Lévy processes : decomposition and properties.

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Definitions.

Definition

A stochastic process $(X_t)_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{R}^d$, is a Lévy process if

1. $X_0 = 0$ a.s.
2. its increments are independent : for every increasing sequence $t_0, \ldots, t_n$, the r.v. $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent ;
3. its increments are stationnary : the law of $X_{t+h} - X_t$ does not depend on t ;
4. X satisfies the property called stochastic continuity : for any $\varepsilon > 0$, $\lim_{h \to 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$
5. there exists a subset Ω_0 s.t. $\mathbb{P}(\Omega_0) = 1$ and for every $\omega \in \Omega_0$, $t \mapsto X_t(\omega)$ is RCLL.

DEFINITIONS.

If a filtration $(\mathcal{F}_t)_{t\geq 0}$ is already given on $(\Omega, \mathcal{F}, \mathbb{P})$

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2. its increments are independent : for any $s \leq t$, the r.v. $X_t - X_s$ is independent of $\mathcal{F}_s$;
3. its increments are stationnary ;
4. X satisfies the property called stochastic continuity ;
5. a.s. $t \mapsto X_t(\omega)$ is RCLL .

PROPOSITION

Let $(X_t)_{t\geq 0}$ be a Lévy process in $\mathbb{R}^d$. Then there exists a function $\psi : \mathbb{R}^d \to \mathbb{R}$ called characteristic exponent of X s.t. :

$$\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)}.$$

JUMP-DIFFUSION PROCESSES.

$$\boxed{X_t = MW_t + \gamma t + \sum_{0<s\leq t} \Delta X_s.}$$

CHARACTERISTIC EXPONENT : for any $z \in \mathbb{R}^d$:

$$\begin{aligned}
\psi_X(z) &= -\frac{1}{2}\langle z, MM^* z\rangle + i\langle z, \gamma\rangle + \lambda \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\pi(dx) \\
&= -\frac{1}{2}\langle z, MM^* z\rangle + i\langle z, \gamma\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx).
\end{aligned}$$

M^* is the transpose matrix of M.

CHARACTERISTIC TRIPLE : $(A = MM^*, \nu, \gamma)$.

Outline

1. Law and decomposition of a Lévy process
   - Infinitely divisible distributions
   - Decomposition of the process

2. Properties of a Lévy process
   - Sample path properties
   - Moments
   - Densities
   - Markov processes, martingales

OUTLINE

1. LAW AND DECOMPOSITION OF A LÉVY PROCESS
   - Infinitely divisible distributions
   - Decomposition of the process

2. PROPERTIES OF A LÉVY PROCESS
   - Sample path properties
   - Moments
   - Densities
   - Markov processes, martingales

Infinitely divisible distributions.

Let μ be a probability measure on $\mathbb{R}^d$. Denote by

- $\hat{\mu}(z) = \int_{\mathbb{R}^d} e^{izx}\mu(dx)$ the Fourier transform of μ.
- μ^n the convolution product of μ n times μ with herself :

$$\mu^n = \underbrace{\mu * \ldots * \mu}_{n \text{ times}} \Longleftrightarrow \widehat{\mu^n} = (\hat{\mu})^n.$$

Definition

A probability measure on $\mathbb{R}^d$ is infinitely divisible if for every $n \in \mathbb{N}^*$, there exists a probability μ_n s.t. $\mu = \mu_n^n \Longleftrightarrow \widehat{\mu} = (\hat{\mu}_n)^n$.

Examples

Gaussian, Cauchy, Poisson, compound Poisson, exponential, gamma, geometric distributions.

PROPOSITION

If X is a Lévy processs in law, the law of X_t is infinitely divisible.

LEMMA

The convolution product between two infinitely divisible distributions is infinitely divisible.

EXAMPLE

Using a Gaussian and a compound Poisson distributions, and denoting by $\hat{\mu}$ its characteristic function

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle\gamma, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1)\nu(dx)\right].$$

$D = \{x \in \mathbb{R}^d, \ |x| \leq 1\}$.

THEOREM (FIRST PART)

If μ is infinitely divisible on $\mathbb{R}^d$, then

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x))\nu(dx)\right]$$

with

- $A \in \mathcal{S}_d^+(\mathbb{R})$,
- ν measure on $\mathbb{R}^d$ s.t.

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < +\infty, \tag{1}$$

- $\gamma \in \mathbb{R}^d$.

THEOREM (LAST PART)

The representation of $\hat{\mu}$ is unique.

Conversely if $A \in \mathcal{S}_d^+(\mathbb{R})$, if ν is a measure satisfying (1), and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible law μ on $\mathbb{R}^d$ with characteristic function given by the previous formula.

DEFINITION

(A, ν, γ) is called the characteristic triple of μ. A is the Gaussian covariance matrix, ν the Lévy measure.

PROPOSITION

If μ is given by its triple (A, ν, γ), the characteristic triple of μ^t is $(tA, t\nu, t\gamma)$.

Remark on γ (1).

Let $c : \mathbb{R}^d \to \mathbb{R}$ be a bounded function s.t.

$$\begin{cases} c(x) = 1 + o(|x|) & \text{when } |x| \to 0, \\ c(x) = O(1/|x|) & \text{when } |x| \to +\infty. \end{cases} \tag{2}$$

Then

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_c, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle c(x))\nu(dx) \right]$$

with

$$\gamma_c = \gamma + \int_{\mathbb{R}^d} x(c(x) - \mathbf{1}_D(x))\nu(dx).$$

Definition

The triple is denoted by $(A, \nu, \gamma_c)_c$ and the previous formula is also a Lévy-Khintchine decomposition of μ.

Remark on γ (2).

- If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, with $c \equiv 0$,

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_0, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\nu(dx)\right].$$

Definition

γ_0 is the drift of μ.

- If $\int_{|x|\geq 1} |x|\nu(dx) < \infty$, with $c \equiv 1$,

$$\hat{\mu}(z) = \exp\left[-\frac{1}{2}\langle z, Az\rangle + i\langle \gamma_1, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle)\nu(dx)\right].$$

Definition

γ_1 is the center of μ. And $\gamma_1 = \int_{\mathbb{R}^d} x\mu(dx)$.

EXAMPLES.

- $\nu \equiv 0$ if and only if μ is Gaussian.
- If μ compound Poisson, $A = 0$, $\nu = \lambda\pi$ and $\gamma_0 = 0$.
- If $d = 1$ and μ Poisson, $A = 0$, $\nu = \lambda\delta_1$, $\gamma_0 = 0$.
- If μ is the Γ distribution with parameters c and α

$$\text{density : } \frac{\alpha^c}{\Gamma(c)} x^{c-1} e^{-\alpha x} \mathbf{1}_{\mathbb{R}_+^*}(x),$$

then $A = 0$, $\gamma_0 = 0$ and $\nu(dx) = c\dfrac{e^{-\alpha x}}{x}\mathbf{1}_{\mathbb{R}_+^*}(x)$.

- If μ is the stable distribution with index 1/2,

$$\text{density : } \frac{c}{\sqrt{2\pi}} e^{-c^2/(2x)} x^{-3/2} \mathbf{1}_{\mathbb{R}_+^*}(x),$$

then $A = 0$, $\gamma_0 = 0$ and $\nu(dx) = \dfrac{c}{\sqrt{2\pi}} x^{-3/2}\mathbf{1}_{\mathbb{R}_+^*}(x)$.

Distribution of Lévy processes.

Theorem

1. If X is a Lévy process (in law), the law of X_t is given by μ^t where μ is the law of X_1.
2. If μ is infinitely divisible on $\mathbb{R}^d$, then there exists a Lévy process in law s.t. $\mathbb{P}_{X_1} = \mu$.

Proposition

Let $(X_t)_{t\geq 0}$ be a Lévy process in law. Then there exists $\psi : \mathbb{R}^d \to \mathbb{R}$, characteristic exponent of X s.t. :

$$\forall z \in \mathbb{R}^d, \quad \mathbb{E}\left(e^{i\langle z, X_t\rangle}\right) = e^{t\psi(z)},$$

with characteristic triple (A, ν, γ) s.t.

$$\psi(z) = -\frac{1}{2}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbf{1}_D(x))\nu(dx).$$

Classification of the Lévy processes.

Let X be a Lévy process with triple (A, ν, γ).

Definition

X is called of

- *type A if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$,*
- *type B if $A = 0$, $\nu(\mathbb{R}^d) = \infty$ and $\int_{|x| \leq 1} |x| \nu(dx) < +\infty$,*
- *type C if $A \neq 0$ or $\int_{|x| \leq 1} |x| \nu(dx) = +\infty$.*

Outline

1. **Law and decomposition of a Lévy process**
   - Infinitely divisible distributions
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2. Properties of a Lévy process
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DECOMPOSITION.

Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (A, ν, γ).

$D(a, b] = \{x \in \mathbb{R}^d,\ a < |x| \leq b\}$ and $D(a, +\infty) = \{x \in \mathbb{R}^d,\ |x| > a\}$.

THEOREM

1. There exists Ω_1 s.t. $\mathbb{P}(\Omega_1) = 1$ and s.t. for any $\omega \in \Omega_1$,

$$\begin{aligned} X_t^1(\omega) &= \lim_{\varepsilon \downarrow 0} \sum_{0<s\leq t} \left[\Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1} - \mathbb{E}(\Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1})\right] \\ &+ \sum_{0<s\leq t} \Delta X_s \mathbf{1}_{|\Delta X_s| \geq 1} \end{aligned}$$

   is defined for every $t \in \mathbb{R}_+$ with uniform time convergence in time on any compact set.
   The process X^1 is a Lévy process with triple $(0, \nu, 0)$.

Remark on X^1.

$$\begin{aligned} X_t^1(\omega) &= \lim_{\varepsilon \downarrow 0} X_t^\varepsilon + \sum_{0<s\le t} \Delta X_s \mathbf{1}_{|\Delta X_s| \ge 1} \\ &= \lim_{\varepsilon \downarrow 0} X_t^\varepsilon + Y_t, \end{aligned}$$

with

- X^ε is a compensated compound Poisson process

$$X_t^\varepsilon = \sum_{0<s\le t} \Delta X_s \mathbf{1}_{\varepsilon < |\Delta X_s| < 1} - t \int_{D(\varepsilon,1]} x\nu(dx), \quad \mathbb{E}(X_t^\varepsilon) = 0;$$

- Y is a compound Poisson process with jumps size greater than 1.

DECOMPOSITION.

Let $(X_t)_{t\geq 0}$ be a Lévy process with characteristic triple (A, ν, γ).

THEOREM

1. The process X^1 is a Lévy process with triple $(0, \nu, 0)$.
2. Denoting $X_t^2 = X_t - X_t^1$, there exists a set Ω_2 s.t. $\mathbb{P}(\Omega_2) = 1$ and s.t. for any $\omega \in \Omega_2$, X^2 is a continuous Lévy process with characteristic triple $(A, 0, \gamma)$.
3. X^2 is a Brownian motion with covariance matrix A and drift γ.
4. The processes X^1 and X^2 are independent.

DEFINITION

X^1 is the jump part and X^2 the continuous part of X :

$$X_t^2 = MW_t + \gamma t.$$

Outline

Outline

PROPOSITION

A Lévy process is continuous if and only if $\nu = 0$. In that case it is a Brownian motion with drift.

PROPOSITION

A Lévy process is piecewise constant if and only if it is a compound Poisson process or if it is of type A with $\gamma_0 = 0$, i.e.

- $A = 0$ *and* $\int_{\mathbb{R}^d} \nu(dx) < +\infty$,
- $\gamma = \int_{|x|\leq 1} x\nu(dx)$;

or

$$\psi(x) = \int_{\mathbb{R}^d} (e^{iux} - 1)\nu(dx), \quad \text{with } \nu(\mathbb{R}^d) < +\infty.$$

JUMPS OF THE PROCESS.

THEOREM (JUMPS REPARTITION)

If $\nu(\mathbb{R}^d) = +\infty$, then a.s. the jumping times are countable and dense in $\mathbb{R}_+$.

If $0 < \nu(\mathbb{R}^d) < +\infty$, there is an infinite countable jumping times, but only a finite number on any bounded interval. Moreover the first jumping time has an exponential distribution with parameter $\nu(\mathbb{R}^d)$.

THEOREM

A Lévy process is of bounded variation if and only if it is of type A or B.

In this case :

$$X_t = \gamma_0 t + \int_{[0,t]\times\mathbb{R}^d} x J_X(ds \times dx) = \gamma_0 t + \sum_{s\in[0,t]}^{\Delta X_s \neq 0} \Delta X_s.$$

Characteristic function :

$$\mathbb{E}\left(e^{i\langle z.X_t\rangle}\right) = \exp t\left[i\langle\gamma_0, z\rangle + \int_{\mathbb{R}^d}(e^{i\langle x,z\rangle} - 1)\nu(dx)\right].$$

PROPOSITION

A Lévy process is non-decreasing if and only if

- $A = 0$ *and* $\nu(]-\infty, 0]) = 0$,
- $\displaystyle\int_0^1 x\nu(dx) < +\infty$ *with* $\gamma_0 \geq 0$.

In this case use the Laplace transform : for $u \geq 0$

$$\mathbb{E}(e^{-uX_t}) = \exp\left[t\int_0^{+\infty}(e^{-ux} - 1)\nu(dx) - t\gamma_0 u\right].$$

REMARK : if $A = 0$, $\nu(]-\infty, 0]) = 0$ and $\displaystyle\int_0^1 x\nu(dx) = +\infty$, the process has just non-negative jumps, but whatever γ, it is not non-decreasing. It has infinite negative drift !

EXAMPLE.

PROPOSITION

Let $(X_t)_{t\geq 0}$ be a Lévy process on $\mathbb{R}^d$ and let $f : \mathbb{R}^d \to [0, \infty[$ be a positive function such that $f(x) = O(|x|^2)$ when $x \to 0$. Then the process $(S_t)_{t\geq 0}$ defined by

$$S_t = \sum_{s\leq t, \Delta X_s \neq 0} f(\Delta X_s),$$

is a subordinator.

For $f(x) = |x|^2$, the sum of the squared jumps

$$S_t = \sum_{s\leq t, \Delta X_s \neq 0} |\Delta X_s|^2$$

is a non decreasing Lévy process.

Outline

SUBMULTIPLICATIVE FUNCTIONS.

DEFINITION

A function on $\mathbb{R}^d$ is under-multiplicative if it is non-negative and if there exists a constant $a > 0$ s.t.

$$\forall (x, y) \in (\mathbb{R}^d)^2, \; g(x + y) \leq a g(x) g(y).$$

LEMMA

1. *The product of two submultiplicative functions is submultiplicative.*
2. *If g is submultiplicative on $\mathbb{R}^d$, then so is $g(cx + \gamma)^\alpha$ with $c \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$ and $\alpha > 0$.*
3. *Let $0 < \beta \leq 1$. Then the following functions are submultiplicative.*

$$|x| \vee 1 = \max(|x|, 1), \quad \exp(|x|^\beta), \; \ln(|x| \vee e), \; \ln \ln(|x| \vee e^e).$$

4. *If g is submultiplicative and locally bounded, then $g(x) \leq b e^{c|x|}$.*

THEOREME

Let g be a under-multiplicative function, locally bounded on $\mathbb{R}^d$. Then equivalence between

- there exists $t > 0$ s.t. $\mathbb{E}(g(X_t)) < +\infty$
- for any $t > 0$, $\mathbb{E}(g(X_t)) < +\infty$.

Moreover $\mathbb{E}(g(X_t)) < +\infty$ if and only if $\int_{|x|\geq 1} g(x)\nu(dx) < +\infty$.

Hence $\mathbb{E}(|X_t|^n) < \infty$ if and only if $\int_{|x|\geq 1} |x|^n \nu(dx) < \infty$. In particular

$$\mathbb{E}(X_t) = t\left(\gamma + \int_{|x|\geq 1} x\nu(dx)\right) = t\gamma_1,$$

and

$$(\text{Var } X_t)_{ij} = t\left(A_{ij} + \int_{\mathbb{R}^d} x_i x_j \nu(dx)\right).$$

EXPONENTIAL MOMENTS.

THEOREM

Let X be a Lévy process with triple (A, ν, γ). Let

$$C = \left\{ c \in \mathbb{R}^d, \int_{|x|\geq 1} e^{\langle c,x \rangle} \nu(dx) < +\infty \right\}.$$

1. C is convex and contains 0.
2. $c \in C$ if and only if $\mathbb{E}(e^{\langle c,X_t \rangle}) < +\infty$ for some $t > 0$ or equivalently for any $t > 0$.
3. If $w \in \mathbb{C}^d$ is s.t. $\mathrm{Re}(w) \in C$, then

$$\psi(w) = \frac{1}{2}\langle w, Aw \rangle + \langle \gamma, w \rangle + \int_{\mathbb{R}^d} (e^{\langle w,x \rangle} - 1 - \langle w, x \rangle \mathbf{1}_D(x)) \nu(dx)$$

has a sense, $\mathbb{E}(e^{\langle w,X_t \rangle}) < +\infty$ and $\mathbb{E}(e^{\langle c,X_t \rangle}) = e^{t\psi(w)}$.

OUTLINE

CONTINUITY OF THE LAW.

LEMMA

If X is a compound Poisson process, then $\mathbb{P}(X_t = 0) \geq e^{-\lambda t}$.

THEOREM

Let X be a Lévy process with triple (A, ν, γ). Equivalence :

1. $\mathbb{P}(X_t)$ is continuous for every $t > 0$,
2. $\mathbb{P}(X_t)$ is continuous for one $t > 0$,
3. $A \neq 0$ or $\nu(\mathbb{R}^d) = +\infty$.

COROLLARY

Equivalence between :

1. $\mathbb{P}(X_t)$ *is discrete for every* $t > 0$,
2. $\mathbb{P}(X_t)$ *is discrete for one* $t > 0$,
3. X *of type* A *and* ν *discrete.*

Existence of a density.

Proposition

Let X be a d-dimensional Lévy process with triple (A, ν, γ) with A of rank d. Then the law of X_t, $t > 0$ is absolutely continuous.

Theorem (for $d = 1$)

Let X be a Lévy process with triple (A, ν, γ).

1. If $A \neq 0$ or if $\nu(\mathbb{R}) = +\infty$, X_t has a continuous density on $\mathbb{R}$.
2. If the Lévy measure satisfies :

$$\exists \beta \in]0, 2[, \quad \liminf_{\varepsilon \downarrow 0} \varepsilon^{-\beta} \int_{-\varepsilon}^{\varepsilon} |x|^2 \nu(dx) > 0$$

   then for every $t > 0$, X_t has a density of class C^∞ and all derivatives of this density go to zero when $|x|$ goes to $+\infty$.

Outline

PROCESSES WITH INDEPENDENT INCREMENTS AND MARTINGALES.

PROPOSITION

Let X be a process with independent increments. Then

1. *for every $u \in \mathbb{R}^d$, $\left(\dfrac{e^{i\langle u, X_t \rangle}}{\mathbb{E}(e^{i\langle u, X_t \rangle})} \right)_{t \geq 0}$ is a martingale.*
2. *If $\mathbb{E}(e^{\langle u, X_t \rangle}) < \infty$, $\forall t \geq 0$, then $\left(\dfrac{e^{\langle u, X_t \rangle}}{\mathbb{E}(e^{\langle u, X_t \rangle})} \right)_{t \geq 0}$ is a martingale.*
3. *If $\mathbb{E}(|X_t|) < \infty$, $\forall t \geq 0$, then $M_t = X_t - \mathbb{E}(X_t)$ is a martingale (with independent increments).*
4. *In dimension 1, if $Var(X_t) < +\infty$, $\forall t \geq 0$, then $(M_t)^2 - \mathbb{E}((M_t)^2)$ is a martingale.*

Martingales.

Proposition

A Lévy process is a MARTINGALE *if and only if* $\int_{|x|\geq 1} |x|\nu(dx) < +\infty$ *and*

$$\gamma + \int_{|x|\geq 1} x\nu(dx) = 0.$$

In dimension 1, $\exp(X)$ *is a* MARTINGALE *if and only if* $\int_{|x|\geq 1} e^x \nu(dx) < +\infty$ *and*

$$\frac{A}{2} + b + \int_{-\infty}^{+\infty} (e^x - 1 - x\mathbf{1}_{|x|\leq 1})\nu(dx) = 0.$$

Markov property.

Theorem

Let μ be an infinitely divisible distribution on $\mathbb{R}^d$ and X the associated Lévy process. Then X is a Markov process with transition function

$$P_t(x, B) = \mu^t(B - x).$$

Conversely : every time homogeneous Markov process, with space homogeneous transition function, is a Lévy process in law.

Proposition

Let $(X_t)_{t\geq 0}$ be a Lévy process (in law). Then for every $s \geq 0$, the process $(X_{t+s} - X_s)_{t\geq 0}$ is a Lévy process with the same distribution as $(X_t)_{t\geq 0}$. And the two processes are independent.

Theorem (Strong Markov)

Let X be a Lévy process in law and $\mathcal{F}$ its completed filtration. Let τ be an a.s. finite $\mathcal{F}$-stopping time. Then the process $(X_{t+\tau} - X_\tau)_{t\geq 0}$ is independent of $\mathcal{F}_\tau$ and with the same law as X.

Infinitesimal generator :

$$\begin{aligned}\mathcal{L}f(x) &= \frac{1}{2}\sum_{j,k=1}^{d} A_{jk}\frac{\partial^2 f}{\partial x_j \partial x_k}(x) + \langle \gamma, \nabla f(x)\rangle \\ &+ \int_{\mathbb{R}^d}\left[f(x+y) - f(x) - \langle y, \nabla f(x)\rangle \mathbf{1}_{|y|\leq 1}\right]\nu(dy).\end{aligned}$$