## CONTINUOUS TIME MODELS WITH CONTINUOUS TRAJECTORIES.

- **Black-Scholes model.** Scale invariance of the Brownian motion.
- **Local volatility models.** Possible perfect hedging.
- **Stochastic volatility models.** Difficulty to obtain heavy tails, no large sudden moves.

## CONTINUOUS TIME MODELS WITH DISCONTINUOUS TRAJECTORIES.

- **Market crash.**
- **Credit risk.**
- **High-frequency trading.** Aït-Sahalia & Jacod, High-frequency Financial Econometrics.
- **Insurance and ruin theory.**

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# Outline of the whole lectures.

- **Part 1 : description of the Lévy processes.**
  - Properties, examples.
  - Simulation.
- **Part 2 : stochastic calculus with jumps.**
  - Stochastic integral.
  - Itô's formula.
  - Change of measures.
- **Part 3 : applications to finance.**

# Lévy processes : definitions and first examples.

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**DEFINITION**

A stochastic process $(X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is a Lévy process if

1. $X_0 = 0$ a.s.
2. its increments are independent : for every increasing sequence $t_0, \ldots, t_n$, the r.v. $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent ;
3. its increments are stationnary : the law of $X_{t+h} - X_t$ does not depend on $t$ ;
4. $X$ satisfies the property called stochastic continuity : for any $\varepsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$
5. there exists a subset $\Omega_0$ s.t. $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, $t \mapsto X_t(\omega)$ is RCLL.

If a filtration $(\mathcal{F}_t)_{t\geq 0}$ is already given on $(\Omega, \mathcal{F}, \mathbb{P})$

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3. its increments are stationnary ;
4. $X$ satisfies the property called stochastic continuity ;
5. a.s. $t \mapsto X_t(\omega)$ is RCLL .

# Remarks.

### Remarks on the definitions :

- If $\mathcal{F}_t = \mathcal{F}_t^X$, the two definitions are equivalent.
- If $\{\mathcal{F}_t\}$ is a larger filtration than ($\mathcal{F}_t^X \subset \mathcal{F}_t$) and if $X_t - X_s$ is independent of $\mathcal{F}_s$, then $\{X_t;\ 0 \leq t < +\infty\}$ is a Lévy process under the large filtration.

### Remarks on the hypotheses :

- If we remove Assumption 5, we speak about *Lévy process in law*.
- If we remove Assumption 3, we obtain an *additive process*.
- Dropping Assumptions 3 and 5, we have an *additive process in law*.

**Theorem**

A Lévy process (or an additive process) in law has a RCLL modification.

We can also prove that 2, 3 and 5 imply 4.

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**THEOREM**

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We can also prove that 2, 3 and 5 imply 4.

# ENLARGED FILTRATION.

For a process $X = \{X_t;\ t \geq 0\}$, we define

- $\mathcal{N}_\infty = \mathcal{N}$ the set of $\mathbb{P}$-negligible events.
- For any $0 \leq t \leq \infty$, augmented filtration : $\mathcal{F}_t = \sigma(\mathcal{F}_t^X \cup \mathcal{N})$.

**THEOREM**

Let $X = \{X_t;\ t \geq 0\}$ be a Lévy process. Then

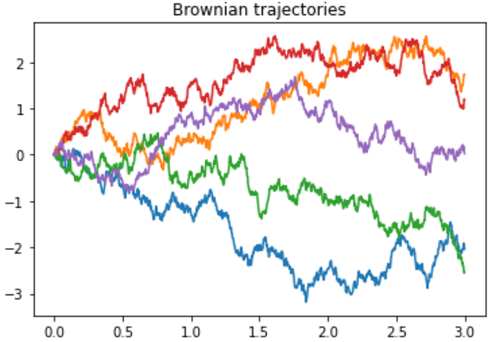
- the augmented filtration $\{\mathcal{F}_t\}$ is right-continuous.
- With respect to the enlarged filtration, $\{X_t,\ t \geq 0\}$ is still a Lévy process.

A Brownian motion is a Lévy process satisfying

1. for every $t > 0$, $X_t$ is Gaussian with mean vector zero and covariance matrix $t \, \mathrm{Id}$ ;
2. the process $X$ has continuous sample paths a.s.

Characteristic function :

$$\mathbb{E}(e^{i\langle z, B_t\rangle}) = \exp(-t|z|^2/2).$$



**PROPOSITION**

*Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Then there exists a function $\psi : \mathbb{R}^d \to \mathbb{R}$ called characteristic exponent of $X$ s.t. :*

$$\boxed{\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z,X_t\rangle}\right) = e^{t\psi(z)}.}$$

# OUTLINE

1. A FIRST CLASS OF LÉVY PROCESSES
   - Compound Poisson process
   - Jump-diffusion processes

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## DEFINITION

*A stochastic process $(X_t)_{t\geq 0}$, with values in $\mathbb{R}$, is a Poisson process with intensity $\lambda > 0$ if it is a Lévy process s.t. for every $t > 0$, $X_t$ has a Poisson law with parameter $\lambda t$.*

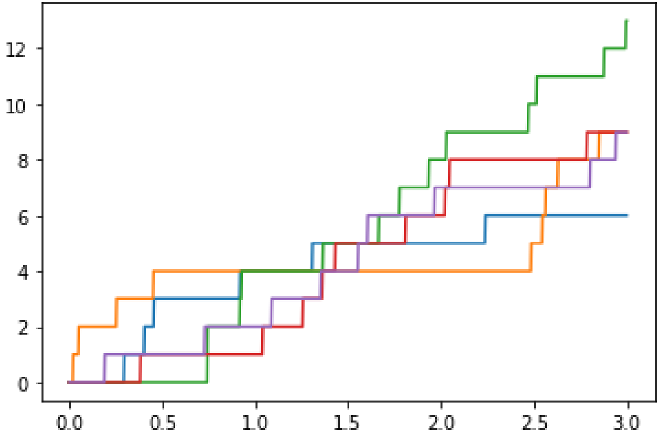
## PROPOSITION (CONSTRUCTION)

*If $(T_n)_{n\in\mathbb{N}}$ is a random walk on $\mathbb{R}$ s.t. for every $n \geq 1$, $T_n - T_{n-1}$ is exponentially distributed with parameter $\lambda$ (with $T_0 = 0$), then the process $(X_t)_{t\geq 0}$ defined by*

$$X_t = n \Longleftrightarrow T_n \leq t < T_{n+1}$$

*is a Poisson process with intensity $\lambda$.*

# POISSON PROCESS.



We consider a Poisson process $(P_t)_{t\geq 0}$ with intensity $\lambda$ and jump times $T_n$, and a sequence $(Y_n)_{n\in\mathbb{N}^*}$ of $\mathbb{R}^d$-valued r.v. s.t.

1. $Y_n$ are i.i.d. with distribution measure $\pi$ ;
2. $(P_t)_{t\geq 0}$ and $(Y_n)_{n\in\mathbb{N}^*}$ are independent.

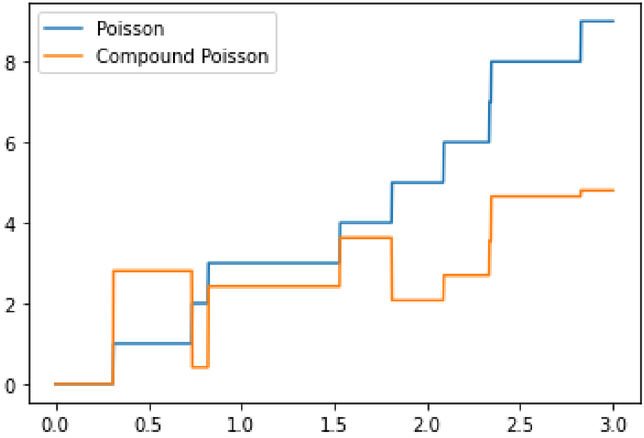
Define

$$X_t = \sum_{n=1}^{P_t} Y_n = \sum_{n=1}^{+\infty} Y_n \mathbf{1}_{[0,t]}(T_n).$$

**DEFINITION**

*The process $(X_t)_{t\geq 0}$ is a compound Poisson processes with intensity $\lambda$ and jump distribution $\pi$.*

Compound Poisson processes with Gaussian jumps.



# Compound Poisson processes.

## Proposition

*The process $(X_t)_{t\geq 0}$ is a Lévy process, with piecewise constant trajectories and characteristic function :*

$$\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = \exp\left(t\lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx)\right) = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right);$$

## Definition

*$\nu$ is a finite measure defined on $\mathbb{R}^d$ by : $\nu(A) = \lambda\pi(A)$, $A \in \mathcal{B}(\mathbb{R}^d)$. $\nu$ is called the Lévy measure of the compound Poisson process. Moreover*

$$\nu(A) = \mathbb{E}\left[\#\{t \in [0,1], \quad \Delta X_t \neq 0, \ \Delta X_t \in A\}\right].$$

# COMPOUND POISSON PROCESSES.

### DEFINITION

*The law $\mu$ of $X_1$ is called compound Poisson distribution and has a characteristic function given by :* $\hat{\mu}(z) = \exp\left(\lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx)\right)$.

### PROPOSITION

*Let $X$ be a compound Poisson process and $A$ and $B$ two disjointed subsets of $\mathbb{R}^d$. Then :*

$$Y_t = \sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in A} \quad \text{and} \quad Z_t = \sum_{s\leq t} \Delta X_s \mathbf{1}_{\Delta X_s \in B}$$

*are two independent compound Poisson processes.*

# OUTLINE

1. A FIRST CLASS OF LÉVY PROCESSES
   - Compound Poisson process
   - Jump-diffusion processes

**DEFINITION**

*A* *jump-diffusion process* *$X$ is the sum of a Brownian motion and of a independent compound Poisson process. Therefore a jump-diffusion process is a Lévy process.*

In other words

- a $k$-dimensional Brownian motion $(W_t)_{t\geq 0}$, a $d \times k$ matrix $M$,
- a $d$-dimensional vector $\gamma$,
- a Poisson process $(P_t)_{t\geq 0}$ with intensity $\lambda$ and jump times $T_n$, and a sequence $(Y_n)_{n\in\mathbb{N}^*}$ of $\mathbb{R}^d$-valued r.v.

such that

1. $Y_n$ are i.i.d. with distribution measure $\pi$ ;
2. $(W_t)_{t\geq 0}$, $(P_t)_{t\geq 0}$ and $(Y_n)_{n\in\mathbb{N}^*}$ are independent.

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$$X_t = MW_t + \gamma t + \sum_{n=1}^{P_t} Y_n = MW_t + \gamma t + \sum_{n=1}^{+\infty} Y_n \mathbf{1}_{[0,t]}(T_n).$$

# DEFINITION.

$$X_t = MW_t + \gamma t + \sum_{0<s\leq t} \Delta X_s.$$

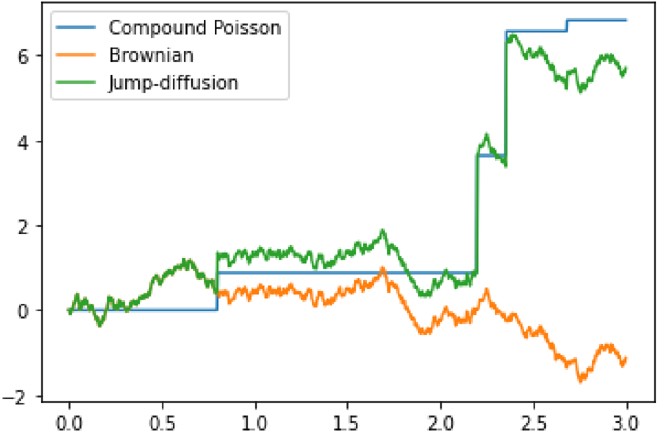
CHARACTERISTIC EXPONENT : for any $z \in \mathbb{R}^d$ :

$$\begin{aligned} \psi_X(z) &= -\frac{1}{2}\langle z, MM^* z\rangle + i\langle z, \gamma\rangle + \lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx) \\ &= -\frac{1}{2}\langle z, MM^* z\rangle + i\langle z, \gamma\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx). \end{aligned}$$

$M^*$ is the transpose matrix of $M$.

CHARACTERISTIC TRIPLE : $(A = MM^*, \nu, \gamma)$.

Jump-diffusion process with Gaussian jumps (used in the Merton model).



Kou model where the jump sizes are given by a non symmetric Laplace distribution.