FINANCE: EXP-LÉVY MODELS.

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OUTLINE

1. EQUIVALENCE OF MEASURES FOR LÉVY PROCESSES

2. EUROPEAN OPTIONS IN EXP-LÉVY MODELS
   - European options
   - PIDE method
   - A Fourier transform method due to Carr & Madan

1. Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be Brownian motions on $(\Omega, \mathcal{F}_T)$ with volatilities $\sigma^{\mathbb{P}} > 0$ and $\sigma^{\mathbb{Q}} > 0$ and drifts $\mu^{\mathbb{P}}$ and $\mu^{\mathbb{Q}}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\sigma^{\mathbb{P}} = \sigma^{\mathbb{Q}}$. In this case the density is

$$\exp\left[\frac{\mu^{\mathbb{Q}} - \mu^{\mathbb{P}}}{\sigma^2} X_T - \frac{1}{2}\frac{(\mu^{\mathbb{Q}})^2 - (\mu^{\mathbb{P}})^2}{\sigma^2} T\right].$$

2. Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be compound Poisson processes on $(\Omega, \mathcal{F}_T)$ with Lévy measures $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$. $\mathbb{P}$ and $\mathbb{Q}$ are equivalent if and only if $\nu^{\mathbb{P}}$ and $\nu^{\mathbb{Q}}$ are equivalent. In this case the density is

$$\exp\left[(\lambda^{\mathbb{P}} - \lambda^{\mathbb{Q}})T + \sum_{0<s\leq T} \phi(\Delta X_s)\right],$$

where $\lambda^{\mathbb{P}} = \nu^{\mathbb{P}}(\mathbb{R})$, $\lambda^{\mathbb{Q}} = \nu^{\mathbb{Q}}(\mathbb{R})$ and $\phi = \ln \frac{d\nu^{\mathbb{Q}}}{\nu^{\mathbb{P}}}$.

GENERAL CASE.

Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be two Lévy processes on $\mathbb{R}^d$ with characteristic triplets (A, ν, γ) and (A', ν', γ').

THEOREM

$\mathbb{P}|_{\mathcal{F}_t}$ and $\mathbb{Q}|_{\mathcal{F}_t}$ are equivalent for all t (or equivalently for one $t > 0$) if and only if the following conditions are satisfied :

1. $A = A'$.
2. The Lévy measures are equivalent with

$$\int_{\mathbb{R}^d} (\exp(\phi(x)/2) - 1)^2 \, \nu(dx) < \infty$$

   where $\phi(x) = \ln \left(\frac{d\nu'}{d\nu} \right)$.

3. $\gamma' - \gamma - \int_{|x| \leq 1} x(\nu' - \nu)(dx) = A\eta$ for some $\eta \in \mathbb{R}^d$.

Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be two Lévy processes on $\mathbb{R}^d$ with characteristic triplets (A, ν, γ) and (A', ν', γ').

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then $\dfrac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t}$ with

$$\begin{aligned} U_t &= \langle \eta, X_t^c \rangle - \frac{t}{2} \langle \eta, A\eta \rangle - t \langle \eta, \gamma \rangle \\ &\quad + \lim_{\varepsilon \downarrow 0} \left(\sum_{0 < s \leq t, |\Delta X_s| > \varepsilon} \phi(\Delta X_s) - t \int_{|x| > \varepsilon} (e^{\phi(x)} - 1) \nu(dx) \right). \end{aligned}$$

Here X^c is the continuous part of X and η is s.t.

$$\gamma' - \gamma - \int_{|x| \leq 1} x(\nu' - \nu)(dx) = A\eta$$

if $A \neq 0$ and zero if $A = 0$.

Let $(X, \mathbb{P})$ and $(X, \mathbb{Q})$ be two Lévy processes on $\mathbb{R}^d$ with characteristic triplets (A, ν, γ) and (A', ν', γ').

When $\mathbb{P}$ and $\mathbb{Q}$ are equivalent, then $\dfrac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t}$ and under $\mathbb{P}$, U is a Lévy process on $\mathbb{R}$ with triplet (A_U, ν_U, γ_U) given by :

$$
\begin{aligned}
A_U &= \langle \eta, A\eta \rangle, \\
\nu_U &= \nu \phi^{-1}|_{\mathbb{R}\setminus\{0\}}, \\
\gamma_U &= -\frac{1}{2}\langle \eta, A\eta \rangle - \int_{\mathbb{R}} (e^y - 1 - y\mathbf{1}_{|y|\leq 1})(\nu\phi^{-1})(dy).
\end{aligned}
$$

The Esscher transform.

Let

- X be a Lévy process with triplet $(0, \nu, \gamma)$,
- θ a real number s.t. $\displaystyle\int_{|x|\geq 1} e^{\theta x}\nu(dx) < \infty$,
- $\phi(x) = \theta x$,
- $\tilde{\nu}(dx) = e^{\theta x}\nu(dx)$.

Method (Esscher transform)

Then we obtain an equivalent probability $\mathbb{Q}$ under which X is a Lévy process with zero Gaussian component, Lévy measure $\tilde{\nu}$ and drift

$$\tilde{\gamma} = \gamma + \int_{-1}^{1} x(e^{\theta x} - 1)\nu(dx).$$

The derivative is given by $\displaystyle\frac{d\mathbb{Q}|_{\mathcal{F}_t}}{d\mathbb{P}|_{\mathcal{F}_t}} = \frac{e^{\theta X_t}}{\mathbb{E}(e^{\theta X_t})} = \exp\left(\theta X_t + f(\theta)t\right)$*, with* $f(\theta) = -\ln \mathbb{E}\exp(\theta X_1)$*.*

CONSEQUENCE :

PROPOSITION

Let $(X, \mathbb{P})$ be a Lévy process. If the trajectories of X are neither almost surely increasing nor almost surely decreasing, then the exp-Lévy model given by $S_t = e^{rt+X_t}$ is arbitrage-free : there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ s.t. $(e^{-rt}S_t)_{t\in[0,T]}$ is a $\mathbb{Q}$-martingale, where r is the interest rate.

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Risk-neutral dynamics of an asset price :

$$S_t = S_0 \exp(rt + X_t)$$

where X is a Lévy process with triplet (σ^2, ν, γ) s.t.

- $\displaystyle\int_{|x|\geq 1} e^x \nu(dx) < \infty$,
- $\displaystyle\gamma + \frac{\sigma^2}{2} + \int (e^y - 1 - y\mathbf{1}_{|y|\leq 1})\nu(dy) = 0.$

X is a Lévy process s.t. $\mathbb{E}(e^{X_t}) = 1$ for all t.

Outline

CALL OPTION. Let $C_t(T,K)$ be the price at time t of the Call with strike K :

$$C_t(T,K) = e^{-r(T-t)}\mathbb{E}\left[(S_T-K)^+\middle|\mathcal{F}_t\right] = C(t,S_t;T,K).$$

Recall the call-put parity relation :

$$C_t(T,K) - P_t(T,K) = S_t - e^{-r(T-t)}K.$$

With $\tau = T - t$

$$\begin{aligned} C(t,y;T,K) &= e^{-r\tau}\mathbb{E}[(S_T-K)^+|\mathcal{F}_t] \\ &= e^{-r\tau}\mathbb{E}[(ye^{r\tau+X_\tau}-K)^+] = Ke^{-r\tau}\mathbb{E}(e^{x+X_\tau}-1)^+, \end{aligned}$$

COMPUTATIONS.

MAIN PROBLEM : compute the price given by

$$C(t, y; T, K) = e^{-r\tau}\mathbb{E}[(ye^{r\tau + X_\tau} - K)^+].$$

More generally for $H(S_T)$ the payoff of a financial derivative, the price at time t :

$$\boxed{\Pi_t(T, K) = e^{-r(T-t)}\mathbb{E}\left[H(S_T)\middle|\mathcal{F}_t\right] = e^{-r\tau}\mathbb{E}[H(S_t e^{r\tau + X_\tau})] = \Pi(\tau, S_t)}$$

with $\tau = T - t$ and $\Pi(\tau, y) = e^{-r\tau}\mathbb{E}[H(ye^{r\tau + X_\tau})]$.

SEVERAL METHODS :

- Monte Carlo simulations. Works always, in any dimension, but is quite slow. If X cannot be exactly simulated, use an approximation and control the error.
- Numerical scheme for PIDE.
- Fourier transform.

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VALUE $C_t = c(t, S(t))$

$$c(t, y) = \mathbb{E}[e^{-r(T-t)} H(S_T) | S_t = y].$$

PROPOSITION

If

- *the payoff $H(S_T)$ satisfies :*

$$|H(y) - H(x)| \leq K|x - y|.$$

- *either* $\sigma > 0$ *or :* $\exists \beta \in (0, 2), \liminf_{\varepsilon \downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 \nu(dx) > 0,$

then the value of a European call with terminal payoff $H(S_T)$ is given by $c : [0, T] \times (0, \infty) \to \mathbb{R}$:

1. $c \in C([0, T] \times [0, \infty)) \cap C^{1,2}((0, T) \times (0, \infty))$ *;*
2. $\forall y > 0, c(T, y) = H(y)$ *;*
3.

Pricing of an European option.

VALUE $C_t = c(t, S(t))$

$$c(t,y) = \mathbb{E}[e^{-r(T-t)} H(S_T) | S_t = y].$$

Proposition

then the value of a European call with terminal payoff $H(S_T)$ is given by $c : [0,T] \times (0,\infty) \to \mathbb{R}$:

1.
2.
3. *c satisfies the equation on $(0,T) \times (0,\infty)$.*

$$\frac{\partial c}{\partial t}(t,y) + ry\frac{\partial c}{\partial y}(t,y) + \frac{\sigma^2 y^2}{2}\frac{\partial^2 c}{\partial y^2}(t,y) - rc(t,y) \tag{1}$$
$$+ \int_{\mathbb{R}} \left[c(t,ye^z) - c(t,y) - y(e^z-1)\frac{\partial c}{\partial y}(t,y) \right] \nu(dz) = 0.$$

To solve numerically the PIDE, a finite difference method can be used.

4 MAIN STEPS :

1. Localise the original space domain (as for the Black-Scholes model).
2. Localise the integration domain $\mathbb{R}$ of $\mathbb{L}_X$.
   - Since $S_t \in L^2$, the localization error decays exponentially with respect to the truncation bound.
3. Approximate the small jumps by a Brownian motion.
4. Compute the solution at discrete grid points and replace the derivatives by finite differences.
   - MAIN PROBLEM : the finite difference methods on the derivatives induces a sparse matrix whereas the integral part induces a densely populated matrix.

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Recall.

For a function f

- Fourier transform : $\mathcal{F}f(v) = \int_{-\infty}^{\infty} e^{ixv} f(x) dx$.
- Inverse Fourier transform : $\mathcal{F}^{-1}f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixv} f(v) dx$.
- For $f \in L^2(\mathbb{R})$, $\mathcal{F}^{-1}\mathcal{F}f = f$.

Method of Carr and Madan.

Assumptions :

- $S_0 = 1$,
- for some $\alpha > 0$, $\mathbb{E}(S_T^{1+\alpha}) < \infty \iff \int_{|y|\geq 1} e^{(1+\alpha)y}\nu(dy) < \infty$.

Aim : compute $C(k) = e^{-rT}\mathbb{E}((e^{rT+X_T} - e^k)^+)$ via

- express its Fourier transform in strike,
- find the prices for a range of strikes by Fourier inversion.

Problem : $C(k)$ is not integrable ! Put

$$z_T(k) = e^{-rT}\mathbb{E}((e^{rT+X_T} - e^k)^+) - (1 - e^{k-rT})^+.$$

Proposition

$$\zeta_T(v) = \mathcal{F}z_T(v) = e^{ivrT}\frac{\Phi_T(v-i) - 1}{iv(1+iv)}$$

where Φ_T is the characteristic function of X_T.

Method of Carr and Madan.

Now $z_T(k) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ivk} \zeta_T(v) dv$.

Remarks :

- $\zeta_t(v) \sim |v|^{-2}$ at infinity.
- Truncation error in the numerical evaluation of $z_T(k)$ will be large.

Improvement :

$$\tilde{z}_T(k) = e^{-rT} \mathbb{E}((e^{rT+X_T} - e^k)^+) - C^{\sigma}_{BS}(k)$$

$$\tilde{\zeta}_T(v) = \mathcal{F}\tilde{z}_T(k) = e^{ivrT} \frac{\Phi_T(v-i) - \Phi^{\sigma}_T(v-i)}{iv(1+iv)}$$

where $\Phi^{\sigma}_T(v) = \exp(-\frac{\sigma^2 T}{2}(v^2 + iv))$.

Remarks :

- $|v|^{\beta} \tilde{\zeta}_t(v) \to 0$ for any β. Inverse Fourier transform : converges very fast.
- Depends on the choice of σ.

In both cases need to compute the inverse Fourier transform. But

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixv} f(v)dx \approx \frac{1}{2\pi}\int_{-R}^{R} e^{-ixv} f(v)dv$$
$$\approx \frac{1}{2\pi}\sum_{j=0}^{N-1} \omega_j f(v_j) e^{-ixv_j}$$

with discretisation step $\delta v = 2R/(N-1)$, $v_j = -R + j\delta v$ and suitable weights ω_j.

- Therefore we compute a discrete Fourier transform, which needs a priori $\mathcal{O}(N^2)$ operations.
- Using the so-called Fast Fourier Transform this computational cost can be reduced to $\mathcal{O}(N \log N)$.