X^\varepsilon_t\rangle)\left(\mathbb{E} \exp i\langle z, \tilde{R}^\varepsilon_t\rangle\right).$$

And we can find some z and some t such that $|\mathbb{E} \exp i\langle z, X_t\rangle| > 0$. This means that $|\,(\mathbb{E} \exp i\langle z, X^\varepsilon_t\rangle)\,|$ is bounded from below by a positive number which does not depend on ε. By the formula (2.3), we obtain

$$\left| \exp\left(t \int_{\varepsilon \leq |x| < 1} (e^{izx} - 1)\nu(dx) \right) \right| \geq C > 0,$$

which implies that

$$\int_{\varepsilon \leq |x| < 1} (1 - \cos(zx))\nu(dx) \leq \tilde{C} < +\infty.$$

Letting ε tend to zero, we obtain the desired result. □

Definition 2.9 *The Lévy measure ν satisfies*

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1], \quad \Delta X_t \in A\}\right], \ A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}).$$

In particular $\mathbb{E}(J_X([0,t] \times A)) = t\nu(A)$. Define $D(a,b] = \{x \in \mathbb{R}^d, \ a < |x| \leq b\}$ and $D(a, +\infty) = \{x \in \mathbb{R}^d, \ |x| > a\}$.

Theorem 2.4 (Lévy-Khintchine decomposition) *Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (A, ν, γ).*

1. *There exists Ω_1 s.t. $\mathbb{P}(\Omega_1) = 1$ and s.t. for any $\omega \in \Omega_1$,*

$$\begin{aligned} X^1_t(\omega) &= \lim_{\varepsilon \downarrow 0} \int_{]0,t] \times D(\varepsilon,1]} [x J_X(\omega, dsdx) - x ds \nu(dx)] \\ &+ \int_{]0,t] \times D(1,+\infty)} x J_X(\omega, dsdx) \end{aligned}$$

   is defined for every $t \in \mathbb{R}_+$ with uniform time convergence in time on any compact set.

   The process X^1 is a Lévy process with triple $(0, \nu, 0)$.

2. *Denoting $X^2_t = X_t - X^1_t$, there exists a set Ω_2 s.t. $\mathbb{P}(\Omega_2) = 1$ and s.t. for any $\omega \in \Omega_2$, X^2 is a continuous Lévy process with characteristic triple $(A, 0, \gamma)$.*

3. *X^2 is a Brownian motion with covariance matrix A and drift γ.*

4. *The processes X^1 and X^2 are independent.*

Definition 2.10 *X^1 is the* jump part *and X^2 the* continuous part *of X.*

We can X^1 in an other way:

$$\begin{aligned}
X_t^1(\omega) &= \lim_{\varepsilon \downarrow 0} \int_{]0,t]\times D(\varepsilon,1]} [xJ_X(\omega, dsdx) - xds\nu(dx)] \\
&+ \int_{]0,t]\times D(1,+\infty)} xJ_X(\omega, dsdx) \\
&= \lim_{\varepsilon \downarrow 0} \tilde{X}_t^\varepsilon + Y_t;
\end{aligned}$$

where

- $\tilde{X}^\varepsilon$ is a compound Poisson process with jumps size (in norm) between ε and 1, *compensated*: $\mathbb{E}(\tilde{X}_t^\varepsilon) = 0$ (see paragraph 2.4.2 on the moments of a Lévy process);
- Y is a compound Poisson process with jumps size greater than 1.

If we compare with the simple case of the above introduction, we see that the novelty is that we add to X the limit of compensated compound Poisson processes.

Proof. We just sketch the proof. We have seen that

$$X_t = Y_t + X_t^\varepsilon + R_t^\varepsilon,$$

where Y is a compound Poisson process with jump size greater than 1, X^ε is a compound Poisson process with jump size between ε and 1, and R^ε is defined by $X - Y - X^\varepsilon$. Since ν can have a singularity at zero, there can be infinitely many small jumps and their sum does not necessarily converge. This prevents us from making ε go to 0 directly in Equation (2.8). In order to obtain convergence we have to center the remainder term:

$$\tilde{X}_t^\varepsilon = X_t^\varepsilon - \mathbb{E}(X_t^\varepsilon) = X_t^\varepsilon - \int_{]0,t]\times D(\varepsilon,1]} xds\nu(dx).$$

Let us consider a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ and define

$$Z^n = \tilde{X}_t^{\varepsilon_{n+1}} - \tilde{X}_t^{\varepsilon_n}.$$

$(Z^n)_{n\geq 0}$ is a sequence of centered random variables and (2.5) ensures that $\sum_n \mathrm{Var}\,(Z^n) < +\infty$. Hence, by Kolmogorov's three series Theorem, $\sum_n Z^n$ converges almost surely, which means that $\tilde{X}_t^\varepsilon$ converges almost surely as $\varepsilon \to 0$. Using Kolmogorov's maximum inequality, one can show that the convergence is uniform in t.

To complete the proof, consider the process $X_t^c = X_t - Y_t - \lim \tilde{X}_t^\varepsilon$. It is a Lévy process which is independent from $Y_t + \lim_\varepsilon \tilde{X}_t^\varepsilon$. It is continuous, because $\lim \tilde{X}_t^\varepsilon$ converges uniformly in t and therefore one can interchange the limits. Finally, the Feller central limit Theorem implies that it is also Gaussian. □

Moreover one can prove that $\tilde{X}_t^\varepsilon$ is a martingale, converging as $\varepsilon \to 0$ in L^2 to a martingale. Now let us consider the following particular case.

Theorem 2.5 (Lévy process with drift) *Let X be a Lévy process with triple (A, ν, γ) s.t. $\int_{|x|\leq 1} |x|\nu(dx) < +\infty$. Let γ_0 be the drift of X.*

1. *There exists a set Ω_3 s.t. $\mathbb{P}(\Omega_3) = 1$ and for any $\omega \in \Omega_3$, the process*

$$X_t^3(\omega) = \int_{]0,t]\times\mathbb{R}^d\setminus\{0\}} x J_X(\omega, dsdx)$$

is well defined for every $t \geq 0$. It is a Lévy process on $\mathbb{R}^d$ s.t.

$$\mathbb{E}\left(e^{i\langle z, X_t^3\rangle}\right) = \exp\left[t\int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1)\nu(dx)\right].$$

2. *Moreover $X_t^4 = X_t - X_t^3$ continuous Lévy process s.t.*

$$\mathbb{E}\left(e^{i\langle z, X_t^4\rangle}\right) = \exp\left[-t\frac{1}{2}\langle z, Az\rangle + it\langle\gamma_0, z\rangle\right].$$

3. *X^3 and X^4 are independent.*

The difference between this theorem and theorem 2.4 is that if $\displaystyle\int_{|x|\leq 1} |x|\nu(dx) < +\infty$, there is no more limit in the decomposition of the process X. In other words, we can integrate x w.r.t. J_X on the entire set $\mathbb{R}^d \setminus \{0\}$. But be careful: X^3 is not a compound Poisson process because we do not have $\nu(\mathbb{R}^d \setminus \{0\}) < +\infty$ in general.

2.4 Properties of a Lévy process

2.4.1 Sample path properties

The main application of theorem 2.4 (or theorem 2.5) is that from the decomposition we can deduce the properties of the trajectories of a Lévy process X. Moreover since the decomposition is given when the characteristic triple is specified, we can "read" the properties of the trajectories directly on the triple.

Let us begin with two extreme cases.

Proposition 2.8 *A Lévy process is* continuous *if and only if $\nu = 0$. In that case it is a Brownian motion with drift.*

Proof. Remark that the jumps of X are counted by the jumps measure J_X. X is continuous if and only if $J_X = 0$, if and only if the intensity of J_X defined by ν is zero. □

Proposition 2.9 *A Lévy process is* piecewise constant *if and only if it is a compound Poisson process or if it is of type A with $\gamma_0 = 0$, i.e.*

1. *$A = 0$ and $\displaystyle\int_{\mathbb{R}^d} \nu(dx) < +\infty$,*
2. *$\gamma = \int_{|x|\leq 1} x\nu(dx)$;*

or

$$\psi(x) = \int_{\mathbb{R}^d} (e^{iux} - 1)\nu(dx), \quad \textit{with } \nu(\mathbb{R}^d) < +\infty.$$

Proof. A compound Poisson process is piecewise constant. Conversely if X is piecewise constant, then we define the process N by

$$N_t = \sharp\{0 < s \leq t, \ \Delta X_s \neq 0\}.$$

We can easily prove that N is a Poisson process. Now we compute the jump sizes $Y_n = X_{S_n} - X_{S_n^-}$ where $S_n = \inf\{t \geq 0, N_t \geq n\}$. To prove that X is a compound Poisson process it remains to prove that the random variables Y_n are i.i.d.

First we would like to show that the increments of X conditionally on the trajectory of N are independent. Let $t > s$ and consider the following four sets:

$$A_1 \in \sigma(X_s), \quad B_1 \in \sigma(N_r, r \leq s), \quad A_2 \in \sigma(X_t - X_s), \quad B_2 \in \sigma(N_r - N_s, r > s).$$

such that $\mathbb{P}(B_1) > 0$ and $\mathbb{P}(B_2) > 0$. The independence of increments of X implies that processes $(X_r - X_s, r > s)$ and $(X_r, r \leq s)$ are independent. Hence,

$$\mathbb{P}[A_1 \cap B_1 \cap A_2 \cap B_2] = \mathbb{P}[A_1 \cap B_1]\mathbb{P}[A_2 \cap B_2].$$

Moreover

- A_1 and B_1 are independent from B_2;
- A_2 and B_2 are independent from B_1;
- B_1 and B_2 are independent from each other.

Therefore, the conditional probability of interest can be expressed as follows:

$$\begin{aligned}\mathbb{P}[A_1 \cap A_2 | B_1 \cap B_2] &= \frac{\mathbb{P}[A_1 \cap B_1]\mathbb{P}[A_2 \cap B_2]}{\mathbb{P}(B_1)\mathbb{P}(B_2)} \\ &= \frac{\mathbb{P}[A_1 \cap B_1 \cap B_2]\mathbb{P}[A_2 \cap B_1 \cap B_2]}{\mathbb{P}(B_1)^2\mathbb{P}(B_2)^2} = \mathbb{P}[A_1 | B_1 \cap B_2]\mathbb{P}[A_2 | B_1 \cap B_2].\end{aligned}$$

This proves that $X_t - X_s$ and X_s are independent conditionally on the trajectory of N. In particular, choosing $B_1 = \{N_s = 1\}$ and $B_2 = \{N_t - N_s = 1\}$, we obtain that Y_1 and Y_2 are independent. Since we could have taken any number of increments of X and not just two of them, this proves that $(Y_n)_{n \geq 1}$ are independent.

Finally, to prove that the jump sizes have the same law, observe that the two-dimensional process (X_t, N_t) has stationary increments. Therefore, for every $n \geq 0$ and for every $s > h > 0$,

$$\mathbb{E}[f(X_h) | N_h = 1, N_s - N_h = n] = \mathbb{E}[f(X_{s+h} - X_s) | N_{s+h} - N_s = 1, N_s - N_h = n],$$

where f is any bounded Borel function. This entails that for every $n \geq 0$, Y_1 and Y_{n+2} have the same law which completes the proof of the first part.

To finish the proof, remark that the characteristic function of a Lévy process is given by (1.2):

$$\mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle} - 1)\nu(dx)\right) = \exp(t\psi(z)).$$

Compare this with (2.7):

$$\psi(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x))\nu(dx),$$

Therefore $\nu(\mathbb{R}^d) < +\infty$, $A = 0$, and

$$\gamma - \int_{\mathbb{R}^d} (x\mathbf{1}_D(x))\nu(dx) = 0,$$

which achieves the proof.

□

Remark that if we remove the second condition, we obtain a piecewise affine trajectory because X will be a compound Poisson process with drift.

Theorem 2.6 (Jumps repartition)

1. *If $\nu(\mathbb{R}^d) = +\infty$, then a.s. the jumping times are countable and dense in $\mathbb{R}_+$.*
2. *If $0 < \nu(\mathbb{R}^d) < +\infty$, there is an infinite countable jumping times, but only a finite number on any bounded interval. Moreover the first jumping time has an exponential distribution with parameter $\nu(\mathbb{R}^d)$.*

Proof. We admit the result in the case $\nu(\mathbb{R}^d) = +\infty$. In the other case, X is the sum of a Brownian motion with drift and a compound Poisson process with intensity $\lambda = \nu(\mathbb{R}^d)$. Hence the first jumping time τ satisfies

$$\mathbb{P}(\tau \leq t) = \mathbb{P}(N_t \geq 1) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-\lambda t}.$$

We deduce the law of τ. □

We recall that the total variation of a function $f : [a, b] \to \mathbb{R}^d$ is defined by

$$TV(f) = \sup \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a, b]$. If $TV(f) < +\infty$, f is of finite variation. In particular, in one dimension every increasing or decreasing function is of finite variation and every function of finite variation is a difference of two increasing functions. A Lévy process is said to be of finite variation if its trajectories are functions of finite variation with probability 1.

Theorem 2.7 (Finite variations) *A Lévy process is of bounded variation if and only if it is of type A or B.*

In this case:

$$X_t = \gamma_0 t + \int_{[0,t]\times\mathbb{R}^d} x J_X(ds \times dx) = \gamma_0 t + \sum_{s\in[0,t]}^{\Delta X_s \neq 0} \Delta X_s.$$

The characteristic function becomes:

$$\mathbb{E}\left(e^{i\langle z.X_t\rangle}\right) = \exp t \left[i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle x,z\rangle} - 1)\nu(dx) \right].$$

Proof. Assume that X is of type A or B:

$$A = 0, \qquad \int_{|x|\leq 1} |x|\nu(dx) < +\infty.$$

Then we use Theorem 2.5. X can be written as follows

$$X_t = bt + \int_{|x|\geq 1, s\in[0,t]} xJ_X(ds\times dx) + \lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon$$

with

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon\leq|x|<1, s\in[0,t]} xJ_X(ds\times dx).$$

The first two terms are of finite variation, therefore we only need to consider the third term. Its variation on the interval $[0,t]$ is

$$TV(\tilde{X}_t^\varepsilon) = \int_{\varepsilon\leq|x|<1, s\in[0,t]} |x|J_X(ds\times dx).$$

Since the integrand in the right-hand side is positive, we obtain, using Fubini's theorem

$$\mathbb{E}(TV(\tilde{X}_t^\varepsilon)) = t\int_{\varepsilon\leq|x|<1} |x|\nu(dx),$$

which converges to a finite value when $\varepsilon\to 0$. Therefore $\mathbb{E}[TV(\lim_{\varepsilon\to 0}\tilde{X}_t^\varepsilon)] < \infty$, which implies that the variation of X_t is almost surely finite.

Conversely consider the Lévy decomposition of X_t. Since the variation of any cadlag function is greater or equal to the sum of its jumps, we have for every $\varepsilon > 0$:

$$\begin{aligned} TV(X_t) &\geq \int_{\varepsilon\leq|x|<1, s\in[0,t]} |x|J_X(ds\times dx) \\ &= t\int_{\varepsilon\leq|x|<1} |x|\nu(dx) + \int_{\varepsilon\leq|x|<1, s\in[0,t]} (|x|(J_X(ds\times dx) - \nu(dx)ds). \end{aligned}$$

Using the exponential formula (2.3) one can show that the variance of the second term is equal to $t\int_{\varepsilon\leq|x|<1} |x|^2\nu(dx)$. Hence, by the same argument that was used in the proof of Lévy decomposition, the second term converges almost surely to something finite. Therefore, if the condition $\int(|x|\wedge 1)\nu(dx) < \infty$ is not satisfied, the first term will diverge and the variation of Xt will be infinite. Suppose now that this condition is satisfied. This means that X_t may be written as

$$X_t = X_t^c + \int_{[0,t]\times\mathbb{R}^d} xJ_X(ds\times dx),$$

where the second term is of finite variation. Since trajectories of Brownian motion are almost surely of infinite variation, if A is nonzero, X_t will also have infinite variation. Therefore we must have A $= 0$.

□

Proposition 2.10 (Monotone process) *A Lévy process is non-decreasing if and only if*

1. $X_t \geq 0$, *a.s. for all* t;

2. *if and only if*

   - $A = 0$ *and* $\nu(]-\infty, 0]) = 0$,

- $\displaystyle\int_0^1 x\nu(dx) < +\infty$ *avec* $\gamma_0 \geq 0$.

Proof. If X is non decreasing, since $X_0 = 0$, $X_t \geq X_0 = 0$. Now assume that $X_t \geq 0$ for some $t > 0$. For every n, X_t is the sum of n i.i.d. random variables $X_{t/n}$, $X_{2t/n} - X_{t/n}, \ldots, X_t - X_{(n-1)t/n}$. This means that all these variables are almost surely nonnegative. With the same logic we can prove that for any two rationals p and q such that $0 < p < q$, $X_{qt} - X_{pt} \geq 0$ a.s. Since the trajectories are right-continuous, this entails that they are nondecreasing.

Since the trajectories are nondecreasing, they are of finite variation. Therefore, $A = 0$ and $\int_{\mathbb{R}} (x \wedge 1)\nu(dx) < \infty$. For trajectories to be nonincreasing, there must be no negative jumps, hence $\nu(]-\infty, 0]) = 0$. If a function is nondecreasing then after removing some of its jumps, we obtain another nondecreasing function. When we remove all jumps from a trajectory of X_t, we obtain a deterministic function $\gamma_0 t$ which must therefore be nondecreasing. This allows to conclude that $\gamma_0 \geq 0$.

Conversely under the conditions on (A, ν, γ) the process is of finite variation, therefore equal to the sum of its jumps plus an increasing linear function. For every trajectory the number of negative jumps on any fixed interval is a Poisson random variable with intensity 0, hence, almost surely zero. This means that almost every trajectory is nondecreasing.

□

The surprising fact is that if $A = 0$, $\nu(] - \infty, 0]) = 0$ and $\displaystyle\int_0^1 x\nu(dx) = +\infty$, the process has just non-negative jumps, but whatever γ, it is not non-decreasing. It has infinite negative drift !

Definition 2.11 *A non-decreasing Lévy process is called a* subordinator*.*

In this case the Laplace transform of X can be written: for $u \geq 0$

$$\mathbb{E}(e^{-uX_t}) = \exp\left[t \int_0^{+\infty} (e^{-ux} - 1)\nu(dx) - t\gamma_0 u\right].$$

As an example we can cite the following result.

Proposition 2.11 *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$ *and let* $f : \mathbb{R}^d \to [0, \infty[$ *be a positive function such that* $f(x) = O(|x|^2)$ *when* $x \to 0$. *Then the process* $(S_t)_{t\geq 0}$ *defined by*

$$S_t = \sum_{s \leq t, \Delta X_s \neq 0} f(\Delta X_s),$$

is a subordinator.

For $f(x) = |x|^2$, the sum of the squared jumps

$$S_t = \sum_{s \leq t, \Delta X_s \neq 0} |\Delta X_s|^2$$

is a non decreasing Lévy process.

Proof. We will just proved that the sum converges. Indeed we can write:

$$S_t = \sum_{s \leq t, \Delta X_s \neq 0, |\Delta X_s| \leq \eta} f(\Delta X_s) + \sum_{s \leq t, \Delta X_s \neq 0, |\Delta X_s| > \eta} f(\Delta X_s).$$

The second term is in fact a finite sum, hence converges. The first one can be bounded by

$$\sigma_t = K \sum_{s \leq t, \Delta X_s \neq 0, |\Delta X_s| \leq \eta} |\Delta X_s|^2.$$

But

$$\mathbb{E}\sigma_t = Kt \int_{s \leq t, |x| \leq \eta} |x|^2 \nu(dx) < +\infty.$$

Therefore σ_t is finite almost surely.

$\square$

2.4.2 Moments

It is well known that a Brownian motion with drift $X_t = bt + AW_t$ has exponential moment:

$$\forall u \in \mathbb{R}^d, \quad \mathbb{E} \exp(\langle u, X_t \rangle) < +\infty.$$

So by independance between the continuous part and the jump part of a Lévy process, integrability problem can just arise because of the Lévy measure ν.

Proposition 2.12 *Let $(X_t)_{t \geq 0}$ be a Lévy process in $\mathbb{R}^d$ with triple (A, ν, γ). If ν is of bounded support, then X has an exponential moment, thus moments of all orders.*

Proof. Let C be a bound for the jumps pf X. Define the stopping times

$$T_1 = \inf\{t, |X_t| \geq C\}, \quad T_{n+1} = \inf\{t > T_n,\ |X_t - X_{T_n}| \geq C\}.$$

Since the paths are right continuous, the stopping times $(T_n)_{n \geq 1}$ form a increasing sequence. Moreover $|\Delta X_T| \leq C$ for any stopping time T. Therefore by recursion $\sup_s |X_{s \wedge T_n}| \leq 2nC$. From the independence of the increments, $T_n - T_{n-1}$ is independent of $\sigma(X_s,\ 0 \leq s \leq T_n)$ and from the stationnarity $T_n - T_{n-1}$ has the same law as T_1. Therefore

$$\mathbb{E}e^{-T_n} = (\mathbb{E}e^{-T_1})^n = \alpha^n$$

for some $\alpha < 1$. And

$$\mathbb{P}(|X_t| > 2nC) \leq \mathbb{P}(T_n < t) \leq \frac{\mathbb{E}e^{-T_n}}{e^{-t}} \leq e^t \alpha^n.$$

This implies that $\mathbb{E} \exp(\lambda |X_t|) < +\infty$ for $\lambda < -\ln(1/\alpha)$. $\square$

Definition 2.12 *A function on $\mathbb{R}^d$ is under-multiplicative if it is non-negative and if there exists a constant $a > 0$ s.t.*

$$\forall (x, y) \in (\mathbb{R}^d)^2,\ g(x+y) \leq a g(x) g(y).$$

Lemma 2.4

1. *The product of two submultiplicative functions is submultiplicative.*
2. *If g is submultiplicative on $\mathbb{R}^d$, then so is $g(cx+\gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$ and $\alpha > 0$.*

3. *Let $0 < \beta \leq 1$. Then the following functions are submultiplicative.*

$$|x| \vee 1 = \max(|x|, 1), \quad |x_j| \vee 1,\ x_j \vee 1, \quad \exp(|x|^\beta),\ \exp(|x_j|^\beta),$$

$$\exp((x_j \vee 0)^\beta), \quad \ln(|x| \vee e),\ \ln(|x_j| \vee e),\ \ln(x_j \vee e),$$

$$\ln\ln(|x| \vee e^e),\ \ln\ln(|x_j| \vee e^e),\ \ln\ln(x_j \vee e^e).$$

Here x_j is the jth component of x.

4. *If g is submultiplicative and locally bounded, then $g(x) \leq be^{c|x|}$ for some $b > 0$ and $c > 0$.*

Theorem 2.8 (Moments of a Lévy process) *Let g be a under-multiplicative function, locally bounded on $\mathbb{R}^d$. Then there is an equivalence between*

- *there exists $t > 0$ s.t. $\mathbb{E}(g(X_t)) < +\infty$*
- *for any $t > 0$, $\mathbb{E}(g(X_t)) < +\infty$.*

Moreover $\mathbb{E}(g(X_t)) < +\infty$ if and only if $\displaystyle\int_{|x|\geq 1} g(x)\nu(dx) < +\infty$.

Hence $\mathbb{E}(|X_t|^n) < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$. In particular

$$\mathbb{E}(X_t) = t\left(\gamma + \int_{|x|\geq 1} x\nu(dx)\right) = t\gamma_1, \tag{2.9}$$

and

$$(\mathrm{Var}\ X_t)_{ij} = t\left(A_{ij} + \int_{\mathbb{R}^d} x_i x_j \nu(dx)\right). \tag{2.10}$$

Theorem 2.9 (Exponential moments) *Let X be a Lévy process with triple (A, ν, γ). Let*

$$C = \left\{c \in \mathbb{R}^d,\ \int_{|x|\geq 1} e^{\langle c,x\rangle}\nu(dx) < +\infty\right\}.$$

1. *C is convex and contains 0.*
2. *$c \in C$ if and only if $\mathbb{E}(e^{\langle c,X_t\rangle}) < +\infty$ for some $t > 0$ or equivalently for any $t > 0$.*
3. *If $w \in \mathbb{C}^d$ is s.t. $Re(w) \in C$, then*

$$\psi(w) = \frac{1}{2}\langle w, Aw\rangle + \langle \gamma, w\rangle + \int_{\mathbb{R}^d} (e^{\langle w,x\rangle} - 1 - \langle w, x\rangle \mathbf{1}_D(x))\nu(dx)$$

has a sense, $\mathbb{E}(e^{\langle w,X_t\rangle}) < +\infty$ and $\mathbb{E}(e^{\langle w,X_t\rangle}) = e^{t\psi(w)}$.

2.4.3 Densities

A first problem is the continuity of the law. Now clearly

Lemma 2.5 *If X is a compound Poisson process, then $\mathbb{P}(X_t = 0) = e^{-\lambda t} > 0$.*

So a compound Poisson process can not have a density (w.r.t. the Lebesgue measure).

Theorem 2.10 *Let X be a Lévy process with triple (A, ν, γ). Equivalence between:*

1. *$\mathbb{P}(X_t)$ is continuous for every $t > 0$,*
2. *$\mathbb{P}(X_t)$ is continuous for one $t > 0$,*
3. *X is of type B or C (i.e. $A \neq 0$ or $\nu(\mathbb{R}^d) = +\infty$).*

Corollary 2.3 *Let X be a Lévy process with triple (A, ν, γ). Equivalence between:*

1. *$\mathbb{P}(X_t)$ is discrete for every $t > 0$,*
2. *$\mathbb{P}(X_t)$ is discrete for one $t > 0$,*
3. *X is of type A and ν discrete.*

Now to prove that X has a density w.r.t. Lebesgue measure, it is not easy.

Proposition 2.13 *Let X be a d-dimensional Lévy process with triple (A, ν, γ) with A of rank d. Then the law of X_t, $t > 0$, is absolutely continuous.*

In the previous proposition, the density is "given" by the Brownian part of X. There is no result in general, except in dimension 1.

Theorem 2.11 (Sufficient conditions for $d = 1$) *Let X be a Lévy process with triple (A, ν, γ).*

1. *If $A \neq 0$ or if $\nu(\mathbb{R}) = +\infty$, X_t has a continuous density on $\mathbb{R}$.*
2. *If the Lévy measure satisfies:*

$$\exists \beta \in]0, 2[, \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 \nu(dx) > 0$$

   then for every $t > 0$, X_t has a density of class C^∞ and all derivatives of this density go to zero when $|x|$ goes to $+\infty$.

2.5 Lévy processes, martingales and Markov processes.

Theorem 2.12 (Markov property) *Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ and X the associated Lévy process. Then X is a Markov process with transition function*

$$P_t(x, B) = \mu^t(B - x).$$

Conversely one can prove that every time homogeneous Markov process, with space homogeneous transition function, is a Lévy process in law.

Proposition 2.14 *Let $(X_t)_{t\geq 0}$ be a Lévy process (in law). Then for every $s \geq 0$, the process $(X_{t+s} - X_s)_{t\geq 0}$ is a Lévy process with the same distribution as $(X_t)_{t\geq 0}$. And the two processes are independent.*

Remember that a strong technical assumption in the stochastic calculus is the right-continuity of the filtration generated by a Brownian motion. The same holds for a Lévy process in general.

Theorem 2.13 (Lévy filtration) *Let X be a Lévy process in law and $\mathcal{F}$ its completed filtration. Then $\mathcal{F}$ is right-continuous.*

Theorem 2.14 (Strong Markov) *Let X be a Lévy process in law and $\mathcal{F}$ its completed filtration. Let τ be a a.s. finite $\mathcal{F}$-stopping time. Then the process $(X_{t+\tau} - X_\tau)_{t\geq 0}$ is independent of $\mathcal{F}_\tau$ and with the same law as X.*

Since a Lévy process X is a strong Markov process, X has a infinitesimal generator. If (A, ν, γ) is the characteristic triple of X, then the infinitesimal generator of X is given by:

$$\begin{aligned} \mathcal{L}f(x) &= \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \langle \gamma, \nabla f(x)\rangle \\ &+ \int_{\mathbb{R}^d} \left[f(x+y) - f(x) - \langle y, \nabla f(x)\rangle \mathbf{1}_{|y|\leq 1}\right] \nu(dy). \end{aligned} \tag{2.11}$$

∇ is the gradient of f. Remark that the two first terms are the generator of the Brownian motion with covariance matrix A and a drift γ. The novelty is the third non local term.

Now let us come to the martingale property.

Proposition 2.15 *Let X be a process with independent increments. Then*

1. *for every $u \in \mathbb{R}^d$, $\left(\dfrac{e^{i\langle u, X_t\rangle}}{\mathbb{E}(e^{i\langle u, X_t\rangle})}\right)_{t\geq 0}$ is a martingale.*
2. *If $\mathbb{E}(e^{\langle u, X_t\rangle}) < \infty$, $\forall t \geq 0$, then $\left(\dfrac{e^{\langle u, X_t\rangle}}{\mathbb{E}(e^{\langle u, X_t\rangle})}\right)_{t\geq 0}$ is a martingale.*
3. *If $\mathbb{E}(|X_t|) < \infty$, $\forall t \geq 0$, then $M_t = X_t - \mathbb{E}(X_t)$ is a martingale (with independent increments).*

4. In dimension 1, if $Var(X_t) < +\infty$, $\forall t \geq 0$, then $(M_t)^2 - \mathbb{E}((M_t)^2)$ is a martingale.

Proof. Left to the reader. □

Proposition 2.16 (Martingales) *A Lévy process is a martingale if and only if*

$$\int_{|x|\geq 1} |x|\nu(dx) < +\infty, \quad \text{and} \quad \gamma + \int_{|x|\geq 1} x\nu(dx) = 0.$$

In dimension 1, $\exp(X)$ *is a martingale if and only if* $\int_{|x|\geq 1} e^x \nu(dx) < +\infty$ *and*

$$\frac{A}{2} + b + \int_{-\infty}^{+\infty} (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu(dx) = 0.$$

Proof. Use the previous proposition, Equation (2.9) and Theorem 2.9. □

2.6 Exercises

Lois infiniment divisibles

Exercice 2.1

1. Calculer la fonction caractéristique de la loi de Laplace de paramètre $\lambda > 0$ de densité $f(x) = \dfrac{\lambda}{2}\exp(-\lambda|x|)$.
2. Par transformée de Fourier inverse, calculer la fonction caractéristique d'une loi de Cauchy de paramètre $c > 0$ de densité $g(x) = \dfrac{c}{\pi(c^2 + x^2)}$.
3. En déduire que la loi de Cauchy est infiniment divisible.

Exercice 2.2 Soit X v.a. de loi Γ de paramètres c et α donnée par sa densité :

$$f_{c,\alpha}(x) = \frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x} \mathbf{1}_{\mathbb{R}_+^*}(x).$$

1. Calculer la transformée de Laplace de X : $\mathbb{E}(e^{-uX})$ avec $u \geq 0$.
2. En déduire sa fonction caractéristique.
3. Montrer alors que les lois Γ et exponentielles sont infiniment divisibles.
4. Calculer leurs triplets caractéristiques.

Exercice 2.3 Soit X variable aléatoire de loi de densité :

$$f_c(x) = \frac{c}{\sqrt{2\pi}} e^{-c^2/(2x)} x^{-3/2} \mathbf{1}_{\mathbb{R}_+^*}(x).$$

1. Calculer la transformée de Laplace de X : $\mathcal{L}(u) = \mathbb{E}(e^{-uX})$ avec $u \geq 0$. On pourra chercher une équation différentielle satisfaite par $\mathcal{L}$ en faisant dans la dérivée de $\mathcal{L}$ le changement de variable $ux = c^2/(2y)$.

2. En déduire sa fonction caractéristique :

$$\forall z \in \mathbb{R},\ \hat{\mu}(z) = \exp\left(-c|z|^{1/2}(1 - i\text{sgn }(z))\right).$$

   On pourra se contenter de vérifier qu'elle coïncide avec la transformée de Laplace sur le bon ensemble.

3. Montrer alors que la loi de X est infiniment divisible.

4. Prouver que

$$\int_0^\infty (e^{-ux} - 1) x^{-3/2} dx = -2\sqrt{\pi u}.$$

   On rappelle que $\Gamma(1/2) = \displaystyle\int_0^\infty s^{-1/2} e^{-s} ds = \sqrt{\pi}$ et que $e^{-ux} - 1 = -u \displaystyle\int_0^x e^{-uy} dy$.

5. En déduire le triplet caractéristique de cette loi. Est-ce une loi de Poisson composé ?

Processus de Lévy

Exercice 2.4 On se donne une mesure ν sur $\mathbb{R}^*$ de densité par rapport à la mesure de Lebesgue :

$$\nu(x) = \frac{c_1}{|x|^{1+\alpha_1}} e^{-\lambda_1 |x|} \mathbf{1}_{x<0} + \frac{c_2}{x^{1+\alpha_2}} e^{-\lambda_2 x} \mathbf{1}_{x>0},$$

où c_1 et c_2 sont des constantes positives ou nulles, λ_1 et λ_2 sont strictement positives, tandis que α_1 et α_2 sont des réels strictement inférieurs à 2.

1. Montrer que ν est une mesure de Lévy.

2. À quelle condition sur α_1 et α_2, le processus de Lévy de triplet caractéristique (A, ν, γ) est-il un processus de Poisson composé ? Dans ce cas déterminer l'intensité du processus de Poisson sous-jacent et la distribution des sauts.

3. À quelle condition sur α_1 et α_2, ce même processus est-il à variation finie ?

4. À quelle condition, ce même processus est-il croissant ?

5. Montrer qu'un processus ayant pour mesure de Lévy peut être représenté comme un mouvement brownien changé de temps par un subordinateur si et seulement si $c_1 = c_2$ et $\alpha_1 = \alpha_2 = \alpha \geq -1$.

6. À quelle condition sur ν, peut-on affirmer que X admet une densité de classe C^∞ ?

Exercice 2.5 (Problème de l'examen de 2010-2011) Au début des années 2000, W. Schoutens a proposé de modéliser des cours d'actifs via le processus de Meixner (avec application au Nikkei-225 ou S&P 500). Celui-ci, noté $X = (X_t,\ t \geq 0)$ dans la suite, a une structure simple, stable par changement de probabilité, et donne des formules semi-fermées, comme pour le modèle de Black-Scholes.

Le processus de Meixner est déterminé par sa fonction caractéristique :

$$\forall t \geq 0, \quad \Phi_t(u) = \mathbb{E}(e^{iuX_t}) = \left(\frac{\cos(b/2)}{\text{ch}\ (\frac{au-ib}{2})} \right)^{2dt} e^{imut}.$$

Les paramètres de ce modèle vérifient :

$$a > 0, \quad d > 0, \quad -\pi < b < \pi, \quad m \in \mathbb{R}.$$

La loi de X_1 est appelée loi de Meixner et notée $M(a,b,d,m)$.
Partie 1 : propriétés de ces lois.

1. Montrer que la loi $M(a,b,d,m)$ est infiniment divisible.

2. Calculer la moyenne μ, la variance σ^2, son skewness $\mathbb{E}\left(\frac{X_1-\mu}{\sigma}\right)^3$ et son kurtosis défini par $\mathbb{E}\left(\frac{X_1-\mu}{\sigma}\right)^4$. On rappelle que le skewness de la loi normale est toujours nul tandis que son kurtosis vaut toujours 3. Que constate-t-on ici ?

   Indication : une fois la moyenne μ calculée, on pourra poser $f(u) = \mathbb{E}(e^{iu(X_1-\mu)})$ et constater que $f'(u) = f(u)g(u)$ avec $g(0) = 0$.

On note (A, ν, γ) le triplet de Lévy-Khintchine de cette loi et on pose

$$\Psi(u) = 2d \ln\left(\frac{\cos(b/2)}{\cos(\frac{au+b}{2})} \right) + mu, \qquad u \in \left] \frac{-\pi-b}{a}, \frac{\pi-b}{a} \right[.$$

Ψ est la fonction génératrice des cumulants : $\Phi_t(-iu) = \exp(t\Psi(u))$.

3. Montrer que
$$\Psi''(u) = d \int_{-\infty}^{+\infty} \frac{x \exp\left(\frac{b}{a}x\right)}{\text{sh}\left(\frac{\pi}{a}x\right)} e^{xu} dx.$$
   On utilisera les propriétés de la fonction Γ d'Euler données à la fin de l'énoncé.

4. En déduire que $A = 0$ et que ν admet une densité par rapport à la mesure de Lebesgue donnée par
$$\forall x \in \mathbb{R},\ x \neq 0, \quad \nu(dx) = \left(d \frac{\exp\left(\frac{b}{a}x\right)}{x\ \text{sh}\left(\frac{\pi}{a}x\right)} \right) dx.$$

5. Vérifier que ν satisfait bien les conditions pour être une mesure de Lévy.
6. Montrer qu'alors : $\gamma = m + ad \tan \frac{b}{2} - 2d \int_1^\infty \frac{\mathrm{ch}\, \frac{bx}{a}}{\mathrm{sh} \frac{\pi x}{a}} dx.$

Partie 2 : propriétés du processus X.

1. Montrer que c'est un processus de type C et en déduire qu'il est à variations infinies.
2. Montrer qu'il existe $\beta \in]0, 2[$ tel que $\liminf_{\varepsilon \downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 \nu(dx) > 0$. En déduire que X_t admet une densité de classe C^∞ pour tout $t > 0$.
3. En utilisant la fonction caractéristique de X, ainsi que la relation (2.12) (cf. fin de l'énoncé), montrer que X_t a pour densité :

$$\forall x \in \mathbb{R}, \qquad \phi(t, x) = \frac{(2 \cos \frac{b}{2})^{2dt}}{2\pi a \Gamma(2dt)} \exp\left(\frac{b}{a}(x - mt)\right) \left|\Gamma\left(dt + i\frac{x - mt}{a}\right)\right|^2.$$

4. Montrer que pour $u > 0$, $\mathbb{E}(\exp(uX)) < \infty$ si et seulement si $u < \frac{\pi - b}{a}$.

Rappels : ch (resp. sh) désigne le cosinus (resp. sinus) hyperbolique :

$$\mathrm{ch}\,(x) = (e^x + e^{-x})/2, \qquad \mathrm{sh}(x) = (e^x - e^{-x})/2.$$

De plus

$$\cos(x) = (e^{ix} + e^{-ix})/2, \qquad \sin(x) = (e^{ix} - e^{-ix})/(2i).$$

Propriétés de la fonction Γ d'Euler :
(2.12)

$$\int_{-\infty}^{+\infty} |\Gamma(\delta + ix)|^2 e^{izx} dx = 2\pi\Gamma(2\delta) \left(\frac{1}{2\mathrm{ch}\, \frac{z}{2}}\right)^{2\delta}, \qquad \delta > 0, \quad z \in \mathbb{C}, \ -\pi < \mathrm{Im}(z) < \pi;$$

$$|\Gamma(1 + ix)|^2 = \frac{\pi x}{\mathrm{sh}(\pi x)}.$$

Chapter 3

Simulation of Lévy processes

For the simulation, the dimension can 1 or d, the simulation is the same. For the Brownian part $X_t = \sigma W_t + \gamma t$, one simulation is based on the fact that W is a sum of Gaussian increments. Thus the procedure is the following:

1. split the interval $[0, t]$ by a grid $t_0 = 0 < t_1 < \ldots < t_n = t$,
2. simulate n standard Gaussian r.v. Z_i,
3. define $\Delta X_i = \sigma\sqrt{t_i - t_{i-1}} Z_i + b(t_i - t_{i-1})$,
4. put $X_i = \sum_{k=1}^{i} \Delta X_i$.

The simulation is exact on the grid in the sense that X_i has the same law as X_{t_i}. Between X_i and X_{i+1}, one can used a linear interpolation. Of course, other methods exist to simulate W (Brownian bridges, Fourier decomposition, random walk approximations, etc.).

The simulation of the compound Poisson process is based on Proposition 1.2. The jump times of a Poisson process knowing the value at time t have the same distribution as the order statistics of uniform r.v. on $[0, t]$. Hence to simulate the compound Poisson part on the interval $[0, T]$, the algorithm is:

1. simulate a Poisson r.v. N with parameter λT,
2. simulate N independent r.v. U_i with uniform law on $[0, T]$,
3. simulate the jumps: N independent r.v. Y_i with distribution $\nu(dx)/\lambda$,
4. put $X_t = \sum_{i=1}^{N} \mathbf{1}_{U_i < t} Y_i$.

As before X_t, $t \in [0, T]$, has the asked law without any error.

Hence for jump-diffusion processes we know how to simulate exactly the process on a time grid. Let us concentrate now on **infinite activity models**. The underlying Lévy measure ν is infinite.

3.1 Approximation

For a general Lévy process, we can proceed by approximation. Indeed if $X = (X_t)_{t \geq 0}$ is a Lévy process with infinite activity and triple $(0, \nu, \gamma)$, then

$$X_t = \gamma t + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} + \lim_{\varepsilon \downarrow 0} N_t^\varepsilon,$$

where

$$N_t^\varepsilon = \sum_{s \leq t} \Delta X_s \mathbf{1}_{\varepsilon \leq |\Delta X_s| < 1} - t \int_{\varepsilon \leq |x| < 1} x \nu(dx).$$

Therefore if we define the process X^ε by

$$X_t^\varepsilon = \gamma t + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} + N_t^\varepsilon,$$

X^ε is a compound Poisson process, and

$$\boxed{\lim_{\varepsilon \to 0} X_t^\varepsilon = X_t}$$

uniformly w.r.t. t. Moreover X^ε is easy to simulate. The residual term:

$$R_t^\varepsilon = -N_t^\varepsilon + \lim_{\varepsilon \downarrow 0} N_t^\varepsilon$$

is a Lévy process

- with characteristic triple $(0, \mathbf{1}_{|x| \leq \varepsilon} \nu(dx), 0)$,
- with infinite activity, with bounded jumps, thus with finite variance,
- $\mathbb{E}(R_t^\varepsilon) = 0$,
- $\operatorname{Var} R_t^\varepsilon = t \int_{|x| < \varepsilon} x^2 \nu(dx) = t \sigma^2(\varepsilon)$.

For example for the Gamma process, ona can prove that $\sigma(\varepsilon) \sim \varepsilon$.

Proposition 3.1 *If f is a differentiable function s.t. $|f'(x)| \leq C$, then*

$$|\mathbb{E} f(X_T^\varepsilon + R_T^\varepsilon) - \mathbb{E} f(X_T^\varepsilon)| \leq C \sigma(\varepsilon) \sqrt{T}.$$

Proof. The difference in question can be estimated as follows:

$$|\mathbb{E} f(X_T^\varepsilon + R_T^\varepsilon) - \mathbb{E} f(X_T^\varepsilon)| = \left| \mathbb{E} \left(R_T^\varepsilon \int_0^1 f'(X_T^\varepsilon + u R_T^\varepsilon) du \right) \right| \leq C \mathbb{E} |R_T^\varepsilon|.$$

For any random variable Z, Jensen's inequality yields: $\mathbb{E}[|Z|]^2 \leq E[|Z|^2] = E[Z^2]$. Applying this to R_T^ε, we conclude that $\mathbb{E}|R_T^\varepsilon| \leq \sigma(\varepsilon) \sqrt{T}$. $\square$

The only problem with this approximation is the term $\sqrt{T}$ in the previous inequality. If we deal with long maturity contract, the approximation with compound Poisson processes is not enough.

To remove this dependence, we give a new approximation:

$$\tilde{X}_t^\varepsilon = X_t^\varepsilon + \sigma(\varepsilon) W_t,$$

where W is a Brownian process independant of X^ε. This is just justified by the following theorem.

Theorem 3.1 (Asmussen and Rosinski) $\lim_{\varepsilon \to 0} \sigma(\varepsilon)^{-1} R^\varepsilon = W$ *in law if and only if for every* $k > 0$

$$\lim_{\varepsilon \to 0} \frac{\sigma(k\sigma(\varepsilon) \wedge \varepsilon)}{\sigma(\varepsilon)} = 1.$$

Since the condition is not easy to prove, we often work with the following sufficient condition:

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty.$$

Let us give some examples.

- The condition is satisfied for a process X with Lévy measure $\nu(x) \underset{x=0}{\sim} 1/|x|^{\alpha+1}$. Indeed $\sigma(\varepsilon) \sim \varepsilon^{1-\alpha/2}$.
- For a compound Poisson processes, $\sigma(\varepsilon) = o(\varepsilon)$.
- For a Gamma process $\sigma(\varepsilon) \sim \varepsilon$.

Proposition 3.2 *If* f *is a differentiable function s.t.* $|f'(x)| \leq C$, *then*

$$|\mathbb{E} f(X_T^\varepsilon + R_T^\varepsilon) - \mathbb{E} f(X_T^\varepsilon + \sigma(\varepsilon) W_T)| \leq A \rho(\varepsilon) C \sigma(\varepsilon),$$

with $A < 16,5$ *and* $\rho(\varepsilon) = \dfrac{1}{\sigma^3(\varepsilon)} \displaystyle\int_{-\varepsilon}^{\varepsilon} |x|^3 \nu(dx) < \dfrac{\varepsilon}{\sigma(\varepsilon)}$.

Proof. Admitted here. □

3.2 Exact simulation on a grid

The approximation method works with a general Lévy process. But for some processes, a exact simulation on a grid can be done. Let us give a time grid $t_1, \ldots, t_n$. We want to compute $X(t_1), \ldots, X(t_n)$ without any error (at least in law). We have done this for a jump-diffusion process.

3.2.1 Stable processes

A process X is called stable if the Lévy measure is given by

$$\nu(x) = \frac{A}{x^{\alpha+1}}\mathbf{1}_{x>0} + \frac{B}{|x|^{\alpha+1}}\mathbf{1}_{x<0}$$

with $0 < \alpha < 2$ and the characteristic function is:

$$\psi_X(z) = \exp\left\{-\sigma^\alpha |z|^\alpha + i\mu z\right\},$$

with $0 < \alpha \leq 2$, $\sigma \geq 0$, and $\mu \in \mathbb{R}$ (the shift). The law of such process is denoted by $S_\alpha(\sigma, \mu)$.

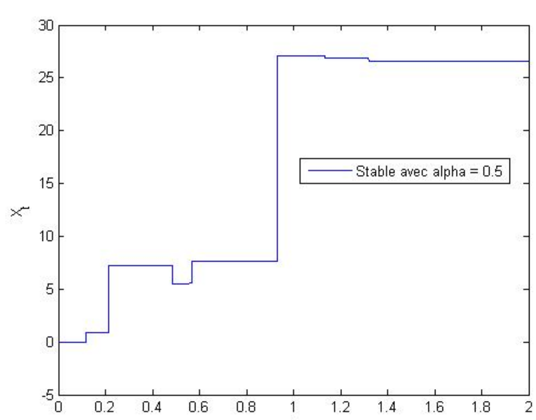
We admit that if X is distributed like $S_\alpha(1, 0)$, then $\sigma X + \mu$ has law $S_\alpha(\sigma, \mu)$. The algorithm for a $S_\alpha(1, 0)$ process is the following:

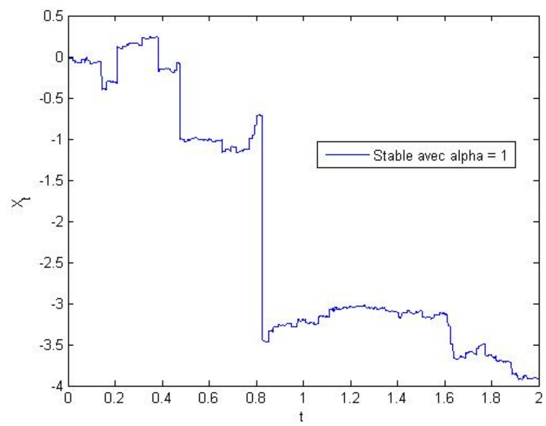
1. simulate n i.i.d. r.v. U_i with uniform law on $[-\pi/2, \pi/2]$ and n i.i.d. r.v. E_i with exponential distribution with parameter 1;

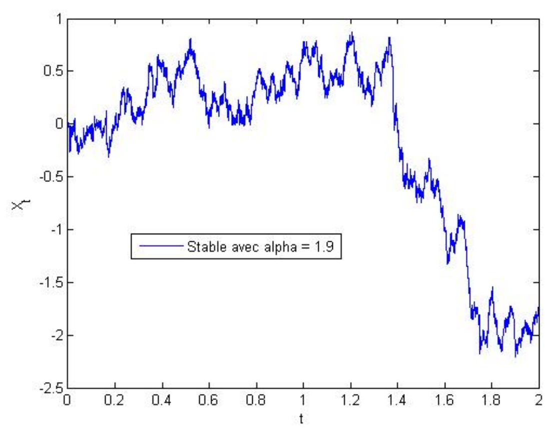
2. define
$$\Delta X_i = (t_i - t_{i-1})^{1/\alpha} \frac{\sin(\alpha U_i)}{(\cos U_i)^{1/\alpha}} \left(\frac{\cos((1-\alpha)U_i)}{E_i}\right)^{(1-\alpha)/\alpha}$$
with $t_0 = 0$;

3. put $X_{t_i} = \displaystyle\sum_{k=1}^{i} \Delta X_k$.

Let us draw the stable process in three cases.







3.2.2 Subordinated processes.

Let us begin with the following time-change result.

Theorem 3.2 *Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ with characteristic triple (A, ν, γ) and characteristic exponent Ψ. Let $(S_t)_{t\geq 0}$ be a subordinator with Lévy measure ρ, drift $\beta_0 \geq 0$, and Laplace transform $\mathcal{L}$. Then the process $Y(t, \omega) = X(S(t, \omega), \omega)$ is a Lévy process s.t. $\mathbb{E}(e^{i\langle u, Y_t\rangle}) = e^{it\mathcal{L}(\Psi(u))}$. Moreover the triple of Y is*

- $A^Y = \beta_0 A$,
- $\nu^Y(B) = \beta_0 \nu(B) + \displaystyle\int_0^\infty p_s^X(B)\rho(ds)$, $\forall B \in \mathcal{B}(\mathbb{R}^d)$,
- $\gamma^Y = \beta_0\gamma + \displaystyle\int_0^\infty \rho(ds) \int_{|x|\leq 1} x p_s^X(dx)$,

where p_t^X is the distribution of X_t.

If $\beta_0 = 0$ and $\int_{]0,1]} s^{1/2}\rho(ds) < \infty$, then Y is of type A or B.

Definition 3.1 (Subordination) *The previous transform is called* subordination *with the subordinator S. Every process equal in law to Y is called* subordinated *at X.*

Remark that a subordinated subordinator is a subordinator (we can "compose" or iterate the subordination).

Moreover as an immediate consequence of this result, it allows us to create multidimensionnal Lévy processes. Indeed using X a d-dimensional Brownian motion, we just have to subordinate it to a process S.

The following result can be extended in dimension $n \geq 2$.

Theorem 3.3 *Let ν be a Lévy measure on $\mathbb{R}$ and $\mu \in \mathbb{R}$. There exists $(X_t)_{t\geq 0}$ with Lévy measure ν s.t. $X_t = W(S_t) + \mu S_t$ for some subordinator $(S_t)_{t\geq 0}$ and a Brownian motion $(W_t)_{t\geq 0}$, independent of S if and only if:*

1. *ν is absolutely continuous with density $x \mapsto \nu(x)$,*

2. *$\nu(x)e^{-\mu x} = \nu(-x)e^{\mu x}$,*

3. *$f : u \mapsto \nu(\sqrt{u})e^{-\mu\sqrt{u}}$ is completely monotone on $]0, +\infty[$.*

Completely monotone means that all derivatives exist and for any $k \geq 1$, $(-1)^k \frac{d^k f}{du^k}(u) > 0$. The jump structure is described as a time changed Brownian motion with drift. If S is a subordinator without drift and Lévy measure ρ, then $X = W(S) + \mu S$ has the following Lévy measure:

$$\nu(x) = \int_0^{+\infty} e^{-\frac{(x-\mu t)^2}{2t}} \frac{\rho(dt)}{\sqrt{2\pi t}}.$$

If $X = W(S)$ is a Brownian motion with volatility σ, drift b, with the time change induced by the subordinator S, the algorithm to simulate X on a grid is the following.

1. simulate the increments of subordinator $\Delta S_i = S_{t_i} - S_{t_{i-1}}$ where $S_0 = 0$,

2. simulate n standard Gaussian r.v. $N_1, \ldots, N_n$,

3. define $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + b\Delta S_i$,

4. put $X_{t_i} = \sum_{k=1}^{i} \Delta X_k$.

3.2.3 Tempered stable processes.

Definition 3.2 *A tempered stable process X is a Lévy process with Lévy measure:*

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} \mathbf{1}_{x<0} + \frac{c_+}{x^{1+\alpha_+}} e^{-\lambda_+ x} \mathbf{1}_{x>0}$$

with $\alpha_+ < 2$ and $\alpha_- < 2$, the other parameters are positive.

The characteristic exponent can be calculated as a function of Γ.

Proposition 3.3 *It is subordinated Brownian motion if $c_- = c_+$ and $\alpha_- = \alpha_+ \geq -1$.*

Such processes are used in finance to modelize some risks (scaling property) or stochastic volatility.

A tempered stable subordinator S has a Lévy measure given by:

$$\rho(x) = \frac{ce^{-\lambda x}}{x^{\alpha+1}} \mathbf{1}_{\mathbb{R}_+^*}(x), \quad c > 0,\ \lambda > 0,\ 0 \leq \alpha < 1.$$

- c controls the intensity of all jumps,
- λ is the rate of decrease of the big jumps,
- α detemines the importance of the small jumps.

The Laplace exponent of S is

$$\mathcal{L}(u) = -c\Gamma(\alpha)\{(\lambda + u)^\alpha - \lambda^\alpha\}$$

if $\alpha \neq 0$ and $\mathcal{L}(u) = -c\ln(1 + u/\lambda)$ si $\alpha = 0$. The density of X is explicitely known if $\alpha = 1/2$ (inverse Gaussian subordinator) or $\alpha = 0$ (gamma subordinator).

Lemma 3.1 *If $(S_t(\alpha, \lambda, c))$ is tempered stable subordinator, then for every $r > 0$, $(rS_t(\alpha, \lambda, c))$ has the same distribution as $(S_{r^\alpha t}(\alpha, \lambda/r, c))$.*

This scaling property and the scaling property of the Brownian motion allow us to just consider the subordinators s.t. $\mathbb{E}(S_t) = t$. Indeed if $\gamma > 0$ is the expectation of S_1, we have

$$X_t = W(S_t) = W(\gamma(S_t/\gamma)) = \sqrt{\gamma}W(S_t/\gamma),$$

and we define $\tilde{S}_t = S_t/\gamma$ such that $\mathbb{E}\tilde{S}_t = \mathbb{E}(S_t/\gamma) = t$ and $\tilde{X}_t = W(\tilde{S}_t)$. $t \mapsto \tilde{S}_{\gamma^\alpha t}$ is still a tempered stable subordinator and $X_t = \sqrt{\gamma}\tilde{X}_t$.

Now it is well known that $\mathbb{E}(S_1) = \mathcal{L}'(0) = -c\Gamma(\alpha)\alpha\lambda^{\alpha-1}$. Therefore c is a function of λ and α, and if we put $\kappa = (1-\alpha)/\lambda$, κ is the variance at time 1 and we define S with the new parameters:

$$\rho(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{1-\alpha}{\kappa}\right)^{1-\alpha} \frac{e^{-(1-\alpha)x/\kappa}}{x^{1+\alpha}},$$

where

- α is still the stability index,
- κ is equal to the subordinator variance at time 1.

Definition 3.3 (Normal tempered stable process) *A* normal tempered stable process *is a Brownian motion (with variance σ^2 and drift θ) subordinated by a tempered stable subordinator.*

Proposition 3.4 *The characteristic exponent of a normal tempered stable process is given by:*

$$\Psi(u) = \frac{1-\alpha}{\kappa\alpha}\left\{1 - \left(1 + \frac{\kappa(u^2\sigma^2/2 - i\theta u)}{1-\alpha}\right)^{\alpha}\right\}, \ \textit{if } \alpha \neq 0;$$

and

$$\Psi(u) = -\frac{1}{\kappa}\ln\left\{1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u\right\}, \ \textit{if } \alpha = 0.$$

Proof. Apply Theorem 3.2 with $\Psi(u) = u^2\sigma^2/2 - i\theta u$. □

Let us detail the two cases $\alpha = 0$ and $\alpha = 1/2$. Recall that these are the only cases with known density. Let us start with $\alpha = 0$.

Gamma process The subordinator S is a gamma process with density at time t

$$p_t(x) = \frac{1}{\kappa^{t/\kappa}\Gamma(t/\kappa)} x^{t/\kappa - 1} e^{-x/\kappa} \mathbf{1}_{\mathbb{R}_+^*}(x).$$

The other parameters are:

- σ and θ resp. volatility and drift of the Brownian motion,
- κ variance of the subordinator.

The resulting normal process $X = W(S)$ is called a variance gamma process and

- is of bounded variation with infinite activity (but relatively weak),
- has a Lévy measure:

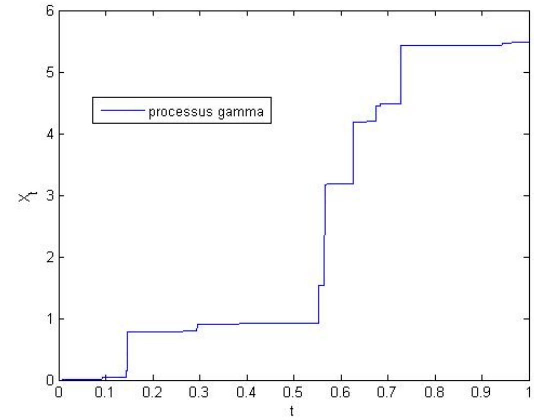
$$\nu(x) = \frac{1}{\kappa|x|} e^{Ax - B|x|}, \quad A = \frac{\theta}{\sigma^2}, \quad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2},$$

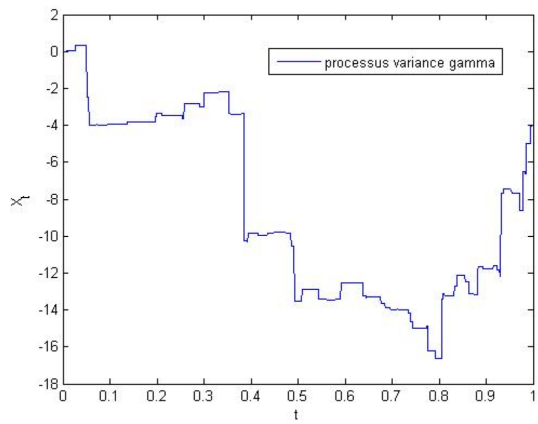
- and a characteristic exponent:

$$\psi(u) = -\frac{1}{\kappa}\ln(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u).$$

- Moreover $\mathbb{E}(X_t) = \theta t$ and $\operatorname{Var} X_t = \sigma^2 t + \theta^2\kappa t$.

If we draw the gamma process and the variance gamma process we obtain the following pictures.





Inverse Gaussian process Now we take $\alpha = 1/2$. Thus the subordinator S is an inverse Gaussian process with density

$$p_t(x) = \sqrt{\frac{t^2/\kappa}{2\pi x^3}} \exp\left(-\frac{t^2/\kappa}{2t^2 x}(x-t)^2\right) \mathbf{1}_{\mathbb{R}_+^*}(x).$$

As before the otherp arameters are:

- σ and θ resp. volatility and drift of the Brownian motion,
- κ variance of the subordinator.

The subordinated process $X = W(S)$ is

- of unbounded variation with stable behaviour of the small jumps.
- The Lévy measure can be written with the Bessel functions:

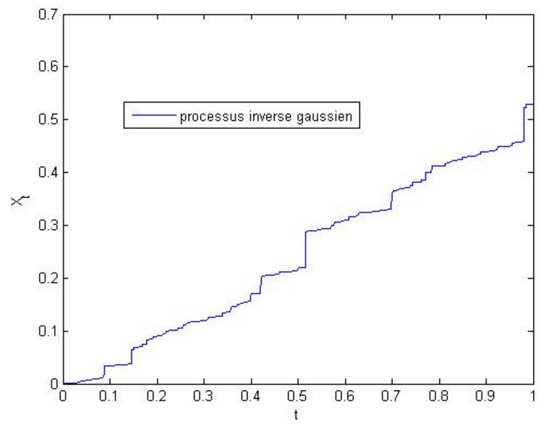
$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|),\ C = \frac{\sqrt{\theta^2 + \sigma^2/\kappa}}{2\pi\sigma\sqrt{\kappa}},\ A = \frac{\theta}{\sigma^2},\ B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}.$$

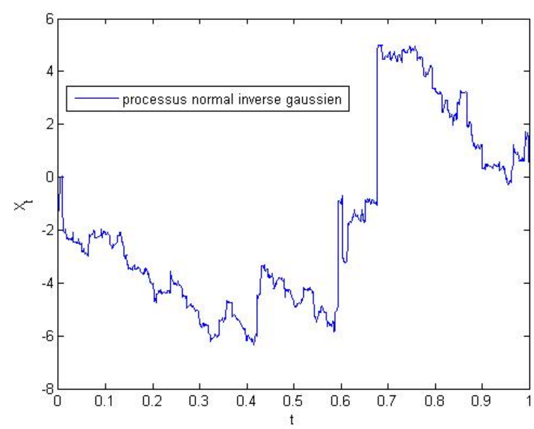
- The characteristic exponent is:

$$\psi(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 + u^2\sigma^2\kappa - 2i\theta\kappa u}.$$

- Finally $\mathbb{E}(X_t) = \theta t$ and $\operatorname{Var} X_t = \sigma^2 t + \theta^2 \kappa t$.

The typical trajectory of S and X are:





Part II

Stochastic calculus for Lévy processes

Chapter 4

Stochastic integral

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t,\ t \geq 0))$. The filtration is supposed to be right-continuous and complete. All processes are supposed to be adapted to this filtration.

First we know that if W is a Brownian motion and if ϕ is a predictable process verifying

$$\mathbb{E}\int_0^T |\phi_t|^2 dt < +\infty,$$

then the Itô integral $\int_0^t \phi_u dW_u$ is a square integrable martingale and

$$\mathbb{E}\int_0^T \phi_u dW_u = 0, \qquad \mathbb{E}\left|\int_0^T \phi_u dW_u\right|^2 = \mathbb{E}\int_0^T |\phi_u|^2 du.$$

But remark that:

- ϕ cannot be interpreted as a trading strategy because it is not LCRL (i.e. left continuous with right limits);
- its integral cannot necessarily be represented as a limit of Riemann sums.

4.1 Stochastic integral w.r.t. a pure-jump process

Definition 4.1 (Pure jump process) *A* pure jump process *is a process with*

- *piecewise constant trajectories,*
- *and a finite number of jumps on every finite time interval.*

Proposition 4.1 *A pure jump Lévy process is a compound Poisson process.*

Definition 4.2 (Stochastic integral) *Let J be a right-continuous pure-jump process and Φ an adapted process in $\mathbb{L}$. The* stochastic integral *of Φ w.r.t. J is defined by*

$$\int_0^t \Phi_s dJ_s = \sum_{0<s\leq t} \Phi_s \Delta J_s.$$

Proof. We consider in Section 4.2 the general case. Here J is a semi martingale. Therefore if Φ is in $\mathbb{L}$, the stochastic integral of Φ w.r.t. J is well defined. The only thing to prove is that if $\Phi \in \mathbb{S}$,

$$\Phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^{n} \phi_i \mathbf{1}_{]T_i, T_{i+1}]}(t),$$

then

$$\int_0^t \Phi_s dJ_s = \phi_0 X_0 + \sum_{i=0}^{n} \phi_i (J_{T_{i+1}\wedge t} - J_{T_i \wedge t}) = \sum_{0<s\leq t} \Phi_s \Delta J_s.$$

□

Proposition 4.2 (Martingale property) *Assume that J is a martingale, that the integrand Φ is adapted and left-continuous. Then the stochastic integral $\left(\int_0^t \Phi_s dX_s\right)_{t\geq 0}$ is a (local) martingale.*

Example. Let us finish with an example.

- $X_t = N_t - \lambda t$ is a compensated Poisson process;
- $\Phi_t = \Delta N_t$ is the jump at time t;
- $\Psi_t = \mathbf{1}_{[0,S_1]}(t)$ where S_1 is the time of the first jump of N.

Then X is a square integrable martingale,

$$I_t = \int_0^t \Phi_s dX_s = N_t, \qquad J_t = \int_0^t \Psi_s dX_s = \mathbf{1}_{[S_1,+\infty]}(t) - \lambda(t \wedge S_1).$$

Hence I is not a martingale, but J is a martingale. Remark that if we take $\Theta_t = \mathbf{1}_{[0,S_1[}(t)$, then $\int_0^t \Theta_s dX_s = -\lambda(t\wedge S_1)$ is not a martingale.

Second let us consider a jump-diffusion process:

$$X_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds + J_t,$$

with

1. X_0 a deterministic initial condition;
2. $I_t = \int_0^t \Gamma_s dW_s$ the Itô's integral of Γ w.r.t. to the Brownian motion W;
3. $R_t = \int_0^t \Theta_s ds$ the Lebesgue's integral of a process Θ;

4. J a right-continuous pure-jump process.

If Φ is an adapted process in $\mathbb{L}$, the stochastic integral of Φ w.r.t. J is defined by

$$\int_0^t \Phi_s dJ_s = \int_0^t \Phi_s \Gamma_s dW_s + \int_0^t \Phi_s \Theta_s ds + \sum_{0<s\leq t} \Phi_s \Delta J_s.$$

4.2 Stochastic integral w.r.t. a semi-martingale

Definition 4.3 (Simple process) *A stochastic process $(\phi_t)_{t\geq 0}$ is called a* simple (predictable) process *if it can be represented as*

$$\phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^n \phi_i \mathbf{1}_{]T_i,T_{i+1}]}(t),$$

where $T_0 = 0 < T_1 < \ldots < T_n < T_{n+1}$ are non-anticipating random times and each ϕ_i is bounded $\mathcal{F}_{T_i}$-measurable random variable.

The set of simple processes is denoted by $\mathbb{S}$.

Let $X = (X_t = (X_t^1, \ldots, X_t^d))_{t\geq 0}$ be a d-dimensional adapted RCLL process. We define for $0 \leq t$, and j s.t. $T_j < t \leq T_{j+1}$

$$\begin{aligned} G_t(\phi) &= \phi_0 X_0 + \sum_{i=0}^{j-1} \phi_i (X_{T_{i+1}} - X_{T_i}) + \phi_j (X_t - X_{T_j}) \\ &= \phi_0 X_0 + \sum_{i=0}^{n} \phi_i (X_{T_{i+1}\wedge t} - X_{T_i \wedge t}). \end{aligned}$$

Definition 4.4 (Integral of simple processes) *The process $G_t(\phi)$ is the* stochastic integral *of ϕ w.r.t. X and is denoted by:*

$$G_t(\phi) = \int_0^t \phi_u dX_u.$$

Proposition 4.3

1. *If X is a martingale, then for any simple process ϕ, the stochastic integral G is also a martingale.*

2. *Assume that X is a real-valued RCLL process. Let ϕ and ψ be real-valued simple processes. Then $Y_t = \int_0^t \phi_u dX_u$ is an adapted RCLL process and*

$$\int_0^t \psi_u dY_u = \int_0^t \psi_u \phi_u dX_u.$$

Proof. For the first part, for $0 \le s < t$, we have

$$\begin{aligned}\mathbb{E}(G_t(\phi)|\mathcal{F}_s) &= \mathbb{E}\left(\phi_0 X_0 + \sum_{i=0}^n \phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t}) \middle| \mathcal{F}_s\right) \\ &= \phi_0 X_0 + \sum_{i=0}^n \mathbb{E}\left(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t}) \middle| \mathcal{F}_s\right).\end{aligned}$$

For any $i = 0, \dots, n$, we have to distinguish four cases.

- If $T_{i+1} \le s$,

$$\begin{aligned}\mathbb{E}\left(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t})|\mathcal{F}_s\right) &= \mathbb{E}\left(\phi_i (X_{T_{i+1}} - X_{T_i})|\mathcal{F}_s\right) \\ &= \phi_i (X_{T_{i+1}} - X_{T_i}) = \phi_i (X_{T_{i+1}\wedge s} - X_{T_i\wedge s}).\end{aligned}$$

- If $s < T_i$, then

$$\mathbb{E}\left(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t})|\mathcal{F}_s\right) = \mathbb{E}\left[\mathbb{E}\left(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t})|\mathcal{F}_{T_i}\right) \middle| \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[\phi_i \mathbb{E}\left((X_{T_{i+1}\wedge t} - X_{T_i\wedge t})|\mathcal{F}_{T_i}\right) \middle| \mathcal{F}_s\right] = \mathbb{E}(\phi_i \times 0|\mathcal{F}_s) = 0 = \phi_i (X_{T_{i+1}\wedge s} - X_{T_i\wedge s})$$

  because the stopped martingale $X_{.\wedge t}$ is still a martingale.

- If $T_i \le s < T_{i+1}$,

$$\mathbb{E}\left(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i\wedge t})|\mathcal{F}_s\right) = \mathbb{E}(\phi_i (X_{T_{i+1}\wedge t} - X_{T_i})|\mathcal{F}_s) = \phi_i \mathbb{E}\left((X_{T_{i+1}\wedge t} - X_{T_i}) \middle| \mathcal{F}_s\right)$$
$$= \phi_i \left[\mathbb{E}(X_{T_{i+1}\wedge t}|\mathcal{F}_s) - X_{T_i}\right] = \phi_i (X_s - X_{T_i}) = \phi_i (X_{T_{i+1}\wedge s} - X_{T_i\wedge s}).$$

Hence in every case

$$\mathbb{E}(G_t(\phi)|\mathcal{F}_s) = \phi_0 X_0 + \sum_{i=0}^n \phi_i (X_{T_{i+1}\wedge s} - X_{T_i\wedge s}) = G_s(\phi).$$

The second part can be proved as above and is left to the reader. □

Definition 4.5 (Semi-martingale) *A adapted RCLL process X is a* semi-martingale *if the stochastic integral of simple processes w.r.t. X verifies the following continuity property: for every ϕ^n and ϕ in $\mathbb{S}$ if*

$$\lim_{n\to+\infty} \sup_{(t,\omega)\in\mathbb{R}_+\times\Omega} |\phi^n_t(\omega) - \phi_t(\omega)| = 0, \tag{4.1}$$

then in probability:

$$\int_0^T \phi^n_u dX_u \underset{n\to+\infty}{\longrightarrow} \int_0^T \phi_u dX_u = G_T(\phi).$$

Denote by

- $\mathbb{S}_u$ the set $\mathbb{S}$ with uniform convergence w.r.t. (t,ω);
- $\mathbb{L}^0$ the set of finite random variables with convergence in probability.

The definition is equivalent to: X is a semi-martingale if $G : \mathbb{S}_u \to \mathbb{L}^0$ is continuous.

Lemma 4.1 *The set of semi-martingales is a vector space.*

Let us start with examples.

Proposition 4.4

- *A finite variation process,*
- *a (locally) square integrable (local) martingale*

are a semi-martingale.

Proof. Assume that X is of finite variation. Then for any simple process ϕ we have

$$|G_t(\phi)| \leq \left(\sup_{(t,\omega)} |\phi(t,\omega)| \right) TV_t(X),$$

where $TV_t(X)$ is the total variation of X on the interval $[0,t]$.

If X is a square integrable martingale, then

$$\begin{aligned}
\mathbb{E}(G_t(\phi)^2) &= \mathbb{E}\left[\left(\phi_0 X_0 + \sum_{i=0}^{n} \phi_i (X_{T_{i+1}\wedge t} - X_{T_i \wedge t}) \right)^2 \right] \\
&= \mathbb{E}\left[\phi_0^2 X_0^2 + \sum_{i=0}^{n} \phi_i^2 (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})^2 \right] \\
&\quad +2 \sum_{i<j} \mathbb{E} \phi_i \phi_j (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})(X_{T_{j+1}\wedge t} - X_{T_j \wedge t})
\end{aligned}$$

The second term on the right-side is equal to zero. Indeed for $i < j$:

$$\begin{aligned}
&\mathbb{E}\left[\phi_i \phi_j (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})(X_{T_{j+1}\wedge t} - X_{T_j \wedge t}) \right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\phi_i \phi_j (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})(X_{T_{j+1}\wedge t} - X_{T_j \wedge t}) \middle| \mathcal{F}_{T_j} \right) \right] \\
&= \mathbb{E}\left[\phi_i \phi_j (X_{T_{i+1}\wedge t} - X_{T_i \wedge t}) \mathbb{E}\left((X_{T_{j+1}\wedge t} - X_{T_j \wedge t}) \middle| \mathcal{F}_{T_j} \right) \right] \\
&= 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}(G_t(\phi)^2) &= \mathbb{E}\left[\phi_0^2 X_0^2 + \sum_{i=0}^{n} \phi_i^2 (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})^2 \right] \\
&\leq \left(\sup_{(t,\omega)} |\phi(t,\omega)|^2 \right) \mathbb{E}\left[X_0^2 + \sum_{i=0}^{n} (X_{T_{i+1}\wedge t} - X_{T_i \wedge t})^2 \right] \\
&= \left(\sup_{(t,\omega)} |\phi(t,\omega)|^2 \right) \mathbb{E}(X_t^2).
\end{aligned}$$

Now if X is a locally square integrable local martingale, just use the following lemma.

Lemma 4.2 *Let $(T_n)_{n\geq 0}$ a sequence of non negative random variables such that $\lim_{n\to+\infty} T_n = +\infty$ a.s. and for every n, $X^n = X^{T_n}$ is a semi martingale. Then X is a semi martingale.*

□

Definition 4.6 (Decomposable process) *An adapted RCLL process X is* decomposable *if $X_t = X_0 + M_t + A_t$, where $M_0 = A_0 = 0$, M is locally square integrable martingale, and A is RCLL, adapted, with paths of finite variation on compacts.*

Proposition 4.5 *A decomposable process is a semi-martingale. Hence all Lévy processes are semi-martingales.*

Proof. Use Proposition 4.4. The second part comes from the Lévy-Khintchine decomposition of a Lévy process X. □

Now we give some technical results. We denote by

- $\mathbb{D}$ the set of adapted RCLL processes;
- $\mathbb{L}$ (resp. $b\mathbb{L}$), the set of adapted RLLC (resp. RLLC and bounded) processes.
- $\phi^*_t = \sup_{0\leq s\leq t} |\phi_s|$.

Definition 4.7 (Ucp topology) *A sequence (ϕ^n) of processes* converges uniformly on compact sets in probability *(ucp in short) to ϕ if:*

$$\forall t > 0, \quad (\phi^n - \phi)^*_t \underset{n\to+\infty}{\longrightarrow} 0 \text{ in probability.}$$

Lemma 4.3

1. *The set $\mathbb{S}$ is dense in $\mathbb{L}$ for the ucp topology.*
2. *For X semi-martingale, $G : \mathbb{S}_{ucp} \to \mathbb{D}_{ucp}$ is continuous.*

Proof. Let Y be in $\mathbb{L}$ and $R_n = \inf\{t \geq 0,\ |Y_t| > n\}$. Then R_n is a stopping time and the process $Y^n_t = Y_{R_n \wedge t} \mathbf{1}_{R_n > 0}$ is in $b\mathbb{L}$ and converges ucp to Y: for every $\varepsilon > 0$

$$\mathbb{P}((Y^n - Y)^*_t > \varepsilon) \leq \mathbb{P}(R_n = 0) + \mathbb{P}\left(\sup_{0 < R_n \leq s \leq t} |Y_s - Y_{R_n}| > \varepsilon \right) \leq \mathbb{P}(R_n \leq t).$$

Now if $Y \in b\mathbb{L}$, then the process $Z_t = \lim_{u\to t,\ u>t} Y_u$ is in $\mathbb{D}$. For $\varepsilon > 0$, we define the sequence $T^\varepsilon_0 = 0$, and

$$T^\varepsilon_{n+1} = \inf\left\{t \geq 0,\ t > T^\varepsilon_n,\ |Z_t - Z_{T^\varepsilon_n}| > \varepsilon\right\}.$$

Since $Z \in \mathbb{D}$, this sequence of stopping times is increasing and a.s. $\lim_{n\to+\infty} T^\varepsilon_n = +\infty$. Let Z^ε define by

$$Z^\varepsilon_t = \sum_{n\in\mathbb{N}} Z_{T^\varepsilon_n} \mathbf{1}_{[T^\varepsilon_n, T^\varepsilon_{n+1}[}(t).$$

This process is bounded (because Y and Z are bounded), is in $\mathbb{D}$, and converges uniformly to Z as ε goes to zero. In order to obtain Y, we put

$$Y^\varepsilon_t = Y_0 \mathbf{1}_{\{0\}}(t) + \sum_{n\in\mathbb{N}} Z_{T^\varepsilon_n} \mathbf{1}_{]T^\varepsilon_n, T^\varepsilon_{n+1}]}(t) \underset{n\to+\infty}{\longrightarrow} Y_0 \mathbf{1}_{\{0\}}(t) + Z_{t^-} = Y_t.$$

Finally if we define

$$Y_t^{N,\varepsilon} = Y_0 \mathbf{1}_{\{0\}}(t) + \sum_{n=0}^{N} Z_{T_n^\varepsilon} \mathbf{1}_{]T_n^\varepsilon \wedge N, T_{n+1}^\varepsilon \wedge N]}(t),$$

$Y^{N,\varepsilon}$ is in $\mathbb{S}$ and converges to Y.

For the second part, assume first that ϕ^n is in $\mathbb{S}$, converges to 0 uniformly in $\mathbb{S}$ and is uniformly bounded. Let $\delta > 0$ and $T^n = \inf\{t \geq 0,\ |G_t(\phi^n)| \geq \delta\}$. Then $\phi^n \mathbf{1}_{[0,T^n]}$ is in $\mathbb{S}$ and converges to 0. Moreover

$$\begin{aligned} \mathbb{P}(G_t(\phi^n)^* > \delta) &\leq \mathbb{P}\left(\left| \int \phi_s^n \mathbf{1}_{[0,T^n]}(s) dX_s \right| \geq \delta \right) \\ &\to 0 \end{aligned}$$

because X is a semi-martingale. We have proved that G is a continuous mapping from $\mathbb{S}$ with the uniform convergence w.r.t. t and ω to the set $\mathbb{D}$. Now for the general case, for any $\eta > 0$, we define $R_n = \inf\{t \geq 0,\ (\phi^n)_t^* > \eta$ and $\tilde{\phi}^n = \phi^n \mathbf{1}_{[0,R_n]} \mathbf{1}_{R_n > 0}$. Then $\tilde{\phi}^n \in \mathbb{S}$ and by the left continuity, $\tilde{\phi}^n$ is uniformly bounded by η. If $\mathbb{R}_n \geq T$, then $G_t(\tilde{\phi}^n)^* = G_t(\phi^n)^*$, and we obtain

$$\begin{aligned} \mathbb{P}(G_t(\phi^n)^* > \delta) &\leq \mathbb{P}(G_t(\tilde{\phi}^n)^* > \delta) + \mathbb{P}(R_n < t) \\ &\leq \mathbb{P}(G_t(\tilde{\phi}^n)^* > \delta) + \mathbb{P}((\phi^n)_t^* > \eta) \\ &\leq \varepsilon \end{aligned}$$

for any $\varepsilon > 0$ and n large enough, because $\lim_{n \to +\infty} \mathbb{P}((\phi^n)_t^* > \eta) = 0$. This achieves the proof. □

Definition 4.8 (Stochastic integral for LCRL process) *Let X be a semi-martingale. The continuous linear mapping $G = G_X : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ obtained as the extension of $G : \mathbb{S} \to \mathbb{D}$ is called the* stochastic integral.

Recall that X^T is the stopped process $X_t^T = X_{T \wedge t}$.

Theorem 4.1

1. *Let T be a stopping time. Then*

$$G(\phi)^T = (G(\phi)_{t \wedge T})_{t \geq 0} = G(\phi \mathbf{1}_{[0,T]}) = G_{X^T}(\phi).$$

2. *The jump process $\Delta(G(\phi))$ is indistinguishable from $\phi(\Delta X)$.*

Proof. These two properties can be proved for $\phi \in \mathbb{S}$ and then extended to the general case. □

Theorem 4.2 *If X is a semi-martingale, and if ϕ is an adapted LCRL process then*

- $Y_t = \displaystyle\int_0^t \phi_u dX_u$ *is a semi-martingale.*
- *If ψ is another adapted LCRL process, then*

$$\int_0^t \psi_u dY_u = \int_0^t \psi_u \phi_u dX_u.$$

- *If X is a (locally) square-integrable (local) martingale, Y is a (locally) square-integrable (local) martingale.*

Proof. Same trick: prove it for simple processes and in the general case troncate with adequate stopping times and pass to the limit. □

Let us make some remarks on these two theorems. The jumps of the stochastic integral only occur at jump times of the process X. Therefore if X is continuous, whatever is ϕ, $G(\phi) = \int \phi dX$ is a continuous process. Moreover if X is a martingale and if $\phi \in \mathbb{L}$, then $G(\phi)$ is a local martingale. We will see in the next section that this can be false if ϕ is in $\mathbb{D}$.

Proposition 4.6 *Let X be a semi-martingale, ϕ be an adapted LCRL process and $\Pi^n = (T_0^n = 0 < T_1^n < \ldots < T_{n+1}^n = T)$ a sequence of random partitions of $[0,T]$ s.t. $\|\Pi^n\| = \sup |T_k^n - T_{k-1}^n| \to 0$ a.s. when $n \to \infty$. Then in probability*

$$\phi_0 S_0 + \sum_{k=1}^{n} \phi_{T_k^n}(X(T_{k+1}^n \wedge t) - X(T_k^n \wedge t)) \underset{\|\Pi^n\|\to 0}{\longrightarrow} \int_0^t \phi(u^-) dX_u.$$

4.3 Integral w.r.t. a Poisson random measure

Let us detail the notion of integration w.r.t. a Poisson measure. It is sometimes convenient to write processes as a stochastic integral w.r.t. a Poisson measure. In fact it is a particular case of integration w.r.t. a semi martingale.

If M is a Poisson measure on $[0,T] \times \mathbb{R}^d$ with intensity $\mu(dt, dx)$, the process $\widetilde{M}_t^A = M([0,t] \times A) - \mu([0,t] \times A)$ is a martingale for any A. And if $A \cap B = \emptyset$ then $\widetilde{M}^A$ and $\widetilde{M}^B$ are independent. In this framework a simple predictable process ϕ is

$$\phi(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} \phi_{ij} \mathbf{1}_{]T_i, T_{i+1}]}(t) \mathbf{1}_{A_j}(y)$$

where

- $T_1 \leq T_2 \leq \ldots \leq T_n$ are adapted random times,
- ϕ_{ij} bounded $\mathcal{F}_{T_i}$-measurable random variables,
- A_j disjoint subsets with $\mu([0,T] \times A_j) < +\infty$.

The stochastic integral of ϕ w.r.t. M or $\widetilde{M}$ is defined by:

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,y) M(ds, dy) = \sum_{i,j=1}^{n,m} \phi_{ij} \left[M_{T_{i+1}\wedge t}(A_j) - M_{T_i \wedge t}(A_j) \right]$$

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,y) \widetilde{M}(ds, dy) = \sum_{i,j=1}^{n,m} \phi_{ij} \left[\tilde{M}_{T_{i+1}\wedge t}(A_j) - \tilde{M}_{T_i \wedge t}(A_j) \right]$$

Proposition 4.7 *For any simple process ϕ, the process $(X_t)_{t\in[0,T]}$*

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y)\widetilde{M}(ds,dy)$$

is a square integrable martingale and verifies the isometry formula

$$\mathbb{E}\,|X_t|^2 = \mathbb{E}\left[\int_0^t \int_{Re^{\;d}} |\phi(s,y)|^2 \mu(ds,dy)\right].$$

For a random function ϕ s.t. $\mathbb{E}\displaystyle\int_0^T \int_{\mathbb{R}^d} |\phi(s,y)|^2 \mu(ds,dy) < \infty$, there exists a sequence of simple processes ϕ^n s.t.

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\phi(s,y) - \phi^n(s,y)|^2 \mu(ds,dy) \underset{n\to+\infty}{\longrightarrow} 0.$$

Proposition 4.8 *For any RLLC process ϕ s.t.*

$$\mathbb{E}\int_0^T \int_{\mathbb{R}^d} |\phi(s,y)|^2 \mu(ds,dy) < \infty,$$

the process $(X_t)_{t\in[0,T]}$

$$X_t = \int_0^t \int_{\mathbb{R}^d} \phi(s,y)\widetilde{M}(ds,dy)$$

is a square integrable martingale and verifies the isometry formula

$$\mathbb{E}\,|X_t|^2 = \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} |\phi(s,y)|^2 \mu(ds,dy)\right].$$

If M is the Poisson measure J_X of some Lévy process X

$$M = J_X(\omega,.) = \sum_{t\in[0,T]}^{\Delta X_t \neq 0} \delta_{(t,\Delta X_t)},$$

then

$$\int_0^t \int_{\mathbb{R}^d} \phi(s,y) M(ds,dy) = \sum_{t\in[0,T]}^{\Delta X_t \neq 0} \phi(t,\Delta X_t).$$

Chapter 5

The Itô formula

The construction follows the same scheme as the stochastic calculus for the Brownian motion or for a continuous square integrable martingale.

5.1 Quadratic variation

The framework is the following:

- X is a semi-martingale, adapted RCLL process with $X_0 = 0$,
- we have a time grid $\pi = \{t_0 = 0 < t_1 < t_2 < \ldots < t_{n+1} = T\}$.

The realized variance is:

$$\begin{aligned} V_X(\pi) &= \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})^2 \\ &= X_T^2 - 2\sum_{i=0}^{n} X_{t_i}(X_{t_{i+1}} - X_{t_i}) \end{aligned}$$

Hence if the mesh a the time grid goes to zero, $V_X(\pi)$ converges in probability to:

$$[X, X]_T = X_T^2 - 2\int_0^T X_{u^-} dX_u.$$

Definition 5.1 (Quadratic variation) *The* quadratic variation process *of a semi-martingale X is the adapted RCLL process defined by:*

$$[X, X]_t = |X_t|^2 - 2\int_0^t X_{u^-} dX_u.$$

$[X, X]$ can be denoted also $[X]$.

Properties 5.1

- $X_0^2 + \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})^2 \underset{\|\pi\| \to 0}{\longrightarrow} [X, X]_T$ *in ucp.*

- *$([X,X]_t)_{t\in[0,T]}$ is a non-decreasing process with $[X,X]_0 = X_0^2$.*
- *The jumps of $[X,X]$ are: $\Delta[X,X]_t = |\Delta X_t|^2$.*
- *If X is continuous and has paths of finite variation, then $[X,X] = 0$.*

Proof. The first property is a consequence of the definition of the stochastic integral w.r.t. a semi martingale. Moreover from this convergence, we immediately obtain the second one. Now

$$\begin{aligned}
|\Delta X_t|^2 &= |X_t - X_{t^-}|^2 = |X_t|^2 + |X_{t^-}|^2 - 2X_t X_{t^-} \\
&= |X_t|^2 - |X_{t^-}|^2 + 2X_{t^-}(X_{t^-} - X_t) = \Delta(X^2)_t - 2X_{t^-}\Delta X_t.
\end{aligned}$$

And

$$\Delta J_t = \Delta\left(\int X_{s^-} dX_s\right)_t = X_{t^-}\Delta X_t.$$

Therefore

$$|\Delta X_t|^2 = \Delta(X^2 - 2J)_t = \Delta([X,X])_t.$$

If X is continuous and has paths of finite variation, then

$$\begin{aligned}
\sum_{i=0}^n (X_{t_{i+1}} - X_{t_i})^2 &\leq \left(\sup_{0\leq k\leq n} |X_{t_{k+1}} - X_{t_k}|\right) \sum_{i=0}^n |X_{t_{i+1}} - X_{t_i}| \\
&\leq \left(\sup_{0\leq k\leq n} |X_{t_{k+1}} - X_{t_k}|\right) TV_t(X).
\end{aligned}$$

Using Heine Theorem, X is uniformly continuous on $[0,t]$, hence

$$\lim_{\|\pi\|\to 0}\left(\sup_{0\leq k\leq n} |X_{t_{k+1}} - X_{t_k}|\right) = 0 \Longrightarrow [X,X]_t = 0.$$

□

Now if we precise that X is given by:

$$X_t = X_t^c + J_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds + J_t,$$

where J is a pure-jump process, then $[X]_T = [X,X]_T = \displaystyle\int_0^T \Gamma_s^2 ds + \sum_{0<s\leq T} (\Delta J_s)^2$.

Proof. We compute

$$\begin{aligned}
\sum_{i=0}^n (X_{t_{i+1}} - X_{t_i})^2 &= \sum_{i=0}^n (X^c_{t_{i+1}} - X^c_{t_i})^2 + \sum_{i=0}^n (J_{t_{i+1}} - J_{t_i})^2 \\
&+ 2\sum_{i=0}^n (X^c_{t_{i+1}} - X^c_{t_i})(J_{t_{i+1}} - J_{t_i}),
\end{aligned}$$

and

$$\left|\sum_{i=0}^n (X^c_{t_{i+1}} - X^c_{t_i})(J_{t_{i+1}} - J_{t_i})\right| \leq \left[\sup_{0\leq i\leq n} |X^c_{t_{i+1}} - X^c_{t_i}|\right] \sum_{0\leq s\leq t} |\Delta(J)_s|.$$

Now

$$\begin{aligned}\sum_{i=0}^{n}(X^c_{t_{i+1}}-X^c_{t_i})^2 &= \sum_{i=0}^{n}(I_{t_{i+1}}-I_{t_i})^2+\sum_{i=0}^{n}(R_{t_{i+1}}-R_{t_i})^2 \\ &+ 2\sum_{i=0}^{n}(I_{t_{i+1}}-I_{t_i})(R_{t_{i+1}}-R_{t_i}),\end{aligned}$$

with

$$\left|\sum_{i=0}^{n}(I_{t_{i+1}}-I_{t_i})(R_{t_{i+1}}-R_{t_i})\right| \leq \left[\sup_{0\leq i\leq n}|I_{t_{i+1}}-I_{t_i}|\right] TV(R)_t,$$

and

$$\left|\sum_{i=0}^{n}(R_{t_{i+1}}-R_{t_i})^2\right| \leq \left[\sup_{0\leq i\leq n}|R_{t_{i+1}}-R_{t_i}|\right] TV(R)_t.$$

Letting the mesh going to zero, we have

$$[X]_T = \lim \sum_{i=0}^{n}(I_{t_{i+1}}-I_{t_i})^2 + \lim \sum_{i=0}^{n}(J_{t_{i+1}}-J_{t_i})^2 = \int_0^T \Gamma_s^2 ds + \sum_{0<s\leq T}(\Delta J_s)^2.$$

□

Definition 5.2 (Cross variation) *Given two semi-martingales X and Y, the cross variation process $[X,Y]$ is:*

$$[X,Y]_t = X_tY_t - X_0Y_0 - \int_0^t X_{s^-}dY_s - \int_0^t Y_{s^-}dX_s.$$

Proposition 5.1

- *$[X,Y]$ is an adapted RCLL process with paths of finite variations.*
- *Polarization identity:*

$$[X,Y] = \frac{1}{2}([X+Y,X+Y]-[X,X]-[Y,Y]).$$

- *$[X,Y]_0 = X_0Y_0$ and $\Delta[X,Y] = \Delta X\Delta Y$.*
- *Ucp convergence:* $X_0Y_0 + \sum_{i=0}^{n}(X_{t_{i+1}}-X_{t_i})(Y_{t_{i+1}}-Y_{t_i}) \underset{\|\pi\|\to 0}{\longrightarrow} [X,Y]_T$.

Proof. Indeed, one can write

$$\begin{aligned}&2X_0Y_0 + 2\sum_{i=0}^{n}(X_{t_{i+1}}-X_{t_i})(Y_{t_{i+1}}-Y_{t_i}) \\ &= (X_0+Y_0)^2 - X_0^2 - Y_0^2 + \sum_{i=0}^{n}(X_{t_{i+1}}+Y_{t_{i+1}}-(X_{t_i}+Y_{t_i}))^2 \\ &\quad -\sum_{i=0}^{n}(X_{t_{i+1}}-X_{t_i})^2 - \sum_{i=0}^{n}(Y_{t_{i+1}}-Y_{t_i})^2.\end{aligned}$$

If we pass to the limit, we obtain for the right-hand side: $[X+Y, X+Y]_t - [X,X]_t - [Y,Y]_t$. This process is the difference of RCLL adapted and non decreasing processes, therefore is of finite variations.

Moreover

$$\begin{aligned} & X_0 Y_0 + \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \\ & = X_0 Y_0 - \sum_{i=0}^{n} X_{t_i}(Y_{t_{i+1}} - Y_{t_i}) - \sum_{i=0}^{n} Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + \sum_{i=0}^{n} (X_{t_{i+1}} Y_{t_{i+1}} - X_{t_i} Y_{t_i}) \\ & = X_t Y_t - \sum_{i=0}^{n} X_{t_i}(Y_{t_{i+1}} - Y_{t_i}) - \sum_{i=0}^{n} Y_{t_i}(X_{t_{i+1}} - X_{t_i}). \end{aligned}$$

Then passing though the limit, we have

$$[X,Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s^-} dY_s - \int_0^t Y_{s^-} dX_s.$$

This achieves the proof. □

Theorem 5.1 *Consider $X^{(i)}$, $i = 1, 2$, two jump processes:*

$$X_t^{(i)} = X_0^{(i)} + I_t^{(i)} + R_t^{(i)} + J_t^{(i)} = X_0^{(i)} + \int_0^t \Gamma_s^{(i)} dW_s + \int_0^t \Theta_s^{(i)} ds + J_t^{(i)}.$$

Then

$$[X^{(1)}, X^{(2)}]_T = \int_0^T \Gamma_s^{(1)} \Gamma_s^{(2)} ds + \sum_{0<s\leq T} \Delta J_s^{(1)} \Delta J_s^{(2)}.$$

Corollary 5.1 *Let W be a Brownian motion and $M = N - \lambda.$ a compensated Poisson process, relative to the same filtration. Then $[W, M]_t = 0$ for every $t \geq 0$.*

Proof. M is a pure jump martingale: $[M,M]_t = N_t$, hence

$$[W,M]_t = W_0 M_0 + \sum_{0<s\leq t} \Delta W_s \Delta M_s = 0.$$

□

Now for the stochastic integral we can extend the previous results. For $i = 1, 2$ let $X^{(i)}$ be jump processes: $X_t^{(i)} = X_0^{(i)} + I_t^{(i)} + R_t^{(i)} + J_t^{(i)}$. Let $\tilde{X}_0^{(i)}$ be two constants and $\Phi^{(i)}$ adapted processes. We define

$$\begin{aligned} \tilde{X}_t^{(i)} &= \tilde{X}_0^{(i)} + \int_0^t \Phi_s^{(i)} dX_s^{(i)} \\ &= \tilde{X}_0^{(i)} + \int_0^t \Phi_s^{(i)} \Gamma_s^{(i)} dW_s + \int_0^t \Phi_s^{(i)} \Theta_s^{(i)} ds + \sum_{0<s\leq t} \Phi_s^{(i)} \Delta J_s^{(i)}. \end{aligned}$$

Then we have

$$\begin{aligned}[\tilde{X}^{(1)}, \tilde{X}^{(2)}]_t &= \int_0^t \Phi_s^{(1)}\Phi_s^{(2)}\Gamma_s^{(1)}\Gamma^{(2)}ds + \sum_{0<s\leq t} \Phi_s^{(1)}\Phi_s^{(2)}\Delta J_s^{(1)}\Delta J_s^{(2)} \\ &= \int_0^t \Phi_s^{(1)}\Phi_s^{(2)} d[X^{(1)}, X^{(2)}]_s.\end{aligned}$$

Definition 5.3 *For a semi-martingale X, the process $[X,X]^c$ denotes the* path-by-path continuous part *of $[X,X]$.*

X is called quadratic pure jump *if $[X,X]^c = 0$.*

Proposition 5.2 *If X is adapted, RCLL, with paths of finite variations on compacts, then X is a quadratic pure jump semi-martingale.*

Proof. With the Stieltjes-Lebesgue integral, we can write

$$X_t^2 = \int_0^t X_{s^-}dX_s + \int_0^t X_s dX_s = 2\int_0^t X_{s^-}dX_s + [X,X]_t,$$

and

$$\int_0^t X_s dX_s = \int_0^t X_{s^-}dX_s + \int_0^t \Delta(X)_s dX_s = \int_0^t X_{s^-}dX_s + \sum(\Delta(X)_s)^2.$$

By identification

$$[X,X]_t = \sum(\Delta(X)_s)^2$$

which finishes the proof. □

Proposition 5.3 *Let X be a local martingale with continuous paths that are not everywhere constant. Then $[X,X]$ is not the constant process X_0^2 and $X^2 - [X,X]$ is a continuous local martingale. Moreover if $[X,X]_t = 0$ for all t, then $X_t = 0$ for all t.*

Proof. Note that a continuous local martingale is a semimartingale. We have $X^2 - [X,X] = 2\displaystyle\int X_{s^-}dX_s$, and by the martingale preservation property (Theorem 4.2) we have that $2\int X_{s^-}dX_s$ is a local martingale. Moreover

$$\Delta\left(2\int X_{s^-}dX_s\right)_t = 2X_{t^-}\Delta(X)_t,$$

and since X is continuous, $\Delta(X)_t = 0$, and thus $2\int X_{s^-}dX_s$ is a continuous local martingale, hence locally square integrable. Thus $X^2 - [X,X]$ is a locally square integrable local martingale.

By stopping, we can suppose X is a square integrable martingale. Assume further $X_0 = 0$. Next assume that $[X,X]$ actually were constant. Then $[X,X]_t = [X,X]_0 = X_0^2 = 0$, for all t. Since $X^2 - [X,X]$ is a local martingale, we conclude X^2 is a non-negative local martingale, with $X_0 = 0$. Thus $X_t = 0$, for all t. This is a contradiction. If X_0 is not identically 0, we set $\tilde{X}_t = X_t - X_0$ and the result follows. □

Corollary 5.2 *Let X be a continuous local martingale, and $S \leq T \leq +\infty$ be stopping times. If X has paths of finite variation on the stochastic interval (S, T), then X is constant on $[S, T]$. Moreover if $[X, X]$ is constant on $[S, T] \cap [0, +\infty)$ then X is constant there too.*

Corollary 5.3 *Let X and Y be two locally square integrable martingales. Then $[X, Y]$ is the unique adapted RCLL process A with paths on finite variation on compacts satisfying the two properties:*

1. *$XY - A$ is a local martingale;*
2. *$\Delta A = \Delta X \Delta Y$, $A_0 = X_0 Y_0$.*

Corollary 5.4 *Let M be a local martingale. Then M is a martingale with $\mathbb{E}(M_t)^2 < \infty$, $t \geq 0$ if and only if $\mathbb{E}([M, M]_t) < \infty$. In that case $\mathbb{E}(M_t^2) = \mathbb{E}([M, M]_t)$.*

The next theorem and the following results will be admitted.

Theorem 5.2 (Kunita-Watanabe inequality) *Let X and Y be two semi-martingales and ϕ, ψ be two measurable processes. Then one has a.s.*

$$\int_0^\infty |\phi_s||\psi_s| d[X, Y]_s \leq \left(\int_0^\infty \phi_s^2 d[X, X]_s \right) \left(\int_0^\infty \psi_s^2 d[Y, Y]_s \right).$$

Proposition 5.4 *Let X be a quadratic pure jump semi-martingale. Then for any semi-martingale Y,*

$$[X, Y]_t = X_0 Y_0 + \sum_{0 < s \leq t} \Delta X_s \Delta Y_s.$$

Proposition 5.5 *Let X, Y be two semi-martingales and let ϕ, ψ be in $\mathbb{L}$. Then*

$$\left[\int \phi dX, \int \psi dY \right]_t = \int_0^t \phi_s \psi_s d[X, Y]_s.$$

Proposition 5.6 *Let ϕ be in $\mathbb{D}$ and let X, Y be two semi-martingales. Let σ_n be a sequence of random partitions tending to the identity. Then*

$$\sum \phi_{T_i^n} (X^{T_{i+1}^n} - X^{T_i^n})(Y^{T_{i+1}^n} - Y^{T_{i+1}^n}) \underset{n \to +\infty,\ ucp}{\longrightarrow} \int \phi(s^-) d[X, Y]_s.$$

Examples.

Method 5.1 *Consider a stochastic integral w.r.t. a Poisson random measure:*

$$X_t = \int_0^t \int_{\mathbb{R}} \phi(s, y) J(ds, dy).$$

Then

$$[X]_T = [X, X]_T = \int_0^t \int_{\mathbb{R}} (\phi(s, y))^2 J(ds, dy).$$

Method 5.2 *If X is a Lévy process with characteristic triplet (σ^2, ν, γ), then*

$$[X, X]_t = \sigma^2 t + \sum_{0<s\leq t} |\Delta X_s|^2 = \sigma^2 t + \int_0^t \int_{\mathbb{R}} y^2 J_X(ds, dy).$$

Moreover $[X, X]$ is a subordinator.

Proof. Remember that X can be decompose into two independent parts: $X_t = X_t^1 + X_t^2$, where X^2 is a Brownian motion with variance σ^2 and drift γ and X^1 is the jump part of X. Therefore $[X, X]_t = \sigma^2 t + [X^1, X^1]_t$. Moreover X^1 is the sum of a compound Poisson process with jump size greater than one and of a limit of compound Poisson processes with jump size between $\varepsilon > 0$ and 1. Thus using Proposition 2.11, Proposition 5.2 and Corollary 5.3, we obtain

$$[X^1, X^1]_t = \sum_{0<s\leq t} |\Delta X_s|^2.$$

We deduce that the quadratic variation of a Lévy process is again a non decreasing Lévy process: it is a subordinator. □

Remark that in particular, if X is a symmetric α-stable Lévy process, which has infinite variance, the quadratic variation is a well-defined process, even though the variance is not defined.

5.2 The Itô formula

Recall that for Γ and Θ adapted, if:

$$X_t^c = X_0 + I_t + R_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds,$$

and if f is a function of class $C^2(\mathbb{R})$, then the Itô formula gives:

$$\begin{aligned} f(X_t^c) &= f(X_0) + \int_0^t f'(X_s^c) dX_s^c + \frac{1}{2} \int_0^t f''(X_s^c) d\left[X^c\right]_s \\ &= f(X_0) + \int_0^t f'(X_s^c) \Gamma_s dW_s + \int_0^t f'(X_s^c) \Theta_s ds + \frac{1}{2} \int_0^t f''(X_s^c) \Gamma_s^2 ds \end{aligned}$$

or in differential notation

$$df(X_t^c) = f'(X_s^c) \Gamma_s dW_s + f'(X_s^c) \Theta_s ds + \frac{1}{2} f''(X_s^c) \Gamma_s^2 ds.$$

5.2.1 For a jump-diffusion process

Let us extend the Itô formula to jump-diffusion processes, that is when there is a finite number of jumps in a finite time interval.

Theorem 5.3 *Let X be a jump-diffusion process and f of class $C^2(\mathbb{R})$:*

$$X_t = X_t^c + J_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds + J_t.$$

Then

$$\boxed{\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)\Gamma_s^2 ds \\ &\quad + \sum_{0<s\leq t} \left[f(X_s) - f(X_{s^-})\right]. \end{aligned}} \tag{5.1}$$

Proof. Let us fix ω and $0 < T_1 < T_1 < \ldots < T_{n-1} < t$ the jump times of X in the interval $[0, t[$. We note $T_0 = 0$ and $T_n = t$ (T_0 is not a jump size, but T_n can be). Now we apply the Itô formula between the times u and v with $T_i < u < v < T_{i+1}$:

$$f(X_v) = f(X_u) + \int_u^v f'(X_s)dX_s^c + \frac{1}{2}\int_u^v f''(X_s)d\left[X^c\right]_s.$$

Letting u tend to T_i and v to T_{i+1} and using X $RCLL$, we obtain

$$f\left(X_{T_{i+1}^-}\right) = f(X_{T_i}) + \int_{T_i}^{T_{i+1}} f'(X_s)dX_s^c + \frac{1}{2}\int_{T_i}^{T_{i+1}} f''(X_s)d\left[X^c\right]_s.$$

Note that if we put dX_S instead of dX_s^c we would have

$$\lim_{v\uparrow T_{i+1}} \int_u^v f'(X_s)dX_s \neq \int_u^{T_{i+1}} f'(X_s)dX_s.$$

Therefore

$$\begin{aligned} f(X_{T_{i+1}}) - f(X_{T_i}) &= \int_{T_i}^{T_{i+1}} f'(X_s)dX_s^c + \frac{1}{2}\int_{T_i}^{T_{i+1}} f''(X_s)d\left[X^c\right]_s \\ &\quad + \left[f(X_{T_{i+1}}) - f\left(X_{T_{i+1}^-}\right)\right]. \end{aligned}$$

To finish we just have to sum over i:

$$\begin{aligned} f(X_t) - f(X_0) &= \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)d\left[X^c\right]_s \\ &\quad + \sum_{i=0}^{n}\left[f(X_{T_{i+1}}) - f\left(X_{T_{i+1}^-}\right)\right] \\ &= \int_0^t f'(X_s)dX_s^c + \frac{1}{2}\int_0^t f''(X_s)d\left[X^c\right]_s + \sum_{0<s\leq t}\left[f(X_s) - f\left(X_{s^-}\right)\right] \end{aligned}$$

□

As two applications, we have the two propositions.

Proposition 5.7 *We consider the geometric Poisson process*

$$S_t = S_0 \exp\left(N_t \log(\sigma+1) - \lambda\sigma t\right) = S_t e^{-\lambda\sigma t}(\sigma+1)^{N_t},$$

where N is a Poisson process with intensity λ and $\sigma > -1$. Then S is a martingale:

$$S_t = S_0 + \sigma\int_0^t S(u^-)dM_u = S(0) + \sigma\int_0^t S(u^-)d(N_u - \lambda u).$$

Proposition 5.8 *Let W be a Brownian motion and N a Poisson process with intensity $\lambda > 0$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ relative to the same filtration $\{\mathcal{F}_t,\ t \geq 0\}$. Then W and N are independent.*

In the multidimensional framework

Theorem 5.4 *Let $X = (X^{(1)}, \ldots, X^{(d)})$ with $X^{(i)}$, $i = 1, \ldots, d$, be jump-diffusion processes and f of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Then*

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s) d(X^{(i)})^c_s \\ &\quad + \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d[(X^{(i)})^c, (X^{(j)})^c]_s \\ &\quad + \sum_{0<s\leq t} [f(s, X_s) - f(s, X_{s^-})] . \end{aligned}$$

The integration by parts formula becomes:

Proposition 5.9 *Consider X_1, X_2 two jump processes. Then*

$$\begin{aligned} X_1(t)X_2(t) &= X_1(0)X_2(0) + \int_0^t X_1(s) dX_2^c(s) + \int_0^t X_2(s) dX_1^c(s) \\ &\quad + [X_1^c, X_2^c]_t + \sum_{0<s\leq t} \left[X_1(s)X_2(s) - X_1(s^-)X_2(s^-) \right] \\ &= X_1(0)X_2(0) + \int_0^t X_1(s-) dX_2(s) + \int_0^t X_2(s-) dX_1(s) \\ &\quad + [X_1, X_2]_t. \end{aligned}$$

Proposition 5.10 (Doléans-Dade exponential) *Let X be a jump-diffusion process. The* Doléans-Dade exponential *of X is defined by*

$$Z_t^X = \exp\left\{ X_t^c - \frac{1}{2}[X^c, X^c]_t \right\} \prod_{0<s\leq t} (1 + \Delta X_s) .$$

This process is solution of the following stochastic differential equation with initial condition $Z^X(0) = 1$:

$$Z_t^X = 1 + \int_0^t Z_{s^-}^X \ dX_s.$$

Proof. Let $X^{(1)}$ and $X^{(2)}$ be two jump-diffusion processes, and $J^{(1)}$ and $J^{(2)}$ their pure jump parts. We apply the multidimensional Itô formula with $f(x_1, x_2) = x_1 x_2$ or the definition of the cross variation:

$$X_t^{(1)} X_t^{(2)} = X_0^{(1)} X_0^{(2)} + \int_0^t X_{s^-}^{(1)} dX_s^{(2)} + \int_0^t X_{s^-}^{(2)} dX_s^{(1)} + [X^{(1)}, X^{(2)}]_t.$$

But

$$\begin{aligned}
&\int_0^t X_{s^-}^{(1)} dX_s^{(2)} + \int_0^t X_{s^-}^{(2)} dX_s^{(1)} + [X^{(1)}, X^{(2)}]_t \\
&\quad = \int_0^t X_{s^-}^{(1)} d(X^{(2)})_s^c + \int_0^t X_{s^-}^{(2)} d(X^{(1)})_s^c + \int_0^t X_{s^-}^{(1)} dJ_s^{(2)} + \int_0^t X_{s^-}^{(2)} dJ_s^{(1)} \\
&\qquad\quad + [(X^{(1)})^c, (X^{(2)})^c]_t + \sum_{0<s\leq t} \Delta J_s^{(1)} \Delta J_s^{(2)} \\
&\quad = \int_0^t X_{s^-}^{(1)} d(X^{(2)})_s^c + \int_0^t X_{s^-}^{(2)} d(X^{(1)})_s^c + [(X^{(1)})^c, (X^{(2)})^c]_t \\
&\qquad\quad + \sum_{0<s\leq t} \left[X_{s^-}^{(1)} \Delta X_s^{(2)} + X_{s^-}^{(2)} \Delta X_s^{(1)} + \Delta X_s^{(1)} \Delta X_s^{(2)} \right].
\end{aligned}$$

Remark that

$$X_{s^-}^{(1)} \Delta X_s^{(2)} + X_{s^-}^{(2)} \Delta X_s^{(1)} + \Delta X_s^{(1)} \Delta X_s^{(2)} = X_s^{(1)} X_s^{(2)} - X_{s^-}^{(1)} X_{s^-}^{(2)}.$$

Hence

$$\begin{aligned}
X_t^{(1)} X_t^{(2)} &= X_0^{(1)} X_0^{(2)} + \int_0^t X_{s^-}^{(1)} d(X^{(2)})_s^c + \int_0^t X_{s^-}^{(2)} d(X^{(1)})_s^c + [(X^{(1)})^c, (X^{(2)})^c]_t \\
&+ \sum_{0<s\leq t} \left[X_s^{(1)} X_s^{(2)} - X_{s^-}^{(1)} X_{s^-}^{(2)} \right].
\end{aligned}$$

Now X is a jump-diffusion process written: $X_t = X_t^c + J_t = X_0 + \int_0^t \theta_s ds + \int_0^t \Gamma_s dB_s + J_t$. We define

$$Y_t = \exp\left(\int_0^t \theta_s ds + \int_0^t \Gamma_s dB_s - \frac{1}{2} \int_0^t \Gamma_s^2 ds \right).$$

Then the Itô formula for continuous process shows that

$$dY_t = Y_t dX_t^c = Y_{t^-} dX_t^c.$$

We put

$$K_t = \prod_{0<s\leq t} (1 + \Delta X_s)$$

with $K_t = 1$ before the first jump. Moreover

$$\Delta K_t = K_t - K_{t^-} = K_{t^-}(1 + \Delta X_t) - K_{t^-} = K_{t^-} \Delta X_t.$$

By definition $Z_t = Y_t K_t$ and from the previous result

$$\begin{aligned}
Z_t &= Y_0 K_0 + \int_0^t K_{s^-} dY_s + \sum_{0<s\leq t} [Y_s K_s - Y_{s^-} K_{s^-}] \\
&= 1 + \int_0^t K_{s^-} Y_{s^-} dX_s^c + \sum_{0<s\leq t} Y_{s^-} \Delta K_s \\
&= 1 + \int_0^t K_{s^-} Y_{s^-} dX_s^c + \sum_{0<s\leq t} Y_{s^-} K_{s^-} \Delta X_s \\
&= 1 + \int_0^t K_{s^-} Y_{s^-} dX_s = 1 + \int_0^t Z_{s^-} dX_s.
\end{aligned}$$

□

From this, we can conclude that if X is a martingale, Z is a local martingale.

Definition 5.4 (Doléans-Dade exponential) *$Z = \mathcal{E}(X)$ is called the* Doléans-Dade exponential *(or stochastic exponential) of X.*

5.2.2 The general case

If

- $X_t = \sigma W_t + \mu t + J_t$ where J is a compound Poisson process and W is a Brownian motion;
- $f \in C^2(\mathbb{R})$,

then the Itô formula (5.1) can be written:

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s)dX_s^c + \frac{\sigma^2}{2}\int_0^t f''(X_s)ds \\ &\quad + \sum_{0<s\leq t} [f(X_s) - f(X_{s^-})] \\ &= f(X_0) + \int_0^t f'(X_s)dX_s + \frac{\sigma^2}{2}\int_0^t f''(X_s)ds \\ &\quad + \sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})]\,. \end{aligned}$$

This last expression can be extended to semi-martingales.

Theorem 5.5 *Let X be an n-tuple of semi-martingales, and $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ a $C^{1,2}$ function. Then $f(.,X)$ is again a semi-martingale, and the following formula holds:*

$$\boxed{\begin{aligned} f(t,X_t) &= f(0,X_0) + \int_0^t \frac{\partial f}{\partial t}(s,X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s,X_{s^-})dX_s^i \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X_s)d[X^i,X^j]_s^c \\ &\quad + \sum_{0<s\leq t} \left[f(s,X_s) - f(s,X_{s^-}) - \sum_{i=1}^d \Delta X_s^i \frac{\partial f}{\partial x_i}(s,X_{s^-}) \right]. \end{aligned}} \tag{5.2}$$

Proof. We make the proof in dimension 1 with f depending only on x. The general case is a straightforward extension.

Remember that from Propositon 2.11 for any $t > 0$, a.s.

$$\sum_{0<s\leq t} |\Delta X_s|^2 \leq [X,X]_t < +\infty.$$

Let $\varepsilon > 0$ and $t > 0$. We define two sets $A = A(\varepsilon,t)$ and $B = B(\varepsilon,t)$ such that

- $A \cup B$ is the set of all jump times of X on the interval $]0,t]$;
- $A \cap B = \emptyset$;

- $\sum_{s \in B} |\Delta X_s|^2 \leq \varepsilon^2$;
- A is a.s. finite.

Then for any $0 = T_0^n < T_1^n < \ldots < T_n^n = t$,

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=0}^{n-1} f(X_{T_{i+1}^n}) - f(X_{T_i^n}) \\ &= f(X_0) + \sum_{i,A} f(X_{T_{i+1}^n}) - f(X_{T_i^n}) + \sum_{i,B} f(X_{T_{i+1}^n}) - f(X_{T_i^n}), \end{aligned} \tag{5.3}$$

where $\sum_{i,A} \alpha_i = \sum_{i=0}^{n-1} \alpha_i \mathbf{1}_{A \cap]T_i^n, T_{i+1}^n] \neq \emptyset}$. When n tends to $+\infty$,

$$\lim_{n \to +\infty} \sum_{i,A} f(X_{T_{i+1}^n}) - f(X_{T_i^n}) = \sum_{s \in A} f(X_s) - f(X_{s^-}).$$

For the second sum, we use Taylor's expansion to obtain:

$$\begin{aligned} &\sum_{i,B} f(X_{T_{i+1}^n}) - f(X_{T_i^n}) \\ &= \sum_{i=0}^{n-1} f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n})^2 \\ &- \sum_{i,A} \left[f'(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n}) + \frac{1}{2} f''(X_{T_i^n})(X_{T_{i+1}^n} - X_{T_i^n})^2 \right] \\ &+ \sum_{i,B} R(T_i^n, T_{i+1}^n). \end{aligned}$$

The first two sums converge respectively to $\int_0^t f'(X_{s^-}) dX_s$ and $\frac{1}{2} \int_0^t f''(X_{s^-}) d[X,X]_s$. The third will converge to

$$- \sum_{s \in A} \left[f'(X_{s^-}) \Delta X_s + \frac{1}{2} f''(X_{s^-})(\Delta X_s)^2 \right].$$

Now if for some constant K, $|X_s| \leq K$ for any $0 \leq s \leq t$, since f'' is uniformly continuous on any compact set, and since X is RCLL, we have:

$$\limsup_{n \to +\infty} \sum_{i,B} |R(T_i^n, T_{i+1}^n)| \leq r(\varepsilon^+)[X,X]_t,$$

where

$$r(\varepsilon^+) = \limsup_{\delta \downarrow \varepsilon} r(\delta), \quad r : \mathbb{R}_+ \to \mathbb{R}_+, \text{ increasing with } \lim_{\delta \to 0} r(\delta) = 0.$$

Coming back to (5.3), we obtain:

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{s \in A} f(X_s) - f(X_{s^-}) + \int_0^t f'(X_{s^-}) dX_s \\ &+ \frac{1}{2} \int_0^t f''(X_{s^-}) d[X,X]_s - \sum_{s \in A} \left[f'(X_{s^-}) \Delta X_s + \frac{1}{2} f''(X_{s^-})(\Delta X_s)^2 \right] + \phi(\varepsilon) \\ &= f(X_0) + \int_0^t f'(X_{s^-}) dX_s + \frac{1}{2} \int_0^t f''(X_{s^-}) d[X,X]_s^c \\ &+ \sum_{s \in A} [f(X_s) - f(X_{s^-}) - f'(X_{s^-}) \Delta X_s] + \phi(\varepsilon). \end{aligned}$$

Now let ε go to zero. Of course $\phi(\varepsilon) \to 0$ and on $[-K, K]$, $|f(y) - f(x) - (y-x)f'(x)| \leq C(y-x)^2$, thus

$$\sum_{s \in A} |f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s| \leq C \sum_{0<s\leq t} (\Delta X_s)^2 \leq C[X,X]_t.$$

Hence

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s^-}) dX_s + \frac{1}{2}\int_0^t f''(X_{s^-}) d[X,X]^c_s \\ &+ \sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s]. \end{aligned}$$

To finish the proof, for any K, we define $\tau_K = \inf\{s \geq 0, \ |X_s| \geq K\} \wedge t$ and we apply the previous result on the semi-martingale $X\mathbf{1}_{[0,\tau_K[}$ and we let K going to $+\infty$. □

When $X = (X^1, \ldots, X^d)$ is a d-dimensional Lévy process with characteristic triplet (A, ν, γ), the continuous quadratic variation is given by the matrix A. Hence we have for any $C^{1,2}$ function $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_{s^-}) dX^i_s \\ &+ \frac{1}{2}\sum_{i,j=1}^d \int_0^t A_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) ds \\ &+ \sum_{0<s\leq t} \left[f(s, X_s) - f(s, X_{s^-}) - \sum_{i=1}^d \Delta X^i_s \frac{\partial f}{\partial x_i}(s, X_{s^-}) \right]. \end{aligned}$$

From this formula, we deduce

Proposition 5.11 *Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) and $f : \mathbb{R} \to \mathbb{R}$ a C^2 function such that f and its two derivatives are bounded by a constant C. Then $Y_t = f(X_t) = M_t + V_t$ where M is the martingale part given by:*

$$M_t = f(X_0) + \int_0^t f'(X_s)\sigma dW_s + \int_0^t \int_{\mathbb{R}} \tilde{J}_X(ds, dy)(f(X_{s^-} + y) - f(X_{s^-})),$$

and V a continuous finite variation process:

$$\begin{aligned} V_t &= \frac{\sigma^2}{2}\int_0^t f''(X_s) ds + \gamma \int_0^t f'(X_s) ds \\ &+ \int_0^t \int_{\mathbb{R}} (f(X_{s^-} + y) - f(X_{s^-}) - yf'(X_s)\mathbf{1}_{|y|\leq 1}) ds\nu(dy). \end{aligned}$$

Proof. Indeed remember that X can be written as follows

$$X_t = \gamma t + \sigma W_t + Y_t + \widetilde{X}_t = \gamma t + \sigma W_t + Y_t + \lim_{\varepsilon \downarrow 0} \widetilde{X}^\varepsilon_t,$$

where W is a Brownian motion, Y is a compound Poisson process with jump size greater that 1, $\widetilde{X}^\varepsilon$ is a compensated Poisson process with jump size between ε and 1. And W, $\widetilde{X}^\varepsilon$ and $\widetilde{X}$ are martingales. Thus

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s^-})dW_s + \int_0^t f'(X_{s^-})\gamma ds + \int_0^t f'(X_{s^-})dY_s + \int_0^t f'(X_{s^-})d\widetilde{X}_s \\ &+\frac{1}{2}\sigma^2 \int_0^t f''(X_s)ds + \sum_{0<s\leq t} [f(X_s) - f(X_{s^-}) - \Delta X_s f'(X_{s^-})]\,. \end{aligned}$$

Since Y and $\widetilde{X}$ are independent, they do not jump at the same time. Thus in the previous equality, since Y is a compound Poisson process, we have

$$\int_0^t f'(X_{s^-})dY_s = \sum_{0<s\leq t} \Delta Y_s f'(X_{s^-})$$

and

$$\begin{aligned} f(X_t) &= f(X_0) + \frac{1}{2}\sigma^2 \int_0^t f''(X_s)ds + \gamma \int_0^t f'(X_{s^-})ds + \int_0^t f'(X_{s^-})dW_s \\ &+ \sum_{0<s\leq t,\ \Delta Y_s \neq 0} [f(X_{s^-} + \Delta Y_s) - f(X_{s^-})] \\ &+ \int_0^t f'(X_{s^-})d\widetilde{X}_s + \sum_{0<s\leq t,\ \Delta \widetilde{X}_s \neq 0} \left[f(X_{s^-} + \Delta \widetilde{X}_s) - f(X_{s^-}) - \Delta \widetilde{X}_s f'(X_{s^-})\right]. \end{aligned}$$

Remark that

$$\sum_{0<s\leq t,\ \Delta Y_s \neq 0} [f(X_{s^-} + \Delta Y_s) - f(X_{s^-})] = \int_0^t \int_{\mathbb{R}} [f(X_{s^-} + y) - f(X_{s^-})]\, J_Y(ds, dy).$$

Define $\widetilde{J}_Y(ds, dy) = J_Y(ds, dy) - ds\mathbf{1}_{|y|\geq 1}\nu(dy)$. $\widetilde{J}_Y$ is a compensated Poisson measure, and for any subset A, the process $\widetilde{J}_Y([0,t]\times A)$ is a martingale. Therefore

$$\begin{aligned} \sum_{0<s\leq t,\ \Delta Y_s \neq 0} [f(X_{s^-} + \Delta Y_s) - f(X_{s^-})] &= \int_0^t \int_{\mathbb{R}} [f(X_{s^-} + y) - f(X_{s^-})]\, \widetilde{J}_Y(ds, dy) \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s^-} + y) - f(X_{s^-})]\, ds\mathbf{1}_{|y|\geq 1}\nu(dy). \end{aligned}$$

The first integral is a martingale (see Proposition 4.8). The same trick can be done for $\widetilde{X}$. Indeed for any $\varepsilon > 0$, $\widetilde{X}^\varepsilon$ is a compound Poisson process with a continuous drift. Hence we have

$$\begin{aligned} &\int_0^t f'(X_{s^-})d\widetilde{X}_s + \sum_{0<s\leq t,\ \Delta \widetilde{X}_s \neq 0} \left[f(X_{s^-} + \Delta \widetilde{X}_s) - f(X_{s^-}) - \Delta \widetilde{X}_s f'(X_{s^-})\right] \\ &= -\int_0^t \int_{\mathbb{R}} f'(X_{s^-}) y \mathbf{1}_{|y|\leq 1}\nu(dy)ds + \int_0^t \int_{\mathbb{R}} [f(X_{s^-} + y) - f(X_{s^-})]\, \widetilde{J}_{\widetilde{X}}(ds, dy) \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s^-} + y) - f(X_{s^-})]\, ds\mathbf{1}_{|y|\leq 1}\nu(dy). \end{aligned}$$

Here the compensation is already done in $\widetilde{X}$. We conclude:

$$\begin{aligned} f(X_t) &= f(X_0) + \frac{1}{2}\sigma^2 \int_0^t f''(X_s)ds + \gamma \int_0^t f'(X_{s^-})ds + \int_0^t f'(X_{s^-})dW_s \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s^-}+y) - f(X_{s^-})] \widetilde{J}_Y(ds,dy) + \int_0^t \int_{\mathbb{R}} [f(X_{s^-}+y) - f(X_{s^-})] ds \mathbf{1}_{|y|\geq 1}\nu(dy) \\ &+ \int_0^t \int_{\mathbb{R}} [f(X_{s^-}+y) - f(X_{s^-})] \widetilde{J}_{\widetilde{X}}(ds,dy) + \int_0^t \int_{\mathbb{R}} [f(X_{s^-}+y) - f(X_{s^-})] ds \mathbf{1}_{|y|\geq 1}\nu(dy) \\ &- \int_0^t \int_{\mathbb{R}} f'(X_{s^-}) y \mathbf{1}_{|y|\leq 1}\nu(dy)ds. \end{aligned}$$

The decomposition of $f(X_t)$ follows. □

5.3 Stochastic exponentials vs. ordinary exponentials

From the Itô formula, we obtain the following result.

Proposition 5.12 *Let X be a (σ^2, ν, γ) Lévy process s.t. $\int_{|y|\geq 1} e^y \nu(dy) < \infty$. Then $Y_t = \exp(X_t)$ is a semi-martingale with decomposition $Y_t = M_t + A_t$ where the martingale part is given by*

$$M_t = 1 + \int_0^t Y_{s^-}\sigma dW_s + \int_0^t \int_{\mathbb{R}} Y_{s^-}(e^z - 1)\tilde{J}_X(ds, dz);$$

and the continuous finite variation drift part by

$$A_t = \int_0^t Y_{s^-} \left[\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{|z|\leq 1})\nu(dz) \right] ds.$$

Proof. We apply Proposition 5.11 with $f(x) = e^x$. □

This shows again that Y is a martingale if and only if

$$\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{|z|\leq 1})\nu(dz) = 0.$$

Proposition 5.13 *Let X be a (σ^2, ν, γ) Lévy process. There exists a unique RCLL process Z such that:*

$$dZ_t = Z_{t^-} dX_t, \quad Z_0 = 1. \tag{5.4}$$

Z is given by:

$$\boxed{Z_t = \exp\left(X_t - \frac{\sigma^2 t}{2}\right) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}.} \tag{5.5}$$

If $\int_{-1}^{1}|x|\nu(dx)<\infty$, *the jumps of* X *have finite variation and the stochastic exponential of* X *can be expressed as*

$$Z_t=\exp\left(X_t^c-\frac{\sigma^2 t}{2}\right)\prod_{0<s\le t}(1+\Delta X_s).$$

Proof. Let

$$V_t=\prod_{0<s\le t}(1+\Delta X_s)e^{-\Delta X_s}.$$

The first step is to show that this process exists and is of finite variation. We decompose V into a product of two terms: $V_t=V_t'V_t''$, where

$$V_t'=\prod_{0<s\le t,\ |\Delta X_s|\le 1/2}(1+\Delta X_s)e^{-\Delta X_s},\qquad V_t''=\prod_{0<s\le t,\ |\Delta X_s|>1/2}(1+\Delta X_s)e^{-\Delta X_s}.$$

V'' for every t is a product of finite number of factors, so it is clearly of finite variation and there are no existence problems. V' is positive and we can consider its logarithm.

$$\ln V_t'=\sum_{0<s\le t,\ |\Delta X_s|\le 1/2}\left[\ln(1+\Delta X_s)-\Delta X_s\right].$$

Note that each term of this sum satisfies

$$0>\ln(1+\Delta X_s)-\Delta X_s>-(\Delta X_s)^2.$$

Therefore, the series is decreasing and bounded from below by $-\sum_{0<s\le t}(\Delta X_s)^2$, which is finite which is finite for every Lévy process (see Proposition 2.11). Hence, (lnV_t') exists and is a decreasing process. This entails that V exists and has trajectories of finite variation.

The second step is to apply the Itô formula for semi-martingales to the function

$$Z_t=f(t,X_t,V_t)=e^{X_t-\sigma^2 t/2}V_t.$$

This yields

$$\begin{aligned}Z_t &= 1-\int_0^t\frac{\sigma^2}{2}Z_s ds+\int_0^t Z_{s-}dX_s+\int_0^t e^{X_{s-}-\sigma^2 s/2}dV_s\\ &+ \frac{\sigma^2}{2}\int_0^t Z_{s-}ds+\sum_{0<s\le t}\left[Z_s-Z_{s-}-(Z_{s-})\Delta X_s-e^{X_{s-}-\sigma^2 s/2}\Delta V_s\right].\end{aligned}$$

Now observe that since V_t is a pure jump process,

$$dV_t=\Delta V_t=V_{t-}\left(e^{\Delta X_t}(1+\Delta X_t)-1\right).$$

Thus: $\Delta Z_s=1+\Delta X_s$. Substituting this into the above equality and making all the cancellations yields the Equation (5.4). □

Definition 5.5 (Doléans-Dade exponential) $Z=\mathcal{E}(X)$ *is called the* Doléans-Dade exponential *(or stochastic exponential) of* X*.*

Remark that the previous proof works for semi-martingales too. We can prove the next result.

Proposition 5.14 *If X is a Lévy process and a martingale, then its stochastic exponential $Z = \mathcal{E}(X)$ is also a martingale.*

Until now we have two different exponentials. And therefore two ways to modelize market prices. But let X be a Lévy process with triplet (σ^2, ν, γ) and $Z = \mathcal{E}(X)$ its stochastic exponential. If $Z > 0$ a.s., there exists another Lévy process L such that $Z = \exp(L)$ where:

$$L_t = \ln Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0<s\leq t} (\ln(1 + \Delta X_s) - \Delta X_s). \tag{5.6}$$

Its characteristic triplet $(\sigma_L^2, \nu_L, \gamma_L)$ is given by:

$$\begin{aligned}
\sigma_L &= \sigma, \\
\nu_L(A) &= \int \mathbf{1}_A(\ln(1+x))\nu(dx), \\
\gamma_L &= \gamma - \frac{\sigma^2}{2} + \int \left[\ln(1+x)\mathbf{1}_{[-1,1]}(\ln(1+x)) - x\mathbf{1}_{[-1,1]}(x)\right] \nu(dx).
\end{aligned}$$

Indeed if $Z > 0$, then $\Delta X_s > -1$ for all s a.s., so taking the logarithm is justified here. In the proof of Proposition 5.13 we have seen that the sum $\sum_{0<s\leq t} \ln(1 + \Delta X_s) - \Delta X_s$ converges and is a finite variation process. Then it is clear that L is a Lévy process and that $\sigma_L = \sigma$. Moreover, $\Delta L_s = \ln(1 + \Delta X_s)$ for all s. This entails that

$$J_L([0,t] \times A) = \int_{[0,t]\times\mathbb{R}} \mathbf{1}_A(\ln(1+x)) J_X(dsdx),$$

and also $\nu_L(dx) = \mathbf{1}_A(\ln(1+x))\nu(dx)$. It remains to compute γ_L. Substituting the Lévy decomposition for L_t and X_t into (5.6), we obtain

$$\begin{aligned}
\gamma_L t - \gamma t + \frac{\sigma^2 t}{2} \quad &+ \int_{s\in[0,t],|x|\leq 1} x\widetilde{J}_L(dsdx) + \int_{s\in[0,t],|x|>1} xJ_L(dsdx) \\
&- \int_{s\in[0,t],|x|\leq 1} x\widetilde{J}_X(dsdx) - \int_{s\in[0,t],|x|>1} xJ_X(dsdx) \\
&- \sum_{0<s\leq t} [\ln(1 + \Delta X_s) - \Delta X_s] = 0.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_{s\in[0,t],|x|\leq 1} x(J_X(dsdx) - J_L(dsdx)) \\
&= \sum_{0<s\leq t} \left[\Delta X_s \mathbf{1}_{[-1,1]}(\Delta X_s) - \ln(1+\Delta X_s)\mathbf{1}_{[-1,1]}(\ln(1+\Delta X_s))\right],
\end{aligned}$$

converges, we can split the above expression into jump part and drift part, both of which must be equal to zero. For the drift part we obtain:

$$\gamma_L - \gamma + \frac{\sigma^2}{2} - \int_{-1}^{1} [x\nu_L(dx) - x\nu(dx)] = 0,$$

which yields the correct formula for γ_L after a change of variable.

Conversely if L is a Lévy process with triple $(\sigma_L^2, \nu_L, \gamma_L)$ and $S_t = \exp(L_t)$ its exponential. The jumps of S_t are given by $\Delta S_t = S_{t^-}(\exp(\Delta L_t)?1)$. If there exists a Lévy process X such that S is the stochastic exponential of X: $S = \mathcal{E}(X)$, then since $dS_t = S_{t^-} dX_t$, $\Delta S_t = S_{t^-}\Delta X_t$, so $\Delta X_t = \exp(\Delta L_t)?1$. Hence ν is given by:

$$\nu(A) = \int \mathbf{1}_A(e^x - 1)\nu_L(dx).$$

In particular $\Delta X_t > -1$ a.s. and it is easily verified that $\ln \mathcal{E}(X)$ is a Lévy process with characteristics matching those of L only if X has characteristics given by

$$\begin{aligned}
&\sigma = \sigma_L, \\
&\nu(A) = \int \mathbf{1}_A(e^x - 1)\nu_L(dx) \\
&\gamma = \gamma_L + \frac{\sigma_L^2}{2} + \int \left[(e^x - 1)\mathbf{1}_{[-1,1]}(e^x - 1) - x\mathbf{1}_{[-1,1]}(x)\right] \nu_L(dx).
\end{aligned}$$

Moreover

$$X_t = L_t + \frac{\sigma^2 t}{2} + \sum_{0<s\le t} \left[1 + \Delta L_s - e^{\Delta L_s}\right].$$

Conversely if X is a Lévy process with the previous characteristics, using (5.5) we can verify as above that $\mathcal{E}(X)_t = \exp L_t$.

This shows that the choice of one exponential to modelize a market price is not important and in both cases, one speaks about exponential Lévy model.

To finish this part, let us prove Proposition 5.14.

Proof. Let $(X_t)_{t\ge 0}$ be a Lévy process with characteristic triplet (σ^2, ν, γ) such that $\gamma + \int_{|x|\ge 1} x\nu(dx) = 0$ (this is the martingale condition). First, suppose that $|\Delta X_s| \le \varepsilon < 1$ a.s. Then there exists a Lévy process L such that $e^{L_t} = Z_t$. Moreover, this process has bounded jumps and therefore admits all exponential moments. Again we can write:

$$\begin{aligned}
\gamma_L + \frac{\sigma_L^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{|z|\le 1})\nu_L(dz) &= \gamma + \int_{-1}^{1} [z\nu_L(dz) - z\nu(dz)] \\
+ \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{|z|\le 1})\nu_L(dz) &= \int_{-\infty}^{\infty} [(e^z - 1)\nu_L(dz) - z\nu(dz)] = 0,
\end{aligned}$$

because $\Delta X_s = e^{\Delta L_s} - 1$ for all s. Therefore by Proposition 5.12, $Z_t = e^{L_t}$ is a martingale.

The second step is to prove the proposition when X is a compensated compound Poisson process. In this case, the stochastic exponential has a very simple form:

$$Z_t = e^{bt} \prod_{0<s\le t} (1 + \Delta X_s),$$

where $b = -\int_{-\infty} +\infty x\nu(dx)$. Denoting the intensity of X by λ, we obtain

$$\mathbb{E}(Z_t) = e^{-\lambda t + bt} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} (1 + \mathbb{E}(\Delta X))^n = 1.$$

Together with the independent increments property of X this proves that Z is a martingale.

Now let X be an arbitrary martingale Lévy process. It can be decomposed into a sum of a compensated compound Poisson process X' and an independent martingale Lévy process with jumps smaller than ε, denoted by X''. Since these two processes never jump together, $\mathcal{E}(X' + X'') = \mathcal{E}(X')\mathcal{E}(X'')$. Moreover, each of the factors is a martingale and they are independent, therefore we conclude that $\mathcal{E}(X' + X'')$ is a martingale.

$\square$

5.4 Exercises

Exercice 5.1 Soit $(N_t)_{t\geq 0}$ un processus de Poisson d'intensité λ, et soient $(Y_n)_{n\geq 1}$ une suite de v.a.r indépendantes et identiquement distribuées définies sur le même espace de probabilité filtré $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. On suppose aussi que les v.a.r $Y_1, Y_2, \ldots$ sont indépendantes du processus $(N_t)_{t\geq 0}$. On définit par $Q_t := \sum_{i=1}^{N_t} Y_i$.

1. Quelle est la nature du processus $(Q_t)_{t\geq 0}$? Donner l'expression de son saut à l'instant $t \geq 0$.

On suppose dans la suite que pour tout $i \geq 1$, chaque v.a.r. Y_i prend ses valeurs dans $\{y_1, y_2, \ldots, y_m\}$ où $m \in \mathbb{N}^*$. On note par $p(y_k)$ la probabilité que le saut est de taille y_k c'est-à-dire $p(y_k) := \mathbb{P}(Y_i = y_k)$ pour $k = 1, \ldots, m$. Cette probabilité ne dépend pas de i car les v.a. Y_i sont identiquement distribuées. On suppose que $p(y_k) > 0$ pour tout k et on rappelle qu'on a $\sum_{k=1}^m p(y_k) = 1$. On note par N_t^k le nombre de sauts du processus Q_t de taille y_k sur l'intervalle de temps $[0, t]$, ce qui permet d'écrire

$$N_t = \sum_{k=1}^m N_t^k \quad \text{et} \quad Q_t = \sum_{k=1}^m y_k N_t^k.$$

2. Justifier que N^1, N^2,..., N^m sont des processus de Poisson indépendants d'intensités $\lambda_1 = \lambda p(y_1)$, $\lambda_2 = \lambda p(y_2)$,...., $\lambda_m = \lambda p(y_m)$.

3. Soit $\widetilde{\lambda}_1$,...., $\widetilde{\lambda}_m$ des réels strictement positifs. On définit

$$Z_k(t) := e^{(\lambda_k - \widetilde{\lambda}_k)t} \left(\frac{\widetilde{\lambda}_k}{\lambda_k}\right)^{N_t^k} \quad \text{et} \quad Z(t) := \prod_{k=1}^m Z_k(t)\,. \tag{5.7}$$

   Montrer que Z_k vérifie l'EDS suivante :

$$dZ_k(t) = \frac{\widetilde{\lambda}_k - \lambda_k}{\lambda_k} Z_k(t-) dM_k(t)$$

   où $M_k(t) = N_t^k - \lambda_k t$ pour $k = 1, ..., m$.

4. Montrer que Z_k est une martingale et que le crochet $[Z_k, Z_{k'}] = 0$ pour $k \neq k'$.

5. Montrer que $Z_1 Z_2$ et $Z_1 Z_2 Z_3$ sont des martingales. En déduire que Z est une martingale et que $\mathbb{E}\big[Z(t)\big] = 1$ pour tout $t \geq 0$.

6. Montrer que $Z_t = e^{(\lambda - \widetilde{\lambda})t} \prod_{k=1}^{N_t} \frac{\widetilde{\lambda}\widetilde{p}(Y_k)}{\lambda p(Y_k)}$.

Exercice 5.2 Soit $(P_t)_{t\geq 0}$ un processus de Poisson d'intensité λ et $(Z_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires indépendantes et identiquement distribuées de loi $\nu(dz)$ sur $\mathbb{R}$. On suppose que la suite $(Z_n)_{n\in\mathbb{N}}$ et le processus $(P_t)_{t\geq 0}$ sont indépendants. On définit le processus de Poisson composé

$$Y_t = \sum_{k=1}^{P_t} Z_k.$$

1. Quel est le triplet caractéristique de Y ?
2. Écrire la décomposition de Lévy de Y en définissant correctement chacun des termes de la formule. En déduire que

$$Y_t = \int_{\mathbb{R}} zN(dr, dz)$$

pour une mesure aléatoire de Poisson N appropriée.

3. On suppose que l'on a une fonction $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ de classe C^2 sur $[0,T] \times \mathbb{R}$, avec dérivées bornées sur $[0,T] \times \mathbb{R}$, telle que

$$\frac{\partial u}{\partial t}(t,x) = \int_{\mathbb{R}} (u(t,x+z) - u(t,x))\lambda\nu(dz)$$

avec condition initiale $u(0,x) = g(x)$. On fixe $t \in [0,T]$ et $x \in \mathbb{R}$. Le but est d'exprimer $u(t,x)$ en fonction du processus Y.

(a) Montrer que le processus

$$Z_t = \int_0^t \int_{\mathbb{R}} [u(t-r, x+Y_r+z) - u(t-r, x+Y_r)]\, \widetilde{N}(dr, dz)$$

est intégrable et d'espérance nulle. $\widetilde{N}$ est la mesure de Poisson compensée.

(b) Montrer que l'on a :

$$u(t,x) = \mathbb{E}g(x+Y_t).$$

Indication : appliquer la formule d'Itô à la fonction $(r,y) \mapsto u(t-r,y)$ et au processus de Lévy $(r, x+Y_r)$.

Exercice 5.3 Soit X un processus de Lévy-Itô de la forme

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{|z|>1} \gamma_s(z) N(ds,dz) + \int_0^t \int_{|z|\leq 1} \gamma_s(z)\tilde{N}(ds,dz)$$

avec $\int_{|x|\leq 1} \gamma_t(x)^2 \nu(dx)$ localement borné. On suppose que

$$\mu_t + \frac{\sigma_t^2}{2} + \int_{\mathbb{R}} \left(e^{\gamma_t(x)} - 1 - \gamma_t(x)\mathbb{1}_{|x|\leq 1}\right)\nu(dx) = 0$$

p.s. pour tout t. En appliquant la formule d'Itô, montrer que e^{X_t} s'écrit sous la forme :

$$e^{X_t} = \int_0^t \alpha_s dW_s + \int_0^t \int_{\mathbb{R}} \beta_s(z) \tilde{N}(ds, dz)$$

avec les coefficients α et β à préciser.

En supposant que σ_t et $\int_{\mathbb{R}} (e^{\gamma_t(x)} - 1)^2 \nu(dx)$ sont bornés p.s. par une constante C, montrer, en utilisant le lemme de Gronwall, que (e^{X_t}) est une martingale de carré intégrable.

Lemme de Gronwall. Soit ϕ une fonction positive localement bornée sur $\mathbb{R}_+$ telle que

$$\phi(t) \leq a + b \int_0^t \phi(s) ds$$

pour tout t et deux constantes a et $b \geq 0$. Alors $\phi(t) \leq a e^{bt}$.

Exercice 5.4 (Extrait de l'examen 2008-2009) Dans cet exercice, la maturité est $T > 0$ et on suppose que le prix de l'actif sans risque est donné par

$$dS_t^0 = S_t^0 r dt, \quad S_0^0 = 1;$$

tandis que le prix de l'actif risqué est donné par l'équation suivante :

$$dS_t = S_{t_-}(bdt + \sigma dW_t + \delta dM_t), \quad S_0 > 0.$$

Ici W est un mouvement brownien standard, M un processus de Poisson compensé, i.e. $M_t = N_t - \lambda t$, avec N processus de Poisson d'intensité $\lambda > 0$, indépendant de W. Tous les processus sont définis sur le même espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ et sont adaptés à la filtration. Les hypothèses sur les paramètres sont :

$$r > 0, \quad b \in \mathbb{R}, \quad \sigma \in \mathbb{R}_+, \quad \delta \in]-1, +\infty[\setminus \{0\}.$$

On rappelle que les exponentielles de Doléans-Dade sont

$$\begin{aligned} \mathcal{E}(\sigma W)(t) &= \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right), \\ \mathcal{E}(\delta M)(t) &= \exp\left(\ln(1+\delta) M_t - \lambda t(\delta - \ln(1+\delta))\right) = \exp\left(\ln(1+\delta) N_t - \lambda \delta t\right), \end{aligned}$$

1. Quelles sont les équations vérifiées par $\mathcal{E}(\sigma W)$ et $\mathcal{E}(\delta M)$? On appliquera la formule d'Itô en justifiant son emploi.

2. Exprimer S_t uniquement en fonction des paramètres du modèle.

3. Montrer que pour tout $a \in \mathbb{R}$,

$$(S_t)^a = (S_0)^a \mathcal{E}(a\sigma W)(t) \mathcal{E}(\delta_a M)(t) \exp\left[\frac{1}{2} a(a-1)\sigma^2 t + abt + \lambda t(\delta_a - a\delta)\right],$$

avec $\delta_a = (1+\delta)^a - 1$.

4. En déduire $\mathbb{E}(S_t^a)$ pour tout $t \geq 0$.

Exercice 5.5 Soit $(P_t)_{t\geq 0}$ un processus de Poisson d'intensité λ et $(Z_n)_{n\in\mathbb{N}}$ une suite de variables aléatoires indépendantes et identiquement distribuées de loi $\nu(dz)$ sur $\mathbb{R}$. On suppose que la suite $(Z_n)_{n\in\mathbb{N}}$ et le processus $(P_t)_{t\geq 0}$ sont indépendants. On définit le processus de Poisson composé

$$Y_t = \sum_{k=1}^{P_t} Z_k.$$

1. Quel est le triplet caractéristique de Y ?

2. Écrire la décomposition de Lévy de Y en définissant correctement chacun des termes de la formule. En déduire que

$$Y_t = \int_{\mathbb{R}} zN(dr, dz)$$

   pour une mesure aléatoire de Poisson N appropriée.

3. On suppose que l'on a une fonction $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ de classe C^2 sur $[0,T] \times \mathbb{R}$, avec dérivées bornées sur $[0,T] \times \mathbb{R}$, telle que

$$\frac{\partial u}{\partial t}(t,x) = \int_{\mathbb{R}} (u(t,x+z) - u(t,x))\lambda\nu(dz)$$

   avec condition initiale $u(0,x) = g(x)$. On fixe $t \in [0,T]$ et $x \in \mathbb{R}$. Le but est d'exprimer $u(t,x)$ en fonction du processus Y.

   (a) Montrer que le processus

$$Z_t = \int_0^t \int_{\mathbb{R}} [u(t-r, x+Y_r+z) - u(t-r, x+Y_r)]\, \widetilde{N}(dr, dz)$$

   est intégrable et d'espérance nulle. $\widetilde{N}$ est la mesure de Poisson compensée.

   (b) Montrer que l'on a :

$$u(t,x) = \mathbb{E}g(x+Y_t).$$

   Indication : appliquer la formule d'Itô à la fonction $(r,y) \mapsto u(t-r,y)$ et au processus de Lévy $(r, x+Y_r)$.

Part III

Application in finance

Chapter 6

Equivalence of measures

6.1 Pricing rules and martingales measures

This short section resumes the theory of arbitrage for semi-martingales models (the notion of semi-martingale will be defined in Chapter 5). Define a market with

- underlying assets described by an adapted semi-martingale:
$$(S_t = (S_t^0, S_t^1, \dots, S_t^d),\ t \in [0,T]),$$
- S^0: numeraire (for example $S_t^0 = \exp(rt)$),
- discount factor: $B(t,T) = S_t^0/S_T^0$.

A contingent claim is represented by its terminal payoff H, a $\mathcal{F}_T$-measurable random variable. The set of contingent claims of interest will be denoted by $\mathcal{H}$.

A pricing rule is a procedure which attributes to each $H \in \mathcal{H}$ a value $\Pi_t(H)$ at each time with the following requirements:

- Adaptativity: $\Pi_t(H)$ is an adapted process (and a semi-martingale).
- Positiveness: $H \geq 0 \Rightarrow \Pi_t(H) \geq 0$.
- Linearity (false for large portfolios in practice).

For any event $A \in \mathcal{F}$, the random variable $\mathbf{1}_A$ represents the payoff of a contingent claim which pays 1 at T if A occurs and zero otherwise: it is a bet on A (also called a lottery). We will assume that $\mathbf{1}_A \in \mathcal{H}$: such contingent claims are priced on the market. In particular $\mathbf{1}_\Omega = 1$ is just a zero-coupon bond paying 1 at T. Its value $\Pi_t(1)$ represents the present value of 1 unit of currency paid at T, i.e., the discount factor:
$$\Pi_t(1) = e^{-r(T-t)}.$$

Define now $\mathbb{Q} : \mathcal{F} \to \mathbb{R}$ by
$$\mathbb{Q}(A) = \frac{\Pi_0(\mathbf{1}_A)}{\Pi_0(1)} = e^{rT}\Pi_0(\mathbf{1}_A).$$

$\mathbb{Q}(A)$ is thus the value of a bet of nominal $\exp(rT)$ on the event A. Then, the linearity and positiveness of Π entail the following properties for $\mathbb{Q}$:

- $1 \geq \mathbb{Q}(A) \geq 0$, since $1 \geq \mathbf{1}_A \geq 0$.

- If A, B are disjointed events $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$ so by linearity of the valuation operator: $\mathbb{Q}(A \cup B) = \mathbb{Q}(A) + \mathbb{Q}(B)$.

If one extends the linearity condition to infinite sums, then $\mathbb{Q}$ defined by is nothing else but a probability measure over the scenario space $(\Omega, \mathcal{F})$! So, starting from a valuation rule Π, we have constructed a probability measure $\mathbb{Q}$ over scenarios. Conversely, Π can be retrieved from $\mathbb{Q}$ in the following way: for random payoffs of the form $H = \sum_i c_i \mathbf{1}_{A_i}$ which means, in financial terms, portfolios of cash-or-nothing options, by linearity of Π we have $\Pi_0(H) = \mathbb{E}^{\mathbb{Q}}[H]$. Now if Π verifies an additional continuity property (i.e., if a dominated convergence theorem holds on $\mathcal{H}$) then we can conclude that for any random payoff $H \in \mathcal{H}$,

$$\Pi_0(H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[H].$$

Therefore there is a one-to-one correspondence between linear valuation rules Π verifying the properties above and probability measures $\mathbb{Q}$ on scenarios: they are related by

$$\Pi_0(H) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[H], \text{ and } \mathbb{Q}(A) = e^{rT}\Pi_0(\mathbf{1}_A). \tag{6.1}$$

The relation (6.1) is sometimes called a risk-neutral pricing formula: the value of a random payoff is given by its discounted expectation under $\mathbb{Q}$. We have shown above that any linear valuation rule Π verifying the properties above is given by a Òrisk-neutralÓ pricing rule: there are no others! It is important to understand that $\mathbb{Q}$ has nothing to do with the actual/objective probability of occurrence of scenarios: in fact, we have not defined any objective probability measure on the scenarios yet! $\mathbb{Q}$ is called a risk-neutral measure or a pricing measure. Although it is, mathematically speaking, a probability measure on the set of scenarios,$\mathbb{Q}(A)$ should not be interpreted as the probability that A happens in the real world but as the value of a bet on A. A risk-neutral measure is just a convenient representation of the pricing rule Π: it is not obtained by an econometric analysis of time series or anything of the sort, but by looking at prices of contingent claims at $t = 0$. Similarly for each t, $A \mapsto A = e^{rt}\Pi_t(\mathbf{1}_A)$ defines a probability measure over scenarios between 0 and t, i.e., a probability measure $\mathbb{Q}_t$ on $(\Omega, \mathcal{F}_t)$. If we require that the pricing rule Π is time consistent, i.e., the value at 0 of the payoff H at T is the same as the value at 0 of the payoff $\Pi_t(H)$ at t, then $\mathbb{Q}_t$ should be given by the restriction of Q, defined above, to $\mathcal{F}_t$ and $\Pi_t(H)$ is given by the discounted conditional expectation with respect to $\mathbb{Q}$:

$$\Pi_t(H) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}t]. \tag{6.2}$$

Therefore we have argued that any time consistent linear pricing rule Π verifying some continuity property is given by a discounted conditional expectation with respect to some probability measure $\mathbb{Q}$. We will now consider such a pricing rule given by a probability measure $\mathbb{Q}$ and examine what restrictions are imposed on $\mathbb{Q}$ by the requirement of absence of arbitrage.

Assume now that, in addition to the market scenarios $(\Omega, \mathcal{F})$ and the information flow $\mathcal{F}_t$, we know something about the probability of occurrence of these scenarios, represented by a probability measure $\mathbb{P}$. $\mathbb{P}$ represents here either the objective probability of future scenarios or the subjective view of an investor. What additional constraints should a pricing rule given by (6.2) verify in order to be compatible with this statistical view of the future evolution of the market? A fundamental requirement for a pricing rule is that it does not generate arbitrage opportunities. An arbitrage opportunity is a self-financing strategy ϕ which can lead to a positive terminal gain, without any probability of intermediate loss:

$$\mathbb{P}(\forall t \in [0,T],\ V_t(\phi) \geq 0) = 1, \quad \mathbb{P}(V_T(\phi) > V_0(\phi)) \neq 0.$$

Of course such strategies have to be realistic, i.e., of the form of a simple process to be of any use. Note that the definition of an arbitrage opportunity involves $\mathbb{P}$ but $\mathbb{P}$ is only used to specify whether the profit is possible or impossible, not to compute its probability of occurring: only events with probability 0 or 1 are involved in this definition. Thus the reasoning in the sequel will not require a precise knowledge of probabilities of scenarios. The self-financing property is important: it is trivial to exhibit strategies which are not self-financing verifying the property above, by injecting cash into the portfolio right before maturity. A consequence of absence of arbitrage is the law of one price: two self-financing strategies with the same terminal payoff must have the same value at all times, otherwise the difference would generate an arbitrage. Consider now a market where prices are given by a pricing rule as in (6.2) represented by some probability measure $\mathbb{Q}$. Consider an event A such that $\mathbb{P}(A) = 0$ and an option which pays the holder 1 (unit of currency) if the event A occurs. Since the event A is considered to be impossible, this option is worthless to the investor. But the pricing rule defined by $\mathbb{Q}$ attributes to this option a value at t $= 0$ equal to

$$\Pi_0(\mathbf{1}_A) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A] = e^{-rT}\mathbb{Q}(A).$$

So the pricing rule $\mathbb{Q}$ is coherent with the views of the investor only if $\mathbb{Q}(A) = 0$. Conversely if $\mathbb{Q}(A) = 0$ then the option with payoff $\mathbf{1}_A \geq 0$ is deemed worthless; if $\mathbb{P}(A) \neq 0$ then purchasing this option (for free) would lead to an arbitrage. So the compatibility of the pricing rule $\mathbb{Q}$ with the stochastic model $\mathbb{P}$ means that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent probability measures: they define the same set of (im)possible events

$$\mathbb{P} \sim \mathbb{Q}: \qquad \forall A \in \mathcal{F},\ \mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0. \tag{6.3}$$

Consider now an asset S^i traded at price S^i_t. This asset can be held until T, generating a terminal payoff S^i_T, or be sold for S^i_t: the resulting sum invested at the interest rate r will then generate a terminal wealth of $e^{r(T-t)}S^i_t$. These two buy-and-hold strategies are self-financing and have the same terminal payoff so they should have the same value at t:

$$\mathbb{E}^{\mathbb{Q}}(S^i_T|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}(e^{r(T-t)}S^i_t|\mathcal{F}_t) = e^{r(T-t)}S^i_t.$$

Dividing by $S^0_T = e^{rT}$, we have

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{S^i_T}{S^0_T}\middle|\mathcal{F}_t\right) = \frac{S^i_t}{S^0_t} \Leftrightarrow \mathbb{E}^{\mathbb{Q}}\left(\widetilde{S}^i_T\middle|\mathcal{F}_t\right) = \widetilde{S}^i_t. \tag{6.4}$$

Therefore absence of arbitrage implies that discounted values $\widetilde{S}_t^i = e^{-rt} S_t^i$ of all traded assets are martingales with respect to the probability measure $\mathbb{Q}$. A probability measure verifying (6.3) and (6.4) is called an equivalent martingale measure. We have thus shown that any arbitrage-free pricing rule is given by an equivalent martingale measure.

Conversely, it is easy to see that any equivalent martingale measure $\mathbb{Q}$ defines an arbitrage-free pricing rule via (6.2). Consider a self-financing strategy ϕ. Of course a realistic strategy must be represented by a simple (piecewise constant) predictable process. Since $\mathbb{Q}$ is a martingale measure $\widetilde{S}$ is a martingale under $\mathbb{Q}$ so, as observed in Chapter 5, the value of the portfolio $V_t(\phi) = V_0 + \int_0^t \phi_u d\widetilde{S}_u$ is a martingale so in particular $\mathbb{E}^{\mathbb{Q}}[\int_0^t \phi_u d\widetilde{S}_u] = 0$. The random variable $\int \phi d\widetilde{S}$ must therefore take both positive and negative values: $\mathbb{Q}(V_T(\phi) - V_0 \geq 0) \neq 1$. Since $\mathbb{P} \sim \mathbb{Q}$, this entails $\mathbb{P}(\int \phi_t d\widetilde{S}_t \geq 0) \neq 1$: ϕ cannot be an arbitrage strategy. There is hence a one-to-one correspondence between arbitrage-free pricing rules and equivalent martingale measures.

Proposition 6.1 *In a market described by a probability measure $\mathbb{P}$ on scenarios, any arbitrage-free linear pricing rule Π can be represented as*

$$\Pi_t(H) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[H|\mathcal{F}_t],$$

where $\mathbb{Q}$ is an equivalent martingale measure: a probability measure on the market scenarios such that

$$\mathbb{Q} \sim \mathbb{P}, \qquad \mathbb{E}^{\mathbb{Q}}\left(\widetilde{S}_T^i | \mathcal{F}_t\right) = \widetilde{S}_t^i.$$

Up to now we have assumed that such an arbitrage-free pricing rule/equivalent martingale measures does indeed exist, which is not obvious in a given model. The above arguments show that if an equivalent martingale measure exists, then the market is arbitrage-free. The converse result, more difficult to show, is sometimes called the Fundamental theorem of asset pricing:

Theorem 6.1 (Fundamental theorem) *The market model defined by $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and asset prices $(S_t)_{t\in[0,T]}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted assets $(\widetilde{S}_t)_{t\in[0,T]}$ are martingales w.r.t. $\mathbb{Q}$.*

And one can precise that

Theorem 6.2 *The market model is complete if and only if there exists a unique martingale measure $\mathbb{Q} \sim \mathbb{P}$.*

This theorem establishes the equivalence between the financial notion of market completeness (the possibility to perfectly hedge any contingent claim) and the uniqueness of equivalent martingale measure, which is a mathematical property of the underlying stochastic model. The theorem holds as stated above in discrete time models. In continuous time models one has to carefully define the set of admissible strategies, contingent claims and the notion of "martingale measure". Unfortunately in the case where S has unbounded jumps, which is the case of most exponential-Lévy models, a rigorous formulation is quite difficult. Moreover while most stochastic models used in option pricing are arbitrage-free, only a few of these models are complete: stochastic volatility models and as we will see shortly, exponential-Lévy models, jump-diffusion models fall into the category of incomplete models.

In mathematical terms, completeness means that for any random variable $H \in \mathcal{H}$ depending on the history of S between 0 and T , H can be represented as the sum of a constant and a stochastic integral of a predictable process with respect to $\widetilde{S}$. If this property holds for all terminal payoffs with finite variance, i.e., any $H \in L^2(\mathcal{F}_T, \mathbb{Q})$ can be represented as

$$H = \mathbb{E}[H] + \int_0^T \phi_s d\widetilde{S}_t$$

for some predictable process ϕ, the martingale $(\widetilde{S}_t)_{t\in[0,T]}$ is said to have the predictable representation property. Thus market completeness is often identified with the predictable representation property, which has been studied for many classical martingales. The predictable representation property can be shown to hold when $\widetilde{S}$ is (geometric) Brownian motion or a Brownian stochastic integral, but it fails to hold for most discontinuous models used in finance. For example, it is known to fail for all non-Gaussian Lévy processes except the (compensated) Poisson process. We will show in this chapter that this property also fails in exponential-Lévy models by a direct computation.

Even if the predictable representation property holds it does not automatically lead to "market completeness": as argued in Chapter 5, any predictable process ϕ cannot be interpreted as a trading strategy. For this interpretation to hold we must be able, in some way, to approximate its value process using an implementable (piecewise constant in time) portfolio, so predictable processes which can be reasonably interpreted as"trading strategies" are simple predictable processes or caglad processes.

Finally let us note that we are looking for a representation of H in terms of a stochastic integral with respect to $\widetilde{S}$. In fact the following theorem shows that when the source of randomness is a Brownian motion W and a Poisson random measure M, a random variable with finite variance can be always represented as a stochastic integral:

$$H = \mathbb{E}(H) + \int_0^t \phi_s dW_s + \int_0^t \int_{\mathbb{R}^d} \psi(s,y)\widetilde{M}(dsdy).$$

This property is also called a predictable representation property by many authors but has nothing to do with market completeness. Even when S is driven by the same sources of randomness W and M and $M = J_S$ represents the jump measure of the process S, the previous expression cannot be represented as a stochastic integral with respect to S. Such representations can be nevertheless useful for discussing hedging strategies.

6.2 Equivalence of measures for Lévy processes

The previous section shows that if we use jump-diffusion processes to modelize the market prices, we must know how to change the probability measure.

6.2.1 For a compound Poisson process

Poisson process. Assume that

- N is a Poisson process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ relative to a filtration $\{\mathcal{F}_t\}$;
- λ is its intensity;
- $M_t = N_t - \lambda t$ denotes the associated compensated Poisson process. Remember, It is a martingale under $\mathbb{P}$.
- $\tilde{\lambda}$ is any positive real number.

Lemma 6.1 *The process Z defined by*

$$Z(t) = \exp\left((\lambda - \tilde{\lambda})t\right) \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t} \tag{6.5}$$

satisfies

$$dZ(t) = \frac{\tilde{\lambda} - \lambda}{\lambda} Z(t-) dM_t.$$

Therefore Z is a martingale under $\mathbb{P}$ and $\mathbb{E}(Z(t)) = 1$, $\forall t \geq 0$.

Proof. Define $X_t = \dfrac{\tilde{\lambda} - \lambda}{\lambda} M_t$. The continuous part of X is

$$X_t^c = (\lambda - \tilde{\lambda})t \Rightarrow [X]_t^c = 0.$$

The jump part is: $J_t = \dfrac{\tilde{\lambda} - \lambda}{\lambda} N_t$. Therefore $1 + \Delta X_t = \left(\dfrac{\tilde{\lambda}}{\lambda}\right)^{\Delta N_t}$. Now we use Propostion 5.13 with a Lévy process X of finite variations to obtain:

$$Z_t = \mathcal{E}(X)_t = \exp((\lambda - \tilde{\lambda})t) \prod_{0<s\leq t} (1 + \Delta X_s) = \exp\left((\lambda - \tilde{\lambda})t\right) \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}$$

and since M is a martingale, Proposition 5.14 shows that Z is a martingale. $\square$

For some $T > 0$, let us define

$$\boxed{\widetilde{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A Z(T)) \ \text{ for } A \in \mathcal{F}_T.} \tag{6.6}$$

Theorem 6.3 *Under the probability $\widetilde{\mathbb{P}}$, the process N is a Poisson process with intensity $\tilde{\lambda}$.*

Proof. We use the Laplace transform of N under $\widetilde{\mathbb{P}}$: for any $u \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}^{\widetilde{\mathbb{P}}}(e^{uN_t}) &= \mathbb{E}(e^{uN_t} Z_t) = e^{(\lambda - \tilde{\lambda})t} \mathbb{E}\left[e^{uN_t} \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}\right] = e^{(\lambda - \tilde{\lambda})t} \mathbb{E}\left[\exp\left[\left(u + \ln\left(\frac{\tilde{\lambda}}{\lambda}\right)\right) N_t\right]\right] \\
&= e^{(\lambda - \tilde{\lambda})t} \exp\left[\lambda t \left(e^{u + \ln\left(\frac{\tilde{\lambda}}{\lambda}\right)} - 1\right)\right] \\
&= \exp(\tilde{\lambda}(e^u - 1)).
\end{aligned}$$

This proves exactly that under $\widetilde{\mathbb{P}}$, the process N is a Poisson process with intensity $\tilde{\lambda}$. □

Now if for

- $\lambda > 0$,
- $\sigma > -1$, $\sigma \neq 0$,
- $\alpha \in \mathbb{R}$,

the process S

$$S_t = S_0 \exp\left[\alpha t + N_t \ln(1+\sigma) - \lambda \sigma t\right] = S_0 e^{(\alpha - \sigma\lambda)t}(\sigma+1)^{N_t}$$

is the price of a asset, then S satisfies

$$dS_t = \alpha S_t dt + \sigma S(t^-) dM_t = \alpha S_t dt + \sigma S(t^-) d(N_t - \lambda t).$$

This can be obtained using the Itô formula, or using Proposition 5.13. S is called a geometric Poisson process. Now assume that under a probability measure $\widetilde{\mathbb{P}}$, N is a Poisson process with intensity $\tilde{\lambda} > 0$. This probability is risk-neutral if under $\widetilde{\mathbb{P}}$, S satisfies

$$dS_t = rS_t dt + \sigma S(t^-) d(N_t - \tilde{\lambda} t),$$

where r is the riskless interest rate. Therefore

$$dS_t = \alpha S_t dt + \sigma S(t^-) d(N_t - \lambda t) = rS_t dt + \sigma S(t^-) d(N_t - \tilde{\lambda} t),$$

which is possible if and only if

$$\alpha - \sigma\lambda = r - \sigma\tilde{\lambda} \iff \tilde{\lambda} = \lambda - \frac{\alpha - r}{\sigma}. \tag{6.7}$$

Lemma 6.2 *Assume that S is geometric Poisson process with intensity λ and drift α, and r is the riskless interest rate.*

- *If $\lambda > \dfrac{\alpha - r}{\sigma}$, we define $\tilde{\lambda} = \lambda - \dfrac{\alpha - r}{\sigma} > 0$, Z by Equation (6.5) and $\widetilde{P}$ by (6.6). Under $\widetilde{\mathbb{P}}$:*

$$dS_t = rS_t dt + \sigma S(t^-) d\tilde{M}_t = rS_t dt + \sigma S(t^-) d(N_t - \tilde{\lambda} t).$$

Hence the discounted price is a martingale and $\widetilde{\mathbb{P}}$ is a martingale risk measure. There is no arbitrage.

- *If $\lambda \leq \dfrac{\alpha - r}{\sigma}$, there is an arbitrage !*

Proof. The first case is an application of Theorem 6.3. For the second case assume that $\sigma > 0$. Then a.s.

$$S_t \geq S_0 e^{rt}(\sigma+1)^{N_t} \geq S_0 e^{rt}.$$

Therefore buy the stock and borrow the price at rate r is an arbitrage. If $\sigma < 0$, then $r \geq \alpha - \lambda\sigma$. Thus

$$S_t \leq S_0 e^{rt}(\sigma+1)^{N_t} \leq S_0 e^{rt}.$$

Sell the stock and deposit the amount at rate r is an arbitrage. □

Compound Poisson process with discrete jumps. We deal now with a compound Poisson process Q with

- N a Poisson process with intensity λ;
- $Y_1, Y_2, \dots$ sequence of i.i.d. discrete random variables, independent of N

$$\mathbb{P}(Y_i = y_m) = p(y_m) > 0,\ m = 1, \dots, M; \quad \text{and } \sum_{m=1}^{M} p(y_m) = 1;$$

- $Q_t = \sum_{i=1}^{N_t} Y_i$.

We decompose the process Q as follows.

- Denote by N^m the number of jumps of Q with size y_m. These are independent Poisson processes with intensity $\lambda_m = \lambda p(y_m)$.
- Write $N_t = \sum_{m=1}^{M} N_t^m$, $Q_t = \sum_{m=1}^{M} y_m N_t^m$.

Let $\tilde{\lambda}_m$, $m = 1, \dots, M$, be positive numbers and

$$Z_m(t) = \exp\left((\lambda_m - \tilde{\lambda}_m)t\right)\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_t^m}, \qquad Z(t) = \prod_{m=1}^{M} Z_m(t).$$

Lemma 6.3 *The process Z is a martingale under $\mathbb{P}$. In particular $\mathbb{E}(Z(t)) = 1$.*

Proof. Left as an exercise (see Exercise 5.1). □

For some $T > 0$, we define as before a new probability: $\widetilde{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A Z(T))$ for $A \in \mathcal{F}_T$.

Theorem 6.4 *Under $\tilde{\mathbb{P}}$, Q is a compound Poisson process with intensity $\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m$ and Y_i, $i \in \mathbb{N}^*$ are i.i.d. with $\tilde{\mathbb{P}}(Y_i = y_m) = \tilde{p}(y_m) = \dfrac{\tilde{\lambda}_m}{\tilde{\lambda}}$.*

Proof. Once again we use the Laplace transform. For any $u \in \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}}(e^{uQ_t}) &= \mathbb{E}(e^{uQ_t}Z_t) = \mathbb{E}\left[\exp\left(u\sum_{m=1}^{M} y_m N_t^m\right)\prod_{m=1}^{M} e^{(\lambda_m-\tilde{\lambda}_m)t}\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_t^m}\right] \\
&= \prod_{m=1}^{M} e^{(\lambda_m-\tilde{\lambda}_m)t}\mathbb{E}\exp\left[\left(uy_m + \ln\frac{\tilde{\lambda}_m}{\lambda_m}\right)N_t^m\right] \\
&= \prod_{m=1}^{M} e^{(\lambda_m-\tilde{\lambda}_m)t}\exp\left\{\lambda_m t\left[\exp\left(uy_m + \ln\frac{\tilde{\lambda}_m}{\lambda_m}\right) - 1\right]\right\} \\
&= \prod_{m=1}^{M}\exp\left\{\tilde{\lambda}_m t\left[\exp\left(uy_m\right) - 1\right]\right\} = \prod_{m=1}^{M}\exp\left(\tilde{\lambda}t\tilde{p}(y_m)e^{uy_m} - \tilde{\lambda}_m t\right) \\
&= \exp\left[\tilde{\lambda}t\left(\sum_{m=1}^{M}\tilde{p}(y_m)e^{uy_m} - 1\right)\right].
\end{aligned}$$

This achieves the proof. □

Remark that we can write

$$Z(t) = \prod_{m=1}^{M} Z_m(t) = \prod_{m=1}^{M}\exp\left((\lambda_m - \tilde{\lambda}_m)t\right)\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_t^m} = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{p}(Y_i)}{\lambda p(Y_i)}.$$

General case. We can extend the previous result to the density case. Assume that Y_i has a density f.

- Let $\tilde{f}$ an other density such that $f(y) = 0 \Rightarrow \tilde{f}(y) = 0$,
- and $\tilde{\lambda} > 0$.

We put

$$Z(t) = e^{(\lambda-\tilde{\lambda})t}\prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}.$$

Theorem 6.5 *Z is a martingale under $\mathbb{P}$. Under the probability measure $\tilde{\mathbb{P}}$ defined by (6.6) with the appropriated Z, Q is a compound Poisson process with intensity $\tilde{\lambda}$ and Y_i, $i \in \mathbb{N}^*$ are i.i.d. with density $\tilde{f}$.*

Proof. First prove that Z is a martingale. Define

$$J_t = \prod_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} = J_{t-}\frac{\tilde{\lambda}\tilde{f}(Y_{N_t})}{\lambda f(Y_{N_t})} = J_{t-}\frac{\tilde{\lambda}\tilde{f}(\Delta Q_t)}{\lambda f(\Delta Q_t)}.$$

Now we consider the following compound Poisson process:

$$H_t = \sum_{i=1}^{N(t)}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}.$$

We have: $\Delta H_t = \dfrac{\tilde{\lambda}\tilde{f}(\Delta Q_t)}{\lambda f(\Delta Q_t)}$ and

$$\mathbb{E}\frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} = \frac{\tilde{\lambda}}{\lambda}\int_{\mathbb{R}} \frac{\tilde{f}(y)}{f(y)} f(y)dy = \frac{\tilde{\lambda}}{\lambda}.$$

Therefore $M_t = H_t - \tilde{\lambda}t$ is a martingale and

$$\Delta J_t = J_{t^-}\left(\frac{\tilde{\lambda}\tilde{f}(\Delta Q_t)}{\lambda f(\Delta Q_t)} - 1\right) = J_{t^-}(\Delta H_t - \Delta N_t).$$

Now the Itô formula implies:

$$\begin{aligned} Z_t &= Z_0 + \int_0^t J_{s^-}(\lambda - \tilde{\lambda})e^{(\lambda-\tilde{\lambda})s}ds + \int_0^t e^{(\lambda-\tilde{\lambda})s}dJ_s \\ &= Z_0 + \int_0^t J_{s^-}(\lambda - \tilde{\lambda})e^{(\lambda-\tilde{\lambda})s}ds + \int_0^t e^{(\lambda-\tilde{\lambda})s}J_{s^-}(dH_s - dN_s) \\ &= 1 + \int_0^t Z_{s^-}d(H_s - \tilde{\lambda}s) - \int_0^t Z_{s^-}d(N_s - \lambda s). \end{aligned}$$

Hence Z is a martingale with

$$\Delta Z_t = Z_{t^-}\Delta H_t - Z_{t^-}\Delta N_t.$$

Now we have to prove that

$$\mathbb{E}^{\tilde{\mathbb{P}}} e^{uQ_t} = \exp\left(\tilde{\lambda}t(\tilde{\phi}_Y(u) - 1)\right), \qquad \tilde{\phi}_Y(u) = \int_{\mathbb{R}} e^{uy}\tilde{f}(y)dy.$$

We define the process

$$X_t = \exp\left(uQ_t - \tilde{\lambda}t(\tilde{\phi}_Y(u) - 1)\right),$$

and we prove that XZ is a $\mathbb{P}$-martingale. We apply the Itô formula to obtain:

$$X_tZ_t = 1 + \int_0^t X_{s^-}dZ_s + \int_0^t Z_{s^-}dX_s + [X,Z]_t. \tag{6.8}$$

On the right hand side, the first term is a martingale. Now remark that $\Delta X_t = X_{t^-}\left(e^{u\Delta Q_t} - 1\right)$. Hence

$$\int_0^t Z_{s^-}dX_s = -\tilde{\lambda}(\tilde{\phi}_Y(u) - 1)\int_0^t X_{s^-}Z_{s^-}ds + \sum_{0<s\leq t} X_{s^-}Z_{s^-}\left(e^{u\Delta Q_s} - 1\right).$$

Moreover

$$[X,Z]_t = \sum_{0<s\leq t}\Delta X_s\Delta Z_s = \sum_{0<s\leq t} X_{s^-}Z_{s^-}\left(e^{u\Delta Q_s} - 1\right)(\Delta H_s - \Delta N_s).$$

Thus

$$\begin{aligned} \int_0^t Z_{s^-}dX_s + [X,Z]_t &= -\tilde{\lambda}(\tilde{\phi}_Y(u) - 1)\int_0^t X_{s^-}Z_{s^-}ds + \sum_{0<s\leq t} X_{s^-}Z_{s^-}\left(e^{u\Delta Q_s} - 1\right) \\ &\quad + \sum_{0<s\leq t} X_{s^-}Z_{s^-}\left(e^{u\Delta Q_s} - 1\right)(\Delta H_s - \Delta N_s) \\ &= \sum_{0<s\leq t} X_{s^-}Z_{s^-}e^{u\Delta Q_s}\Delta H_s - \tilde{\lambda}\tilde{\phi}_Y(u)\int_0^t X_{s^-}Z_{s^-}ds \\ &\quad - \sum_{0<s\leq t} X_{s^-}Z_{s^-}\Delta H_s + \tilde{\lambda}\int_0^t X_{s^-}Z_{s^-}ds. \end{aligned}$$

The last equality can be written as follows:

$$\text{(6.9)} \qquad \int_0^t Z_{s^-} dX_s + [X,Z]_t = \int_0^t X_{s^-} Z_{s^-} \left(dV_s - \tilde{\lambda}\tilde{\phi}_Y(u) ds \right) - \int_0^t X_{s^-} Z_{s^-} \left(dH_s - \tilde{\lambda} ds \right)$$

with $M_t = H_t - \tilde{\lambda} t$ martingale and $V_t = \sum_{i=1}^{N(t)} e^{uY_i} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}$. Indeed $\Delta V_t = e^{u\Delta Q_t} \Delta H_t$. Finally $\mathbb{E}\left(e^{uY_i} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)} \right) = \frac{\tilde{\lambda}}{\lambda} \tilde{\Phi}(u)$. Combining Equations (6.8) and (6.9), we obtain that XZ is a $\mathbb{P}$-martingale, hence $\mathbb{E}^{\tilde{\mathbb{P}}}(X_t) = \mathbb{E}(X_t Z_t) = 1$. □

The compound Poisson case can be resumed as follows.

Proposition 6.2 *Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be compound Poisson processes on $(\Omega, \mathcal{F}_T)$ with Lévy measures $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$ are equivalent. In this case the density is*

$$\exp\left[(\lambda^{\mathbb{P}} - \lambda^{\mathbb{Q}})T + \sum_{0<s\leq T} \phi(\Delta X_s) \right],$$

where $\lambda^{\mathbb{P}} = \nu^{\mathbb{P}}(\mathbb{R})$, $\lambda^{\mathbb{Q}} = \nu^{\mathbb{Q}}(\mathbb{R})$ and $\phi = \ln \frac{d\nu^{\mathbb{Q}}}{d\nu^{\mathbb{P}}}$.

Proof. The if part is a consequence of the previous results. Now the only if part can be proved as follows. Assume that $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$ are not equivalent. Then we can find either a set B such that $\nu^{\mathbb{P}}(B) > 0$ and $\nu^{\mathbb{Q}}(B) = 0$ or a set B' such that $\nu^{\mathbb{P}}(B') = 0$ and $\nu^{\mathbb{Q}}(B') > 0$. Suppose that we are in the first case. Then the set of trajectories having at least one jump the size of which is in B has positive $\mathbb{P}$-probability and zero $\mathbb{Q}$-probability, which shows that these two measures are not equivalent. □

6.2.2 For a jump-diffusion process

Now let us add a Brownian component to our compound Poisson process. First recall that due to the Girsanov theorem we have:

Proposition 6.3 (Girsanov theorem) *Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be Brownian motions on $(\Omega, \mathcal{F}_T)$ with volatilities $\sigma^{\mathbb{P}} > 0$ and $\sigma^{\mathbb{Q}} > 0$ and drifts $\mu^{\mathbb{P}}$ and $\mu^{\mathbb{Q}}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\sigma^{\mathbb{P}} = \sigma^{\mathbb{Q}}$. In this case the density is*

$$\exp\left[\frac{\mu^{\mathbb{Q}} - \mu^{\mathbb{P}}}{\sigma^2} X_T - \frac{1}{2} \frac{(\mu^{\mathbb{Q}})^2 - (\mu^{\mathbb{P}})^2}{\sigma^2} T \right].$$

Assume now that on the same space $(\Omega, \mathcal{F}, \mathbb{P})$

- W is a Brownian motion;

- $Q_t = \sum_{i=1}^{N_t} Y_i$ is a compound Poisson process with N Poisson process with intensity λ, and Y_i, $i \in \mathbb{N}^*$are i.i.d. random variables with density f.

Let $\tilde{f}$ be a density such that $f(y) = 0 \Rightarrow \tilde{f}(y) = 0$, $\tilde{\lambda} > 0$, and Θ an adapted process. We define

$$\begin{aligned} Z_t^1 &= \exp\left(-\int_0^t \Theta_u dW_u - \frac{1}{2}\int_0^t \Theta_u^2 du\right), \\ Z_t^2 &= e^{(\lambda-\tilde{\lambda})t} \prod_{i=1}^{N(t)} \frac{\tilde{\lambda}\tilde{f}(Y_i)}{\lambda f(Y_i)}, \\ Z_t &= Z_t^1 Z_t^2. \end{aligned}$$

Lemma 6.4 *Z is a martingale under $\mathbb{P}$. In particular $\mathbb{E}(Z_t) = 1$, $\forall t \geq 0$.*

Proof. The proof is straightforward if Θ just depends on W. Recall that from Proposition 5.8, the two processes W and Q are independent. Therefore Z^1 and Z^2 are two independent martingales, and thus Z is a martingale.

Now in general let us use the Itô formula to obtain:

$$Z_t = 1 + \int_0^t Z_{s^-}^2 dZ_s^1 + \int_0^t Z_{s^-}^1 dZ_s^2 + [Z^1, Z^2]_t.$$

Now since Z^1 is continuous and Z^2 is a pure jump quadratic martingale, $[Z^1, Z^2]_t = 0$. $\square$

For some $T > 0$, we put $\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A Z(T))$ for $A \in \mathcal{F}_T$.

Theorem 6.6 *Under $\tilde{\mathbb{P}}$,*

$$\tilde{W}_t = W_t + \int_0^t \Theta_u du$$

is a Brownian motion and Q is a compound Poisson process with intensity $\tilde{\lambda}$ and Y_i, $i \in \mathbb{N}^*$ i.i.d. with intensity $\tilde{f}$. Furthermore $\tilde{W}$ and Q are independent.

Proof. Here we want to show that

$$\mathbb{E}^{\tilde{\mathbb{P}}} e^{u_1 \tilde{W}_t + u_2 Q_t} = \exp\left(\frac{1}{2}u_1^2 t\right) \exp\left(\tilde{\lambda} t(\tilde{\phi}_Y(u_2) - 1)\right).$$

We define

$$\begin{aligned} X_t^1 &= \exp\left(u_1 \tilde{W}_t - \frac{1}{2}u_1^2 t\right), \\ X_t^2 &= \exp\left(u_2 Q_t - \tilde{\lambda} t(\tilde{\phi}_Y(u_2) - 1)\right). \end{aligned}$$

From the proof of Theorem 6.5, we know that $X^2 Z^2$ is a $\mathbb{P}$-martingale. Moreover

$$d(X^1 Z^1)_t = (u_1 - \Theta_t) X_t^1 Z_t^1 dW_t,$$

which implies that $X^1 Z^1$ is a martingale. Since $[X^1 Z^1, X^2 Z^2] = 0$,

$$(X^1 X^2 Z^1 Z^2)_t = 1 + \int_0^t (X^1 Z^1)_{s^-} d(X^2 Z^2)_s + \int_0^t (X^2 Z^2)_{s^-} d(X^1 Z^1)_s$$

which proves that $X^1X^2Z^1Z^2$ is a $\mathbb{P}$-martingale. $\square$

6.2.3 The general case

The previous section shows that if we use Lévy processes to modelize the market prices, we must know how to change the probability measure. This has been done for jump-diffusion processes. Now we come to the general case. Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be two Lévy processes on $\mathbb{R}^d$ with characteristic triplets (A, ν, γ) and (A', ν', γ').

Theorem 6.7 *$\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are equivalent for all t (or equivalently for one $t > 0$) if and only if the following conditions are satisfied:*

1. $A = A'$.
2. *The Lévy measures are equivalent with*
$$\int_{\mathbb{R}^d} (\exp(\phi(x)/2) - 1)^2 \nu(dx) < \infty$$
where $\phi(x) = \ln\left(\dfrac{d\nu'}{d\nu}\right)$.
3. $\gamma' - \gamma - \displaystyle\int_{|x|\leq 1} x(\nu' - \nu)(dx) = A\eta$, *for some* $\eta \in \mathbb{R}^d$.

We will admit this result. If $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then $\dfrac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t}$ with

$$\begin{aligned} U_t &= \langle \eta, X_t^c \rangle - \frac{t}{2}\langle \eta, A\eta \rangle - t\langle \eta, \gamma \rangle \\ &\quad + \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \leq t, |\Delta X_s| > \varepsilon} \phi(\Delta X_s) - t \int_{|x| > \varepsilon} (e^{\phi(x)} - 1)\nu(dx) \right). \end{aligned}$$

Here X^c is the continuous part of X and η is such that

$$\gamma' - \gamma - \int_{|x|\leq 1} x(\nu' - \nu)(dx) = A\eta$$

if $A \neq 0$ and zero if $A = 0$. So under $\mathbb{P}$, U is a Lévy process on $\mathbb{R}$ with triplet (A_U, ν_U, γ_U) given by:

$$\begin{aligned} A_U &= \langle \eta, A\eta \rangle, \\ \nu_U &= \nu\phi^{-1}|_{\mathbb{R}\setminus\{0\}}, \\ \gamma_U &= -\frac{1}{2}\langle \eta, A\eta \rangle - \int_{\mathbb{R}} (e^y - 1 - y\mathbf{1}_{|y|\leq 1})(\nu\phi^{-1})(dy). \end{aligned}$$

Remark that from the definition of a density, $\mathbb{E}(e^{U_t}) = \mathbb{E}^{\mathbb{P}}(e^{U_t}) = 1$.

Let us describe the Esscher transform. In fact this is a special case of Theorem 6.7. Let

- X be a Lévy process with triplet $(0, \nu, \gamma)$,
- θ a real number such that $\displaystyle\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty$.
- Define the function $\phi(x) = \theta x$ and the measure $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$.

Method 6.1 (Esscher transform) *Then we obtain an equivalent probability $\mathbb{Q}$ under which X is a Lévy process with zero Gaussian component, Lévy measure $\tilde{\nu}$ and drift*

$$\tilde{\gamma} = \gamma + \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx).$$

The derivative is given by

$$\frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \frac{e^{\theta X_t}}{\mathbb{E}(e^{\theta X_t})} = \exp\left(\theta X_t + f(\theta)t\right),$$

with $f(\theta) = -\ln \mathbb{E}\exp(\theta X_1)$.

Proof. The first part is an immediate application of Theorem 6.7. Since $\eta = 0$, the process U is given by:

$$U_t \quad = \quad \lim_{\varepsilon \downarrow 0}\left(\sum_{0<s\leq t, |\Delta X_s|>\varepsilon} \theta \Delta X_s - t\int_{|x|>\varepsilon}(e^{\theta x} - 1)\nu(dx)\right).$$

Recall that X can be written like:

$$X_t = \gamma t + \sum_{0<s\leq t, |\Delta X_s|>1} \Delta X_s + \lim_{\varepsilon \downarrow 0}\left(\sum_{0<s\leq t, \varepsilon<|\Delta X_s|\leq 1} \Delta X_s - t\int_{\varepsilon<|x|\leq 1} x\nu(dx)\right).$$

Hence

$$U_t = \theta X_t - \theta\gamma t - t\lim_{\varepsilon\downarrow 0}\left(\int_{|x|>\varepsilon}(e^{\theta x} - 1)\nu(dx) - \int_{\varepsilon<|x|\leq 1}\theta x\nu(dx)\right).$$

But remember that from Theorem 2.9, $\mathbb{E}\exp(\theta X_1) = e^{\psi(\theta)}$ where

$$\psi(\theta) = \gamma\theta + \int_{\mathbb{R}}(e^{\theta x} - 1 - \theta x \mathbf{1}_{[-1,1]}(x))\nu(dx).$$

Hence

$$U_t = \theta X_t - t\psi(\theta) = \theta X_t + tf(\theta),$$

which finishes the proof. □

An important consequence of the Esscher transform is a sufficient condition for no arbitrage in exponential Lévy models.

Proposition 6.4 (No arbitrage) *Let $(X, \mathbb{P})$ be a Lévy process. If r is the interest rate and if the trajectories of X are neither almost surely increasing nor almost surely decreasing, then the exp-Lévy model given by $S_t = e^{rt+X_t}$ is arbitrage-free: there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $(e^{-rt}S_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale.*

Proof. Assume that the characteristic triplet of X is (σ^2, ν, γ). If $\sigma^2 > 0$, we can change the drift without changing the Lévy measure, using a Girsanov transform. In fact it is the same trick as the one used in the Black Scholes model. Therefore we assume now that $\sigma^2 = 0$.

First take the function $\phi(x) = -x^2$. We define an equivalent probability $\tilde{\mathbb{P}}$ under which X is a Lévy process with triplet $(0, \tilde{\nu}, \tilde{\gamma})$ with

$$\tilde{\nu}(dx) = e^{-x^2}\nu(dx), \qquad \tilde{\gamma} = \gamma + \int_{|x|\leq 1} x(e^{-x^2} - 1)\nu(dx).$$

Under $\tilde{\mathbb{P}}$, X has exponential moments of any order.

Next use the Esscher transform with some parameter θ. Under $\tilde{\mathbb{P}}$ we can choose $\theta \in \mathbb{R}$ as we want. Therefore we have a probability measure $\mathbb{Q}^\theta$ equivalent to $\mathbb{P}$ such that under $\mathbb{Q}^\theta$, the characteristics of X are 0 and

$$\overline{\nu}(dx) = e^{\theta x}\tilde{\nu}(dx), \qquad \overline{\gamma} = \tilde{\gamma} + \int_{-1}^{1} x(e^{\theta x} - 1)\tilde{\nu}(dx).$$

Now from Proposition 2.16, $\exp(X)$ is a martingale under $\mathbb{Q}^\theta$ if and only if

$$\overline{\gamma} + \int_{-\infty}^{\infty} (e^x - 1 - x\mathbf{1}_{[-1,1]}(x))\overline{\nu}(dx) = 0.$$

Now this equation can be written

$$-\tilde{\gamma} = f(\theta) = \int_{-1}^{1} x(e^{\theta x} - 1)\tilde{\nu}(dx) + \int_{-\infty}^{\infty} (e^x - 1 - x\mathbf{1}_{[-1,1]}(x))e^{\theta x}\tilde{\nu}(dx). \tag{6.10}$$

Therefore $\exp(X)$ is a martingale if and only if the equation $f(\theta) = -\tilde{\gamma}$ has one solution. Let us study the function f. With the dominated convergence theorem, it is obvious that f is a continuous function on $\mathbb{R}$. Moreover

$$f'(\theta) = \int_{-\infty}^{\infty} x(e^x - 1)e^{\theta x}\tilde{\nu}(dx) \geq 0.$$

Hence f is a non decreasing function on $\mathbb{R}$. Let us now distinguish several cases.

- $\nu(]0, +\infty[) > 0$ and $\nu(] - \infty, 0[) > 0$. Since $\tilde{\nu}$ is equivalent to ν, the same holds for $\tilde{\nu}$. In this case, the derivative of f is bounded from below by a constant $C > 0$. Therefore $f(+\infty) = +\infty)$ and $f(-\infty) = -\infty$. Equation (6.10) has one solution.

- $\nu(] - \infty, 0[) = 0$ and $\int_0^1 x\nu(dx) = +\infty$. We still have $f(+\infty) = +\infty$. Now from the dominated convergence theorem,

  $$\lim_{\theta\to-\infty} \int_{-\infty}^{\infty} (e^x - 1 - x\mathbf{1}_{[-1,1]}(x))e^{\theta x}\tilde{\nu}(dx) = 0.$$

  From the second term of f,

  $$\lim_{\theta\to-\infty} \int_{-1}^{1} x(e^{\theta x} - 1)\tilde{\nu}(dx) = -\int_0^1 x\tilde{\nu}(dx) = -\int_0^1 xe^{-x^2}\nu(dx) = -\infty.$$

  And we still have $f(-\infty) = -\infty$.

- $\nu(] - \infty, 0[) = 0$ and $\int_0^1 x\nu(dx) < +\infty$. Here

  $$\lim_{\theta\to-\infty} f(\theta) = -\int_0^1 x\tilde{\nu}(dx) = -\tilde{\gamma} + \tilde{\gamma}_0,$$

  where $\tilde{\gamma}_0$ is the drift of X under $\tilde{\mathbb{P}}$. Hence there is a solution if $\tilde{\gamma}_0 < 0$.

If we summarize the different cases, we see that there is a solution except if:

$$\nu(]-\infty,0[)=0, \quad \int_0^1 x\nu(dx)<+\infty, \quad \tilde{\gamma}_0 \geq 0,$$

i.e. if X is a non decreasing Lévy process. By symmetry we can treat the case of decreasing trajectories and complete the proof. □

From the financial point of view, the main consequences of this proposition and the proof are

- an easy sufficient condition for viability of the market model,
- and the non-uniqueness of the risk-neutral probability in general. In other words, in a financial model involving a general Lévy process, the market is incomplete.

Chapter 7

Pricing and hedging for jump diffusion processes models

7.1 Asset driven by a Poisson process

The model is the following. Assume that

- N is a Poisson process with intensity $\lambda > 0$,
- $M_t = N_t - \lambda t$ is the compensated Poisson process, thus a martingale.

The dynamic of the risky asset S_t is given by:

$$\begin{aligned} S_t &= S_0 \exp\left[\alpha t + N_t \ln(1+\sigma) - \lambda\sigma t\right] \\ &= S_0 e^{(\alpha-\lambda\sigma)t}(1+\sigma)^{N_t}. \end{aligned}$$

In order to avoid arbitrage opportunity, we assume that

$$\boxed{\lambda > \frac{\alpha - r}{\sigma}.}$$

We define $\tilde{\lambda} = \lambda - \dfrac{\alpha - r}{\sigma} > 0$ and

$$Z_t = e^{(\lambda-\tilde{\lambda})t}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{N_t}$$

is a martingale and

$$\forall A \in \mathcal{F}, \quad \boxed{\tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A Z_T)}.$$

$\tilde{\mathbb{P}}$ is the risk-neutral measure and under $\tilde{\mathbb{P}}$:

$$dS_t = rS_t dt + \sigma S(t^-) d\tilde{M}_t \Longleftrightarrow d(e^{-rt}S_t) = \sigma e^{-rt} S(t^-) d\tilde{M}_t$$

with $\tilde{M}_t = N_t - \tilde{\lambda} t$ is a martingale. Equivalently

$$\boxed{S_t = S_0 e^{(r-\tilde{\lambda}\sigma)t}(\sigma+1)^{N_t}}.$$

See Theorem 6.3 and what follows for the details.

Now we consider an European call at time T with strike K:

$$C(T) = (S_T - K)^+.$$

We denote with $C(t)$ the risk-neutral price of the European call:

$$\begin{aligned} e^{-rt}C(t) &= \tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)^+ | \mathcal{F}_t\right] \\ &= \tilde{\mathbb{E}}\left[e^{-rT}(S(t)e^{(r-\tilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N_T - N_t} - K)^+ | \mathcal{F}_t\right] \end{aligned}$$

Therefore $C(t) = c(t, S(t))$ where

$$\boxed{c(t,x) = \sum_{j=0}^{\infty} \left(xe^{-\tilde{\lambda}\sigma(T-t)}(\sigma+1)^j - Ke^{-r(T-t)}\right)^+ \frac{\tilde{\lambda}^j (T-t)^j}{j!} e^{-\tilde{\lambda}(T-t)}.}$$

Proposition 7.1 *The function c satisfies:*

- $c(T,x) = (x-K)^+$ *for every* $x \geq 0$*;*
- *for* $0 \leq t < T$ *and* $x \geq 0$*:*

$$\boxed{-rc(t,x) + \frac{\partial c}{\partial t}(t,x) + (r - \tilde{\lambda}\sigma)x\frac{\partial c}{\partial x}(t,x) + \tilde{\lambda}(c(t,(\sigma+1)x) - c(t,x)) = 0.}$$

If we define

$$\boxed{\delta(t) = \frac{c(t,(\sigma+1)S(t)) - c(t,S(t))}{\sigma S(t)}},$$

then

$$dC(t) = \delta(t^-)dS(t) + r(C(t) - \delta(t)S(t))dt$$

and

$$e^{-rt}C(t) = C(0) + \int_0^t e^{-ru}\left[c(u,(\sigma+1)S(u^-)) - c(u,S(u^-))\right] d\tilde{M}(u).$$

This proves that the model is complete and the risk-neutral probability is unique ! Hence this model is equivalent to the Black-Scholes model for pure jump process.

7.2 Asset driven by a compound Poisson process and a Brownian motion

Now we try to generalize the previous model. We define

- W Brownian motion,
- $N_1, \ldots, N_M$ independent Poisson processes, with intensity $\lambda_m > 0$,
- $-1 < y_1 < \ldots < y_M$ nonzero numbers.

and

$$N_t = \sum_{m=1}^{M} N_m(t), \quad Q_t = \sum_{m=1}^{M} y_m N_m(t) = \sum_{i=1}^{N_t} Y_i, \quad \lambda = \sum_{m=1}^{M} \lambda_m,$$

where $Y_1, Y_2, \ldots$ are i.i.d. random variables with distribution

$$\mathbb{P}(Y_i = y_m) = p(y_m) = \frac{\lambda_m}{\lambda}.$$

Q is a compound Poisson process with expectation equal to $\beta\lambda$ where $\beta = \mathbb{E}(Y_i) = \frac{1}{\lambda}\sum_{m=1}^{M} \lambda_m y_m$. Therefore the process $M_t = Q_t - \beta\lambda t$ is a martingale.

The model for the stock price is now: $S(0) > 0$ and

$$\begin{aligned} dS(t) &= \alpha S(t)dt + \sigma S(t)dW_t + S(t^-)dM_t \\ &= (\alpha - \beta\lambda)S(t)dt + \sigma S(t)dW_t + S(t^-)dQ_t. \end{aligned}$$

Proposition 7.2 *The solution of the previous equation is*

$$S(t) = S(0)\exp\left[\sigma W(t) + \left(\alpha - \beta\lambda - \frac{1}{2}\sigma^2\right)t\right]\prod_{i=1}^{N_t}(Y_i + 1).$$

Proof. Multiply S_t by $\exp[-(\alpha - \beta\lambda)t]$, apply Itô's formula to $X_t = S_t \exp[-(\alpha - \beta\lambda)t]$:

$$dX_t = \sigma X(t)dW_t + X(t^-)dQ_t$$

and use Proposition 5.10. □

We construct risk-neutral measures. We use Theorem 6.6. Let

- $\theta \in \mathbb{R}$,
- $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ be positive constants.
- $$Z_0(t) = \exp\left(-\theta W(t) - \frac{1}{2}\theta^2 t\right), \quad Z_m(t) = e^{(\lambda_m - \tilde{\lambda}_m)t}\left(\frac{\tilde{\lambda}_m}{\lambda_m}\right)^{N_m(t)}$$

The density of the measure change is:

$$Z(t) = Z_0(t)\prod_{m=1}^{M} Z_m(t), \quad \tilde{\mathbb{P}}(A) = \mathbb{E}(\mathbf{1}_A Z(T)).$$

Under $\tilde{\mathbb{P}}$:

- $\tilde{W}(t) = W(t) + \theta t$ is a Brownian motion,

- each N_m is a Poisson process with intensity $\tilde{\lambda}_m$,
- $\tilde{W}$ and $N_1, \ldots, N_m$ are independent of one another.

Now define

$$\tilde{\lambda} = \sum_{m=1}^{M} \tilde{\lambda}_m, \quad \tilde{p}(y_m) = \frac{\tilde{\lambda}_m}{\tilde{\lambda}}.$$

Under $\tilde{\mathbb{P}}$:

- $N_t = \sum_{m=1}^{M} N_m(t)$ is a Poisson process with intensity $\tilde{\lambda}$,
- the jump-size r.v. $Y_1, Y_2, \ldots$ are i.i.d. r.v. with $\tilde{\mathbb{P}}(Y_i = y_m) = \tilde{p}(y_m)$,
- $\tilde{M}(t) = Q(t) - \tilde{\beta}\tilde{\lambda}t$ is a martingale where

$$\tilde{\beta} = \tilde{\mathbb{E}} Y_i = \frac{1}{\tilde{\lambda}} \sum_{m=1}^{M} \tilde{\lambda}_m y_m.$$

Recall that

$$\begin{aligned} dS(t) &= (\alpha - \beta\lambda) S(t) dt + \sigma S(t) dW_t + S(t^-) dQ(t) \\ &= r S(t) dt + \sigma S(t) d\tilde{W}_t + S(t^-) d(Q(t) - \tilde{\beta}\tilde{\lambda} t), \end{aligned}$$

or

$$S(t) = S(0) \exp\left[\sigma \tilde{W}(t) + \left(r - \tilde{\beta}\tilde{\lambda} - \frac{1}{2}\sigma^2\right) t\right] \prod_{i=1}^{N_t} (Y_i + 1).$$

Hence $\tilde{\mathbb{P}}$ is a risk-neutral probability if the *market price of risk equation* is satisfied:

$$\boxed{\alpha - \beta\lambda = r + \sigma\theta - \tilde{\beta}\tilde{\lambda}} \Longleftrightarrow \alpha - r = \sigma\theta + \sum_{m=1}^{M} (\lambda_m - \tilde{\lambda}_m) y_m. \tag{7.1}$$

This equation has many solutions and a choice has to be made. Merton in his seminal article has choosen to let the coefficients λ_m unchanged and to only modify the Brownian drift.
Assumption: we choose $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_M$ and then θ such that (7.1) holds.

Remember that the *Black-Scholes price* of a call with volatility σ, interest rate r, current stock price x, expiration data τ, strike K is:

$$\kappa(\tau, x) = x\mathcal{N}(d_+(\tau, x)) - K e^{-r\tau} \mathcal{N}(d_-(\tau, x)),$$

with

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}} \left[\ln \frac{x}{K} + \left(r \pm \frac{1}{2}\sigma^2\right) \tau\right]$$

and

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} \exp\left(-\frac{1}{2} z^2\right) dz.$$

Theorem 7.1 *For $0 \leq t \leq T$ the risk-neutral price of a call*

$$C(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}(S(T)-K)^+ \middle| \mathcal{F}_t\right]$$

is given by $C(t) = c(t, S(t))$ where

$$\boxed{c(t,x) = \sum_{j=0}^{+\infty} e^{-\tilde{\lambda}(T-t)} \frac{\tilde{\lambda}^j (T-t)^j}{j!} \widetilde{\mathbb{E}}\left[\kappa\left(T-t, xe^{-\tilde{\beta}\tilde{\lambda}(T-t)} \prod_{i=1}^{j}(Y_i+1)\right)\right]}.$$

The price is a convex combinaison of Black-Scholes prices. The function c satisfies $c(T,x) = (x-K)^+$ and the partial integro-differential equation (PIDE in short)

$$\boxed{\begin{aligned} -rc(t,x) + \frac{\partial c}{\partial t}(t,x) + (r - \tilde{\lambda}\tilde{\beta})x\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \\ +\tilde{\lambda}\left[\sum_{m=1}^{M} \tilde{p}(y_m) c(t, (y_m+1)x) - c(t,x)\right] = 0. \end{aligned}}$$

Corollary 7.1 *The call price c satisfies*

$$\begin{aligned} d(e^{-rt}c(t,S(t))) &= e^{-rt}\sigma S(t) \frac{\partial c}{\partial x}(t,S(t)) d\tilde{W}_t \\ &+ \sum_{m=1}^{M} e^{-rt}\left[c(t,(y_m+1)S(t^-)) - c(t,S(t^-))\right] d(N_m(t) - \tilde{\lambda}_m t). \end{aligned}$$

A natural question is: what about hedging ? Define a portfolio by: $X(0) = c(0, S(0))$ and

$$dX(t) = \delta(t^-)dS(t) + r[X(t) - \delta(t)S(t)]dt$$

with the **delta-hedging strategy:** $\boxed{\delta(t) = \frac{\partial c}{\partial x}(t, S(t))}$.

Proposition 7.3

1.

$$\begin{aligned} & d\left[e^{-rt}c(t,S(t)) - e^{-rt}X(t)\right] \\ = & \sum_{m=1}^{M} e^{-rt} d(N_m(t) - \tilde{\lambda}_m t) \\ & \times \left[c(t,(y_m+1)S(t^-)) - c(t,S(t^-)) - y_m S(t^-)\frac{\partial c}{\partial x}(t,S(t^-))\right]. \end{aligned}$$

2. for any $0 \leq t \leq T$ $\tilde{\mathbb{E}}\left[e^{-rt}c(t,S(t))\right] = \tilde{\mathbb{E}}\left[e^{-rt}X(t)\right]$.

Continuous jump distribution. Let us finish with the case of continuous jump distribution, that is, Y_i have a density f with support in $(-1, \infty)$. We denote $\beta = \mathbb{E}(Y_i) = \int_{-1}^{\infty} y f(y) dy$, and we choose θ, $\tilde{\lambda} > 0$ and a density $\tilde{f}$ with support in supp (f) s.t.

$$\boxed{\alpha - r = \sigma\theta + \beta\lambda - \tilde{\beta}\tilde{\lambda}}, \tag{7.2}$$

with $\tilde{\beta} = \tilde{\mathbb{E}}(Y_i) = \int_{-1}^{\infty} y \tilde{f}(y) dy$. Once again to solve (7.2), the *Merton's approach* consists to take $f = \tilde{f}$ and to change only the drift on W.

In this case, Theorem 7.1 holds. But the PIDE becomes:

$$\boxed{\begin{aligned} -rc(t,x) + \frac{\partial c}{\partial t}(t,x) + (r - \tilde{\lambda}\tilde{\beta})x\frac{\partial c}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2}(t,x) \\ + \tilde{\lambda}\left[\int_{-1}^{\infty} c(t,(y+1)x)\tilde{f}(y)dy - c(t,x)\right] = 0. \end{aligned}}$$

If X is a hedging portfolio with $X(0) = c(0, S(0))$ and

$$dX(t) = \delta(t^-)dS(t) + r[X(t) - \delta(t)S(t)]dt, \text{ with } \delta(t) = \frac{\partial c}{\partial x}(t, S(t)),$$

then

$$\begin{aligned} & d\left[e^{-rt}c(t,S(t)) - e^{-rt}X(t)\right] \\ = \; & e^{-rt}\left[c(t,S(t)) - c(t,S(t^-)) - (S(t) - S(t^-))\frac{\partial c}{\partial x}(t,S(t^-))\right] dN_t \\ - \; & e^{-rt}\tilde{\lambda}\int_{-1}^{\infty}\left[c(t,S(t)) - c(t,S(t^-)) - yS(t^-)\frac{\partial c}{\partial x}(t,S(t^-))\right]\tilde{f}(y)dydt. \end{aligned}$$

In that case, the market is always incomplete.

7.3 Exercises

Exercice 7.1 (Extrait de l'examen de 2009-2010) Soit $X = (X_t)_{t\geq 0}$ un processus de Lévy réel de triplet caractéristique (σ^2, ν, γ), tel que ν se décompose en

$$\nu(dx) = A \sum_{n=1}^{\infty} p^n \delta_{-n}(dx) + Bx^{\beta-1}(1+x)^{-\alpha-\beta} e^{-\lambda x} \mathbf{1}_{]0,+\infty[}(x)dx,$$

avec $\alpha > 0$ et $\beta > 0$, $\lambda \geq 0$, $A \geq 0$ et $B \geq 0$, $p \in]0,1[$ et δ_y est la masse de Dirac au point y :

$$\forall A \subset \mathbb{R},\ \delta_Y(A) = 1 \text{ si } y \in A,\ \delta_Y(A) = 0 \text{ si } y \notin A.$$

Préliminaires.

1. Que peut-on dire des sauts négatifs du processus X ?
2. Montrer que si $\beta > 0$ et $\alpha > 0$, alors

$$\int_0^{\infty} x^{\beta-1}(1+x)^{-\alpha-\beta} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

   On pourra utiliser l'indication et le changement de variable $y = \dfrac{x}{1+x}$.

Partie 1 : étude des sauts de X. Dans cette partie, on suppose que $\sigma = 0$. Tout d'abord on va supposer $\lambda = 0$.

1. Montrer que X est un processus de Poisson composé avec dérive ? Quelle est la valeur de cette dérive en fonction de la fonction de répartition d'une loi béta au point 1/2 ?
2. À quelles conditions sur α et β, X_t admet-il un moment d'ordre $n \in \mathbb{N}^*$ (avec $t > 0$ fixé quelconque) ?
3. Calculer $\mathbb{E}(X_t)$ si elle existe, en fonction de la fonction de répartition d'une loi béta au point 1/2.
4. Montrer que si $\beta > 0$, X se décompose comme suit :

$$X_t = \gamma_0 t + \sum_{i=1}^{N_t^1} \frac{Y_i}{1+Y_i} - \sum_{j=1}^{N_t^2} Z_j,$$

   avec

   - $N^1 = (N_t^1)_{t\geq 0}$ et $N^2 = (N_t^2)_{t\geq 0}$ deux processus de Poisson d'intensité respective μ_1 et μ_2 à déterminer,

- les Y_i suivant une loi béta de paramètres α et β,
- les Z_i une loi géométrique de paramètre p,
- N^1, N^2, $Y = (Y_i)_{i\in\mathbb{N}^*}$ et $Z = (Z_i)_{i\in\mathbb{N}^*}$ étant tous indépendants.

5. En admettant que l'on sache simuler des lois béta et géométrique, proposer un algorithme de simulation du processus de Lévy X pour $\beta > 0$. On supposera la dérive γ_0 connue.

On suppose maintenant que $\lambda > 0$.

6. Pour quels $u \in \mathbb{R}$ a-t-on $\mathbb{E}\exp(uX_t) < +\infty$?
7. Si $B = 0$, calculer l'exposant caractéristique Ψ de X et en déduire $\mathbb{E}\exp(uX_t)$.
8. Montrer que si $\beta > 0$, alors X se décompose comme suit :

$$X_t = \gamma_0 t + \sum_{i=1}^{N_t^3} V_i - \sum_{j=1}^{N_t^4} Z_j,$$

   avec

   - $N^3 = (N_t^3)_{t\geq 0}$ et $N^4 = (N_t^4)_{t\geq 0}$ deux processus de Poisson d'intensité respective μ_3 et μ_4 à déterminer,
   - les V_i étant positifs ou nuls de densité f donnée par

$$f(x) = cx^{\beta-1}(1+x)^{-\alpha-\beta}e^{-\lambda x}\mathbf{1}_{]0,+\infty[}(x)$$

     où c est une constante de normalisation

9. Montrer qu'il existe une constante $C = C(\alpha, \beta, \lambda)$ telle que

$$\forall x > 0,\ f(x) \leq Cx^{\beta-1}(1+x)^{-\alpha-\beta}.$$

   En déduire un algorithme de simulation par rejet des V_i.

Indication. *On rappelle qu'une variable aléatoire Y suit une loi béta de paramètres $a > 0$ et $b > 0$ si Y admet pour densité la fonction g :*

$$g(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{b-1}(1-x)^{a-1}\mathbf{1}_{]0,1[}(x).$$

On note par $F(x; a, b)$ la fonction de répartition au point x de la loi béta de paramètres a et b.

Partie 2 : probabilités risque-neutres. On suppose que le prix $S = (S_t)_{0\leq t\leq T}$ d'un actif risqué est de la forme $S_t = S_0\exp(rt + X_t)$. T désigne la maturité du contrat, $r > 0$ le taux sans risque. On suppose que $\lambda > 2$.

1. Montrer que S est bien défini et admet des moments d'ordre 1 et 2.

2. À quelles conditions sur (σ^2, ν, γ) est-on certain d'avoir un modèle sans arbitrage ? On expliquera pourquoi et on prendra soin de bien détailler les conditions.

3. Montrer que si $\sigma^2 > 0$, on peut déterminer une probabilité $\mathbb{Q}$ équivalente à $\mathbb{P}$ sous laquelle les prix actualisés sont martingales en utilisant une transformation de Girsanov (comme dans le modèle de Black-Scholes). Donner la densité de $\mathbb{Q}$ par rapport à $\mathbb{P}$.

Partie 3 : valorisation d'un call. On suppose que $\lambda > 0$ et $\beta > 0$. On rappelle qu'alors X se décompose ainsi

$$X_t = \gamma_0 t + \sigma W_t + Y_t = \gamma_0 t + \sigma W_t + \sum_{i=1}^{N_t^3} V_i - \sum_{j=1}^{N_t^4} Z_j,$$

avec

- W mouvement brownien,
- $N^3 = (N_t^3)_{t\geq 0}$ et $N^4 = (N_t^4)_{t\geq 0}$ deux processus de Poisson d'intensité respective μ_3 et μ_4,
- les V_i étant positifs ou nuls de densité f,
- les Z_i une loi géométrique de paramètre p.

Pour tout $\eta > 0$, $\tilde{\mu} > 0$, et toute densité g sur $]0, +\infty[$ ayant un moment exponentiel d'ordre 1, on pose

$$\Theta(t) = \exp\left(-\eta W_t - \eta^2 t/2\right) e^{(\mu_3 - \tilde{\mu})t} \prod_{i=1}^{N_t^4} \frac{\tilde{\mu} g(V_i)}{\mu_3 f(V_i)}.$$

1. Montrer que Θ est une martingale sous $\mathbb{P}$.

2. Soit $\mathbb{Q}$ la probabilité équivalente à $\mathbb{P}$ de densité Θ_T. Quelle est la loi de $S = (S_t)_{0\leq t\leq T}$ sous $\mathbb{Q}$?

3. Montrer que sous $\mathbb{Q}$, le prix actualisé est une martingale si et seulement si

$$\gamma_0 - \sigma\eta + \sigma^2/2 + \mu_4\left(\frac{(1-p)}{e-p} - 1\right) + \tilde{\mu}\left(\int_0^{+\infty} e^x g(x)dx - 1\right) = 0.$$

4. Exprimer le prix à l'instant initial de l'option d'achat européenne de strike K, de maturité T, comme une série double faisant intervenir les prix Black-Scholes d'options d'achat européennes.

Exercice 7.2 (Problème de l'examen 2012-2013) Soit $(P_t)_{t\geq 0}$ un processus de Poisson composé d'intensité λ et de loi des sauts μ. On suppose de plus que, pour un certain $p \in [0,1]$,

$$\mu(dz) = p\delta_1(dz) + (1-p)\delta_{-1}(dz),$$

c'est-à-dire que le processus n'effectue que des sauts de taille ± 1. On note $(\mathcal{F}_t)_{t\geq 0}$ la filtration engendrée par P.

Partie 1, loi des grands nombres.

1. (a) Montrer que pour tout $t \geq 0$, P_t est intégrable et calculer $\mathbb{E}[P_t]$.

   (b) En déduire que l'on a, $\mathbb{P}$ presque sûrement, $\lim_{n\to+\infty} \frac{P_n}{n} = \lambda(2p-1)$.

2. (a) À l'aide de la formule d'Itô, montrer que

   $$M_t = (P_t - \lambda t(2p-1))^2 - \lambda t$$

   est une martingale $\mathcal{F}_t$-adaptée.

   (b) En déduire que

   $$\mathbb{E}\left[\sup_{t\in[0,1]} |P_t|^2\right] < +\infty.$$

   Indication : utiliser l'inégalité de Doob valable pour des martingales càdlàg :

   $$\mathbb{E}\left(\sup_{t\in[0,T]} |M_t|^2\right) \leq 4 \sup_{t\in[0,T]} \mathbb{E}(|M_t|^2).$$

3. (a) Vérifer que pour tout $n \in \mathbb{N}$ et tout $t \in [n, n+1[$ on a :

   $$\left|\frac{P_t}{t} - \lambda(2p-1)\right| \leq \left|\frac{P_n}{n} - \lambda(2p-1)\right| + \sup_{t\in[n,n+1[} \left|\frac{P_t - P_n}{n}\right| + \left|\frac{P_n}{n(n+1)}\right|.$$

   (b) Vérifier que les variables aléatoires $X_n = \sup_{t\in[n,n+1[} |P_t - P_n|^2$ sont identiquement distribuées et $\mathbb{E}[X_n] < +\infty$ pour tout $n \geq 0$.

   (c) En déduire que la série $\sum_{n\in\mathbb{N}} \frac{X_n}{n^2}$ converge, puis que $\frac{X_n}{n^2}$ tend vers zéro lorsque n tend vers $+\infty$ presque sûrement.

   (d) Montrer que l'on a $\mathbb{P}$ presque sûrement

   $$\lim_{t\to+\infty} \frac{P_t}{t} = \lambda(2p-1).$$

Partie 2, temps d'atteinte.

1. Justifer que le processus P ne prend que des valeurs entières.
2. Pour $m \in \mathbb{N}^*$, on définit

$$\tau^m = \inf\{t > 0,\ P_t \geq m\}.$$

   (a) Montrer que τ^m est un temps d'arrêt.
   (b) Montrer que, sur $\{\tau^m < +\infty\}$, on a $P_{\tau^m} = m$.
   (c) Montrer que si $p > 1/2$, on a $\tau^m < +\infty$ presque sûrement.
3. Pour $u \in \mathbb{R}$, on pose

$$\forall t \geq 0, \quad M_t^u = \exp\left(uP_t - t\phi(u)\right),$$

   où : $\phi(u) = \displaystyle\int_{\mathbb{R}} (e^{uz} - 1)\lambda\mu(dz)$.

   (a) Justifier que $\phi(u)$ est bien défini pour tout $u \in \mathbb{R}$ et est de classe C^∞ sur $\mathbb{R}$.
   (b) Montrer que, pour tout $u \in \mathbb{R}$, M^u est une $\mathcal{F}_t$-martingale.
4. Dans cette question, on suppose $p \geq 1/2$.

   (a) Montrer qu'il existe $\delta > 0$ tel que $\phi(u) > 0$ pour $u \in]0, \delta[$.
   (b) Montrer que, pour $u \in]0, \delta[$, on a :

$$\mathbb{E}\left[e^{-\tau^m\phi(u)}\mathbf{1}_{\tau^m < +\infty}\right] = e^{-um}.$$

   Penser à utiliser le théorème d'arrêt de Doob.
   (c) En déduire que

$$\mathbb{P}(\tau_m < +\infty) = 1,$$

   et en déduire la valeur de $\mathbb{E}[\tau^m]$.
5. Dans cette question, on suppose $0 < p < 1/2$.

   (a) Trouver un $\delta > 0$ tel que $\phi(u) > 0$ pour $u \in]\delta, +\infty[$ et $\phi(\delta) = 0$.
   (b) Montrer que, pour $u \in]\delta, +\infty[$, on a :

$$\mathbb{E}\left[e^{-\tau^m\phi(u)}\mathbf{1}_{\tau^m < +\infty}\right] = e^{-um}.$$

   (c) En déduire que

$$\mathbb{P}(\tau^m < +\infty) = \left(\frac{p}{1-p}\right)^m.$$

Chapter 8

Finance with general Lévy process

8.1 Pricing of European options in exp-Lévy models

The exponential Lévy models assume that the risk-neutral dynamics of an asset price is given by:

$$S_t = S_0 \exp(rt + X_t)$$

where X is a Lévy process with triplet (σ^2, ν, γ) s.t.

- $\displaystyle\int_{|x|\geq 1} e^x \nu(dx) < \infty,$
- $\displaystyle\gamma + \frac{\sigma^2}{2} + \int (e^y - 1 - y\mathbf{1}_{|y|\leq 1})\nu(dy) = 0.$

X is a Lévy process such that $\mathbb{E}(e^{X_t}) = 1$ for all t.

From Section 5.3, this class of models is the same as the following construction:

$$dS_t = rS_t dt + S(t^-)dZ_t, \text{ with } Z \text{ Lévy process.}$$

$e^{-rt}S_t$ martingale if and only if Z is a martingale with $\mathbb{E}(Z_1) = 0$. Moreover we have seen that if $\sigma \neq 0$ and if $\nu \geq 0$, the market is incomplete.

8.1.1 Call options

Recall that if H is a convex payoff function, then

$$H(S_T) = H(0) + H'(0)S_T + \int_0^\infty \rho(dK)(S_T - K)^+,$$

where the measure ρ is the second derivative of H. Moreover one has the call-put parity relation:

$$C_t(T, K) - P_t(T, K) = S_t - e^{-r(T-t)}K.$$

Hence the basic derivative is the call option. Let $C_t(T, K)$ be the price at time t of the call with strike K:

$$\boxed{C_t(T, K) = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+ \middle| \mathcal{F}_t\right] = C(t, S_t; T, K).} \tag{8.1}$$

With $\tau = T - t$

$$\begin{aligned} C(t, y; T, K) &= e^{-r\tau}\mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] \\ &= e^{-r\tau}\mathbb{E}[(ye^{r\tau + X_\tau} - K)^+] = Ke^{-r\tau}\mathbb{E}(e^{x+X_\tau} - 1)^+, \end{aligned}$$

where x is the log forward moneyness

$$x = \ln \frac{y}{K} + r\tau.$$

Therefore entire structure of options prices is given by

$$\boxed{u(\tau, x) = \frac{e^{r\tau}C(t, y; T, K)}{K} = \mathbb{E}(e^{x+X_\tau} - 1)^+ = (\rho_\tau * h)(x),} \tag{8.2}$$

where ρ_τ density of X_τ.

Assume that u is computed, then we can calculate the implied volatility. Recall that the Black-Scholes formula states:

$$C^{BS}(S_t, K, \tau, \sigma) = S_T\mathcal{N}(d_+) - Ke^{-r\tau}\mathcal{N}(d_-),$$

with $\tau = T - t$, $x = \ln \frac{S_t}{K} + r\tau$ and

$$d_\pm = \frac{1}{\sigma\sqrt{\tau}}\left[x \pm \frac{1}{2}\sigma^2\tau\right].$$

Moreover $\sigma \mapsto C^{BS}(\sigma)$ is an increasing function, mapping $]0, \infty[$ into $](S_t - Ke^{-r\tau})^+, S_t[$ (maximal interval allowed by arbitrage bounds).

Now given the market price $C_t^*(T, K)$, the implied volatility is defined by the formula:

$$C^{BS}(S_t, K, \tau, \Sigma_t(T, K)) = C_t^*(T, K).$$

Using moneyness $m = K/S_t$, we define the implied volatility surface

$$I_t(\tau, m) = \Sigma_t(t + \tau, mS(t)).$$

Proposition 8.1 *In the exp-Lévy models, the implied volatility for a given moneyness level $m = K/S_t$ and time to maturity τ does not depend on time*

$$\forall t \geq, \quad I_t(\tau, m) = I_0(\tau, m).$$

Hence $\Sigma_t(T, K) = I_0\left(\frac{K}{S_t}, T - t\right)$. Let us mention some features of implied volatility surfaces.

1. **Skew/smile.**
   - A negatively skew jump distribution implies skew of the surface.
   - And a strong variance of jumps generates curvature (smile).

2. **Short term skew.**

   - Diffusion models produce little skew for short maturities.
   - But in exp-Lévy models there is a strong short term skew.

3. **Flattening of the skew/smile with maturity.** For a Lévy process with finite variance,

$$\frac{X_T - \mathbb{E}X_T}{\sqrt{T}} \underset{T\to+\infty}{\longrightarrow} \text{Gaussian.}$$

   - Long maturities prices are close to Black-Scholes prices.
   - Hence implied volatility smile becomes flat.

The main problem is to compute the price given by (8.1) or (8.2). More generally if $H(S_T)$ is the payoff of a financial derivative, the price at time t is given by:

$$\boxed{\Pi_t(T,K) = e^{-r(T-t)}\mathbb{E}\left[H(S_T)\middle|\mathcal{F}_t\right] = e^{-r\tau}\mathbb{E}[H(S_t e^{r\tau+X_\tau})] = \Pi(\tau, S_t)} \tag{8.3}$$

with $\tau = T - t$ and $\Pi(\tau, y) = e^{-r\tau}\mathbb{E}[H(ye^{r\tau+X_\tau})]$. Let us mention several methods:

- **Monte Carlo simulations.** Using Chapter 3, if we can simulate the Lévy process, we compute the expectation $\Pi(\tau, y)$ by Monte Carlo approximation. The method works always, in any dimension, but is quite slow. The rate of convergence is of the order $1/\sqrt{N}$, where N is the number of simulations.

  Moreover if X cannot be exactly simulated, we can use an approximation and Propositions 3.1 and 3.2.

- **Fourier transform.**

- **Numerical scheme for PIDE.**

8.1.2 Fourier transform methods

Since a Lévy process is well described by its characteristic function, a very powerful method to compute the price is the Fourier transform. Recall that for a function f,

- the Fourier transform is: $\boxed{\mathcal{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x)dx.}$

- The inverse Fourier transform is: $\mathcal{F}^{-1}f(x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-ixv} f(v)dv.$

For $f \in L^2(\mathbb{R})$, $\mathcal{F}^{-1}\mathcal{F}f = f$. This can be extended to a d-dimensional space.

Transformation w.r.t. the log strike price. Let us assume that

- $S_0 = 1$,
- for some $\alpha > 0$, $\mathbb{E}(S_T^{1+\alpha}) < \infty \iff \int_{|y|\geq 1} e^{(1+\alpha)y}\nu(dy) < \infty$.

The aim is to compute $C(k) = e^{-rT}\mathbb{E}((e^{rT+X_T} - e^k)^+)$ via the algorithm:

- express its Fourier transform in strike,
- find the prices for a range of strikes by Fourier inversion.

The problem is that $C(k)$ is not integrable ! Put

$$z_T(k) = e^{-rT}\mathbb{E}((e^{rT+X_T} - e^k)^+) - (1 - e^{k-rT})^+.$$

Proposition 8.2 (Carr and Madan method)

$$\boxed{\zeta_T(v) = \mathcal{F}z_T(v) = e^{ivrT}\frac{\Phi_T(v-i)-1}{iv(1+iv)}}$$

where Φ_T is the characteristic function of X_T.

Now using Fourier inversion, $z_T(k) = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{R}} e^{-ivk}\zeta_T(v)dv$. But the difficulty is that $\zeta_t(v) \sim |v|^{-2}$ at infinity. Therefore truncation error in the numerical evaluation of $z_T(k)$ will be large. One possible improvement consists to deal with

$$\tilde{z}_T(k) = e^{-rT}\mathbb{E}((e^{rT+X_T} - e^k)^+) - C_{BS}^{\sigma}(k).$$

In that case

$$\boxed{\tilde{\zeta}_T(v) = \mathcal{F}\tilde{z}_T(k) = e^{ivrT}\frac{\Phi_T(v-i)-\Phi_T^{\sigma}(v-i)}{iv(1+iv)}}$$

where $\Phi_T^{\sigma}(v) = \exp\left(-\dfrac{\sigma^2 T}{2}(v^2+iv)\right)$. The advantage is that $|v|^{\beta}\tilde{\zeta}_t(v) \to 0$ for any β. Hence the inverse Fourier transform converges very fast. But the inconvenient is the dependence on the choice of σ.

Transformation w.r.t. the log spot price. Lewis has considered another method. Denote $s = \ln S_0$ and f be the payoff function of the option. Then

$$C(s) = e^{-rT}\int_{\mathbb{R}} f(e^{s+x+rT})\rho_T(x)dx.$$

We assume that

- ρ_T is Fourier integrable in some strip S_1;
- $f^*(x) \equiv f(e^{x+rT})$ is Fourier integrable in some strip S_2;

- $S = \bar{S}_1 \cap S_2 = \{z = \Re z + i\Im z;\ \Re z - i\Im z \in S_1\} \cap S_2 \neq \emptyset$.

Definition 8.1 (and Proposition) *g is* Fourier integrable *in a strip (a,b) if*

$$\int_{\mathbb{R}} e^{-au}|g(u)|du < \infty, \quad \int_{\mathbb{R}} e^{-bu}|g(u)|du < \infty.$$

Then $\mathcal{F}g(z) = \int_{\mathbb{R}} e^{iuz} g(u)du$ exists and is analytic for all $z \in \mathbb{C}$ such that $a < Im\ (z) < b$. Moreover for $a < w < b$

$$g(x) = \frac{1}{2\pi} \int_{iw-\infty}^{iw+\infty} e^{-izx} \mathcal{F}g(z)dz.$$

For every $z \in S$:

$$\boxed{\mathcal{F}C(z) = e^{-rT}\Phi_T(-z)\mathcal{F}f^*(z).}$$

For a call option, the payoff is Fourier integrable in the region $\Im z > 1$:

$$\mathcal{F}f^*(z) = \frac{e^{k+iz(k-rT)}}{iz(iz+1)}.$$

Hence ρ_T must be integrable in a strip (a,b) with $a < -1$ and $b > 0$ (because it's a density). Finally

$$\boxed{\mathcal{F}C(z) = \Phi_T(-z)\frac{e^{(1+iz)(k-rT)}}{iz(iz+1)}}$$

and

$$C(x) = \frac{\exp(wx + (1-w)(k-rT))}{2\pi} \int_{\mathbb{R}} \frac{e^{iu(k-rT-x)}\Phi_T(-iw-u)}{(iu-w)(1+iu-w)}du$$

for some $w \in (1, 1+\alpha)$.

In both cases we have to compute the inverse Fourier transform. But remark that

$$\begin{aligned} \mathcal{F}^{-1}(f)(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx \approx \frac{1}{2\pi}\int_{-R}^{R} e^{-ixv} f(v)dv \\ &\approx \frac{1}{2\pi}\sum_{j=0}^{N-1} \omega_j f(v_j) e^{-ixv_j} \end{aligned}$$

with discretisation step $\delta v = 2R/(N-1)$, $v_j = -R + j\delta v$ and suitable weights ω_j. Therefore we compute a discrete Fourier transform, which needs a priori $\mathcal{O}(N^2)$ operations. Using the so-called *Fast Fourier Transform* this computational cost can be reduced to $\mathcal{O}(N \log N)$.

8.1.3 Integro-differential equations

We always consider a exp-Lévy price: $S_t = S_0 \exp(rt+X_t)$ where X is a Lévy process $(\sigma 2, \nu, \gamma)$ such that under some risk-neutral measure $\mathbb{Q}$, $\hat{S}_t = e^{X_t}$ is a martingale. We assume that $S_t \in L^2$ i.e.

$$\int_{|y|\geq 1} e^{2y}\nu(dy) < \infty.$$

Then

$$S_t = S_0 + \int_0^t rS_u du + \int_0^t \sigma S_u dW_u + \int_0^t \int_{\mathbb{R}} (e^x - 1) S(u^-) \tilde{J}_X(du, dx),$$

and

$$\frac{d\hat{S}_t}{\hat{S}_t} = \sigma dW_t + \int_{\mathbb{R}} (e^x - 1)\tilde{J}_X(du, dx), \quad \sup_{t\in[0,T]} \mathbb{E}(\hat{S}_t^2) < \infty.$$

We want to price an European option. The value is

$$c(t, y) = \mathbb{E}^{\mathbb{Q}}[e^{-r(T-t)}H(S_T)|S_t = y]$$

or with $\tau = T - t$, $x = \ln(y/K) + r\tau$, $h(x) = H(Ke^x)/K$ and

$$u(\tau, x) = \frac{e^{r(T_t)}c(t, y)}{K} = \mathbb{E}^{\mathbb{Q}}[h(x + X_\tau)].$$

Proposition 8.3 *We assume that*

- *the payoff $H(S_T)$ satisfies*

$$|H(y) - H(x)| \leq K|x - y|;$$

- *either $\sigma > 0$ or there exists $\beta \in (0, 2)$, such that $\liminf_{\varepsilon\downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2\nu(dx) > 0$.*

Then the value of a European call with terminal payoff $H(S_T)$ is given by $c : [0, T] \times (0, \infty) \to \mathbb{R}$ with:

1. *$c \in C([0, T] \times [0, \infty)) \cap C^{1,2}((0, T) \times (0, \infty))$;*
2. *$\forall y > 0$, $c(T, y) = H(y)$;*
3. *c satisfies on $(0, T) \times (0, \infty)$ the following equation:*

$$\boxed{\begin{aligned} &\frac{\partial c}{\partial t}(t, y) + ry\frac{\partial c}{\partial y}(t, y) + \frac{\sigma^2 y^2}{2}\frac{\partial^2 c}{\partial y^2}(t, y) - rc(t, y) \\ &\quad + \int_{\mathbb{R}} \left[c(t, ye^z) - c(t, y) - y(e^z - 1)\frac{\partial c}{\partial y}(t, y)\right] \nu(dz) = 0. \end{aligned}} \tag{8.4}$$

It implies that the function u satisfies the following PIDE on $(0,T] \times \mathbb{R}$:

$$\boxed{\frac{\partial u}{\partial \tau} = \mathcal{L}_X u, \quad u(0,x) = h(x);} \tag{8.5}$$

where

$$\boxed{\mathcal{L}_X f(x) = \gamma \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_{\mathbb{R}} [f(x+y) - f(x) - y\mathbf{1}_{|y|\le 1}\frac{\partial f}{\partial x}(x)]\nu(dy).} \tag{8.6}$$

Moreover if there is no jump, the measure ν is equal to zero, and Equation (8.4) becomes

$$\frac{\partial c}{\partial t}(t,y) + ry\frac{\partial c}{\partial y}(t,y) + \frac{\sigma^2 y^2}{2}\frac{\partial^2 c}{\partial y^2}(t,y) - rc(t,y) = 0$$

which is the **Black-Scholes equation** and Equation (8.5) becomes the classical heat equation

$$\frac{\partial u}{\partial \tau} = \gamma \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}.$$

The result can be extended to **weak solutions** in some Sobolev space or to **viscosity solutions**, removing the assumption that X has a smooth density.

To solve numerically Equation (8.4), a finite difference method can be used. There are four main steps:

1. The original space domain is localized (as for the Black-Scholes model).
2. The integration domain $\mathbb{R}$ of $\mathcal{L}_X$ in (8.6) must also be localized to a bounded domain.
3. The small jumps mest be approximated by a Brownian motion.
4. The solution is computed at discrete grid points and the derivatives in (8.4) are replaced by finite differences.

For the first two steps, one can prove that since $S_t \in L^2$, the localization error decays exponentially with respect to the truncation bound. The small jumps approximation has been made in Chapter 3. The main difficulty in the last step is that the finite difference methods on the derivatives induces a sparse matrix whereas the integral part induces a densely populated matrix.

8.2 Wiener-Hopf factorization and barrier options

Theorem 8.1 (Wiener-Hopf factorization) *For X a real Lévy process, let Ψ be the characteristic exposant of X. There exist two unique characteristic functions Φ_q^+ and Φ_q^- of ∞-divisible laws, with zero drift, and with support included resp. in $[0,\infty[$ and $]-\infty,0]$ s.t. for any $q > 0$*

$$\frac{q}{q - \Psi(u)} = \Phi_q^+(u)\Phi_q^-(u).$$

Moreover if ρ_t is the law of X at time t:

$$\Phi_q^+(u) = \exp\left[\int_0^\infty t^{-1}e^{-qt}\int_0^\infty (e^{iux}-1)\rho_t(dx)dt\right]$$

$$\Phi_q^-(u) = \exp\left[\int_0^\infty t^{-1}e^{-qt}\int_{-\infty}^0 (e^{iux}-1)\rho_t(dx)dt\right].$$

For X a real Lévy process, define

$$M_t = \sup_{0\le s\le t} X_t, \qquad N_t = \inf_{0\le s\le t} X_t.$$

Theorem 8.2 *Using Wiener-Hopf factorization, we have*

$$q\int_0^\infty e^{-qt}\mathbb{E}\left[\exp\left(izM_t + iw(X_t - M_t)\right)\right]dt = \Phi_q^+(z)\Phi_q^-(w).$$

$$q\int_0^\infty e^{-qt}\mathbb{E}\left[\exp\left(izN_t + iw(X_t - N_t)\right)\right]dt = \Phi_q^+(w)\Phi_q^-(z).$$

Hence in law

$$M_t = X_t - N_t, \quad N_t = X_t - M_t.$$

These results can be apply to barrier (or lookback) options. Assume that the payoff is

$$H = (S_T - e^k)^+\mathbf{1}_{]-\infty,e^b[}\left(\sup_{0\le t\le T} S_t\right),$$

- $S_t = e^{X_t}$ is the spot price at time t of the stock with $S_0 = 1$;
- the riskless rate is zero ;
- we work under a risk neutral probability.

Let $C(T,k,b)$ be the price of the option at time 0.

Proposition 8.4 *Assume that the law of (X_t, M_t) is Fourier integrable. Then for $Im(v) > 0$ et $Im(u) < 0$*

$$q\int_{\mathbb{R}^2} e^{iuk+ivb}\int_0^\infty e^{-qT}C(T,k,b)dTdkdb = \frac{\Phi_q^+(v+u-i)\Phi_q^-(u-i)}{uv(1+iu)}.$$

Therefore the price can be computed using an inverse Fourier transform and the so-called Gaver-Stehfest algorithm to inverse the Laplace transform. A different way to estimate the price consists to use PDE methods like for Call options. If B is the barrier and K the strike, we have to solve for all $(t,y)\in]0,T[\times]0,B[$

$$\frac{\partial c}{\partial t}(t,y) + ry\frac{\partial c}{\partial y}(t,y) + \frac{\sigma^2y^2}{2}\frac{\partial^2 c}{\partial y^2}(t,y) - rc(t,y)$$

$$+\int_{\mathbb{R}}\left[c(t,ye^z) - c(t,y) - y(e^z-1)\frac{\partial c}{\partial y}(t,y)\right]\nu(dz) = 0,$$

with

$$\begin{array}{ll} c(T,y) = (y-K)^+, & \text{for } y\in]0,B[, \\ c(t,y) = 0, & \text{for } t\in[0,T] \text{ and } y\ge B. \end{array}$$

8.3 Exercises

Exercice 8.1 (Extrait de l'examen 2008-2009) Dans cet exercice, la maturité est $T > 0$ et on suppose que le prix de l'actif sans risque est donné par

$$dS_t^0 = S_t^0 r dt, \quad S_0^0 = 1;$$

tandis que le prix de l'actif risqué est donné par l'équation suivante :

$$dS_t = S_{t_-}(bdt + \sigma dW_t + \delta dM_t), \quad S_0 > 0.$$

Ici W est un mouvement brownien standard, M un processus de Poisson compensé, i.e. $M_t = N_t - \lambda t$, avec N processus de Poisson d'intensité $\lambda > 0$, indépendant de W. Tous les processus sont définis sur le même espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ et sont adaptés à la filtration. Les hypothèses sur les paramètres sont :

$$r > 0, \quad b \in \mathbb{R}, \quad \sigma \in \mathbb{R}_+, \quad \delta \in]-1, +\infty[\setminus\{0\}.$$

On rappelle que les exponentielles de Doléans-Dade sont

$$\begin{aligned}
\mathcal{E}(\sigma W)(t) &= \exp\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right), \\
\mathcal{E}(\delta M)(t) &= \exp\left(\ln(1+\delta)M_t - \lambda t(\delta - \ln(1+\delta))\right) = \exp\left(\ln(1+\delta)N_t - \lambda\delta t\right),
\end{aligned}$$

1. Quelles sont les équations vérifiées par $\mathcal{E}(\sigma W)$ et $\mathcal{E}(\delta M)$? On appliquera la formule d'Itô en justifiant son emploi.

2. Exprimer S_t uniquement en fonction des paramètres du modèle.

3. Montrer que pour tout $a \in \mathbb{R}$,

$$(S_t)^a = (S_0)^a \mathcal{E}(a\sigma W)(t)\mathcal{E}(\delta_a M)(t) \exp\left[\frac{1}{2}a(a-1)\sigma^2 t + abt + \lambda t(\delta_a - a\delta)\right],$$

   avec $\delta_a = (1+\delta)^a - 1$.

4. En déduire $\mathbb{E}(S_t^a)$ pour tout $t \geq 0$.

5. Soit $\gamma > -1$. Montrer que $(L_t^\gamma = \mathcal{E}(\psi W)(t)\mathcal{E}(\gamma M)(t))_{0\leq t\leq T}$ est une martingale définissant une probabilité risque-neutre $\mathbb{Q}^\gamma$ équivalente à $\mathbb{P}$, si et seulement si

$$b - r + \sigma\psi + \lambda\delta\gamma = 0.$$

6. Montrer que sous $\mathbb{Q}^\gamma$, on a pour tout $t \in [0, T]$

$$\exp(-rt)S_t = S_0 \mathcal{E}(\sigma W^\gamma)(t)\mathcal{E}(\delta M^\gamma)(t).$$

   On précisera la dynamique de $(W_t^\gamma)_{0\leq t\leq T}$ et $(M_t^\gamma)_{0\leq t\leq T}$ sous $\mathbb{Q}^\gamma$.

On définit alors le prix V^γ de l'option d'achat $(S_T - K)^+$ sous $\mathbb{Q}^\gamma$ par

$$\forall t \in [0,T],\ V_t^\gamma = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}^\gamma}((S_T - K)^+|\mathcal{F}_t) = e^{-r(T-t)}\mathbb{E}^\gamma((S_T - K)^+|\mathcal{F}_t).$$

On définit aussi $H(t, S_t)$ comme le prix Black-Scholes de ce même contrat à l'instant t et $\mathcal{L}H$ la fonction

$$\mathcal{L}H(t,x) = H(t,(1+\delta)x)) - H(t,x) - \delta x \frac{\partial H}{\partial x}(t,x).$$

On supposera (sans le redémontrer) que $H \in C^{1,2}([0,T[\times\mathbb{R})$, que $\dfrac{\partial H}{\partial x}$ est bornée et que H est convexe par rapport à la seconde variable.

7. En appliquant la formule d'Itô, montrer que

$$e^{-rt}V_t^\gamma = e^{-rt}H(t,S_t) + (1+\gamma)\lambda\mathbb{E}^\gamma\left[\int_t^T e^{-rs}\mathcal{L}H(s,S_s)ds\middle|\mathcal{F}_t\right].$$

8. Montrer que pour tout $\gamma \in]-1,+\infty[$,

$$\forall t \in [0,T], \quad H(t,S_t) \le V^\gamma(t) \le S_t.$$

9. Montrer que $|\mathcal{L}H(t,x)| \le 2xC|\delta|$ avec $|\partial H/\partial x(t,x)| \le C$.

10. En déduire que $\lim\limits_{\gamma\to -1} V^\gamma(t) = H(t,S_t)$.

11. Prouver que pour tout $0 < a < 1$ et $t > 0$, $\lim\limits_{\gamma\to+\infty} \mathbb{E}^\gamma((\mathcal{E}(\delta M^\gamma)(t))^a) = 0$.

    Indication : procéder comme à la question 3. On rappelle que sous les hypothèses imposées à a et δ, $(1+\delta)^a - a\delta - 1 < 0$.

12. Montrer que $V_0^\gamma = S_0 - e^{-rT}\mathbb{E}^\gamma\left[G(S_0\mathcal{E}(\delta M^\gamma)(T))\right]$ avec $G(y) = \mathbb{E}^\gamma\left[g(ye^{rT}\mathcal{E}(\sigma W^\gamma)(T))\right]$ et $g(x) = x - (x-K)^+$.

13. En déduire alors que $\lim\limits_{\gamma\to+\infty} \mathbb{E}^\gamma(e^{-rT}(S_T - K)^+) = S_0$ et que $\lim\limits_{\gamma\to+\infty} V^\gamma(t) = S_t$.

14. À quoi correspond l'intervalle $[H(t,S_t), S_t]$ en terme de prix ?

Exercice 8.2 (Extrait de l'examen 2011-2012) Soit $X = (X_t)_{t\ge 0}$ un processus de Lévy de triplet caractéristique (σ^2, ν, γ) tel que $\sigma^2 > 0$, $\nu(dz) = f(z)dz$, où f est une fonction continue à support compact inclus dans $]-1;+\infty[$. On écrira X sous la forme $X_t = \gamma_0 t + \sigma B_t + Q_t$, où $\gamma_0 \in \mathbb{R}$, $B = (B_t)_{t\ge 0}$ est un mouvement brownien standard et Q un processus de Poisson composé compensé.

On considère également, pour $S_0 \in \mathbb{R}_+^*$, la solution S de l'EDS

$$\forall t \ge 0, \quad S_t = S_0 + \int_0^t S_{r^-} dX_r.$$

1. Expliquer pourquoi $S_t > 0$, pour tout $t > 0$. Exprimer S_t en fonction de γ_0, B, σ, Q.

   Indication : Appliquer la formule d'Itô pour exprimer $\ln(S_t)$.

2. Montrer que pour $u \in \mathbb{R}$ et $t \geq 0$, l'espérance $\mathbb{E}[e^{uX_t}]$ est bien définie et peut s'écrire sous la forme
$$\mathbb{E}[e^{uX_t}] = e^{t\phi(u)}$$
   où ϕ est une fonction que l'on déterminera.

3. Montrer que, pour tout $u \in \mathbb{R}$, le processus M^u défini par
$$M_t^u = \exp(uX_t - t\phi(u))$$
   est une martingale.

4. On définit une nouvelle mesure de probabilité $\mathbb{Q}^u$ par
$$\frac{d\mathbb{Q}^u}{d\mathbb{P}} = M_T^u.$$
   Montrer que l'on peut trouver $u \in \mathbb{R}$ tel que $(S_t,\ 0 \leq t \leq T)$ soit une martingale sous $\mathbb{Q}^u$?

5. Si S est utilisé pour modéliser l'évolution d'un actif risqué sur un marché financier où le taux d'intérêt sans risque est nul, commentez les résultats obtenus.

Exercice 8.3 (Suite de l'exercice 2.5) Au début des années 2000, W. Schoutens a proposé de modéliser des cours d'actifs via le processus de Meixner (avec application au Nikkei-225 ou S&P 500). Celui-ci, noté $X = (X_t,\ t \geq 0)$ dans la suite, a une structure simple, stable par changement de probabilité, et donne des formules semi-fermées, comme pour le modèle de Black-Scholes.

Le processus de Meixner est déterminé par sa fonction caractéristique :
$$\forall t \geq 0, \quad \Phi_t(u) = \mathbb{E}(e^{iuX_t}) = \left(\frac{\cos(b/2)}{\mathrm{ch}\ (\frac{au-ib}{2})} \right)^{2dt} e^{imut}.$$
Les paramètres de ce modèle vérifient :
$$a > 0, \quad d > 0, \quad -\pi < b < \pi, \quad m \in \mathbb{R}.$$
La loi de X_1 est appelée loi de Meixner et notée $M(a, b, d, m)$.
Partie 3 : modèle financier sans arbitrage. On suppose que le prix d'un actif financier est donné par
$$\forall t \geq 0, \quad S_t = S_0 \exp(X_t), \tag{8.7}$$
où X est un processus de Meixner de paramètres (a, b, d, m). r désigne le taux sans risque du marché.

1. Expliquer (sans démonstration) pourquoi ce modèle sera sans arbitrage.
2. En appliquant la transformation d'Esscher pour un $\theta \in \mathbb{R}$, montrer que X_t suit, sous la nouvelle probabilité, la loi $M(a, b + a\theta, dt, mt)$.
3. Montrer que l'unique θ^* qui rend les prix actualisés martingales, est donné par la formule :
$$\theta^* = -\frac{1}{a}\left(b + 2\text{Arctan }\left(\frac{-\cos(a/2) + \exp((m-r)/(2d))}{\sin(a/2)}\right)\right).$$
4. Une autre façon de définir un processus risque-neutre est de corriger l'exponentielle :
$$S_t^{risk-neutral} = S_0 \exp(X_t)\frac{\exp(rt)}{\mathbb{E}(\exp(X_t))}.$$
Montrer que la loi de $S_1^{risk-neutral}$ est $M(a, b, d, \widetilde{m})$ avec
$$\widetilde{m} = r - 2d \ln\left(\frac{\cos(b/2)}{\cos((a+b)/2)}\right).$$

Partie 4 : pricing d'option. On utilise ici le modèle (8.7) avec transformation d'Esscher via θ^*. Soit $C(T, K)$ le prix du call européen de maturité T et de prix d'exercice K.

1. Montrer que l'on a une formule analogue au modèle de Black-Scholes, à savoir :
$$C(T, K) = S_0\left[1 - F_T(x, \theta^* + 1)\right] + e^{-rT}K\left[1 - F_T(x, \theta^*)\right],$$
avec $x = \ln\frac{K}{S_0}$ et $F_t(., \theta)$ est la fonction de répartition de la loi $M(a, b + a\theta, d, m)$.
2. On souhaite utiliser la méthode de Carr-Madan pour calculer le prix. Montrer qu'il existe $\alpha > 0$ tel que $\mathbb{E}(S_T^{1+\alpha}) < +\infty$ si et seulement si $1 + \alpha < \frac{\pi-b}{a} - \theta^*$.
3. On pose $C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k)\phi(T, s)ds$. Montrer que
$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}h_T(v)dv$$
avec
$$h_T(v) = \int_{-\infty}^{\infty} e^{-ivk}\int_k^\infty e^{\alpha k}e^{-rT}(e^s - e^k)g(T, s)ds.$$
4. En permutant les deux intégrales et en utilisant la définition de la fonction caractéristique, montrer que
$$h_T(v) = \frac{e^{-rT}\Phi_T(v - (1+\alpha)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v}.$$

Bibliography

[1] J. Bertoin, *Lévy Processes*, Cambridge University Press.

[2] R. Cont, P. Tankov, *Financial Modeling with Jump Processes*, Chapman and Hall.

[3] I. Karatzas, S.E. Shreve, *Brownian motion and Stochastic Calculus*, Springer.

[4] N. El Karoui, E. Gobet, *Les outils stochastiques des marchés financiers*, Éditions de l'École Polytechnique.

[5] A. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer.

[6] D. Lamberton, B. Lapeyre, *Introduction au calcul stochastique en finance*, Ellipses.

[7] B. Oksendal, *Stochastic Differential Equations, An introduction with Applications*, Springer.

[8] E. Platen, N. Bruti–Liberati, *Numerical Solution of Stochastic Differential Equations with Jumps in Finance*, Springer.

[9] P. Protter, *Stochastic Integration and Differential Equations*, Springer.

[10] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press.

[11] W. Schoutens, *Lévy Processes in Finance. Pricing Finance Derivatives*, Wiley.