Stochastic calculus for Lévy processes.

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February-March 2023

Outline

OUTLINE

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Simple processes.

Given :

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$.
- All processses are supposed to be adapted w.r.t. this filtration.

Definition

A stochastic process $(\phi_t)_{t\geq 0}$ is called a simple (predictable) process if it can be represented as

$$\phi_t = \phi_0 \mathbf{1}_{t=0} + \sum_{i=0}^{n} \phi_i \mathbf{1}_{]T_i, T_{i+1}]}(t),$$

where $T_0 = 0 < T_1 < \ldots < T_n < T_{n+1}$ are non-anticipating random times and each ϕ_i is bounded $\mathcal{F}_{T_i}$-measurable r.v..

Notation : set of simple processes : $\mathbb{S}$.

INTEGRAL OF SIMPLE PROCESSES.

SETTING : $X = (X_t = (X_t^1, \dots, X_t^d))_{t \geq 0}$ is a d-dimensional adapted RCLL process.

DEFINE : for $0 \leq t$, and j s.t. $T_j < t \leq T_{j+1}$

$$\begin{aligned} G_t(\phi) &= \phi_0 X_0 + \sum_{i=0}^{j-1} \phi_i (X_{T_{i+1}} - X_{T_i}) + \phi_j (X_t - X_{T_j}) \\ &= \phi_0 X_0 + \sum_{i=0}^{n} \phi_i (X_{T_{i+1} \wedge t} - X_{T_i \wedge t}) \end{aligned}$$

DEFINITION

The process $G_t(\phi)$ is the stochastic integral of ϕ w.r.t. X and is denoted by :

$$G_t(\phi) = \int_0^t \phi_u dX_u.$$

SETTING : $X = (X_t = (X_t^1, \dots, X_t^d))_{t \geq 0}$ is a d-dimensional adapted RCLL process.

PROPOSITION

If X is a martingale, then for any simple process ϕ, the stochastic integral G is also a martingale.

PROPOSITION

Assume that X is a real-valued RCLL process. Let ϕ and ψ be real-valued simple processes. Then $Y_t = \int_0^t \phi_u dX_u$ is an adapted RCLL process and

$$\int_0^t \psi_u dY_u = \int_0^t \psi_u \phi_u dX_u.$$

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PROPOSITION

Let ϕ be a predictable process verifying

$$\mathbb{E} \int_0^T |\phi_t|^2 dt < +\infty.$$

Then $\int_0^t \phi_u dW_u$ is a square integrable martingale and

$$\mathbb{E} \int_0^T \phi_u dW_u = 0, \qquad \mathbb{E} \left| \int_0^T \phi_u dW_u \right|^2 = \mathbb{E} \int_0^T |\phi_u|^2 du.$$

REMARK

- ϕ cannot be interpreted as a trading strategy : not LCRL,
- its integral cannot necessarily be represented as a limit of Riemann sums.

DEFINITION

A pure jump process X is a process with

- *piecewise constant trajectories,*
- *and a finite number of jumps on every finite time interval.*

The stochastic integral of Φ w.r.t. X is defined

$$\int_0^t \Phi_s dX_s = \sum_{0<s\leq t} \Phi_s \Delta J_s.$$

EXAMPLE. With

- $X_t = N_t - \lambda t$: compensated Poisson process ;
- $\Phi_t = \Delta N_t$;
- $\Psi_t = \mathbf{1}_{[0,S_1]}(t)$ where S_1 is the time of the first jump of N.

$$I_t = \int_0^t \Phi_s dX_s = N_t, \quad J_t = \int_0^t \Psi_s dX_s = \mathbf{1}_{[S_1,+\infty]}(t) - \lambda(t \wedge S_1).$$

DEFINITION

An adapted RCLL process X is a semi-martingale if the stochastic integral of simple processes w.r.t. X verifies the following continuity property : for every ϕ^n and ϕ in $\mathbb{S}$ if

$$\lim_{n\to+\infty} \sup_{(t,\omega)\in\mathbb{R}_+\times\Omega} |\phi^n_t(\omega) - \phi_t(\omega)| = 0, \qquad (1)$$

then in probability : $\displaystyle\int_0^T \phi^n_u dX_u \underset{n\to+\infty}{\longrightarrow} \int_0^T \phi_u dX_u = G_T(\phi).$

EXAMPLES.

- A finite variation process.
- A (locally) square integrable (local) martingale.
- An adapted RCLL decomposable process X :

$$X_t = X_0 + M_t + A_t,$$

with
  - $M_0 = A_0 = 0$,
  - M locally square integrable martingale,
  - A is RCLL, adapted, with paths of finite variation on compacts.,

CONSEQUENCE : all Lévy processes are semi-martingales.

NOTATIONS :

- $\mathbb{D}$: set of adapted RCLL processes.
- $\mathbb{L}$ (resp. $b\mathbb{L}$) : set of adapted LCRL (resp. bounded) processes.
- $\phi_t^* = \sup_{0 \leq s \leq t} |\phi_s|$.

DEFINITION

A sequence (ϕ^n) of processes converges uniformly on compact sets in probability (ucp in short) to ϕ if :

$$\forall t > 0, \quad (\phi^n - \phi)_t^* \underset{n \to +\infty}{\longrightarrow} 0 \text{ in probability.}$$

LEMMA

1. *The set $\mathbb{S}$ is dense in $\mathbb{L}$ for the ucp topology.*
2. *For X semi-martingale, $G : \mathbb{S}_{ucp} \to \mathbb{D}_{ucp}$ is continuous.*

DEFINITION

Let X be a semi-martingale. The continuous linear mapping $G = G_X : \mathbb{L}_{ucp} \to \mathbb{D}_{ucp}$ obtained as the extension of $G : \mathbb{S} \to \mathbb{D}$ is called the stochastic integral.

THEOREM

1. Let T be a stopping time. Then

$$G(\phi)^T = (G(\phi)_{t\wedge T})_{t\geq 0} = G(\phi \mathbf{1}_{[0,T]}) = G_{X^T}(\phi).$$

2. The jump process $\Delta(G(\phi))$ is indistinguishable from $\phi(\Delta X)$.

Stochastic integral for LCRL process.

Theorem

If X is a semi-martingale, and if ϕ is an adapted LCRL process then

- $Y_t = \int_0^t \phi_u dX_u$: semi-martingale.
- If ψ is another adapted LCRL process, then

$$\int_0^t \psi_u dY_u = \int_0^t \psi_u \phi_u dX_u.$$

- If X is a (locally) square-integrable (local) martingale, Y is a (locally) square-integrable (local) martingale.

OUTLINE

Realized volatility.

Framework :

- X semi-martingale, adapted RCLL process with $X_0 = 0$,
- time grid $\pi = \{t_0 = 0 < t_1 < t_2 < \ldots < t_{n+1} = T\}$.

Realized variance :

$$\begin{aligned} V_X(\pi) &= \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})^2 \\ &= X_T^2 - 2\sum_{i=0}^{n} X_{t_i}(X_{t_{i+1}} - X_{t_i}) \end{aligned}$$

Convergence in probability :

$$[X, X]_T = X_T^2 - 2\int_0^T X_{u^-} dX_u.$$

Quadratic variation.

Definition

The *quadratic variation process* *of a semi-martingale X is the adapted RCLL process defined by :*

$$[X,X]_t = |X_t|^2 - 2\int_0^t X_{u^-} dX_u.$$

Proposition (Properties)

- $X_0^2 + \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})^2 \underset{\|\pi\|\to 0}{\longrightarrow} [X,X]_T$ *in ucp.*
- $([X,X]_t)_{t\in[0,T]}$ *is a non-decreasing process with* $[X,X]_0 = X_0^2$.
- *Jumps of* $[X,X]$: $\Delta[X,X]_t = |\Delta X_t|^2$.
- *If X is continuous and has paths of finite variation, then* $[X,X] = 0$.

Cross variation.

Definition

Given two semi-martingales X, Y, cross variation process $[X, Y]$

$$[X, Y]_t = X_t Y_t - X_0 Y_0 - \int_0^t X_{s^-} dY_s - \int_0^t Y_{s^-} dX_s.$$

Proposition

- $[X, Y]$ *is an adapted RCLL process with finite variations.*
- *Polarization identity :*

$$[X, Y] = \frac{1}{2}([X + Y, X + Y] - [X, X] - [Y, Y]).$$

- $[X, Y]_0 = X_0 Y_0$ *and* $\Delta[X, Y] = \Delta X \Delta Y$*.*
- *Convergence (in probability) :*

$$X_0 Y_0 + \sum_{i=0}^{n} (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \underset{\|\pi\| \to 0}{\longrightarrow} [X, Y]_T.$$

EXAMPLE : consider X a jump process :

$$X_t = X_0 + I_t + R_t + J_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds + J_t.$$

Then $[X]_T = \int_0^T \Gamma_s^2 ds + \sum_{0<s\leq T} (\Delta J_s)^2.$

PROPOSITION

Let X and Y be two locally square integrable martingales. Then $[X, Y]$ is the unique adapted RCLL process A with paths on finite variation on compacts satisfying the two properties :

1. *$XY - A$ is a local martingale ;*
2. *$\Delta A = \Delta X \Delta Y$, $A_0 = X_0 Y_0$.*

Pure jump processes.

Proposition

Let X be a quadratic pure jump semi-martingale. Then for any semi-martingale Y,

$$[X, Y]_t = X_0 Y_0 + \sum_{0<s\leq t} \Delta X_s \Delta Y_s.$$

Theorem

Consider $X^{(i)}$, $i = 1, 2$, two jump processes :

$$X_t^{(i)} = X_0^{(i)} + I_t^{(i)} + R_t^{(i)} + J_t^{(i)} = X_0^{(i)} + \int_0^t \Gamma_s^{(i)} dW_s + \int_0^t \Theta_s^{(i)} ds + J_t^{(i)}.$$

Then

$$[X^{(1)}, X^{(2)}]_T = \int_0^T \Gamma_s^{(1)} \Gamma_s^{(2)} ds + \sum_{0<s\leq T} \Delta J_s^{(1)} \Delta J_s^{(2)}.$$

COROLLARY

Let W be a Brownian motion and $M = N - \lambda.$ a compensated Poisson process, relative to the same filtration. Then $[W, M]_t = 0$ for every $t \geq 0$.

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For Γ and Θ adapted let :

$$X_t^c = X_0 + I_t + R_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds,$$

and f a function of class $C^2(\mathbb{R})$. Then

$$\begin{aligned} f(X_t^c) &= f(X_0) + \int_0^t f'(X_s^c) dX_s^c + \frac{1}{2} \int_0^t f''(X_s^c) d\left[X^c\right]_s \\ &= f(X_0) + \int_0^t f'(X_s^c) \Gamma_s dW_s + \int_0^t f'(X_s^c) \Theta_s ds + \frac{1}{2} \int_0^t f''(X_s^c) \Gamma_s^2 ds \end{aligned}$$

or in differential notation

$$df(X_t^c) = f'(X_s^c) \Gamma_s dW_s + f'(X_s^c) \Theta_s ds + \frac{1}{2} f''(X_s^c) \Gamma_s^2 ds.$$

THEOREM

Let X be a jump process and f of class $C^2(\mathbb{R})$:

$$X_t = X^c_t + J_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds + J_t.$$

Then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX^c_s + \frac{1}{2}\int_0^t f''(X_s)\Gamma^2_s ds + \sum_{0<s\leq t} \left[f(X_s) - f(X_{s-}) \right].$$

PROPOSITION

We consider the geometric Poisson process

$$S_t = S_0 \exp\left(N_t \log(\sigma + 1) - \lambda \sigma t\right) = S_t e^{-\lambda\sigma t} (\sigma + 1)^{N_t},$$

where N is a Poisson process with intensity λ and $\sigma > -1$.
Then S is a martingale :

$$S_t = S_0 + \sigma \int_0^t S(u-) dM_u = S(0) + \sigma \int_0^t S(u-) d(N_u - \lambda u).$$

PROPOSITION

Let W be a Brownian motion and N a Poisson process with intensity $\lambda > 0$, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ relative to the same filtration $\{\mathcal{F}_t,\ t \geq 0\}$. Then W and N are independent.

THÉORÈME

Let $X = (X^{(1)}, \ldots, X^{(d)})$ with $X^{(i)}$, $i = 1, \ldots, d$, jump processes and f of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Then

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_s)d(X^{(i)})^c_s \\ &\quad + \frac{1}{2}\sum_{i=1}^d \sum_{j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s)[(X^{(i)})^c, (X^{(j)})^c]_s ds \\ &\quad + \sum_{0<s\leq t} [f(s, X_s) - f(s, X_{s-})]. \end{aligned}$$

Doléans-Dade exponential.

Proposition

Let X be a jump process. The Doléans-Dade exponential of X is defined by

$$Z^X(t) = \exp\left\{ X^c(t) - \frac{1}{2}[X^c, X^c]_t \right\} \prod_{0<s\leq t} (1 + \Delta X(s)) .$$

This process is solution of the following stochastic differential equation with initial condition $Z^X(0) = 1$:

$$Z^X(t) = 1 + \int_0^t Z^X(s-) dX(s).$$

OUTLINE

Recall.

Setting :

- $X_t = \sigma W_t + \mu t + J_t$ where J compound Poisson process and W Brownian motion ;
- $f \in C^2(\mathbb{R})$.

Formula :

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t f'(X_s) dX_s^c + \frac{\sigma^2}{2} \int_0^t f''(X_s) ds \\
&\quad + \sum_{0<s\leq t} \{ f(X_s) - f(X_{s-}) \} \\
&= f(X_0) + \int_0^t f'(X_s) dX_s + \frac{\sigma^2}{2} \int_0^t f''(X_s) ds \\
&\quad + \sum_{0<s\leq t} \left\{ f(X_s) - f(X_{s-}) - \Delta X_s f'(X_{s-}) \right\}.
\end{aligned}$$

THEOREM

Let X be an n-tuple of semi-martingales, and $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ a $C^{1,2}$ function. Then $f(., X)$ is again a semi-martingale, and the following formula holds :

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \frac{\partial f}{\partial t}(s, X_s) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, X_{s-}) dX_s^i \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_s) d[X^i, X^j]_s^c \\ &+ \sum_{0<s\leq t} \left\{ f(s, X_s) - f(s, X_{s-}) - \sum_{i=1}^d \Delta X_s^i \frac{\partial f}{\partial x_i}(s, X_{s-}) \right\}. \end{aligned}$$

Decomposition.

Proposition

Let X be a Lévy process with characteristic triplet (σ^2, ν, γ) and $f : \mathbb{R} \to \mathbb{R}$ a C^2 function s.t. f and its two derivatives are bounded by a constant C. Then $Y_t = f(X_t) = M_t + V_t$ where M is the martingale part given by :

$$M_t = f(X_0) + \int_0^t f'(X_s)\sigma dW_s + \textit{pure-jump martingale},$$

and V a continuous finite variation process :

$$\begin{aligned} V_t &= \frac{\sigma^2}{2}\int_0^t f''(X_s)ds + \gamma \int_0^t f'(X_s)ds \\ &\quad + \int_0^t \int_{\mathbb{R}} (f(X_{s^-} + y) - f(X_{s^-}) - yf'(X_s)\mathbf{1}_{|y|\leq 1})ds\nu(dy). \end{aligned}$$

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PROPOSITION

Let X be a (σ^2, ν, γ) Lévy process s.t. $\int_{|y|\geq 1} e^y \nu(dy) < \infty$. Then $Y_t = \exp(X_t)$ is a semi-martingale with decomposition $Y_t = M_t + A_t$ where the martingale part is given by

$$M_t = 1 + \int_0^t Y_{s^-} \sigma dW_s + \text{ pure-jump martingale };$$

and the continuous finite variation drift part by

$$A_t = \int_0^t Y_{s^-} \left[\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z\mathbf{1}_{|z|\geq 1}) \nu(dz) \right] ds.$$

Proposition

Let X be a (σ^2, ν, γ) Lévy process. There exists a unique RCLL process Z s.t. :

$$dZ_t = Z_{t-} dX_t, \quad Z_0 = 1.$$

Z is given by :

$$Z_t = \exp\left(X_t - \frac{\sigma^2 t}{2} \right) \prod_{0<s\leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

If $\int_{-1}^{1} |x| \nu(dx) < \infty$, the jumps of X have finite variation and the stochastic exponential of X can be expressed as

$$Z_t = \exp\left(X_t^c - \frac{\sigma^2 t}{2} \right) \prod_{0<s\leq t} (1 + \Delta X_s).$$

Doléans-Dade exponential.

Definition

$Z = \mathcal{E}(X)$ is called the ***Doléans-Dade exponential*** *(or stochastic exponential) of X.*

Proposition

If X is a Lévy process and a martingale, then its stochastic exponential $Z = \mathcal{E}(X)$ is also a martingale.

RELATION BETWEEN THE TWO EXPONENTIALS.

Let X be a Lévy process with triplet (σ^2, ν, γ) and $Z = \mathcal{E}(X)$ its stochastic exponential. If $Z > 0$ a.s., there exists another Lévy process L s.t. $Z = \exp(L)$ where :

$$L_t = \ln Z_t = X_t - \frac{\sigma^2 t}{2} + \sum_{0<s\leq t} (\ln(1 + \Delta X_s) - \Delta X_s).$$

Its characteristic triplet $(\sigma_L^2, \nu_L, \gamma_L)$ is given by :

$$\begin{aligned}
\sigma_L &= \sigma, \\
\nu_L(A) &= \int \mathbf{1}_A(\ln(1+x))\nu(dx), \\
\gamma_L &= \gamma - \frac{\sigma^2}{2} + \int \left[\ln(1+x)\mathbf{1}_{[-1,1]}(\ln(1+x)) - x\mathbf{1}_{[-1,1]}(x)\right] \nu(dx).
\end{aligned}$$

Relation between the two exponentials.

Let L be a Lévy process with triple $(\sigma_L^2, \nu_L, \gamma_L)$ and $S_t = \exp L_t$ its exponential. Then there exists a Lévy process X s.t. S is the stochastic exponential of X : $S = \mathcal{E}(X)$ where

$$X_t = L_t + \frac{\sigma^2 t}{2} + \sum_{0<s\leq t} \left[1 + \Delta L_s - e^{\Delta L_s} \right].$$

The triplet (σ^2, ν, γ) of X is given by :

$$\begin{aligned}
\sigma &= \sigma_L, \\
\nu(A) &= \int \mathbf{1}_A(e^x - 1)\nu_L(dx), \\
\gamma &= \gamma_L + \frac{\sigma_L^2}{2} + \int \left[(e^x - 1)\mathbf{1}_{[-1,1]}(e^x - 1) - x\mathbf{1}_{[-1,1]}(x) \right] \nu_L(dx).
\end{aligned}$$