SIMULATION OF THE LÉVY PROCESSES.

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OUTLINE

1. FOR A JUMP-DIFFUSION PROCESS

2. GENERAL LÉVY PROCESS
   - Approximation by compound Poisson processes
   - Approximation by Brownian motion
   - Exact simulation on a grid

Decomposition.

Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (A, ν, γ).

$D(a,b] = \{x \in \mathbb{R}^d,\ a < |x| \leq b\}$ and $D(a,+\infty) = \{x \in \mathbb{R}^d,\ |x| > a\}$.

Theorem

1. There exists Ω_1 s.t. $\mathbb{P}(\Omega_1) = 1$ and s.t. for any $\omega \in \Omega_1$,

$$\begin{aligned} X^1_t(\omega) &= \lim_{\varepsilon \downarrow 0} \sum_{0<s\leq t} \left[\Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1} - \mathbb{E}(\Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1}) \right] \\ &+ \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} \end{aligned}$$

   is defined for every $t \in \mathbb{R}_+$ with uniform time convergence in time on any compact set.
   The process X^1 is a Lévy process with triple $(0, \nu, 0)$.

Remark on X^1.

$$\begin{aligned} X_t^1(\omega) &= \lim_{\varepsilon \downarrow 0} X_t^\varepsilon + \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} \\ &= \lim_{\varepsilon \downarrow 0} X_t^\varepsilon + Y_t, \end{aligned}$$

with

- X^ε is a compensated compound Poisson process

$$X_t^\varepsilon = \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1} - t \int_{D(\varepsilon,1]} x \nu(dx), \quad \mathbb{E}(X_t^\varepsilon) = 0;$$

- Y is a compound Poisson process with jumps size greater than 1.

DECOMPOSITION.

Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (A, ν, γ).

THEOREM

1. The process X^1 is a Lévy process with triple $(0, \nu, 0)$.
2. Denoting $X_t^2 = X_t - X_t^1$, there exists a set Ω_2 s.t. $\mathbb{P}(\Omega_2) = 1$ and s.t. for any $\omega \in \Omega_2$, X^2 is a continuous Lévy process with characteristic triple $(A, 0, \gamma)$.
3. X^2 is a Brownian motion with covariance matrix A and drift γ.
4. The processes X^1 and X^2 are independent.

DEFINITION

X^1 is the jump part and X^2 the continuous part of X :

$$X_t^2 = MW_t + \gamma t.$$

TWO DIFFERENT CLASSES.

1. JUMP-DIFFUSION MODELS ($\nu(\mathbb{R}^d) < +\infty$).
   - Prices : diffusion process, with jumps at randon times.
   - Jumps : rare events $\longrightarrow$ cracks or large losses.

   1. Advantages :
      - price structure : easy to understand, to describe and to simulate ;
      - then efficient Monte Carlo methods to compute path-depend prices.
      - Very performant to interpolate the implicit volatility smiles.
   2. Inconvenients :
      - unknown closed formula for the densities,
      - statistical estimation or moments/quantiles computations : difficult to realize.
2. INFINITE ACTIVITY MODELS (general Lévy processes).

TWO DIFFERENT CLASSES.

1. JUMP-DIFFUSION MODELS ($\nu(\mathbb{R}^d) < +\infty$).
2. INFINITE ACTIVITY MODELS (general Lévy processes).
   - Models with a infinite number of jumps during any time period.
   - Unnecessary Brownian component.

   1. Advantages :
      - give a more realistic description of the prices at different time scales.
      - often obtained as subordinator of a Brownian motion (time change),
      - hence closed formulas or more tractable than for the jump-diffusion models.
   2. Inconvenients :
      - often more complicated to simulate.
      - Price structure less intuitive.

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BROWNIAN MOTION WITH DRIFT.

$$\forall t \in \mathbb{R}_+, \; X_t = bt + \sigma W_t$$

with :

- γ and σ two constants,
- W Brownian.

SIMULATION :

1. simulate n standard Gaussian r.v. Z_i,
2. define $\Delta X_i = \sigma \sqrt{t_i - t_{i-1}} Z_i + b(t_i - t_{i-1})$,
3. put $X(t_i) = \sum_{k=1}^{i} \Delta X_i$.

Jump distribution for a Poisson process.

For any interval I, we denote by $N(I)$ the number of jumps of X_t, $t \in I$.

Proposition

For $0 < s < t$ and $n \geq 1$, the conditional law of X_s knowing $X_t = n$ is binomial with parameters n and s/t.
For $0 = t_0 < t_1 < \ldots < t_k = t$ and $I_j =]t_{j-1}, t_j]$, the conditional law of $(N(I_1), \ldots, N(I_k))$ knowing $X_t = n$ is multinomial with parameters n, $(t_1 - t_0)/t, \ldots, (t_k - t_{k-1})/t$.

T_i : jump times of the process.

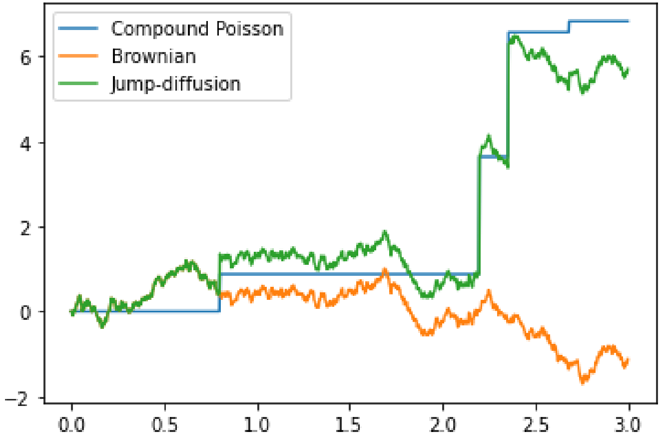
Proposition

Let $n \geq 1$ and $t > 0$. The conditional law of $T_1, \ldots, T_n$ knowing $X_t = n$ coincides with the law of the order statistics $U_{(1)}, \ldots, u_{(n)}$ of n independent variables, uniformly distributed on $[0, t]$.

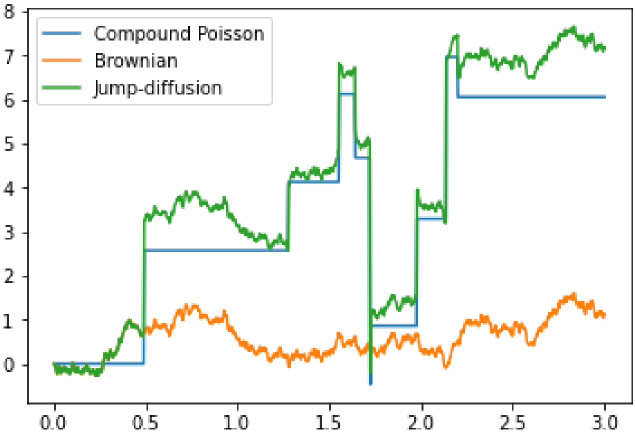
SIMULATION FOR THE COMPOUND POISSON PROCESS.

1. simulate a Poisson r.v. N with parameter λT,
2. simulate N independent r.v. U_i with uniform law on $[0, T]$,
3. simulate the jumps : N independent r.v. V_i with distribution $\nu(dx)/\lambda$,
4. put $Y_t = \sum_{i=1}^{N} \mathbf{1}_{U_i < t} V_i$.

Jump distribution = Gaussian (Merton's model).



Jump distribution = (non symmetric) Laplace (Kou's model).



OUTLINE

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APPROXIMATION BY COMPOUND POISSON.

If $X = (X_t)_{t \geq 0}$ with infinite activity and triple $(0, \nu, \gamma)$, then

$$X_t = \gamma t + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} + \lim_{\varepsilon \downarrow 0} X_t^\varepsilon,$$

where

$$X_t^\varepsilon = \sum_{s \leq t} \Delta X_s \mathbf{1}_{\varepsilon \leq |\Delta X_s| < 1} - t \int_{\varepsilon \leq |x| < 1} x \nu(dx).$$

APPROXIMATION by a compound Poisson

$$Z_t^\varepsilon = bt + \sum_{s \leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} + X_t^\varepsilon$$

with $\lim_{\varepsilon \downarrow 0} Z^\varepsilon = X$.

Approximation by compound Poisson.

Residual term : $R_t^\varepsilon = -X_t^\varepsilon + \lim_{\varepsilon \downarrow 0} X_t^\varepsilon$.

- with characteristic triple $(0, \mathbf{1}_{|x| \leq \varepsilon} \nu(dx), 0)$,
- with infinite activity, with bounded jumps, thus with finite variance,
- $\mathbb{E}(R_t^\varepsilon) = 0$,
- $\text{Var } R_t^\varepsilon = t \int_{|x|<\varepsilon} x^2 \nu(dx) = t\sigma^2(\varepsilon)$.

Exemple

Gamma process : $\sigma(\varepsilon) \sim \varepsilon$.

Proposition

If f is a differentiable function s.t. $|f'(x)| \leq C$, then

$$|\mathbb{E}f(X_T) - \mathbb{E}f(Z_T^\varepsilon)| \leq C\sigma(\varepsilon)\sqrt{T}.$$

Tempered stable processes : approximation.

If $\nu(x) = \frac{c}{x^{\alpha+1}} e^{-\lambda x}$, then approximation

$$X_t^\varepsilon = \gamma t + \sum_{s \leq t} \Delta X_s \mathbf{1}_{\Delta X_s \geq \varepsilon} + \mathbb{E}\left(\sum_{s \leq t} \Delta X_s \mathbf{1}_{\Delta X_s < \varepsilon} \right).$$

- Drift : $\gamma^\varepsilon = \gamma + c \int_0^\varepsilon \frac{e^{-\lambda x}}{x^\alpha} dx$,
- Lévy measure : $\nu^\varepsilon(x) = \frac{c}{x^{\alpha+1}} e^{-\lambda x} \mathbf{1}_{x > \varepsilon}$,
- Intensity : $U(\varepsilon) = c \int_\varepsilon^\infty \frac{e^{-\lambda x}}{x^{\alpha+1}} dx$,
- Jump distribution $p^\varepsilon(x) = \frac{\nu^\varepsilon(x)}{U(\varepsilon)}$.

Simulation by rejection $\forall x \in \mathbb{R},\ p^\varepsilon(x) \leq f^\varepsilon(x) \frac{\varepsilon^{-\alpha} e^{-\lambda \varepsilon}}{\alpha U(\varepsilon)}$.

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Approximation by Brownian motion.

New approximation : $\tilde{Z}^\varepsilon_t = Z^\varepsilon_t + \sigma(\varepsilon) W_t$.

Theorem

$\lim_{\varepsilon \to 0} \sigma(\varepsilon)^{-1} R^\varepsilon = W$ in law if and only if for every $k > 0$

$$\lim_{\varepsilon \to 0} \frac{\sigma(k\sigma(\varepsilon) \wedge \varepsilon)}{\sigma(\varepsilon)} = 1.$$

Sufficient condition

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty.$$

Approximation by Brownian motion.

New approximation : $\tilde{Z}_t^\varepsilon = Z_t^\varepsilon + \sigma(\varepsilon) W_t$.
Sufficient condition

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty.$$

Examples

- Processes with Lévy measure $\nu(x) \underset{x=0}{\sim} 1/|x|^{\alpha+1}$, $\sigma(\varepsilon) \sim \varepsilon^{1-\alpha/2}$.
- Compound Poisson processes, $\sigma(\varepsilon) = o(\varepsilon)$,
- Gamma process $\sigma(\varepsilon) \sim \varepsilon$.

Approximation by Brownian motion.

New approximation : $\tilde{Z}_t^\varepsilon = Z_t^\varepsilon + \sigma(\varepsilon) W_t$.

Sufficient condition

$$\lim_{\varepsilon \to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = +\infty.$$

Proposition

If f is a differentiable function s.t. $|f'(x)| \leq C$, then

$$|\mathbb{E} f(X_T) - \mathbb{E} f(Z_T^\varepsilon + \sigma(\varepsilon) W_T)| \leq A \rho(\varepsilon) C \sigma(\varepsilon),$$

with $A < 16,5$ and $\rho(\varepsilon) = \frac{1}{\sigma^3(\varepsilon)} \int_{-\varepsilon}^{\varepsilon} |x|^3 \nu(dx) < \frac{\varepsilon}{\sigma(\varepsilon)}$.

Outline

STABLE PROCESSES.

PROBLEM : time grid $t_1, \ldots, t_n \longrightarrow$ simulate $X(t_1), \ldots, X(t_n)$?

X stable : Lévy measure :

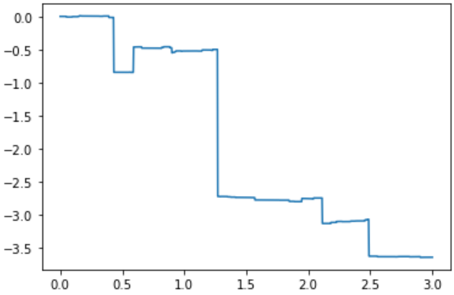
$$\nu(x) = \frac{1}{|x|^{\alpha+1}} \mathbf{1}_{x \neq 0}$$

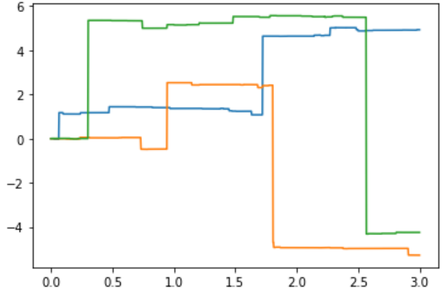
if $0 < \alpha < 2$

ALGORITHM :

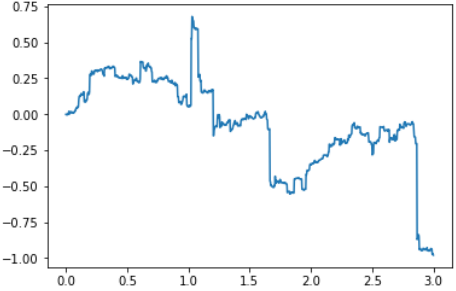
1. simulate n i.i.d. r.v. U_i with uniform law on $[-\pi/2, \pi/2]$ and n i.i.d. r.v. E_i with exponential distribution with parameter 1 ;
2. define
$$\Delta X_i = (t_i - t_{i-1})^{1/\alpha} \frac{\sin(\alpha U_i)}{(\cos U_i)^{1/\alpha}} \left(\frac{\cos((1-\alpha)U_i)}{E_i} \right)^{(1-\alpha)/\alpha}$$
with $t_0 = 0$;
3. put $X_{t_i} = \sum_{k=1}^{i} \Delta X_k$.

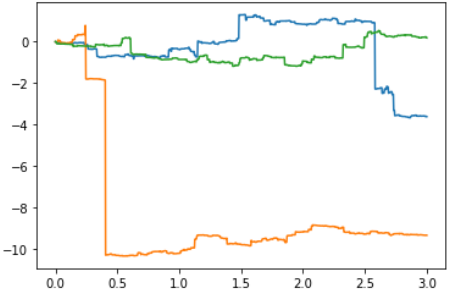
Stable processes.



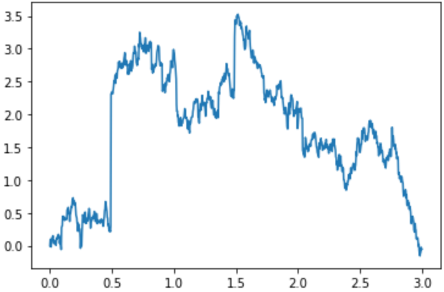


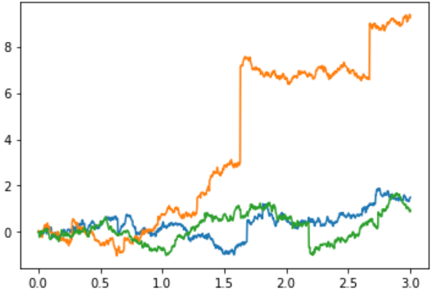
STABLE PROCESSES.





Stable processes.





Exact simulation on a grid.

Problem : time grid $t_1, \dots, t_n \longrightarrow$ simulate $X(t_1), \dots, X(t_n)$?

- $X = \sigma W(S) + \theta S$ is a Brownian motion with volatility σ, drift θ, with the time change induced by the subordinator S.

Algorithm :

1. simulate the increments of subordinator $\Delta S_i = S_{t_i} - S_{t_{i-1}}$ where $S_0 = 0$,
2. simulate n standard Gaussian r.v. $N_1, \dots, N_n$,
3. define $\Delta X_i = \sigma N_i \sqrt{\Delta S_i} + \theta \Delta S_i$,
4. put $X_{t_i} = \sum_{k=1}^{i} \Delta X_k$.

- Subordinator = gamma process with density at time t

$$p_t(x) = \frac{1}{\kappa^{t/\kappa}\Gamma(t/\kappa)} x^{t/\kappa - 1} e^{-x/\kappa} \mathbf{1}_{\mathbb{R}_+^*}(x).$$

- Parameters :
  - σ and θ resp. volatility and drift of the Brownian motion,
  - κ variance of the subordinator.
- Of bounded variation with infinite activity (but relatively weak),
- Lévy measure :

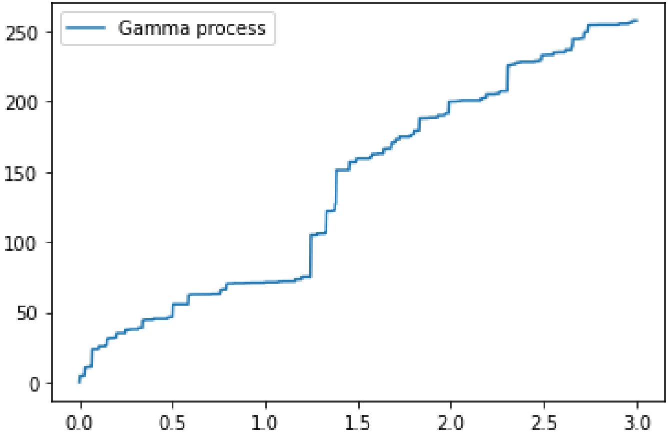
$$\nu(x) = \frac{1}{\kappa|x|} e^{Ax - B|x|}, \quad A = \frac{\theta}{\sigma^2}, \quad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2},$$

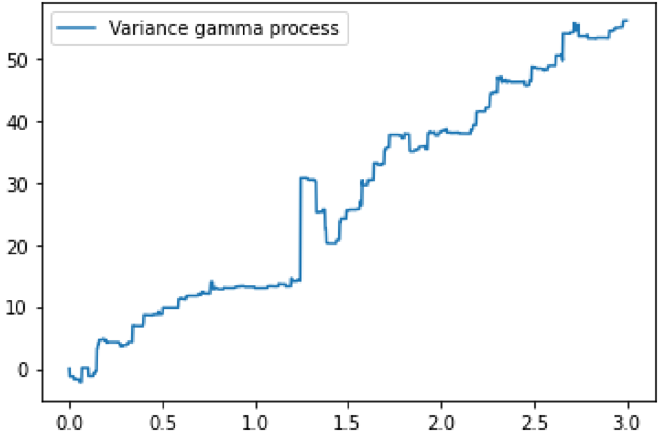
- Characteristic exponent :

$$\psi(u) = -\frac{1}{\kappa} \ln(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u).$$

- $\mathbb{E}(X_t) = \theta t$ and Var $X_t = \sigma^2 t + \theta^2 \kappa t$.

Gamma process.




Variance gamma process
50
40
30
20
10
0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Normal inverse Gaussian process.

- Subordinator = inverse Gaussian process with density

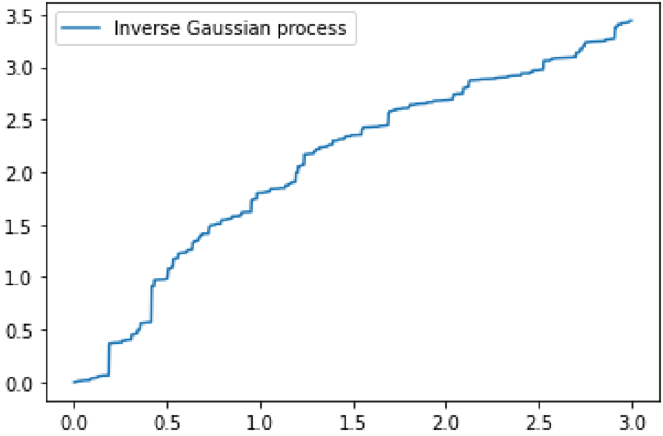
$$p_t(x) = \sqrt{\frac{t^2/\kappa}{2\pi x^3}} \exp\left(-\frac{t^2/\kappa}{2t^2 x}(x-t)^2\right) \mathbf{1}_{\mathbb{R}_+^*}(x).$$

- Parameters :
  - σ and θ resp. volatility and drift of the Brownian motion,
  - κ variance of the subordinator.
- Of unbounded variation with stable behaviour of the small jumps,
- Lévy measure : uses the Bessel functions,
- Characteristic exponent :

$$\psi(u) = \frac{1}{\kappa} - \frac{1}{\kappa}\sqrt{1 + u^2\sigma^2\kappa - 2i\theta\kappa u}.$$

- $\mathbb{E}(X_t) = \theta t$ et Var $X_t = \sigma^2 t + \theta^2 \kappa t$.

INVERSE GAUSSIAN PROCESS.



Normal inverse Gaussian process.