Matching with phantoms*

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Abstract: Searching for partners involves informational persistence that reduces future traders' matching probability. In this paper, traders who are no longer available but who left tracks on the market are called phantoms. We examine a dynamic matching market in which phantoms are a by-product of search activity, no coordination frictions are assumed, and non-phantom traders may lose time trying to match with phantoms. The resulting aggregate matching technology features increasing returns to scale in the short run, but has constant returns to scale in the long run. We embed a generalized version of this matching function in the canonical continuous-time equilibrium search unemployment model. Long-run constant returns to scale imply there is a unique steady state, whereas short-run increasing returns generate excess volatility in the short run and endogenous fluctuations based on self-fulfilling prophecies.

Keywords: Information persistence; Endogenous matching function; Business cycles

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1 Introduction

The matching technology is a popular tool among labor market specialists and macroeconomists. The technology gives the number of jobs formed as an increasing function of the numbers of job-seekers and vacancies. This function is generally assumed to be well-behaved in that it is strictly concave and has constant returns to scale. These properties have empirical relevance (see Petrongolo and Pissarides, 2001) and are associated with strong model outcomes, such as the independence of the unemployment rate vis-à-vis workforce size, and the saddle-path and uniqueness properties of equilibrium under rational expectations. The standard Diamond-Mortensen-Pissarides model portrays the long-run relationships between unemployment, vacancies and the job-finding rate well, but struggles to reproduce the volatility of unemployment and vacancies (see, e.g., Shimer, 2005, Hall, 2005, Hagedorn and Manovskii, 2008, and Pissarides, 2009).

Often, the functional form of the matching technology is specified exogenously and is not derived from elementary principles. As noted by Stevens (2007), several papers rely on an implicit limited mobility assumption with an associated coordination problem.[1] Given that workers cannot readily transfer their attention from one job (or sub-market) to another, lack of coordination generates frictions. However, another property is also involved: matching frictions result from intratemporal congestion externalities. Traders on one side of the market detract from the search prospects for those who are *currently* on the same side, and improve prospects for those who are *currently* on the other side.

This paper emphasizes a different form of market frictions based on the persistence of obsolete information about traders who have already found a match. We refer to these traders as phantom traders, or phantoms for short. Phantoms are a by-product of the search activity: when exiting the market, each trader may leave a trace that disappears over time, i.e. information about them remains visible despite they are no longer available. Phantoms result in a loss of time and resources for future traders who want to find a partner. Our purpose is to analyze this dynamic form of congestion and its implications for labor market dynamics.

We start by introducing motivational evidence (Section 2). There are different possible reasons why obsolete information persists: destroying obsolete ads is generally costly, keeping active ads at zero extra cost is optimal when there is uncertainty on the survival of recent matches, some firms are legally constrained to publish their positions despite already knowing

[1] In mismatch models, workers are imperfectly mobile between sub-markets, and the distribution of traders across sub-markets governs the shape of the aggregate matching technology (see Drèze and Bean, 1990, Lagos, 2000, 2003, and Shimer, 2007). In stock-flow matching models, traders can only match with newcomers (see e.g. Taylor, 1995, Coles and Muthoo, 1998, Coles and Smith, 1998, Coles, 1999, Gregg and Petrongolo, 2005, Coles and Petrongolo, 2008, and Ebrahimy and Shimer, 2010). In urn-ball matching models, buyers independently send one buy order to each seller. Buyers do not coordinate and some sellers receive several buy orders, while others do not receive any order (see, e.g., Butters 1977, Hall 1977, Burdett et al 2001, Albrecht et al 2004, 2006, and Galenianos and Kircher 2009). Stevens (ibid) makes explicit the time-consuming nature of search and endogenizes search investments. She obtains a CES matching function.

who they will hire. Thus it is not surprising to find many statements of frustration from people who respond to ads without obtaining any answer (we have a collection of quotes on our webpages). Though phantoms seem ubiquitous, it is difficult in practice to measure them. We provide indirect evidence based on Craigslist, a major US job board, and show that the density of job postings by vacancy age is uniform. In other words, employers never delete obsolete ads. We also compute a minimum bound on the phantom proportion on this website: at least 37% of the job listings advertise for already filled jobs.

We then single out this particular form of dynamic congestion and show that it can generate a matching technology with constant returns in the long run and increasing returns in the short run (Section 3). We provide a discrete-time model where buyers and sellers try to contact each other at a unique search site. To disentangle the impacts of phantoms on the search market from more standard congestion externalities, each buyer meets one seller at most and vice-versa, whereas every trader on the short side of the market is sure to meet someone. However, that someone may be a phantom trader implying no trade takes place. We assume that the populations of phantoms obey simple flow-stock equations, the inflow of new phantoms being proportional to the past outflow of successful traders, and examine the resulting matching pattern between the two populations of traders.

We refer to the aggregate matching technology as the phantom matching technology, or PMT for short. The PMT features intratemporal and intertemporal externalities. Intratemporal externalities result from the fact that an increase in the number of agents on the long side of the market reduces the proportion of phantom traders. A larger proportion of contacts leads to matches as a result. Intratemporal externalities imply that the PMT displays increasing returns to scale in the short run. Intertemporal externalities result from the fact that current matches fuel phantom traders, thereby lowering the future number of matches. Intratemporal and intertemporal externalities balance each other, and the PMT features constant returns to scale vis-à-vis the whole history of traders in the long run.

We combine information obsolescence with the usual Cobb-Douglas matching function. This leads to a generalized version of the PMT denoted GPMT. Just like the PMT, the GPMT has short-run increasing returns to scale, and long-run constant ones. However, it is more flexible: both types of phantoms can affect matching simultaneously with varying intensity. The GPMT allows us to interpret phantom proportions as determinants of the matching efficiency (the scale parameter) of an otherwise standard Cobb-Douglas function. Therefore, when such proportions increase, the matching efficiency goes down and the number of matches decreases for given numbers of buyers and sellers.

The interaction between intra- and intertemporal externalities seems promising for the study of the business-cycle. Long-run constant returns should preserve all the desirable long-run properties of the standard model, whereas short-run increasing returns should provide additional volatility in the short run. This is why we embed the GPMT in the canonical continuous-time model of equilibrium search unemployment (Section 4). As intuition suggests, the long-run behavior of the model (i.e. existence, uniqueness, and comparative statics of the steady state) is entirely governed by the properties of the long-run matching function, whereas the short-run behavior (i.e. local stability and transition dynamics) depends on the short-run returns to unemployment and vacancies.

Short-run increasing returns have two implications that we develop at length. For parameterizations in which the short-run elasticity with respect to vacancies is less than one, exogenous shocks have larger effects in the short run than in the long run. Consider an unexpected increase in output per worker. Owing to rent-sharing, this shock implies an immediate increase in the supply of vacancies. This reduces the phantom vacancy proportion, thereby improving matching efficiency. Then, unemployment gradually declines, and there is an associated increase in the phantom unemployed and phantom vacancy proportions. The vacancy-to-unemployed ratio overreacts in the short run. The magnitude of overreaction increases with the differential between the short-run and the long-run elasticities of the GPMT with respect to vacancies.

For parameterizations in which the short-run elasticity with respect to vacancies is larger than one, there may be endogenous fluctuations based on self-fulfilling prophecies. When employers believe that the supply of vacancies will be large, they actually expect that the phantom vacancy proportion will be small. Thus they anticipate that the matching efficiency will be high. This leads them to post many vacancies to benefit from a fast matching process, and this confirms the belief. As phantoms progressively accumulate, the process reverts after some time and the system goes back to its steady state, or the trajectory follows a limit cycle.[2]

Therefore information obsolescence not only provides a rationale for the properties of the long-run matching function, but also offers a natural source of aggregate volatility. We illustrate these properties in the case of a limit cycle, which is based on the procyclicality of the nonphantom vacancy proportion induced by information obsolescence. The cycle features the US volatility and covariance of unemployment and vacancy-to-unemployed ratio. It can also reproduce the degree of persistence of such data when the phantom unemployed do not count, the wage is rigid over the business cycle, and the phantom death rates are small.[3]

[2] There already exist several continuous-time equilibrium search models involving indeterminacy. Models with multiple steady states have a continuum of equilibrium trajectories (see, e.g., Mortensen, 1989 and 1999, Ellison et al, 2013, Kaplan and Menzio, 2014). Giammarioli (2003) examines the case of increasing returns to vacancies. There may be indeterminacy even when there is a unique equilibrium. However, the job-filling rate always increases with the number of vacancies, and the steady state has counter-intuitive comparative statics properties. In our model, vacancies have (local) increasing returns in the short-run, but long-run decreasing returns. Thus the model features a standard long-run matching function, and a unique steady state with usual comparative statics properties.

[3] Beaudry et al (2015) provide arguments in favor of the deterministic view of business-cycle fluctuations. Sniekers (2014) calibrates the limit cycle of the Mortensen (1999) model with two steady states (in this model, aggregate productivity increases with employment). This model is formally equivalent to the standard DMP model where the scale parameter of the matching function is *negatively* affected by unemployment. Information obsolescence actually implies the opposite property.

2 Phantoms in markets

Phantom traders are ubiquitous in search markets. Most searchers for a job, a house, or even a partner, will have experienced situations where information regarding the object of the search was clearly outdated. A friend who talks about a job that was already filled, an ad for a dream bike that was already sold, a showcased house that has just been rented. Head-hunters also have specific stories involving obsolete profiles of job-seekers.

Agents who post ads on match-maker websites are usually invited to withdraw them once obsolete. However, this action involves a cost generally without benefits. Thus obsolete ads are rarely deleted. Keeping outdated ads can also be beneficial to the ad maker. Some firms may voluntarily keep outdated job listings because they wait until the new hire has actually started working. This insures them in situations where the person works for a few days and then resigns. In real-estate markets, sellers often keep outdated posts in case a potential buyer does not manage to get bank approval for a mortgage. Though these practices are optimal for one side – keeping an ad at zero marginal cost for a very unlikely event – they generate phantom vacancies that can be misleading for job-seekers and house-buyers.

A clear example of this behavior can be found on Craigslist, a major job board in the US. Craigslist is a classified advertisements website with sections devoted to jobs, housing, personals, for sale, items wanted, services, community, gigs, résumés, and discussion forums. It was created in 1995 by Craig Newmark and initially covered the San Francisco Bay Area; still its most active area today. It later expanded to other US cities and is still growing. The website is organized at a local level; thus ads are posted over a given geographic area and someone who wishes to cover more areas has to duplicate ads on different parts of the website.

As of 2014, most ads are free, except for job listings in 18 major cities, and paid broker apartment listings in New York City. The price for job listings is cheap and varies across locations, ranging from $25 in the majority of cities to $75 in the San Francisco Bay area. Job-seekers choose a location and a job category and then see the different job listings by posting date, starting with the most recent one. All job listings stay on the site for at most 30 days. Recruiters can edit their post and change its content during that period. They can also renew it if they want to see it upfront of the list. In such instances they pay the cost and two ads for the same job will then coexist with different dates. Lastly, employers are invited to delete their obsolete job listings. A phantom vacancy, according to our definition, occurs when a match is formed and remains on Craigslist until the recruiter deletes the ad or the thirty day time limit is reached.

On Wednesday March 26 2014, we computed the distribution of job listings by age for the 23 US cities listed under the heading "Cities" on the Craigslist website. The support of this distribution cannot exceed 30 days, because ads are deleted after this age. Table 1 reports the different quartiles of this distribution in each city. Two consecutive quartiles are systematically separated by 7 or 8 days. The median of the distribution corresponds to day 15 or day 16. Thus the distribution is uniform by quartile. Under the assumption that job creation did not dramatically decline over the month, this shows that very few employers delete obsolete ads. If outdated ads were destroyed, the density of job listings by age would be decreasing, and the quartiles of the distribution would be separated by an increasing number of days.

	Atlanta	Austin	Boston	Chicago	Dallas	Denver	Detroit	Houston
Q1	7	7	8	7	8	8	8	7
Q2	15	15	15	15	15	15	15	15
Q3	22	23	23	23	23	23	23	23
Q4	30	30	30	30	30	30	30	30
# ads	16000	15000	27000	29900	27900	21700	9400	19000
	Las Vegas	Los Angeles	Miami	Minneapolis	New York	Orange cty	Philadelphia	Phoenix
Q1	8	7	7	8	8	8	8	7
Q2	15	15	15	15	15	15	16	15
Q3	23	22	22	23	23	23	23	23
Q4	30	30	30	30	30	30	30	30
# ads	8500	41600	24400	15100	45000	16600	12300	21300
	Portland	Raleigh	Sacramento	San Diego	San Francisco	Seattle	Washington	
Q1	8	8	8	8	7	8	8	
Q2	16	16	15	15	15	16	16	
Q3	23	23	23	23	23	23	23	
Q4	30	30	30	30	30	30	30	
# ads	17500	6400	8700	18200	41600	30200	20900	

Table 1: Quartiles of the distribution of Craigslist's job listings by age

The lines headed Q1 to Q4 report the different daily quartiles of the empirical distribution of vacancies by age. The line # ads gives the total number of job listings rounded to the hundred. Source: Craigslist website, March 26 2014.

Reasonable values for the job-finding probability suggest that the uniform law is very far from the density that would result if employers destroyed ads immediately after they hired someone. Let us compute an example for the years 2000-2008, just before the great recession. Following Shimer (2005), the mean monthly job-finding probability μ_m for the US can be computed by using standard monthly data on unemployment, labor force, and short-term unemployment from the Bureau of Labor Statistics. We obtain $\mu_m = 0.4$. The associated continuous-time job-finding rate is defined by $1 - \exp(-\mu) = \mu_m$, which gives $\mu = 0.5$. Using data from the Job Openings and Labor Turnover Survey over the same period, we can also compute the mean vacancy-to-unemployed ratio, $\theta = v/u$. The JOLTS dataset consists of employers' declarations as to how many vacancies they have at the date of interview, and yields an average value of $\theta \approx 0.5$ for the period 2000-2008. The associated continuous-time rate of filling jobs is $\eta = \mu/\theta$. This gives $\eta = 1.0$.

In order to compute a rough estimate of Craigslist's phantom vacancy proportion, we make three assumptions: (i) the only job-seekers are unemployed workers, (ii) the rate η applies to job listings on Craigslist, (iii) the rate η is the same for the different vacancy ages. Assumption (i) is conservative since it leads to undervaluing Craigslist's job-filling rate. Accounting for employed job-seekers would imply $\eta v > \mu u$, and therefore $\eta > \mu/\theta$. In the same vein, Assumption (iii) minimizes the phantom proportion at given job-filling rate. If workers apply more for recent vacancies than for older ones, phantoms last longer on the website, and so the phantom proportion is larger. As for Assumption (ii), we have no evidence proving that the actual rate should differ much from the mean rate.

The number of phantoms of age a is $v_0(1 - \exp(-\eta a))$, where v_0 is the stationary inflow of job listings. Thus the total number of phantoms is $\int_0^1 v_0(1 - \exp(-\eta a))da$. The resulting phantom proportion is $1-(1-\exp(-\eta))/\eta$. The computation gives 37%, implying that over a third of the job listings on Craigslist correspond to jobs that are already filled.

Another approach is to compute the density of nonphantom vacancies by age. This is the density of job listings by age that would result if employers deleted their obsolete job postings as soon as they recruited someone. This density is $\eta \exp(-\eta a)/[1-\exp(-\eta)]$, i.e. an exponential law truncated at one month. Let a_q denote the q-th quartile of the distribution, $q = 1, 2, 3, 4$. These quartiles a_q can be found by solving

$$a_q = -\frac{1}{\eta} \ln[1 - q + q \exp(-\eta)].$$

We report the result in days, i.e. $d_q = a_q \times 30$. We obtain $d_1 = 5.2$, $d_2 = 11.4$, and $d_3 = 19.3$. In Table 1, the corresponding numbers averaged over all cities are $d_1 = 7.6$, $d_2 = 15.2$ and $d_3 = 22.8$, clearly indicating that phantoms exist and represent a substantial fraction of vacancies.[4]

Other job boards have alternative policies that affect the potential number of phantoms. On Monster.com, an ad lasts between 14 days and two months. Monster offers a discount for longer durations, which is likely to increase the number of phantom vacancies. It also allows recruiters to refresh their postings, manually or automatically. The initial posting then appears on the top of the list at regular time intervals. This increases the posting visibility, but this is true for nonphantom as well as for phantom vacancies if refreshing is automatic. Lastly, the posting can be renewed. Unlike Craigslist, renewing an ad deletes its previous version. This increases the weight of more recent ads in the distribution of vacancies by age. Note, however, that this is not due to phantom destruction. Rather, renewing an ad replaces an active job listing by a newer one. Some websites aggregate information available on job boards or even on firms' websites. This is advantageous for job seekers who can access different sources of information at a single location, but implies that this information is more likely to be obsolete.

It is important to realize that new job listings are not necessarily available to job seekers. Some firms are legally required to post job openings even when they have already chosen

[4] Clearly, we have made strong assumptions. It may be that the rate of filling vacancies is much lower on Craigslist than on other job boards, or that workers search with different intensities across the different vacancy vintages. The implications of alternative scenarii are not straightforward, and we leave them for specific research on the topic.

the person they will hire. Government jobs, for instance, must be advertised officially. Some companies' human-resources departments require them to be listed on a job board or career site for a period of time. This protects employers from discrimination lawsuits. The website of the Wall Street Journal has a an article on this phenomenon[5], and many testimonies of frustration can be found on the web. The phantom phenomenon is magnified in such cases because information on the job only exists because it is already attributed.

Hereafter, we proceed to explore the implications of phantoms in search markets. In Section 3, we study the impact of such phantoms by sketching a simple scenario where information obsolescence is the only source of matching frictions. This scenario generates an aggregate matching function with short-run increasing returns to scale, and long-run constant returns to scale. Section 4 embeds a generalized version of this function in to an equilibrium search unemployment model. The novel dynamic aspects of information obsolescence lead to interesting phenomena such as excess volatility in the short-run, local indeterminacy, and deterministic cycles.

3 The Phantom Matching Technology

3.1 The model

Time is discrete and denoted by $t \geq 0$. A population of nonatomistic buyers and sellers want to trade with each other. But they have to meet before trade takes place. Matching takes place every period. Every time a buyer and a seller meet and agree on match formation, they exit the market.

Let B denote the (mass) number of buyers, S the number of sellers, P^B the number of phantom buyers, and P^S the number of phantom sellers.

The matching mechanism involves two steps. In a first step, each trader on the short side of the market is assigned to a trader on the long side. This results in the following number of contacts:

$$\min \left\{ B_t + P_t^B, S_t + P_t^S \right\}. \tag{1}$$

In a second step, matches are derived from contacts. The rule is that only contacts between non-phantom traders lead to effective trade. The number of matches is

$$M_t = \frac{B_t}{B_t + P_t^B} \frac{S_t}{S_t + P_t^S} \min \left\{ B_t + P_t^B, S_t + P_t^S \right\}. \tag{2}$$

The number of contacts is multiplied by the product of the two proportions of non-phantom traders. We assume that phantoms cannot be distinguished from non-phantoms by the matching mechanism.

[5] Weber, L., Kwoh, L., 2013. Beware the phantom job listing.

The matching probabilities are μ_t for buyers and η_t for sellers. We have

$$\begin{aligned}
\mu_t &= \frac{M_t}{B_t} = \frac{S_t}{S_t + P_t^S} \min\left\{1, \frac{S_t + P_t^S}{B_t + P_t^B}\right\}, \\
\eta_t &= \frac{M_t}{S_t} = \frac{B_t}{B_t + P_t^B} \min\left\{\frac{B_t + P_t^B}{S_t + P_t^S}, 1\right\}.
\end{aligned}$$

We assume that the numbers of phantoms obey the following laws of motion:

$$P_t^B = \beta^B M_{t-1} + \left(1 - \delta^B\right) P_{t-1}^B, \tag{3}$$

$$P_t^S = \beta^S M_{t-1} + \left(1 - \delta^S\right) P_{t-1}^S, \tag{4}$$

with $0 < \beta^j \leq 1$, and $0 < \delta^j \leq 1$, $P_0^i \geq 0$, $j = B, S$. The inflow of new phantoms is proportional to former matches. The parameter β^j can be interpreted as the probability that a match gives birth to a phantom trader, or as the relative search efficiency of phantoms vis-à-vis non-phantoms. The outflow results from constant depreciation at rate δ^j. Phantoms face a constant probability of dying δ^j each period. Their life expectancy is $1/\delta^j$, which measures the degree of persistence of obsolete information.

Proposition 1 (The PMT) In each period t, the number of matches is given by

$$\ln M_t = \ln S_t + \ln B_t - \ln\left[X_t + \beta_t \sum_{k=0}^{t-1} (1 - \delta_t)^k M_{t-k-1} + (1 - \delta_t)^t P_0^t\right], \tag{PMT}$$

with $\begin{cases} \beta_t = \beta^B,\ \delta_t = \delta^B,\ X_t = B_t \text{ and } P_0^t = P_t^B \text{ if } B_t + P_t^B > S_t + P_t^S \\ \beta_t = \beta^S,\ \delta_t = \delta^S,\ X_t = S_t \text{ and } P_0^t = P_t^S \text{ if } B_t + P_t^B < S_t + P_t^S \end{cases}.$

Proof. Suppose that $\min\left\{B_t + P_t^B, S_t + P_t^S\right\} = S_t + P_t^S$. Then $M_t = S_t B_t/(B_t + P_t^B)$. We have

$$P_t^B = \beta^B \sum_{k=0}^{t-1} \left(1 - \delta^B\right)^k M_{t-k-1} + (1 - \delta^B)^t P_0^B.$$

This gives

$$\ln M_t = \ln S_t + \ln B_t - \ln\left[B_t + \beta^B \sum_{k=0}^{t-1} \left(1 - \delta^B\right)^k M_{t-k-1} + (1 - \delta^B)^t P_0^B\right].$$

Now suppose that $\min\left\{B_t + P_t^B, S_t + P_t^S\right\} = B_t + P_t^B$. Then $M_t = B_t S_t/(S_t + P_t^S)$. We have

$$P_t^S = \beta^S \sum_{k=0}^{t-1} \left(1 - \delta^S\right)^k M_{t-k-1} + (1 - \delta^S)^t P_0^S.$$

This gives

$$\ln M_t = \ln S_t + \ln B_t - \ln\left[S_t + \beta^S \sum_{k=0}^{t-1} \left(1 - \delta^S\right)^k M_{t-k-1} + (1 - \delta^S)^t P_0^S\right].$$

The phantom matching technology (PMT) collapses into the usual non-frictional technology whenever $\beta_t = 0$, where β_t is defined in Proposition 1. The novelty comes from the inclusion of the weighted sum of former matches in the last term. The weights depend on survival probabilities $(1-\delta_t)^k$ and entry rate of new phantoms β_t, where δ_t is also defined in Proposition 1.

The role played by market history is parameterized by δ_t. As δ_t tends to 0, phantoms are almost infinite-lived and old phantoms have a large impact on current matches. Conversely, with full depreciation $\delta^B = \delta^S = 1$, phantoms live for one period and the PMT reduces to

$$\ln M_t = \ln S_t + \ln B_t - \ln\left[X_t + \beta_t M_{t-1}\right].$$

3.2 Intratemporal vs intertemporal externalities

We examine the matching externalities featured by the PMT. The combination of intratemporal and intertemporal externalities implies that the technology has increasing returns to scale in the short run and constant returns in the long run.

Proposition 2 (Intratemporal and intertemporal externalities) *The following properties hold.*

(i) Assume that $S_t + P_t^S > B_t + P_t^B$*. Then,*

$$\begin{aligned} d\ln M_t / d\ln B_t &= 1, \\ d\ln M_t / d\ln S_t &= 1-\mu_t, \\ d\ln M_t / d\ln P_t^S &= -(1-\mu_t). \end{aligned}$$

(ii) Assume that $S_t + P_t^S \neq B_t + P_t^B$ *for all* $t \geq 0$ *and there is* $\tau \geq 0$ *such that* $S_t + P_t^S > B_t + P_t^B$ *or* $S_t + P_t^S > B_t + P_t^B$ *for all* $t \geq \tau$*. Then,*

$$\lim_{t\to\infty} \sum_{k=0}^{t} \left\{ d\ln M_t / d\ln B_k + d\ln M_t / d\ln S_k \right\} = 1.$$

Proof. See Appendix A.1.

Part (i) examines intratemporal externalities. It focuses on the case where phantom and nonphantom sellers are on the long side of the market, but similar though inverted results hold when buyers are on the long side. Part (i) shows that the PMT has constant returns with respect to the number of traders on the short side of the market. This property is typical of non-frictional matching models. Meanwhile, the PMT has positive returns with respect to the number of traders on the long side. The reason is that additional traders reduce the proportion of phantom traders. The more phantoms there are, the greater the effect.

Intratemporal externalities imply that the matching technology exhibits increasing returns to scale in the short run. Indeed, $d\ln M_t/d\ln S_t + d\ln M_t/d\ln B_t = 2-\mu_t > 1$. The magnitude of increasing returns to scale is parameterized by β^S (driving phantom births), δ^S (governing phantom deaths), and by the sequence of matching flows $\{M_{t-k-1}\}_{k=0}^{\infty}$ (fueling potential phantoms).

Part (ii) describes intertemporal externalities. Former matches generate phantom traders. In turn, phantoms deteriorate the current matching process. These intertemporal externalities imply that the whole market history of buyers and sellers affects the current number of matches. Part (ii) shows that intratemporal and intertemporal externalities combine so that the matching technology has constant returns to scale in the long run with respect to the history of traders. The restriction $S_t + P_t^S \neq B_t + P_t^B$ is made because the PMT is not differentiable when the two numbers coincide. The other restriction is made for tractability. Though the long side of the market can alternate for some time, it does not fluctuate in the long run.

Current matches may positively or negatively alter future matches. To understand this property, we consider the case where phantoms only last one period, i.e. $\delta^B = \delta^S = 1$. Assuming that phantom and nonphantom sellers are always on the long side, we have

$$d \ln M_{t+k}/d \ln S_t = (-1)^k \prod_{j=0}^{k-1} \frac{\beta^S M_{t+j}}{S_{t+j} + \beta^S M_{t+j}}.$$

The magnitude of this elasticity decreases with horizon period k. Its sign depends on $(-1)^k$, which is negative for even k and positive for odd k. An increase in the number of period-t traders increases the number of period-$t+1$ phantoms, thereby reducing the flow of matches in period $t+1$. For a similar reason, this increases the flow of matches in period $t+2$.

3.3 Stationary phantom matching technology

We assume that whenever a buyer and a seller match they are replaced by a similar pair of agents. This situation corresponds to a steady state where there is a constant inflow of new agents into the pools of buyers and sellers. We show that the phantom matching technology (PMT) converges towards a stationary technology, the stationary phantom matching technology (SPMT).

The numbers of traders are $B_t = B$ and $S_t = S$ for all $t \geq 0$. The number of matches follows the PMT.

Proposition 3 (The SPMT) *Let $\sigma^B = \beta^B/\delta^B$, $\sigma^S = \beta^S/\delta^S$, and $\theta = S/B$.*

(i) If $B + P_t^B > S + P_t^S$ for all $t \geq 0$, then the sequence M_t converges towards the stationary number of matches

$$M = m^B(B, S) = B\frac{-1 + (1 + 4\sigma^B S/B)^{1/2}}{2\sigma^B}, \tag{5}$$

and the elasticity of this function vis-à-vis S is

$$\varepsilon^B(\theta) = \frac{2\sigma^B\theta}{-(1 + 4\sigma^B\theta)^{1/2} + 1 + 4\sigma^B\theta} \in (1/2, 1); \tag{6}$$

(ii) If $B + P_t^B < S + P_t^S$ for all $t \geq 0$, then the sequence M_t converges towards the stationary number of matches

$$M = m^S(B, S) = S \frac{-1 + (1 + 4\sigma^S B/S)^{1/2}}{2\sigma^S}, \tag{7}$$

and the elasticity of this function vis-à-vis S is

$$\varepsilon^S(\theta) = 1 - \frac{2\sigma^S\theta^{-1}}{-\left(1 + 4\sigma^S\theta^{-1}\right)^{1/2} + 1 + 4\sigma^S\theta^{-1}} \in (0, 1/2); \tag{8}$$

Proof. We prove (i) and omit the proof of (ii), which is similar. In steady state, $M_t = M$ and solves $\sigma^B M^2 + BM - BS = 0$. Resolution gives (5). To establish convergence, note that $M_t = BS/(B + P_t)$. This implies that the sequence $\{M_t\}$ converges towards M if and only if the sequence $\{P_t\}$ converges towards $P = \sigma^B M$. But, $P_{t+1} = \phi(P_t)$, with $\phi(x) = \beta^B BS/(B + x) + (1 - \delta^B)x$. As $\phi(0) > 0$ and $0 < \phi'(x) < 1$ for all $x \geq 0$, $\{P_t\}$ converges towards P for all $\beta^B \in (0, 1]$ and all $\delta^B \in (0, 1]$. The elasticity $\varepsilon^B = \theta m_2^B(B, S)/m^B(B, S)$. The computation gives (6).

The SPMT features standard properties. First, it is strictly increasing in the numbers of traders on each market side. Second, it has constant returns to scale. This property results from the constant intertemporal returns to scale discussed by Proposition 2. Third, the elasticity of the matching technology with respect to the ratio of sellers to buyers depends on whether buyers or sellers are on the short side of the market. It is larger than 1/2 when phantom and nonphantom sellers are on the short side, whereas it is lower than 1/2 when buyers are.

At given seller-to-buyer ratio θ, the stationary number of matches decreases with $\sigma^i = \beta^i/\delta^i$. This parameter governs the magnitude of search frictions induced by information obsolescence. It consists of the product of phantoms' life expectancy $1/\delta^i$ by their birth rate β^i. Thus σ^i captures the amount of outdated information and its degree of visibility for match seekers. When σ^i tends to 0, the non-frictional case obtains and the number of matches tends to $m^B(B, S) = S$ or $m^S(B, S) = B$. When σ^i tends to infinity, the number of matches tends to 0 and the elasticity of the matching function ε tends to 1/2.

3.4 Generalized phantom matching technology

The PMT is derived from a single source of frictions. However, the functional form is severly constrained: it has a Leontieff structure as only one of the two phantom proportions can affect the number of matches at a time, and the effect of such phantom proportions is proportional to the number of contacts. The idea here is to consider an ad hoc generalized version of the PMT that retains its main properties, i.e. increasing returns in the short run and constant returns in the long run. We set it in continuous time instead of discrete time to avoid the usual problems associated with time discretization. We consider the following GPMT:

$$M = A\,(\pi_B)^{\lambda_B}\,(\pi_S)^{\lambda_S}\,\left(B + P^B\right)^{1-\alpha}\left(S + P^S\right)^{\alpha}, \tag{9}$$

where $\pi_B \equiv B/(B + P^B)$ and $\pi_S \equiv S/(S + P^S)$ are the nonphantom proportions, $A > 0$, $0 \leq \alpha \leq 1$, $\lambda_B \geq 1 - \alpha$ and $\lambda_S \geq \alpha$. The latter restrictions imply that phantoms have nonpositive effects on the number of matches. With respect to the PMT, the Leontieff contact technology is replaced by a Cobb-Douglas function, both nonphantom proportions affect the number of matches, and their impact is parameterized by λ_B and λ_S.

The GPMT can be rewritten as follows:

$$M = A\,(\pi_B)^{\lambda_B+\alpha-1}\,(\pi_S)^{\lambda_S-\alpha}\,B^{1-\alpha}S^{\alpha}. \tag{10}$$

This shows that phantoms affect the matching efficiency $\pi \equiv A\,(\pi_B)^{\lambda_B+\alpha-1}\,(\pi_S)^{\lambda_S-\alpha}$ of an otherwise standard Cobb-Douglas matching function. The usual Cobb-Douglas case results when $\lambda_B = 1 - \alpha$ and $\lambda_S = \alpha$. The PMT obtains when $\alpha = 0$ and $\lambda_B = \lambda_S = 1$ (sellers are on the long side), or when $\alpha = \lambda_B = \lambda_S = 1$ (buyers are on the long side).

The short-run elasticities of the GPMT are

$$\varepsilon_B^{SR} \equiv d\ln M/d\ln B = 1 - \alpha + (\lambda_B - 1 + \alpha)(1 - \pi_B), \tag{11}$$

$$\varepsilon_S^{SR} \equiv d\ln M/d\ln S = \alpha + (\lambda_S - \alpha)(1 - \pi_S). \tag{12}$$

As $\varepsilon_B^{SR} + \varepsilon_S^{SR} = 1 + (\lambda_B - 1 + \alpha)(1 - \pi_B) + (\lambda_S - \alpha)(1 - \pi_S)$, the GPMT features increasing returns to scale in the short run. The magnitude of such returns depends on the nonphantom proportions π_B and π_S. Unlike the PMT, the GPMT may have increasing returns with respect to each input. Note, however, that such a phenomenon is only local. As the number of traders, say S, further increases, the nonphantom proportion π_S goes up and the elasticity ε_S^{SR} goes down. If the increase in S is sufficiently large, ε_S^{SR} falls below one.

In continuous time, the phantom motions are given by:

$$\dot{P}^i \equiv dP^i/dt = \beta^i M - \delta^i P^i, \tag{13}$$

with $P^i(0) = P_0^i$, $i = B, S$.

As in the PMT case, we can define the stationary GPMT. For B and S constant, the corresponding stationary numbers of phantoms are $P^i = \sigma^i M$, $\sigma^i = \beta^i/\delta^i$, $i = B, S$. The stationary flow of matches is

$$M = A\left(\frac{B}{B + \sigma^B M}\right)^{\lambda_B-1+\alpha}\left(\frac{S}{S + \sigma^S M}\right)^{\lambda_S-\alpha} B^{1-\alpha}S^{\alpha}.$$

This equation implicitly defines the long-run matching function $M = m^{LR}(B, S)$. The function m^{LR} has the properties of a standard matching technology. It is strictly increasing in its arguments, strictly concave, satisfies $m^{LR}(0, S) = m^{LR}(B, 0) = 0$ and $\lim_{S\to\infty} m^{LR}(B, S) =$

$\lim_{B\to\infty} m^{LR}(B,S) = \infty$. Its elasticities with respect to B and S are

$$\varepsilon_B^{LR} \equiv \frac{d\ln M}{d\ln B} = \frac{\bar{\varepsilon}_B^{SR}}{\bar{\varepsilon}_B^{SR} + \bar{\varepsilon}_S^{SR}} \in (0,1), \tag{14}$$

$$\varepsilon_S^{LR} \equiv \frac{d\ln M}{d\ln S} = \frac{\bar{\varepsilon}_S^{SR}}{\bar{\varepsilon}_B^{SR} + \bar{\varepsilon}_S^{SR}} = 1 - \bar{\varepsilon}_B^{LR}, \tag{15}$$

where $\bar{\varepsilon}_B^{SR}$ and $\bar{\varepsilon}_S^{SR}$ are the short-run elasticities of the GPMT evaluated in stationary state.

Like the PMT, the GPMT displays contant returns to scale. This statement holds whatever the magnitude of the short-run returns to each input. The consideration of phantom sellers increases the elasticity of this matching function, whereas the consideration of phantom buyers decreases it.

3.5 Discussions

We discuss several aspects of the PMT: we first provide a specific scenario leading to its formation, then consider the case of atomistic traders, the case of on-the-match search, and the interplay between information obsolescence and coordination frictions. Readers who are not interested can directly skip this part and go to Section 4 where we apply the generalized PMT to the labor market.

A specific scenario.—The PMT results from an abstract mechanism that randomly associates phantom and nonphantom traders so that everyone on the short side is assigned to someone. We now consider a specific scenario that leads to this assignment. Suppose there is a dating agency that arranges dates at some time frequency between singles of two demographic groups. The dating agency holds two lists of singles $\{1,...,B_t\}$ and $\{1,...,S_t\}$ and schedules dates between members of the two lists. In the absence of phantoms, the period-t number of dates is $\min\{B_t, S_t\}$ and equals the number of matches. The names of those who have found a partner are removed from the lists. The next day, the dating agency schedules dates between the B_{t+1} and S_{t+1} singles. However, the dating agency may fail to update its lists. Each time a couple forms, there is a probability β that their names stay on the lists. Then, their names are removed from the lists with per period probability δ. Every period, the lists contain $B_t + P_t^B$ and $S_t + P_t^S$ names. If pairs are formed randomly, then the number of dates is $\min\{B_t + P_t^B, S_t + P_t^S\}$. The proportion of such dates that give birth to matches is $B_t/(B_t + P_t^B)\times\ S_t/(S_t + P_t^S)$. The remaining proportion includes dates between a nonphantom single and a phantom one – then only one person shows up – and dates between two phantoms – then none shows up. The phantom motions are given by (3) and (4) with $\beta_B = \beta_S = \beta$ and $\delta_B = \delta_S = \delta$.

Atomistic traders.—Propositions 1 to 3 apply for nonatomistic agents. We here relax this assumption and examine the case of atomistic agents. The model is unchanged, but B, S, P^B, and P^S are now integers. Let M be the stochastic number of matches. When phantom and nonphantom sellers are on the long side, this number follows a hypergeometric law such that

$$\Pr[M = k] = \frac{\binom{S}{k}\binom{P^S}{B-k}}{\binom{P^S+S}{B}},$$

with $M \leq S$. The denominator is the total number of combinations involving the B nonphantom buyers and the $P^S + S$ sellers, and the numerator is the total number of combinations where exactly k nonphantom buyers are matched with the same number of nonphantom sellers.

The mean and the variance of M are

$$\begin{aligned}
\mathbb{E}(M) &= \mu B, \\
V(M) &= B\mu(1-\mu)\frac{S + P^S - B}{S + P^S - 1}
\end{aligned}$$

The mean coincides with the PMT given by Proposition 1 when phantom and nonphantom sellers are on the long side.

Let $B = bN$, $S = sN$ and $P^S = p^S N$, where N is the population size. As N becomes arbitrarily large, the variance of M/N tends to 0. This implies that Propositions 1 to 3 still hold for a large population of atomistic agents provided we define per capita variables (i.e. B/N, S/N, or P^S/N). For instance, the matching probability M/B tends to the deterministic number μ and the matching probability M/S tends to $\mu B/S$.

The reasoning is the same when buyers are on the long side.

On-the-match search.—On-the-match search occurs when matched traders go on searching for alternative partners. When on-the-match seekers compete with the unmatched traders, they act as phantoms from the perspective of those unmatched. This implies that usual strategies to quantify the impact of on-the-match search on the unmatched may fail to distinguish matched seekers from phantoms.

In what follows, we assume that phantom and nonphantom buyers are on the short side. We denote by E_t the number of matched buyers. Such agents separate with exogenous probability δ^E, and a fraction $\beta^E \leq 1$ search for alternative partners and compete with unmatched buyers. If the matching odds are the same for matched and unmatched traders, then the number of matches between unmatched buyers and sellers is given by the following modified PMT:

$$\ln M_t = \ln S_t + \ln B_t - \ln(S_t + \beta^E E_t + \beta^S P_t^S), \tag{16}$$

where phantoms obey equation (4) and the motion for matched buyers is

$$E_t = M_{t-1} + (1 - \delta^E)E_{t-1}.$$

On-the-match search and information obsolescence have very different implications for match formation. However, it is hard to disentangle their respective impacts on the unmatched. Suppose indeed that the true specification of the matching function is (16). Manipulating equation (16) and adding an error component e_t orthogonal to the other covariates gives

$$\frac{S_t B_t}{M_t} - S_t = \beta^E E_t + \beta^S P_t^S + e_t.$$

Omitting P_t^S should bias upward the estimate of parameter β^E. The problem is that E_t and P_t^S are positively correlated: they both depend on the sequence of previous matches. They are actually identical when $\beta^E = \beta^S$ and $\delta^E = \delta^S$. In other words, we may wrongly attribute the whole effect of β^E to on-the-match search, whereas part of this effect is also due to phantoms. This leads to overestimating the extent of competition between matched and unmatched seekers. For instance, the OLS estimate of parameter β^E is

$$\hat{\beta}^E = \sum_{t=1}^{T} E_t S_t \frac{B_t - M_t}{M_t} \left(\sum_{t=1}^{T} E_t^2 \right)^{-1},$$

where T is the sample size. The expectation of $\hat{\beta}^E$ is

$$\mathbb{E}\left(\hat{\beta}^E\right) = \beta^E + \beta^S \frac{\mathbb{E}\left(E_t P_t^S\right)}{\mathbb{E}\left(E_t^2\right)} > \beta^E.$$

This problem goes beyond the particular specification (16). The statement casts doubt on the interpretation of estimated matching functions for the unemployed that explicitly account for on-the-job search. Such estimates typically find evidence of job competition between employed and unemployed job-seekers (see, e.g., Broersma and van Ours, 1999, and Anderson and Burgess, 2000). The fact that traders of the past affect current recruitments does not prove that employees create congestion effects for the unemployed. Moreover, there are reasons to believe that unemployed and employees compete for different jobs (see Delacroix and Shi, 2006, for a directed search model with homogenous agents in which unemployed and employees search in segmented markets). Crowding-out effects reported in the literature may also result from information obsolescence.[6]

Information obsolescence and coordination frictions.—The GPMT implicitly combines information obsolescence with alternative sources of matching frictions. We here consider an explicit combination that mixes information obsolescence with coordination frictions. We assume that each buyer sends a buy order to one of the sellers, including phantoms. The probability that a particular seller receives a buy order from a particular buyer is $1/\left(S + P^S\right)$. The number of matches is

$$M = S\left[1 - \left(1 - \frac{1}{S + P^S}\right)^B\right].$$

As $B, S, P^S \to \infty$, this gives

$$M = S\left[1 - \exp\left(-\frac{B}{S + P^S}\right)\right].$$

[6] The literature mentioned above already points out the problem of other missing variables like unregistered vacancies or the search intensity of the various job-seekers (see also Sunde, 2007).

This technology still features increasing returns to scale vis-à-vis B and S because increasing S allows the phantom proportion on the sellers' side to be reduced. In the long run, the SPMT is

$$M = S\left[1 - \exp\left(-\frac{B}{S + \beta^S M/\delta^S}\right)\right].$$

This equation implicitly defines $M = m(B, S)$. The SPMT has constant returns to scale.

This scenario could be enriched. Buyers could observe each seller's time spent in the market and use this information to direct their search. Sellers could signal they are still active. We leave these extensions to future research.

4 Labor market phantoms

We embed the generalized version of the phantom matching technology (GPMT) in to a canonical continuous-time equilibrium search unemployment model. We study the local stability properties of the steady state and calibrate the model to US labor market data. We emphasize the role of information obsolescence, which implies excess volatility in the short run and self-fulfilling prophecies.

4.1 The model

The only non-standard element of the model is the matching function. Time is continuous and starts at 0. There is a continuum of homogenous workers whose total size is normalized to unity. These workers are either employed— mass $1-u$—, or unemployed— mass u. Unemployed workers search for jobs, whereas employed workers do not. On the other side of the market, there is a continuum of jobs. These jobs are either vacant— mass v—, or filled— mass $1-u$. Holding a vacancy involves paying the flow cost c, whereas employed workers produce flow output y. All agents are risk-neutral and discount time at rate r.

The unemployed and vacancies are brought together by pair according to the GPMT introduced in Section 3.4. In the terminology of that section, the unemployed are buyers and vacancies are sellers, i.e. $B = u$ and $S = v$. Similarly, $P^B = p^u$ and $P^S = p^v$. Let $\pi_u = u/(u + p^u)$ and $\pi_v = v/(v + p^v)$ be, respectively, the nonphantom unemployed and the nonphantom vacancy proportions. The job-finding rate is $\mu = M/u = \pi\theta^\alpha$, where $\theta \equiv v/u$, $\alpha \in (0,1)$ and $\pi = A(\pi_u)^{\lambda_u+\alpha-1}(\pi_v)^{\lambda_v-\alpha}$, where $A > 0$ is a scale parameter, $\lambda_u \geq 1-\alpha$ and $\lambda_v \geq \alpha$. The rate μ increases with the vacancy-to-unemployed ratio θ and the matching efficiency π. Similarly, the recruitment rate is $\eta = M/v = \pi\theta^{\alpha-1}$, which decreases with θ and increases with π. Parameters λ_u and λ_v size the impact of each nonphantom proportion on the matching efficiency.

Let s be the job separation rate, w the wage, and b the unemployment income. The values of being employed and unemployed respectively are W and U; the values of a vacant job and a filled job are V and J. These values satisfy the standard arbitrage equations:

$$\begin{aligned} rU &= b+\mu(W-U)+\dot{U} && (17)\\ rW &= w+s(W-U)+\dot{W} && (18)\\ rV &= -c+\eta(J-V)+\dot{V} && (19)\\ rJ &= y-w-sJ+\dot{J} && (20)\end{aligned}$$

A dot over a variable signals a derivative with respect to time.

Wages are determined by Nash bargaining. For all $t \geq 0$, we have $\gamma(J-V) = (1-\gamma)(W-U)$, where $\gamma \in (0,1)$ is the worker's exogenous bargaining power.

Free entry drives the value of a vacancy down to zero, i.e. $V = 0$ for all $t \geq 0$, which yields $J = c/\eta$. The bargained wage is

$$w = \gamma y + (1-\gamma)b + \gamma c\theta. \tag{21}$$

4.2 Equilibrium

Putting the wage (21) in equation (20), using $V = 0$ and noting that $\dot{J} = -c\eta'(\theta)/\eta^2\dot{\eta}$, we obtain

$$-\frac{\dot{\eta}}{\eta} = r+s-(1-\gamma)\frac{y-b}{c}\eta+\gamma\mu. \tag{22}$$

Taking the time derivative of the recruitment rate η, we have

$$\frac{\dot{\eta}}{\eta} = -(1-\alpha)\frac{\dot{\theta}}{\theta}+\frac{\dot{\pi}}{\pi}. \tag{23}$$

The growth rate of the recruitment flow probability η linearly decreases with the growth rate of tightness, and linearly increases with the rate of change in matching efficiency.

An equilibrium is a set of four bounded functions θ, u, p^u, p^v that solve the following equations:

$$\begin{aligned} (1-\alpha)\frac{\dot{\theta}}{\theta}-\frac{\dot{\pi}}{\pi} &= r+s-(1-\gamma)\frac{y-b}{c}\pi\theta^{\alpha-1}+\gamma\pi\theta^{\alpha}, && (24)\\ \dot{u} &= s(1-u)-\pi\theta^{\alpha}u, && (25)\\ \dot{p^u} &= \beta^u\pi\theta^{\alpha}u-\delta^u p^u, && (26)\\ \dot{p^v} &= \beta^v\pi\theta^{\alpha}u-\delta^v p^v, && (27)\end{aligned}$$

with $\pi = \pi_u^{\lambda_u-1+\alpha}\pi_v^{\lambda_v-\alpha}$, $\pi_u = u/(u+p^u)$, $\pi_v = \theta u/(\theta u+p^v)$, $u(0) = u_0$, $p^u(0) = p_0^u$ and $p^v(0) = p_0^v$ given.

In the standard model, the matching efficiency π of the GPMT is fixed and equation (24) only involves tightness. This implies that $\theta(0)$ immediately jumps to θ^*, whereas unemployment monotonically converges to u^*. Here π varies over time, which implies that tightness motion depends on the phantom stocks and unemployment rate. It follows that $\theta(0)$ differs from θ^*. More generally the dynamics are richer and involve a full 4-D system.

Steady state.—A steady state is a vector $(\theta^*, u^*, p^{u*}, p^{v*})$ such that $\dot{\theta} = \dot{u} = \dot{p}^u = \dot{p}^v = 0$. The variables θ^* and u^* solve

$$r + s = (1-\gamma)\frac{y-b}{c} m^{LR}(1,\theta^*)/\theta^* - \gamma m^{LR}(1,\theta^*), \tag{28}$$

$$u^* = \frac{s}{s + m^{LR}(1,\theta^*)}, \tag{29}$$

where the long-run matching function $m^{LR}(u,v)$ is implicitly defined by the solution M to

$$M = A\left(\frac{u}{u+\sigma^u M}\right)^{\lambda_u - 1+\alpha}\left(\frac{v}{v+\sigma^v M}\right)^{\lambda_v-\alpha} u^{1-\alpha}v^{\alpha},$$

where $\sigma^u = \beta^u/\delta^u$ and $\sigma^v = \beta^v/\delta^v$ measure the size of obsolete information in steady state.

Equations (28) and (29) are the usual job creation and Beveridge curves of the standard model in which the matching function is replaced by the stationary GPMT. In particular, the Beveridge curve defines the steady-state unemployment rate as the ratio of the job separation rate s to the sum of the separation rate and the steady-state job-finding rate $\mu^* = m^{LR}(1,\theta^*)$. It follows that the different parameters have standard effects on steady-state variables: unemployment increases with bargaining power, discount rate, vacancy posting cost, unemployment income and decreases with output per worker. Moreover, tightness decreases and unemployment increases with parameters σ^u and σ^v.

The following result is useful for calibration purposes. We omit its proof, which is immediate.

Proposition 4 (Normalized steady state) *Let $\mu > 0$ and $\theta > 0$. Then, $\mu^* = \mu$ and $\theta^* = \theta$ if and only if*

$$A = \mu\theta^{-\alpha}(1+\sigma^u\mu)^{\lambda_u-1+\alpha}(1+\sigma^v\mu/\theta)^{\lambda_v-\alpha}, \tag{30}$$

$$c = \frac{(1-\gamma)(y-b)\mu/\theta}{r+s+\gamma\mu}. \tag{31}$$

Suppose μ and θ are the observed stationary job-finding rate and vacancy-to-unemployed ratio. Proposition 4 states that given a subset of parameters $(\alpha, \sigma^u, \sigma^v, \lambda_u, \lambda_v, \gamma, y, b, r, s)$, we can always find A and c so that μ and θ are the predicted steady-state job-finding rate and vacancy-to-unemployed ratio.

Stability.—The dynamic system involves three predetermined variables, u, p^u and p^v, and one forward variable, θ. The steady state is locally a saddle when the Jacobian matrix of the dynamic system (24)-(27) evaluated in steady state has three eigenvalues with a negative real part, whereas the last one has a positive real part. It is a local sink when all eigenvalues have a negative real part, and a local source when at least two eigenvalues have a positive real part.

Proposition 5 (Local stability properties). *Let $\delta^v = \delta^u = s$. Also let $\bar{\varepsilon}_u^{SR} \equiv 1 - \alpha + (\lambda_u - 1 + \alpha)(1-\pi_u^*)$ and $\bar{\varepsilon}_v^{SR} \equiv \alpha + (\lambda_v - \alpha)(1-\pi_v^*)$. The steady state is locally* *(i) a saddle if* $\bar{\varepsilon}_v^{SR} < 1$; *(ii) a sink if* $1 + \frac{\gamma\mu - (\mu+s)\bar{\varepsilon}_u^{SR}}{r+s} > \bar{\varepsilon}_v^{SR} > 1$; *(iii) a source if* $\bar{\varepsilon}_v^{SR} > 1 + \max\{0, \frac{\gamma\mu - (\mu+s)\bar{\varepsilon}_u^{SR}}{r+s}\}$.

Proof. See Appendix A.2.

Studying the local stability properties involves examining the roots of a four-degree polynomial. We restrict the analysis to the particular case where the rates of phantom decay among the unemployed and the vacancies are equal to each other and the separation rate. Then $-s$ is an eigenvalue of multiplicity two and we can simply focus on the sub-system in θ and u. We demonstrate via a series of calibrations below, that relaxing this restriction leads to qualitatively similar properties.

Proposition 5 illucidates a noteworthy feature of the phantom unemployed and vacancies. A marginal increase in the number of vacancies usually diminishes firms' matching odds through increased congestion. However, in this case, it also improves the matching efficiency by raising the nonphantom vacancy proportion. When the latter effect is sufficiently small, the matching function has short-run decreasing returns with respect to vacancies and part (i) shows that the steady state is saddle-path stable. However, the increase in matching efficiency may dominate the congestion effect. Then the matching function has local increasing returns with respect to vacancies and drastically different properties arise; namely local stability, as in part (ii), and local instability, as in part (iii).

We now comment on each of these cases.

Part (i). When $\bar{\varepsilon}_v^{SR}$ is lower than one, all eigenvalues are real, three of them are negative, the fourth one is positive, and the steady state is saddle-path stable as a result. At given (u_0, p_0^u, p_0^v), there is a unique θ_0 implying that the trajectory stays bounded and this trajectory leads to the steady state. This configuration encompasses the PMT case, where the elasticity $\bar{\varepsilon}_v^{SR} = 0$ when phantom and nonphantom vacancies are on the short side, or $\bar{\varepsilon}_v^{SR} = 1$ when they are on the long side. Unlike the standard DMP model, tightness does not immediately jump to θ^* because the short-run elasticities of the GPMT differ from the long-run ones. This property implies there is excess volatility in the short run following exogenous shocks, which we emphasize in sub-section 4.3.

Part (ii). When $\bar{\varepsilon}_v^{SR}$ is greater than one but not too large, two eigenvalues are real and negative, whereas the two others are complex conjugates with a negative real part. Therefore the steady state is a local sink. A marginal increase in the number of vacancies causes a short-run increase in firms' job-filling rate and a continuum of beliefs on the matching efficiency are sustainable. The mechanism behind indeterminacy is as follows: suppose employers think that more vacancies will be posted than in steady state. Owing to locally short-run increasing returns to vacancies, they expect an increase in the recruitment rate. This leads them to post more jobs, thereby confirming the belief. Then phantoms accumulate and unemployment decreases. This contributes to reducing the matching efficiency of the GPMT. After some time, this effect becomes strong and there is a reversal in the number of posted jobs. Unemployment and tightness converge to the steady state with oscillations of declining amplitude.

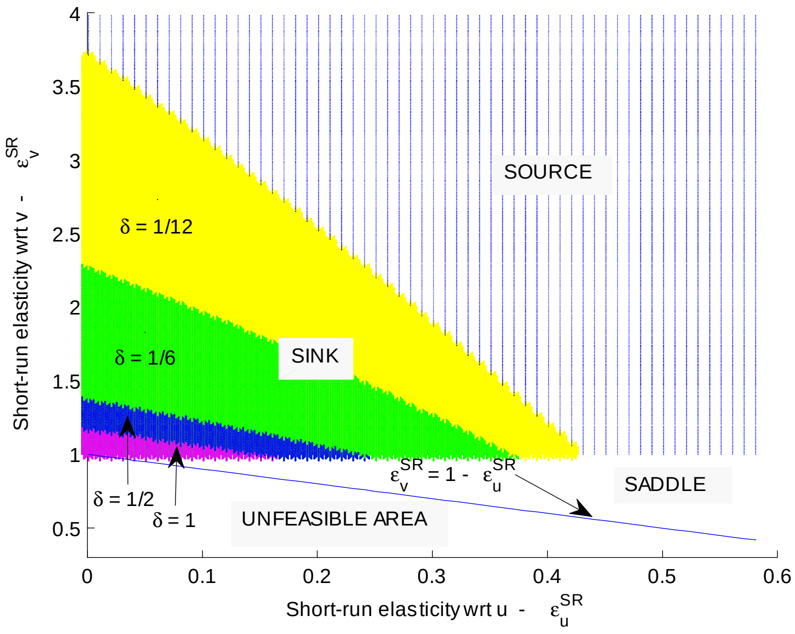

Figure 1: Local stability properties of the normalized steady state for different values of $\beta = \delta$. The other parameter values are given in the body text.

Part (iii). When the elasticity $\bar{\varepsilon}_v^{SR}$ is sufficiently larger than one, there are two negative real eigenvalues, and the two others are complex conjugates with a positive real part. Thus the steady state is a local source. When the elasticity $\bar{\varepsilon}_v^{SR}$ crosses the threshold $[\gamma\mu - (\mu + s)\bar{\varepsilon}_u^{SR}]/(r + s)$, there is a qualitative change in the long-run behavior of the dynamic system. Formally, the real part of the two complex conjugate eigenvalues turns from negative to positive. This situation corresponds to a Hopf bifurcation: as $\bar{\varepsilon}_v^{SR}$ further increases, the steady state becomes a source but a limit cycle emerges and attracts all trajectories. The cycle is based on the procyclicality of the matching efficiency, which boosts job creation in good times and lowers it in bad times. The amplitude of this cycle is zero at the threshold $[\gamma\mu - (\mu + s)\bar{\varepsilon}_u^{SR}]/(r + s)$ and then increases with $\bar{\varepsilon}_v^{SR}$. When $\bar{\varepsilon}_v^{SR}$ becomes too large, the cycle disappears and all trajectories starting out of the steady state diverge.

General case.—We proceed to numerical simulations. The unit of time is the month. As in Section 2, we use JOLTS and BLS data and fix $\theta^* = 0.524$ and $\mu^* = 0.5$. We standardly set $r = 0.003$, $b = 0.7$, $\gamma = 0.5$, $y = 1$, and $s = 0.025$. We then assume $\beta^u = \beta^v = \delta^u = \delta^v = \delta$, i.e. phantoms last one month on average. Six parameters associated with the GPMT remain: λ_u, λ_v, β^u, β^v, α and δ. We feature several δ because this parameter constrains the likelihood of indeterminacy. In each case, we let λ_u, λ_v and α vary, whereas A and c are identified through Proposition 4.

Figure 1 shows the local properties of the steady state in the $(\bar{\varepsilon}_u^{SR}, \bar{\varepsilon}_v^{SR})$ plane when δ varies from 1 to 1/12. Each parameterization is associated with a particular point in this plane. The infeasible area corresponds to cases where there are decreasing returns to scale

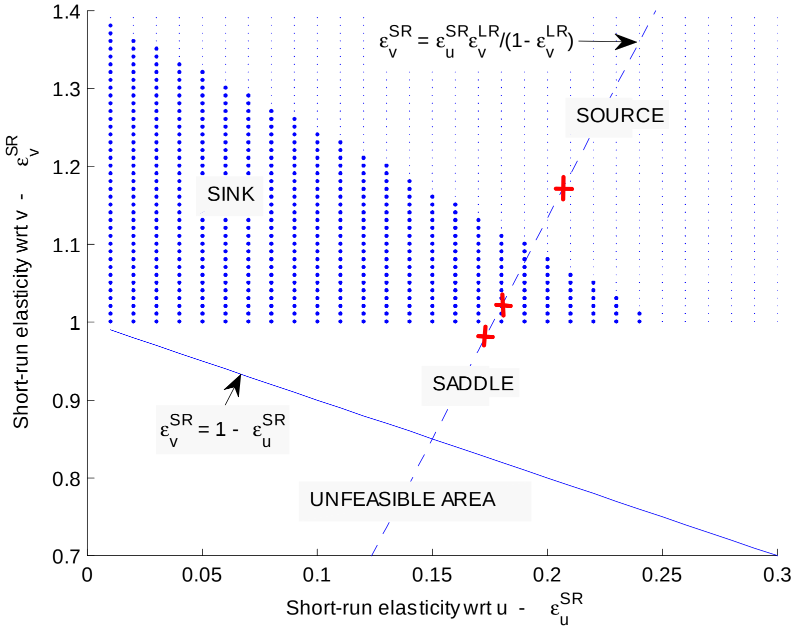

Figure 2: Local stability properties of the normalized steady state when $\delta = \beta = 0.5$. The red crosses show the three parameterizations considered by Sections 4.3 to 4.5. The values of the different parameters are given in the text.

in the short run, i.e. $\bar{\varepsilon}_u^{SR} + \bar{\varepsilon}_v^{SR} < 1$, which is not compatible with information obsolescence. The SADDLE, SINK, and SOURCE zones correspond, respectively, to situations of saddle-path stability, local indeterminacy, and local instability. For all values of δ, the SINK zone involves $\bar{\varepsilon}_v^{SR} > 1$ and $\bar{\varepsilon}_u^{SR}$ not too large. Figure 1, therefore, confirms the main message of Proposition 5: indeterminacy requires that the GPMT displays increasing returns with respect to vacancies in the short run. Moreover, the SINK area shrinks with δ. This parameter decreases the share of new phantoms in the phantom stock, which makes this stock more volatile and, therefore, less likely to originate indeterminacy.

We choose $\beta = \delta = 1/2$ and illustrate the three possible stability regimes through specific parameterizations indicated by crosses '+' on Figure 2. The steady-state properties of the model depend on the long-run elasticity of the matching function with respect to vacancies ε_v^{LR}, whereas the dynamic properties involve the two short-run elasticities $\bar{\varepsilon}_u^{SR}$ and $\bar{\varepsilon}_v^{SR}$. We keep constant the long-run elasticity across parameterizations and change the short-run elasticities as follows: $\bar{\varepsilon}_v^{SR} = \bar{\varepsilon}_u^{SR} \times \varepsilon_v^{LR}/(1-\varepsilon_v^{LR})$.

Indeterminacy requires that ε_v^{LR} is sufficiently large. In Figure 2, the line must cross the SINK area, which implies $\varepsilon_v^{LR} > 0.8$. We set $\varepsilon_v^{LR} = 0.85$ and examine three different combinations of $\bar{\varepsilon}_v^{SR}$ and $\bar{\varepsilon}_u^{SR}$ associated each to a particular stability configuration: saddle (section 4.3), sink (section 4.4) and limit cycle (section 4.5). In terms of structural parameters, we arbitrarily fix $\alpha = 0.850$ (α must be such that $\bar{\varepsilon}_v^{SR} \geq 1-\alpha \geq 1-\bar{\varepsilon}_u^{SR}$) and invert equations (11) and (12) to obtain λ_u and λ_v.

Indeterminacy in equilibrium search unemployment models.—Mortensen (1989) and Gianmarioli (2003) study the emergence of indeterminacy when the matching technology is $M = Au^b v^a$, with $a + b > 1$. This technology implies that the short-run and long-run elasticities of the matching function coincide. Mortensen imposes $a < 1$ and shows that indeterminacy arises when there are multiple steady states. Gianmmarioli allows $a > 1$ and shows that indeterminacy also arises when the steady state is unique. However, the job-filling rate M/v always increases with θ and the steady state has counter-intuitive comparative statics properties. This result borrows from Benhabib and Farmer (1994) who highlight indeterminacy in the competitive model with increasing returns. They interpret their findings in terms of an increasing labor demand curve.

Information obsolescence naturally leads us to disentangle the long-run elasticity ε_v^{LR} from the short-run one ε_v^{SR}. The former elasticity is lower than or equal to one, but the latter elasticity can be larger. Then we show that local indeterminacy requires that the short-run elasticity is larger than one. Indeterminacy, therefore, neither involves multiple steady states nor nonstandard comparative statics properties.

Another branch of literature emphasizes decreasing returns to scale in the matching function as a source of business cycle fluctuations (see Ellison et al, 2013, though they mostly focus on saddle-path stable configurations). Such decreasing returns in matching are formally equivalent to having operating profits increasing in employment as in Kaplan and Menzio (2014), Mortensen (1999) and Snieckers (2014). These models feature multiple steady states and there is a continuum of equilibrium trajectories leading to one of the steady states or even a limit cycle. Information obsolescence differs from such cases as employment exerts negative spillovers on the matching function through the decrease in nonphantom proportions.

4.3 Saddle-path stability and productivity shocks

We start with a parameterization where the steady state is locally saddle-path stable and study the propagation of an unexpected permanent productivity shock. In his critique of the standard DMP model, Shimer (2005) argues that a typical calibration of this model implies that tightness and unemployment insufficiently respond to such shocks. The main cause being the too low predicted elasticity of steady-state tightness vis-à-vis labor productivity. Pissarides (2009) sets the target for this elasticity at 7.56, whereas typical calibrations of the model yield less than four. Unlike the DMP model, our model with information obsolescence implies that tightness is more responsive to shocks in the short run than in the long run. Thus it is possible for productivity shocks to have large effects on tightness although the stationary elasticity remains small.

We set the parameters so that the short-run elasticity of tightness vis-à-vis productivity shocks is in the order of magnitude proposed by Pissarides. The long-run elasticity $\varepsilon_v^{LR} = 0.850$, whereas the short-run elasticity is $\bar{\epsilon}_v^{SR} = 0.980$. This implies that $\bar{\epsilon}_u^{SR} = 0.173$. In Figure 2, this parameterization is indicated by the cross in the SADDLE area. The

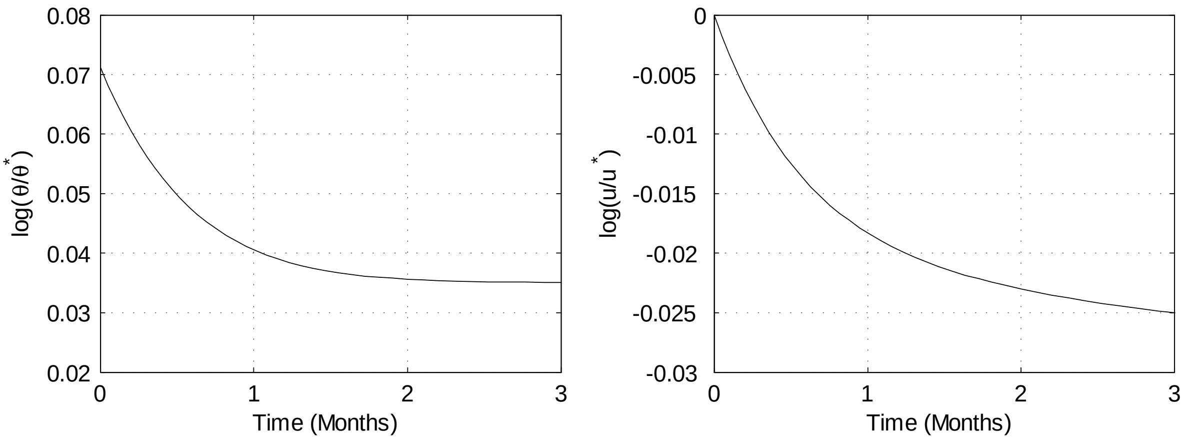

Figure 3: IRF following a 1% permanent increase in output per worker in the saddle case. θ^* and u^* are the steady state values before the shock. Parameter values are given in the body of the text.

corresponding structural parameters of the model are $\lambda_u = 0.196$ and $\lambda_v = 1.049$.

We start from the steady state, and explore output per worker permanently increasing by 1%. Figure 3 reports the dynamics of labor market tightness and unemployment in deviations from the initial steady-state values following this productivity shock.

Tightness overshoots its post-shock steady-state value. Overshooting reflects increasing returns in the short run. The long-run elasticity of steady-state tightness with respect to y is

$$\varepsilon_{\theta y}^{LR} = \frac{y}{y-b} \frac{r+s+\gamma m^{LR}(1,\theta^*)}{(1-\varepsilon_v^{LR})(r+s)+\gamma m^{LR}(1,\theta^*)}.$$

This is the usual formula where the elasticity of the matching function with respect to vacancies has been replaced by ε_v^{LR}. In the short run, tightness is defined by $c/\eta(0) = J(0)$. Under the assumption that $J(0)$ stays close to J^*, the short-run elasticity of tightness with respect to y is

$$\varepsilon_{\theta y}^{SR} \approx \frac{1-\varepsilon_v^{LR}}{1-\varepsilon_v^{SR}} \varepsilon_{\theta y}^{LR} > \varepsilon_{\theta y}^{LR}.$$

The short-run elasticity is larger than the long-run one because the nonphantom proportion jumps upward at the moment of the shock, which increases the matching efficiency and magnifies the effect of the shock. Then phantoms accumulate, which reduces the nonphantom proportions and decreases tightness. The magnitude of overshooting increases with the spread between the short-run elasticity $\bar{\epsilon}_v^{SR}$ and the long-run one ε_v^{LR}.

Excess volatility is a fundamental property induced by information obsolescence. We have illustrated it with a particular type of shock, productivity shocks, under a particular parameterization where $\beta = \delta = 1/2$ and $\varepsilon_v^{LR} = 0.85$. However, all shocks that affect the steady state will have this type of short-run impact. Moreover, excess volatility arises whenever the short-run elasticity ε_v^{SR} is lower than one.

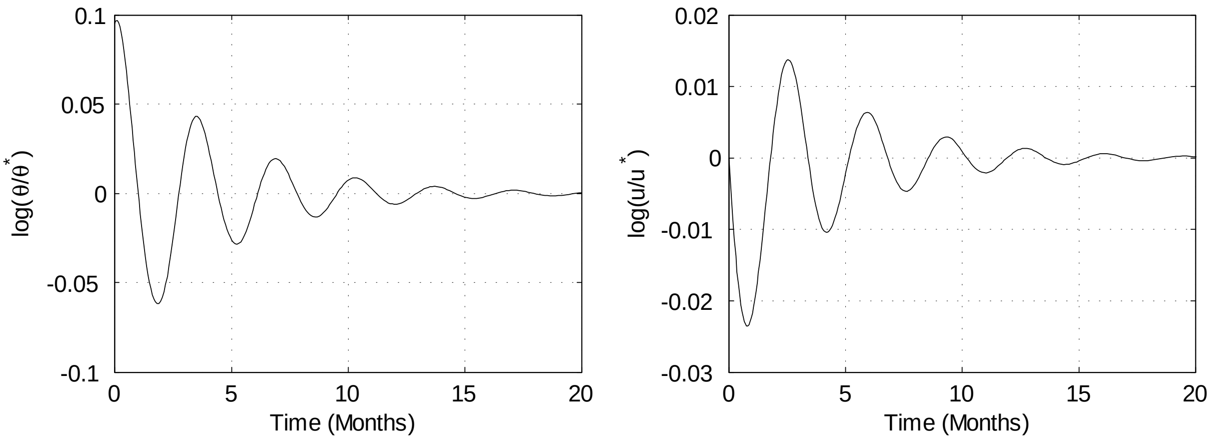

Figure 4: IRF following a 10% transitory increase in tightness in the sink case. θ^* and u^* are the steady state values. Parameter values are given in the body of the text.

4.4 Local indeterminacy and sunspot fluctuations

We now consider a parameterization where the steady state is a local sink. This configuration is associated with the indeterminacy of equilibrium and endogenous fluctuations based on self-fulfilling prophecies. We keep the same long-run elasticity $\varepsilon_v^{LR} = 0.850$ and increase the short-run elasticity to $\bar{\varepsilon}_v^{SR} = 1.020$. This implies that $\bar{\varepsilon}_u^{SR} = 0.180$. In Figure 2, this parameterization is indicated by the cross in the SINK area. The corresponding structural parameters of the model are $\lambda_v = 1.109$ and $\lambda_u = 0.210$.

We start from the steady state and suppose that a belief shock occurs: tightness suddenly increases by 10%. Figure 4 shows the propagation of the belief shock in deviation from the steady state. Though initially in steady state, all agents expect that the number of vacancies increases. As this implies a rise in matching efficiency, firms effectively post more vacancies. A reverting process takes place through phantom accumulation, and aggregate variables converge with decreasing oscillations.

Here aggregate volatility is implied by animal spirits and thus completely disconnected from labor productivity. We now demonstrate an alternative parameterization generating deterministic cycles.

4.5 Limit cycles

When the short-run elasticity $\bar{\varepsilon}_v^{SR}$ crosses a larger-than-one threshold, a Hopf bifurcation occurs and a limit cycle arises. As time tends to infinity, unemployment and tightness tend to periodic functions of time. A parameterization along these lines can, broadly, reproduce key characteristics of US business-cycles in terms of volatility and covariance.

As noted in the previous sub-section, there already exist equilibrium search unemployment models that predict limit cycles. Sniekers (2014) calibrates the Mortensen (1999) model where employment exerts a positive externality on output per worker. Sniekers adds a search

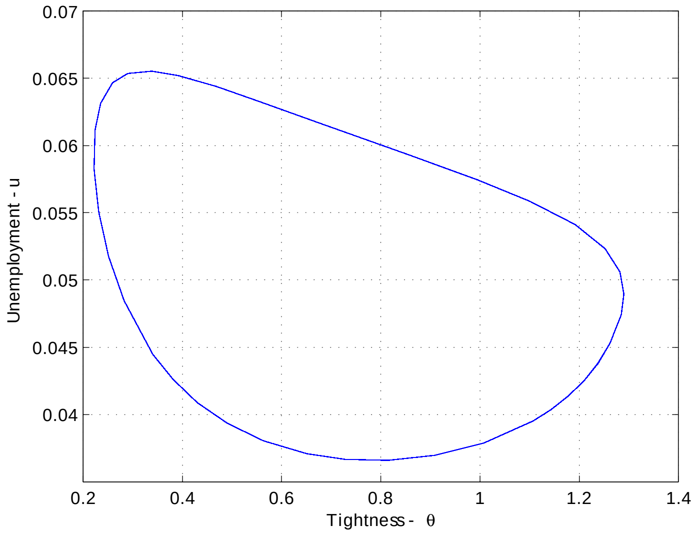

Figure 5: Limit cycle in the (θ, u) plane. The model is simulated from $t = 0$ to $t = 200,000$ with $\theta_0 = 1.1\theta^*$, $u_0 = u^*$, $p_0^u = p^{u*}$ and $p_0^v = p^{v*}$. We only keep observations after $t = 150,000$. The values of the different parameters are given in the body of the text.

intensity margin and focuses on the case where there are two steady states. We here illustrate a different model where endogenous fluctuations are based on information obsolescence and its impacts on the matching efficiency.

We keep the same long-run elasticity $\varepsilon_v^{LR} = 0.850$, and increase $\bar{\varepsilon}_v^{SR}$ until we cross the border of the SINK area. In the neighborhood of the Hopf bifurcation, the limit cycle has zero amplitude. Therefore we further increase $\bar{\varepsilon}_v^{SR}$ until the limit cycle broadly mimics the US quarterly volatility of unemployment and tightness. The short-run elasticities are $\bar{\epsilon}_u^{SR} = 0.206$ and $\bar{\epsilon}_v^{SR} = 1.170$. This parameter combination lies in the SOURCE area, as shown by the highest cross in Figure 2. The corresponding structural parameters of the model are $\lambda_u = 0.263$ and $\lambda_v = 1.338$.

Figure 5 shows the limit cycle in the (θ, u) plane. Tightness varies between 0.25 and 1.3, whereas unemployment varies between 2.5% and 6.5%. The main axis of the ring is downward sloping, which implies that unemployment and tightness are negatively correlated during the business-cycle.

Figure 6 visually confirms that the matching efficiency π is procyclical. Bad times are characterized by a large phantom proportion, whereas good times feature a smaller one. Thus employers have stronger incentive to post jobs in booms than in recessions. Figure 6 also shows that the duration of the cycle is short in this parameterization, around 5 months.

Table 1 compares the properties of the predicted asymptotic business-cyles with the actual US ones. We take logs of quarterly averages and use the hp-filter with parameter 1600 to extract the cycle components of unemployment and labor market tightness. The US statistics are from Hagedorn and Manovskii (2008).

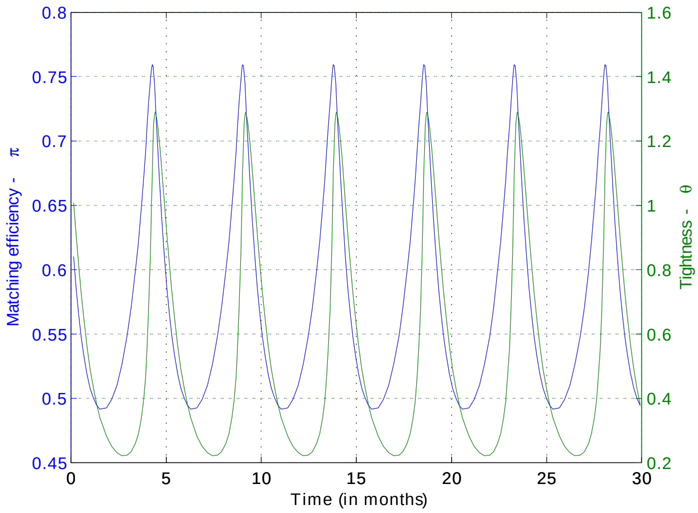

Figure 6: Periodic fluctuations of tightness and matching efficiency. The model is simulated from $t = 0$ to $t = 200,000$ with $\theta_0 = 1.1\theta^*$, $u_0 = u^*$, $p_0^u = p^{u*}$ and $p_0^v = p^{v*}$. We consider a typical 30-month period after $t = 150,000$. The values of the different parameters are given in the body of the text.

Table 1: Business cycle statistics for the limit cycle

	$\sigma(u)$	$\sigma(\theta)$	$corr(\theta, u)$	$ac(u)$	$ac(\theta)$
US data	0.125	0.259	-0.977	0.870	0.896
Model	0.100	0.332	-0.462	-0.669	-0.613

The standard deviation of unemployment $\sigma(u)$ and tightness $\sigma(\theta)$, as well as the correlation coefficient $corr(\theta, u)$ and autocorrelation coefficients $ac(u)$ and $ac(\theta)$ are computed at quarterly frequency after smoothing with the HP filter of parameter 1600.

Tightness and unemployment are as volatile as in the data, but the limit cycle is less satisfactory in terms of covariance generating only half of the predicted correlation between tightness and unemployment. Table 1 also shows that the predicted auto-correlation is negative, which is entirely driven by the small length of the cycle.

Persistence.—The limit cycles of our model feature a trade-off between persistence and volatility. Improving persistence can always be achieved by increasing the short-run elasticity $\bar{\varepsilon}_v^{SR}$. This lengthens the period of the limit cycle, but also increases its amplitude. Therefore, the problem is to find a set of parameters that maximize the period of the cycle conditional on aggregate volatility.

The Hopf bifurcation theory guides us towards parameterizations where the wage is rigid over the business cycle and the long-run elasticity ε_v^{LR} is equal to one. Appendix B analyzes the case $\delta = s$,already featured in Proposition 5, and studies the combination of parameters that maximizes the period of the limit cycle in the neighborhood of the Hopf bifurcation. The reasoning leads us to bind the constraint $\bar{\varepsilon}_u^{SR} \geq 1-\alpha$, set $\alpha = 1$, and workers' bargaining power γ close to 0. This parameterization also strengthens the negative correlation between tightness and unemployment: the condition $\bar{\varepsilon}_u^{SR} = 0$ eliminates the main positive channel linking these two variables.

What is the intuition behind such a parameterization? Indeterminacy requires $\bar{\varepsilon}_v^{SR} > 1$, which implies that the incentive to create jobs increases with the vacancy-to-unemployed ratio in the short run. To have a cycle, two counter-acting forces lead firms to post fewer vacancies in good times: the nonphantom unemployed proportion $\pi_u = u/(u + p^u)$, which decreases when unemployment falls, and the wage $w = \gamma y + (1 - \gamma)b + \gamma c\theta$, which increases with tightness. When both forces are strong, the period of the cycle is short. The condition $\bar{\varepsilon}_u^{SR} = 0$ removes the first force, whereas $\gamma \approx 0$ considerably weakens the second one.

The period of the corresponding limit cycle is $2\pi/\sqrt{(\mu + s)(r + s)} \approx 52$ months, where $\pi = 3.14...$ is here the transcendental number. This period decreases with the job-finding rate μ, which governs the share of new phantoms in the overall phantom stock. The Hopf bifurcation theory only predicts the period of the limit cycle in the neighborhood of the bifurcation. As we further expand $\bar{\varepsilon}_v^{SR}$, this period increases as well.

From the case $\delta = s$, we persist with $\bar{\varepsilon}_u^{SR}$ and γ being low. We set $\bar{\varepsilon}_u^{SR} = 1 - \alpha = 0$ and $\gamma = 0.01$ and consider five values for the phantom death rate δ, ranging from $\delta = 1$ to $\delta = s$. The phantom birth rate $\beta = \delta$ so that the average duration β/δ of obsolete information is one month. In each simulation, we set $\bar{\varepsilon}_v^{SR}$ so that the limit cycle reproduces the US quarterly volatility of tightness.

Table 2 reports the business-cycle statistics for each limit cycle. The calibration strategy effectively increases the predicted auto-correlation of θ and u. This can be seen when $\delta = 1/2$: the auto-correlation parameters are equal to 0 compared with -0.65 in Table 1. As expected, the calibration strategy also strengthens the predicted correation between θ and u: when $\delta = 1/2$ the correlation coefficient is -0.707 compared with -0.462 in Table 1. Finally, persistence decreases with δ. The quarterly auto-correlation parameters remain negative for $\delta = 1$, but are positive for $\delta \leq 1/2$ and larger than in the data when $\delta = s$. The model, therefore, can predict the level of persistence in US data for sufficiently low values of δ.

Table 2: Business-cycle statistics as functions of $\beta = \delta$

	$\sigma(u)$	$\sigma(\theta)$	$corr(\theta, u)$	$ac(u)$	$ac(\theta)$
US data	0.125	0.259	-0.977	0.870	0.896
$\beta = \delta = 1$	0.136	0.252	-0.577	-0.575	-0.575
$\beta = \delta = 1/2$	0.176	0.265	-0.707	0.009	0.016
$\beta = \delta = 1/3$	0.188	0.257	-0.778	0.280	0.290
$\beta = \delta = 1/6$	0.208	0.254	-0.869	0.598	0.605
$\beta = \delta = s = 0.025$	0.215	0.232	-0.976	0.914	0.915

The standard deviation of unemployment $\sigma(u)$ and tightness $\sigma(\theta)$, as well as the correlation coefficient $corr(\theta, u)$ and autocorrelation coefficients $ac(u)$ and $ac(\theta)$ are computed at quarterly frequency after smoothing with the HP filter of parameter 1600.

The implications of such parameterizations are as follows. Having $\bar{\varepsilon}_u^{SR} = 1 - \alpha$ means that unlike phantom vacancies, the phantom unemployed do not alter matching efficiency. Accounting for persistence, therefore, shows that information obsolescence is a stronger problem on the vacancy side than on the unemployed one. Once combined with $\alpha = 1$, this assumption implies that the long-run elasticity $\varepsilon_v^{LR} = 1$. In usual calibrations of equilibrium search unemployment models the elasticity is often set around 1/2. However, our knowledge of empirical matching functions certainly allows cases where this elasticity lies at the upper bound of the unit interval.

Having γ small means that the wage is rigid during the business cycle. The wage is procyclical because it depends on tightness θ. With γ small, this dependence is modest and the wage fluctuates much less than tightness or unemployment. This resembles Shimer's unemployment volatility puzzle. However, aggregate volatility is not an issue here and wage rigidity is only needed to increase persistence.

Having δ small means some of the phantoms persist for a long time. As $\beta = \delta$, this is compensated by a large proportion of phantoms dying instantly so that their average lifetime β/δ is one month. These features of the distribution can be disputed. However, imposing $\beta = \delta > 6$ months seems unrealistic. Thus it is fair to say that the model underpredicts the persistence of aggregate variables for reasonable values of δ.

Do these results extend to other countries? The US job-finding rate is very large by international standards, which implies that obsolete information is mostly composed of recent phantoms. In countries where the job-finding rate is lower, calibrated limit cycles should display more persistence. For instance, in France the job-finding and job separation rates over 2000-2008 are about $\mu = 0.15$ and $s = 0.015$ (see Hairault et al, 2015, Figure 1). When $\delta = s$, the longest period of the limit cycle in the neighborhood of the Hopf bifurcation jumps to 115 months, against 52 months in the US case. Simulations similar to Table 2, not reported here, show that the auto-correlation parameters increase to 0.65 when $\beta = \delta = 1/2$ and 0.75 when $\beta = \delta = 1/3$.

5 Conclusion

Most versions of the aggregate matching function are based on contemporaneous congestion externalities. This paper focuses on an alternative source of market frictions caused by the persistence of obsolete information about traders who have already found a match. The key idea is that each new match gives birth to a pair of phantom traders who haunt the search place for some period, inducing wasted resources from unmatched traders seeking to contact them. When this is the single source of frictions, the resulting aggregate matching technology features increasing returns to scale in the short run, and constant returns in the long run.

We embed a generalized version of this technology into the continuous-time equilibrium search model of unemployment. Long-run constant returns to scale imply that there is a unique steady state with standard comparative statics properties, whereas short-run increasing returns imply excess volatility in the short run. We emphasize two main phenomena. When the short-run elasticity of the matching function with respect to vacancies is smaller than one, exogenous shocks have stronger effects in the short run than in the long run. When this elasticity is larger than one, there may be endogenous fluctuations due to self-fulfilling prophecies. When firms expect the matching process to be very efficient, they have strong incentive to post vacancies. This reduces the proportion of phantom vacancies, thereby validating the initial belief. Then this phantom proportion grows, and there is a reversal in the supply of vacancies. This mechanism implies that all trajectories lead to the steady state, or that a limit cycle emerges.

Research can be extended in two main directions. On the macro side, the model could be calibrated to other countries featuring different labor markets. This would reveal, in each case and for each market, the role played by information obsolescence. Moreover, the interplay between intratemporal and intertemporal externalities should have implications for turnover externalities. On the micro side, the parameters that govern phantom birth and death could be endogenized. Match makers spend time and money to clean their websites or to advertise for available trade partners. Lastly, match-seekers may use time spent in the market as a signal of trader's availability, thereby adjusting search strategies to mitigate the effects of phantom traders.

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A Proofs

A.1 Proof of Proposition 2

Point (i) results from direct computation. To prove (ii), we first show that M_t has constant returns to scale with respect to $S^t = (S_0, S_1, ..., S_t)$, $B_t = (B_0, B_1, ..., B_t)$, P_0^B and P_0^S. Let $H^t = (S^t, B^t, P_0^B, P_0^S)$ be the *market history.* Let also $\mathbf{M}_t : H^t \to \mathbb{R}_+$ and $\mathbf{P}_t^i : H^{t-1} \to \mathbb{R}_+$, $i = B, S$, be such that $M_t = \mathbf{M}_t(H^t)$ and $P_t^i = \mathbf{P}_t^i(H^{t-1})$. We first remark that $\mathbf{M}_t$ has constant returns to scale with respect to H^t if and only if $\mathbf{P}_t^B$ and $\mathbf{P}_t^S$ have constant returns to scale with respect to H^{t-1}. Then, suppose $t = 0$. We have $M_0 = B_0 S_0/(S_0 + P_0^S)$ if $S_0 + P_0^S > B_0 + P_0^B$ and $M_0 = B_0 S_0/(B_0 + P_0^B)$ if $S_0 + P_0^S < B_0 + P_0^B$. Multiplying each term by a similar factor $\lambda > 0$, we obtain that $\lambda S_0 + \lambda P_0^S > \lambda B_0 + \lambda P_0^B$ if and only if $S_0 + P_0^S > B_0 + P_0^B$. Thus the long side of the market stays the same. Moreover, in both cases we obtain $\mathbf{M}_0(\lambda H^0) = \lambda \mathbf{M}_0(H^0)$. So the property is true for $t = 0$. Now suppose the property holds for $t > 0$ and consider $t + 1$. We multiply all components of the market history by the same factor $\lambda > 0$. We have

$$\begin{aligned} \lambda S_{t+1} + \mathbf{P}_{t+1}^S(\lambda H^t) &= \lambda S_{t+1} + \beta^S \mathbf{M}_t(\lambda H^t) + (1-\delta)\mathbf{P}_t^S(\lambda H^t) \\ &= \lambda(S_{t+1} + \mathbf{P}_{t+1}^S(H^t)), \end{aligned} \tag{32}$$

because $\mathbf{M}_t$ and $\mathbf{P}_t$ are homogenous of degree one by assumption. Similarly, we can show that

$$\lambda B_{t+1} + \mathbf{P}_{t+1}^B(\lambda H^t) = \lambda(B_{t+1} + \mathbf{P}_{t+1}^B(H^t)). \tag{33}$$

Thus the difference $\lambda S_{t+1} + \mathbf{P}_{t+1}^S(\lambda H^t) - (\lambda B_{t+1} + \mathbf{P}_{t+1}^B(\lambda H^t))$ has the sign of the difference $S_{t+1} + \mathbf{P}_{t+1}^S(H^t) - (B_{t+1} + \mathbf{P}_{t+1}^B(H^t))$ and the long side of the market is the same again.

If $S_{t+1} + P_{t+1}^S > B_{t+1} + P_{t+1}^B$, then

$$\begin{aligned} \mathbf{M}_{t+1}(\lambda H^{t+1}) &= \frac{\lambda^2 B_{t+1} S_{t+1}}{\lambda S_{t+1} + \beta \mathbf{M}_t(\lambda H^t) + (1-\delta^S)\mathbf{P}_t^S(\lambda H^{t-1})} \\ &= \lambda \mathbf{M}_{t+1}(H^{t+1}), \end{aligned} \tag{34}$$

because $\mathbf{M}_t$ has constant returns to scale with respect to H^t by assumption and this implies that $\mathbf{P}_t^S$ has constant returns to scale with respect to H^{t-1}. The reasoning is very similar when $S_{t+1} + P_{t+1}^S < B_{t+1} + P_{t+1}^B$. Thus the property holds in $t + 1$ and the function $\mathbf{M}_t$ is homogenous of degree one with respect to the market history H^t.

The function $\mathbf{M}_t$ is differentiable by assumption. The Euler theorem implies that

$$\sum_{k=1}^{t} \left\{ \frac{\partial \mathbf{M}_t(H^t)}{\partial B_k} B_k + \frac{\partial \mathbf{M}_t(H^t)}{\partial S_k} S_k \right\} + \frac{\partial \mathbf{M}_t(H^t)}{\partial P_0^B} P_0^B + \frac{\partial \mathbf{M}_t(H^t)}{\partial P_0^S} P_0^S = \mathbf{M}_t(H^t), \tag{35}$$

Now we show that $P_0^B \partial \mathbf{M}_t(H^t)/\partial P_0^B$ and $P_0^S \partial \mathbf{M}_t(H^t)/\partial P_0^S$ tend to 0 as t tends to infinity. Let $I_t = 1$ if $S_t + P_t^S > B_t + P_t^B$ and $I_t = 0$ if $S_t + P_t^S < B_t + P_t^B$. For $j = B, S$, we have

$$\frac{\partial \mathbf{M}_t(H^t)}{\partial P_0^j} = -(1-\mu_t)\left(I_t \frac{\partial P_t^S}{\partial P_0^j} + (1-I_t)\frac{\partial P_t^B}{\partial P_0^j} \right). \tag{36}$$

Let $P_t = (P_t^B, P_t^S)'$ and A_t be the following 2×2 matrix:

$$A_t = \begin{bmatrix} 1-\delta^B-(1-I_t)\frac{\beta^B M_t}{B_t+P_t^B} & -I_t\frac{\beta^B M_t}{S_t+P_t^S} \\ -(1-I_t)\frac{\beta^S M_t}{B_t+P_t^B} & 1-\delta^S-I_t\frac{\beta^S M_t}{S_t+P_t^S} \end{bmatrix}. \tag{37}$$

We have $\partial P_t/\partial P_0^j = \Pi_{k=0}^{t-1} A_k \partial P_0/\partial P_0^j$, with $\partial P_0/\partial P_0^B = (1,0)'$ and $\partial P_0/\partial P_0^S = (0,1)'$. Suppose $I_t = 1$ for all $t \geq \tau$. Then, for all $t > \tau$:

$$\Pi_{k=\tau}^t A_k = \begin{bmatrix} 1-\delta^B & -\frac{\beta^B M_t}{S_t+P_t^S} \\ 0 & 1-\delta^S-\frac{\beta^S M_t}{S_t+P_t^S} \end{bmatrix} \Pi_{k=\tau}^{t-1} A_k.$$

This gives

$$\Pi_{k=\tau}^t A_k = \begin{bmatrix} (1-\delta^B)^{t-\tau} & Z_t \\ 0 & \Pi_{k=\tau}^t\left(1-\delta^S-\frac{\beta^S M_t}{S_t+P_t^S}\right) \end{bmatrix},$$

where $Z_t = (1-\delta^B)Z_{t-1} - \frac{\beta^B M_t}{S_t+P_t^S}\Pi_{k=\tau}^{t-1}\left(1-\delta^S-\frac{\beta^S M_k}{S_t+P_k^S}\right)$. As $M_k/(S_t+P_k^S) < 1$ for all $k \geq \tau$, we have $\lim_{t\to\infty} Z_t = 0$, and all the coefficients of the matrix above tend to 0 when t tends to infinity. It follows that $\partial P_t/\partial P_0^j = \Pi_{k=0}^{t-1} A_k \partial P_0/\partial P_0^j$ tends to $(0,0)$. We can reach the same conclusion when $I_t = 0$ for all $t \geq \tau$. Thus $\lim_{t\to\infty} \partial \mathbf{M}_t(H^t)/\partial P_0^i = 0$ for $i = B, S$ and

$$\lim_{t\to\infty}\sum_{k=1}^t \left\{\frac{\partial \ln \mathbf{M}_t(H^t)}{\partial \ln B_k} + \frac{\partial \ln \mathbf{M}_t(H^t)}{\partial \ln S_k}\right\} = 1. \tag{38}$$

A.2 Proof of Proposition 5

We first rewrite the dynamic system in a more convenient way. Let $x_i = p^i + \beta^i u$, $i = u, v$. We have

$$\begin{aligned} (1-\alpha)\frac{\dot\theta}{\theta} &= \frac{\dot\pi}{\pi} + r + s - (1-\gamma)\frac{y-b}{c}\pi\theta^{\alpha-1} + \gamma\pi\theta^\alpha, &(39)\\ \dot u &= s(1-u) - \pi\theta^\alpha u, &(40)\\ \dot x_i &= \beta_i s + \beta_i(\delta_i - s)u - \delta_i x_i, &(41)\end{aligned}$$

with $\pi = \pi_u^{\lambda_u-1+\alpha}\pi_v^{\lambda_v-\alpha}$, $\pi_u = u/((1-\beta^u)u + x_u)$, $\pi_v = \theta u/(\theta u + x_v - \beta^v u)$.

Let $X = (\theta, u, x_u, x_v)'$. The dynamic system is $\dot X = f(X)$. We study it in the neighborhood of the steady state $X^* = (\theta^*, u^*, x_u^*, x_v^*)'$. Let $Y = X - X^*$. Then the linarized system is

$$\dot Y = JY,$$

where J is the Jacobian matrix of the function f evaluated in steady state. We have

$$J = \begin{bmatrix} \frac{I_{11}}{1-\bar\varepsilon_v^{SR}} & \frac{I_{12}}{1-\bar\varepsilon_v^{SR}} & \frac{I_{13}}{1-\bar\varepsilon_v^{SR}} & \frac{I_{14}}{1-\bar\varepsilon_v^{SR}} \\ -\frac{\mu}{\theta}u\bar\varepsilon_v^{SR} & J_{22} & (\bar\varepsilon_u^{SR}-1+\alpha)/\sigma^u & (\bar\varepsilon_v^{SR}-\alpha)/\sigma^v \\ 0 & \beta^u(\delta^u - s) & -\delta^u & 0 \\ 0 & \beta^v(\delta^v - s) & 0 & -\delta^v \end{bmatrix}, \tag{42}$$

with

$$I_{11} = -\left[(\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha)\right]\bar{\varepsilon}_v^{SR} + (r+s)(1-\bar{\varepsilon}_v^{SR}) + \gamma\mu,$$

$$\begin{aligned} I_{12} &= -\frac{(\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha)\theta}{u\mu}\left[\mu + s + (\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha) + \frac{\delta^u(\delta^u - s)}{\mu+\delta^u}\right] \\ &\quad -\frac{(\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha)\theta}{u\mu}\left[\mu + s + (\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha) + \frac{\delta^v(\delta^v - s)}{\mu+\delta^v}\right] \\ &\quad -\frac{\theta}{u\mu}(r+s)\left[(\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha)\right], \end{aligned}$$

$$I_{13} = \frac{(\bar{\varepsilon}_u^{SR} - 1 + \alpha)\theta}{\mu\sigma^u u}\left\{(\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) - \delta^u + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha) + r + s\right\},$$

$$I_{14} = \frac{(\bar{\varepsilon}_v^{SR} - \alpha)\theta}{\mu\sigma^v u}\left\{(\mu+\delta^u)(\bar{\varepsilon}_u^{SR} - 1 + \alpha) + (\mu+\delta^v)(\bar{\varepsilon}_v^{SR} - \alpha) - \delta^v + r + s\right\},$$

$$J_{22} = -(s+\mu) - (\bar{\varepsilon}_u^{SR} - 1 + \alpha)(\mu+\delta^u) - (\bar{\varepsilon}_v^{SR} - \alpha)(\mu+\delta^v).$$

When $\delta^u = \delta^v = s$, $J_{32} = J_{42} = 0$. The characteristic polynomial of J is $\det(J - \lambda I) = (s+\lambda)^2(\lambda^2 - T\lambda + D)$, where T and D are, respectively, the trace and the determinant of the submatrix

$$\underline{J} = \begin{bmatrix} \frac{I_{11}}{1-\bar{\varepsilon}_v^{SR}} & \frac{I_{12}}{1-\bar{\varepsilon}_v^{SR}} \\ -\frac{\mu}{\theta}u\bar{\varepsilon}_v^{SR} & -(\bar{\varepsilon}_u^{SR} + \bar{\varepsilon}_v^{SR})(s+\mu) \end{bmatrix}. \tag{43}$$

We have

$$\begin{aligned} T &= \frac{-(\mu+s)\bar{\varepsilon}_u^{SR} - (r+s)\bar{\varepsilon}_v^{SR} + r + s + \gamma\mu}{1-\bar{\varepsilon}_v^{SR}}, \\ D &= -\frac{(\mu+s)[\bar{\varepsilon}_u^{SR}(r+s+\gamma\mu) + \bar{\varepsilon}_v^{SR}\gamma\mu]}{1-\bar{\varepsilon}_v^{SR}}. \end{aligned}$$

There are three backward and one forward variables. The characteristic polynomial has one negative real root $-s$ of algebraic multiplicity two. The other roots depend on D and T. Namely, there are two real eigenvalues of opposite signs when $D > 0$, which is equivalent to $\bar{\varepsilon}_v^{SR} < 1$. Thus part (i) is proved. The two eigenvalues have negative real parts when $D > 0$ and $T < 0$. This is equivalent to $1 + \frac{\gamma\mu - (\mu+s)\bar{\varepsilon}_u^{SR}}{r+s} > \bar{\varepsilon}_v^{SR} > 1$, and so part (ii) is proved. Finally, the two eigenvalues have positive real parts when $D > 0$ and $T > 0$. This is equivalent to $\bar{\varepsilon}_v^{SR} > 1 + \max\{0, \frac{\gamma\mu - (\mu+s)\bar{\varepsilon}_u^{SR}}{r+s}\}$, which proves part (iii).

B Maximizing the period of the limit cycle

Consider Figure 1, and focus on the frontier between the SINK and SOURCE areas, i.e. the set of Hopf bifurcations. For all points on this frontier, the Jacobian matrix of the linearized system admits a pair of purely imaginary eigenvalues. Consider one of these points, and suppose that such eigenvalues are $\lambda = \pm i\omega$, $\omega > 0$. In the neighborhood of this point, the Hopf bifurcation theory predicts that the period of the limit cycle is equal to $2\pi/\omega$, where $\pi \approx 3.14...$ is here the transcendental number. Improving persistence thus involves reducing ω.

We cannot compute ω in the four-dimensional general case. Instead, we adopt the restriction made by Proposition 5 and assume $\delta^u = \delta^v = s$. We set $\bar{\varepsilon}_v^{SR} > 1$ and compute the two complex eigenvalues. This gives $T/2 \pm i\sqrt{D}$, where T and D are defined in the proof of Proposition 5. Thus we have to select γ, $\bar{\varepsilon}_u^{SR}$ and $\bar{\varepsilon}_v^{SR}$ to minimize D under the constraints $T = 0$, $\bar{\varepsilon}_v^{SR} > 1$ and $\bar{\varepsilon}_u^{SR} \geq 1 - \alpha$. From the constraint $T = 0$ we can express $\bar{\varepsilon}_v^{SR}$ as a function of the different parameters and the other control variables:

$$\bar{\varepsilon}_v^{SR} = \frac{r + s + \gamma\mu - (\mu + s)\bar{\varepsilon}_u^{SR}}{r + s}.$$

The optimal commands solve the following minimization problem:

$$\min_{\alpha,\gamma,\bar{\varepsilon}_u^{SR}} \left\{ (\mu + s) \frac{\bar{\varepsilon}_u^{SR}(r + s + \gamma\mu)(r + s) + (r + s + \gamma\mu - (\mu + s)\bar{\varepsilon}_u^{SR})\gamma\mu}{\gamma\mu - (\mu + s)\bar{\varepsilon}_u^{SR}} \right\}$$

subject to $\bar{\varepsilon}_u^{SR} \geq 1 - \alpha \equiv \bar{\alpha} \geq 0$ and $\gamma\mu > (\mu + s)\bar{\varepsilon}_u^{SR}$.

The objective strictly decreases with $\bar{\varepsilon}_u^{SR}$ as long as $\gamma\mu > (\mu + s)\bar{\varepsilon}_u^{SR}$. Therefore the constraint $\bar{\varepsilon}_u^{SR} \geq \bar{\alpha}$ binds with $\bar{\alpha}$ small. Let $f(x) = \frac{\bar{\alpha}(r+s+x)(r+s)+(r+s+x-(\mu+s)\bar{\alpha})x}{x-(\mu+s)\bar{\alpha}}$. The derivative of f with respect to x has the sign of the following polynomial: $P(x) = x^2 - 2(\mu + s)\bar{\alpha}x - (\mu+s)\bar{\alpha}[(r-\mu)\bar{\alpha}+r+s] - (r+s)^2\bar{\alpha}$. As $P((\mu+s)\bar{\alpha}) = -(r+s)\bar{\alpha}[(\mu+s)(1+\bar{\alpha})+r+s] < 0$, we have $f'((\mu + s)\bar{\alpha}) < 0$. Thus the function f reaches its minimum when $P(x) = 0$ with $x > (\mu + s)\bar{\alpha}$. This gives

$$x = (\mu + s)\bar{\alpha} + \sqrt{(r + s)\bar{\alpha}[(\mu + s)(1 + \bar{\alpha}) + r + s]}.$$

For $\bar{\alpha}$ sufficiently small, $(\mu + s)\bar{\alpha} < x < \mu$ and so the optimal bargaining power γ is such that $\gamma\mu = x$.

The optimal commands are $\bar{\varepsilon}_u^{SR} = \bar{\alpha} = 0$ and $\gamma > 0$ as small as possible. The period of the limit cycle is then $2\pi/\sqrt{D}$, which tends to $2\pi/\sqrt{(\mu + s)(r + s)}$ when γ tends to 0. Replacing parameters μ and s by their values gives about 52 months.