# Life Cycle Equilibrium Unemployment

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**Abstract**

This paper extends the job creation - job destruction approach to the labor market to take into account a deterministic finite horizon. As forward looking decisions about hirings and separations depend on the time over which to recoup adjustment costs (horizon effect), the life-cycle setting implies age-differentiated labor market flows. Whereas, the search effort of unemployed workers presents an age-decreasing profile, the age-dynamics of the separation rate can be either decreasing or increasing, according to the level of the productivity shocks, the bargaining power of workers and their search effort. The most realistic prediction of the model is a U-shaped profile for the separation rate over the life cycle. Worker heterogeneity in the context of undirected search implies that there is another efficiency gap, due to an intergenerational externality, which is not eliminated by the Hosios condition. We then show that age-specific policies are necessary to reach the first best allocation.

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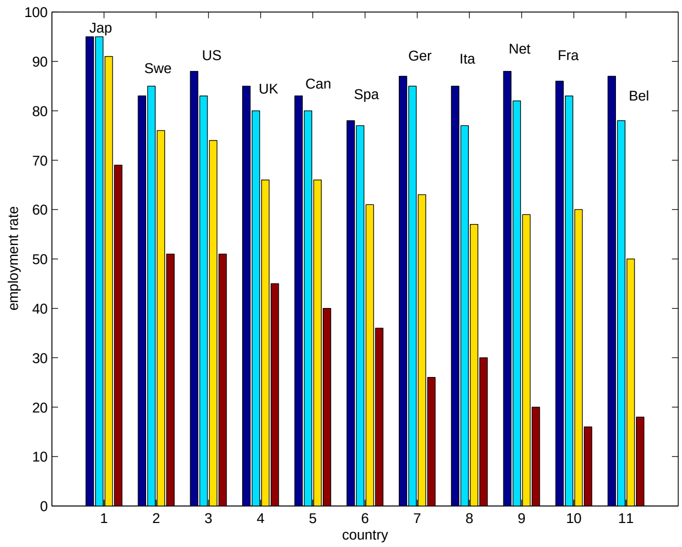
# 1 Introduction

It is now well known that the low employment rate of older workers accounts for half of the European employment gap (see OECD [2006]). Moreover, low older worker employment rates, especially before the early retirement age, are in many countries a matter of concern in the context of an aging population. A first strand of the empirical literature which attempts to explain these features emphasizes the negative role played by labor market institutions (specific insurance programs) on the job-search decisions of older unemployed workers (see for instance Blondal and Scarpetta [1998]). A second strand gives greater importance to skill obsolescence, arguing that older workers suffer from technological progress. Under wage stickiness, this gives firms incentives to send older workers into early retirement (see for instance Hellerstein, Newmark and Troske [1999]).[1]

Figure 1: Employment rates from age 30 to 64 for OECD Countries




Ranking of effective retirement age
(from the highest to the lowest)

Source: OECD data for 1995 (authors' calculation). In each country, each bar refers to employment rates of the age groups : 30 - 49 (first bar on the left), 50 - 54, 55 - 59 and 60 - 64 (last bar on the right)

However, something is missing in this picture. Figure 1 shows that the fall in the employment rate of older workers is steeper when the retirement age gets closer, whatever the country considered. Two country groups emerged very clearly in the mid-nineties: those with high employment rates for workers aged 55-59 (Canada, Great Britain, Japan, the United States and Sweden) and those which experience a huge decrease in employment rates at these ages, around 25 points with

[1]This point has already been put forward by Lazear [1979], from a theoretical standpoint.

respect to the 50-54 age group (Belgium, France, Italy and the Netherlands). As documented by Gruber and Wise [1999], the second group of countries was characterized by a normal retirement age of 60 (versus 65 in the first group). This suggests that the retirement age could affect the employment rate of older workers prior to this age: the later the retirement age, the higher the employment of older workers before 60.

In this paper, we investigate the influence of the distance to retirement on labor market job flows. We propose a canonical labor market equilibrium model with matching frictions and Nash wage bargaining, in the line of Mortensen and Pissarides [1994], characterized by a deterministic exogenous age at which workers exit the labor market, typically an exogenous retirement age. We aim to show that this canonical theoretical setting naturally emphasizes the influence of impending retirement on both job creation and job destruction, as endogenous hirings and separations depend on the expected duration of jobs. Surprisingly enough, despite the recent interest in the life-cycle approach in macroeconomic research, the implications of a deterministic finite horizon for workers in this extensively-used framework have not been yet addressed. From this point of view, our paper fills a gap.

In our model, the only heterogeneity across workers comes from the distance to retirement. Fundamentally, the expected sum of production flows generated by older workers is lower due to the shorter career duration. This horizon effect explains why labor hoarding can be weaker and separations higher at the end of the working life. It can also explain why unemployed older workers search less and then why hirings are lower for older workers. However, as wages depend on the unemployment value in the Nash bargaining setting, a shorter horizon can also make wages lower for older workers and then inverts the age-profile of the separation rate. We show that this may occur when the persistence of the productivity shocks, the bargaining power of workers and the search intensity of unemployed workers are particularly high. The age-dynamics of the search intensity is then key for both the age-profile of job creations and job separations. As it is highly age-decreasing, the natural prediction of our model is a U-shaped profile for the separation rates over the life-cycle. Overall, the search effort decisions explain why older workers simultaneously experience low hiring rates and high separation rates, whereas both the hiring and the separation rates are high for the young workers. This is partially at odds with Menzio, Telyukova and Visschers [2010] who have recently shown on US data that the flows from employment to unemployment are age-decreasing. We then argued that this is the specificity of older worker labor market to generate a specific unemployment status characterized by a so low search intensity, consistently with our model's predictions, that unemployed older workers claim to be retired. Once these population of workers is included into the unemployment pool, the proportion of individuals leaving employment increases almost continuously just before retirement.

The life-cycle equilibrium unemployment theory then intrinsically implies an heterogeneity across workers. Since Davis [2001], it is well known that heterogeneity can have important implications for the efficiency of the matching allocation when there is only one matching function. We assume in this paper that search is undirected, which is then crucial for the normative implications of the horizon effect, whereas the positive implications discussed above are independent of this assumption[2]. Although firms could implement age-directed recruitment policies, as age is an observable characteristic, there is indeed legislation prohibiting age-discrimination in the US and European countries as far as vacancies are concerned. On the other hand, as job productivity is assumed to be not age-specific[3], search equilibrium with separate markets does not exist in our economy. We then assume that firms cannot *ex-ante* age-direct their search, that is vacancies cannot be targeted at a specific age group. However, as we observe ex-post discrimination against older workers (Neumark [2001]), we assume that once the contact with a worker is made, according to the observed productivity of the job-worker pair, the firm can make the choice of not recruiting the worker. The non-directed search assumption explains why the Hosios condition is not enough to restore the social optimality of the life-cycle equilibrium. The impact of each generation of unemployed workers on the average return on vacancies, named hereafter an intergenerational externality, makes the internalization of the search costs for the other generations imperfect through the ex-post Nash wage bargaining. The job search value is then badly evaluated at the market equilibrium. We then emphasize that the horizon heterogeneity across workers implies there is a life-cycle specificity of the efficiency gap created by the non-directed search, leading to an intrinsically sub-optimal labor market tightness. Moreover, we show that there are too many separations and too high search intensity for older workers at the equilibrium. The opposite is true for younger workers. We then derive some age-specific policies in order to restore the social optimality of the decentralized life-cycle equilibrium. They share the same objective of keeping the older workers away from the search sector, either by subsidizing their jobs and/or by taxing their search effort. The latter policy is enough to make both the separation and the search effort margins efficient, as the tax on job search intervenes at the roots of the inefficiency centered on the search value. This emphasizes the crucial role played by the search effort of unemployed workers in reconciliating the equilibrium job separations with their optimal level.

In a last section we extend the life cycle equilibrium unemployment theory to endogenous retirement decisions. We firstly show that a natural implication of the search frictions is that unemployed workers become retired before employed workers. Separations after the retirement age of the unemployed workers are then actually retirement decisions: as the outside opportunities are no longer the unemployment occupation, the inefficiencies, especially the intergenerational externality, led by the search process, disappear and the retirement decisions are optimal. On the other hand, the decisions on search effort and job creation and separation before the retirement

[2] See Chéron, Hairault and Langot [] for an analysis with directed search.

[3] Technology diffusion and standardization, training programmes and worker experience are likely to make age requirements irrelevant to most jobs, except maybe for jobs requiring the use of the most recent wave of technology.

age of unemployed workers are consistent with those analyzed in our benchmark model with an exogenous retirement age.

Section 2 discusses related literature. Section 3 presents the benchmark model and the age-dynamic properties of the equilibrium. Section 4 deals with the social efficiency of the equilibrium and presents some optimal age-policies. Section 5 presents the endogenous retirement extension of the benchmark model and discusses the robustness of the main conclusions. A final section concludes.

## 2 Related literature

The horizon effect has already received some empirical support and some theoretical foundation based on the job search theory (Seater [1977], Ljungqvist and Sargent [2008] and Hairault, Langot and Sopraseuth [2008]). Our paper is the first to examine the horizon effect in the canonical matching model *à la* Mortensen-Pissarides. We then propose a life-cycle equilibrium unemployment theory with persistent productivity shocks, endogenous separations and search effort, then extended to the retirement decision. This generalizes the model proposed in Cheron, Hairault and Langot [2008], which analyzes the age-dependent employment protection in a context where the participation decisions are neglected (search effort and retirement age are exogenous) and the productivity shocks purely transitory.

We believe that this is a natural starting point for a life-cycle view of the labor market flows before incorporating additional features. Adopting a more quantitative approach, some papers use a simplified version of the Mortensen-Pissarides model, but incorporate some additional heterogeneity across ages, especially in human capital and/or in labor market institutions. In this line, Ljungvist and Sargent [2007] or Cheron, Hairault and Langot [2009] explain the differences between the US and French labor market performance. More recently, Menzio, Telyukova and Visschers [2010] have added, beyond human capital accumulation, on-the-job search in order to better match the observed wage and labor market transition age-profile on US data. On the other hand, Bettendorf and Broer [2003] and de la Croix, Pierrard and Sneessens [2010] take into account capital accumulation. By contrast, our paper rests on the canonical matching model and adopt a more analytical approach in order to derive the basics of the horizon effect. Moreover, we show, at least at the qualitative level, that the MP model only modified by the existence of a retirement age is able to generate labor market flows consistent with reality.

From a normative point of view, our analysis is related to the literature incorporating ex-ante heterogeneity into undirected search models. In Davis [2001], Acemoglu [2001] or Bertola and Caballero [1994], heterogenous jobs are allocated by a non-directed matching function to homogeneous workers. In this context, even if the Hosios condition is satisfied, there is an excessive relative supply of bad jobs because firms obtain only a fraction of the extra-surplus associated to upgrading the job quality, whereas incurring all the additional costs. Let us notice that Davis [2001] shows that the result is symetric (excessive supply of low skill workers) if workers are heterogeneous and invest in human capital. Blazquez and Jansen [2008], following Albrecht and Vroman [2002], are the first to analyze the inefficiency of the allocation when two-sided heterogeneity is introduced in the canonical Mortensen-Pissarides. Their paper extends the seminal Sattinger's [1995] paper, where the meeting rate was fixed. We share with Blazquez and Jansen [2008] the failure of the ex-post wage bargaining to internalizing search costs, leading to inefficient job creation decisions. Contrary to Blazquez and Jansen [2008], a directed search in our model is not incentive-compatible from the viewpoint of older workers as they would suffer from a lower probability of contact for the same expected productivity draws if they specialized their search to possible older worker-specific jobs. The shortcoming of the Hosios condition in the case of heterogeneity also concerns the search and participation decisions of workers. This point has been already discussed in Shimer and Smith [2001] in a search model with a fixed number of participants. Albrecht, Navarro and Vroman [2010] have recently extended this result to an endogenous participation margin.

Our paper is the first to address the efficiency issue of the heterogeneity created by the life-cycle matching model when search is undirected. We analyze in an original and unified framework all the efficiency gaps implied by the wrong evaluation of the job surplus traditionally put forward in the literature without life-cycle features, from the vacancy decision to that of separation by passing by those of search effort and participation. We emphasize the specificity of the intergenerational externality created by the age heterogeneity of workers: aging naturally leads workers to move in the distribution of this heterogenous characteristics. This implies that the gap in the job surplus valuation between the market and the planner is not constant over the expected duration of the match. Overall, our paper is the first to focus on the normative implication of age heterogeneity, which is shown to imply a specific efficiency gap due to both the horizon heterogeneity across workers and the aging process.

## 3 Job creation and job destruction when the horizon is finite

Let us consider an economy *à la* Mortensen - Pissarides [1994]. Labor market frictions imply that there is a costly delay in the process of filling vacancies, and endogenous job destructions closely interact with job creations. Unlike the large literature following MP, we consider a life cycle setting characterized by a deterministic age at which workers exit the labor market, interpreted as the retirement age.

## 3.1 Model environment

We consider a discrete time model and assume that at each period the older worker generation retiring from the labor market is replaced by a younger worker generation of the same size (normalized to unity) so that there is no labor force growth in the economy. We denote the worker's age $i$ and $T$ the exogenous age at which workers exit the labor market: they are both perfectly known by employers. There is no other heterogeneity across workers. The economy is at steady-state, and we do not allow for any aggregate uncertainty. We assume that each worker of the new generation enters the labor market as unemployed.

### 3.1.1 Shocks

Firms are small and each has one job. The destruction flows derive from idiosyncratic productivity shocks that hit the jobs at random. At the beginning of each age, a new productivity level is drawn with probability $\lambda \leq 1$ in the distribution $G(\epsilon)$, with $\epsilon \in [0, 1]$. Once a shock arrives, the firm has no choice but either to continue production or to destroy the job. For age $i \in (2, T-1)$, employed workers are faced with layoffs when their job becomes unprofitable. The firms decide to close down any jobs whose productivity is below an (endogenous) productivity threshold (productivity reservation) denoted $R_i$. Because it is assumed that the productivity value is known after firm and worker have met, workers, contacted when they were $i-1$ years old, whose productivity is below the reservation productivity $R_i$, are not hired.

### 3.1.2 Worker flows with non age-directed search

We assume that firms cannot *ex-ante* age-direct their search and that the matching function embodies all unemployed workers. Let $v$ be the number of vacancies, $u$ the number of unemployed workers, and $e_i$ the endogenous search effort for a worker of age $i$. Let us define the effective unemployment rate $\tilde{u} = \sum_{i=1}^{T-1} e_i u_i$. The matching function gives the number of contact, $M(v, \tilde{u})$, where $M$ is increasing and concave in both its arguments, and with constant returns-to-scale. From the perspective of a firm, the contact probability is $q(\theta) = \frac{M(v,\tilde{u})}{v} = M(1, \theta^{-1})$ with $\theta = \frac{v}{\tilde{u}}$ the labor market tightness accordingly defined. The probability for unemployed workers of age $i$ to be employed at age $i+1$ is then defined by $jc_i \equiv e_i p(\theta)[1 - G(R_{i+1})]$. Note that the hiring process is then age-differentiated even if firms are not allowed to age-direct their search. Similarly, we define the job destruction rate for an employed worker of age $i$ as $jd_i = \lambda G(R_i)$. For any age $i$, the flow from employment to unemployment is then equal to $\lambda G(R_i)(1 - u_{i-1})$. The other workers who remain employed $(1 - \lambda G(R_i))(1 - u_{i-1})$ can renegotiate their wage. The age-dynamic of unemployment is then given by:

$$u_{i+1} = u_i \left[1 - e_i p(\theta)(1 - G(R_{i+1}))\right] + \lambda G(R_{i+1})(1 - u_i) \quad \forall i \in (1, T-1) \tag{1}$$

for a given initial condition $u_1 = 1$. The overall level of unemployment is $u = \sum_{i=1}^{T-1} u_i$, so that the average unemployment rate is $u/[T-1]$.

### 3.2 Intertemporal values of firms and workers

**Firms.** Any firm is free to open a job vacancy and engage in hiring. $c$ denotes the flow cost of recruiting a worker and $\beta \in [0,1]$ the discount factor. Let $V$ be the expected value of a vacant position and $J_i(\epsilon)$ the value of a job filled by a worker of age $i$ with productivity $\epsilon$:

$$V = -c + q(\theta)\beta \sum_{i=1}^{T-1} \left( \frac{e_i u_i}{\widetilde{u}} \int_0^1 J_{i+1}(x)dG(x) \right) + (1-q(\theta))\beta V$$

Vacancies are determined according to the expected value of a contact with an unemployed worker. This value is the average of the expected job-worker pair values (hiring values) for the firm over the age distribution of unemployed workers. The expected value of a contact for the firm then depends on the age distribution of the unemployed workers, as uncertainty in the hiring process arises not only from productivity, but also from the age of workers. We will show in Section 5 that heterogeneity across ages in hiring values will imply the existence of intergenerational externalities in the search process.

For a bargained wage $w_i(\epsilon)$, the expected value $J_i(\epsilon)$ of a filled job by a worker of age $i$, $\forall i \in [1, T-1]$, is defined by:

$$J_i(\epsilon) \quad = \quad \max\left\{\epsilon - w_i(\epsilon) + \beta \left[\lambda \int_0^1 J_{i+1}(x)dG(x) + (1-\lambda)J_{i+1}(\epsilon)\right]; V\right\} \tag{2}$$

**Workers.** Values of employed (on a job of productivity $\epsilon$) and unemployed workers of any age $i$, $\forall i \in [1, T-1]$, are respectively given by:

$$\mathcal{W}_i(\epsilon) \quad = \quad \max\left\{w_i(\epsilon) + \beta \left[\lambda \int_0^1 \mathcal{W}_{i+1}(x)dG(x) + (1-\lambda)\mathcal{W}_{i+1}(\epsilon)\right]; \mathcal{U}_i\right\} \tag{3}$$

$$\mathcal{U}_i \quad = \quad \max_{e_i}\left\{b - \phi(e_i) + \beta \left[e_i p(\theta) \int_0^1 \mathcal{W}_{i+1}(x)dG(x) + (1-e_i p(\theta))\mathcal{U}_{i+1}\right]\right\} \tag{4}$$

with $b \geq 0$ denoting the instantaneous opportunity cost of employment and $\phi(.)$ the convex function capturing the disutility of the search effort $e_i$. Let us denote the elasticity of this disutility $\omega(e_i) \equiv \phi'(e_i)\frac{e_i}{\phi(e_i)}$, with $\omega(e_i) > 1$ because $\phi(\cdot)$ is a convex function.

### 3.3 Wage bargaining

The rent associated with a job is divided between the employer and the employee according to a wage rule. Following the most common specification, wages are determined by the Nash solution to a bargaining problem. For a given bargaining power of the workers $\gamma$, considered as constant across ages, the global surplus generated by a job $S_i(\epsilon) \equiv J_i(\epsilon) + \mathcal{W}_i(\epsilon) - \mathcal{U}_i - V$, is divided according to the following sharing rule, which is the solution of the conventional Nash bargaining problem:

$$\mathcal{W}_i(\epsilon) - \mathcal{U}_i = \gamma S_i(\epsilon) \quad \text{and} \quad J_i(\epsilon) = (1-\gamma) S_i(\epsilon) \tag{5}$$

Accordingly to the Nash bargaining, the equilibrium wage rule solves:

$$w_i(\epsilon) = \gamma\epsilon + (1-\gamma)\left[b - \phi(e_i) + \beta e_i p(\theta)\gamma \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)\right] \tag{6}$$

The higher the job surplus generated by the search process, the higher the reservation wage, the higher the bargained wage. On the other hand, it is possible to give another expression for the wage in terms of the search costs saved by the firms.

$$w_i(\epsilon) = \gamma\left[\epsilon + c\theta e_i \tau_i\right] + (1-\gamma)[b - \phi(e_i)] \tag{7}$$

where $\tau_i$ is defined as follows:

$$\tau_i \equiv \frac{\int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)}{\sum_{i=1}^{T-1}\left(\frac{e_i u_i}{\widetilde{u}} \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)\right)} \tag{8}$$

As in the MP model, the way that market tightness enters the wage equation is through the asymmetry between firms and workers in the search process. If the worker has a competitive advantage in the search process, implying $ep(\theta) > q(\theta)$, then the evaluation of the search cost is larger than $c$ in the wage equation. In our life-cycle model, there is an additional asymmetry due to the undirected search process. For the workers, the search cost saved when the match is formed is proportional to their individual hiring value. On the other hand, for the firms, this cost is proportional to the average hiring value, as the result of the non-directed search assumption. This is why the bargained wage depends on $\tau_i$, which gives the value of a worker hired at age $i$ relative to the average value of a job according to the age distribution of unemployed workers. Workers with a higher hiring value than average can capture a larger fraction of these hiring costs than workers with a lower hiring value. Despite the assumption of undirected search, implying a homogenous search cost for each match, wages are age-specific.

### 3.4 The labor market equilibrium

**Definition 1.** *Assuming a free-entry condition, leading to the zero-profit condition $V = 0$, the labor market equilibrium with search frictions in a finite-horizon environment is defined by:*

$$\begin{aligned}
\frac{c}{q(\theta)} &= (1-\gamma)\beta \sum_{i=1}^{T-1} \left( \frac{e_i u_i}{\widetilde{u}} \int_{R_{i+1}}^{1} S_{i+1}(x) dG(x) \right) & (9) \\
R_i &= b - \phi(e_i) - [\lambda - \gamma e_i p(\theta)]\beta \int_{R_{i+1}}^{1} S_{i+1}(x) dG(x) - (1-\lambda)\beta S_{i+1}(R_i) & (10) \\
S_i(\epsilon) &= \max\{\epsilon - R_i + (1-\lambda)\beta[S_{i+1}(\epsilon) - S_{i+1}(R_i)]; 0\} & (11) \\
\phi'(e_i) &= \gamma p(\theta)\beta \int_{R_{i+1}}^{1} S_{i+1}(x) dG(x) & (12) \\
u_{i+1} &= u_i \left[1 - e_i p(\theta)(1 - G(R_{i+1}))\right] + \lambda G(R_{i+1})(1-u_i) & (13)
\end{aligned}$$

*The terminal conditions are $e_{T-1} = 0$ and $R_{T-1} = b$. The initial condition is given by $u_1$.*

Equation (9) derives from the zero-profit condition $V = 0$, and shows that the labor market tightness depends on the average job surplus over the unemployed workers of different age. Equation (10) defines the productivity threshold $R_i$ from the condition $S_i(R_i) = 0$. As in MP, a crucial implication of this rule is that the job destruction is optimal, not only from the firm's point of view but also from that of the worker. $S_i(R_i) = 0$ indeed entails $J_i(R_i) = V$ and $\mathcal{W}_i(R_i) = \mathcal{U}_i$. Equation (11) is the job surplus definition using the expression of the productivity threshold. Equation (12) is the first order condition on the search effort of the unemployed worker. At the end of the life cycle, we have $e_{T-1} = 0$ because at the age $T$, agents are retired. Finally, equation (13) is the unemployment age-dynamics.

Given the homogenous labor market tightness $\theta$, solving for the age-dynamics of the productivity thresholds $R_i$ and of the search intensity $e_i$ determines the age-dynamics of job creations and job destructions, respectively $jc_i = e_i p(\theta)[1-G(R_{i+1})]$ and $jd_i = \lambda G(R_{i+1})$. These age-dynamics are complex as they result from the expectations of aging at any age in an environment characterized by an horizon heterogeneity.

# 4 Age-dynamics of job creations and job destructions

The current and expected values of the job surplus at any ages determine the job creations and job destructions. We first investigate the age-dynamics of the job surplus before solving for the age-dynamics of job creations and job destructions.

## 4.1 The job surplus heterogeneity across ages and the horizon effect

Combining equations (2), (3) and (4), the job surplus can be written at the equilibrium of the labor market as follows:

$$\begin{aligned}
S_i(\epsilon) &= \max \underbrace{\{0; \epsilon - b + \phi(e_i)}_{\text{Current surplus}} \\
&\quad \underbrace{+[\lambda - \gamma e_i p(\theta)]\beta \int_{R_{i+1}}^{1} S_{i+1}(x) dG(x)}_{\text{New opportunities}} + \underbrace{(1-\lambda)\beta S_{i+1}(\epsilon)\}}_{\text{Capitalization}} \quad \forall i < T-1 \\
S_{T-1}(\epsilon) &= \max \quad \underbrace{\{0; \epsilon - b\}}_{\text{Current surplus}}
\end{aligned}$$

The job surplus differs across ages, although there is no heterogeneity across workers, except the distance to retirement. The main difference between old and young workers comes from the larger sensitivity of the job surplus to the current surplus for the older workers. Considering the oldest workers, just before the retirement age, it is obvious that only the current surplus matters. The fundamental feature of older workers is that the remaining time on the labor market is shorter. It has two deep implications for older workers' job surplus. Firstly, the capitalization value, which is measured by the sum of any current productivity flows on the expected remaining career time, is particularly low. The older the worker, the lower the capitalization component: as for any investment, the longer the payoff, the higher the return[4]. Secondly, older workers have less time to have new opportunities through new productivity draws. These new opportunities can be obtained inside the firm with a probability $\lambda$, yielding some value to the labor hoarding strategy. A share $\gamma$ of these new opportunities can also be obtained outside the firm by an unemployed worker with a probability $e_i p(\theta)$, giving some value to the search strategy. The comparison between the arrival rate of productivity draws within the firm and the arrival rate of job offers outside the firm determines the relative value of the labor hoarding and of the search strategies. If $\lambda > \gamma e_i p(\theta)$, the search process is too slow and the bargaining power of the workers too small relative to the probability of having a new opportunity within the firm: the longer horizon of younger workers gives them a higher labor hoarding value. But, it can be the case that the persistence is so high that the search strategy becomes a better strategy than labor hoarding. This may reverse the implication of the horizon differentials in the production sector. Older workers are less likely to leave their jobs, simply because there is less time for them to reap the benefit of finding a better job in the search sector, making a shorter horizon synonymous with a lower reservation wage.

[4] Note that this is all the more true that the shock is highly persistent ($\lambda$ low). By contrast, for idiosyncratic productivity($\lambda = 1$), the older workers would not suffer from a shorter horizon, as long as the capitalization value of the current productivity is concerned.

Overall, the life-cycle equilibrium unemployment framework creates heterogeneity in the job surplus generated at any age. This heterogeneity is perfectly expected at any age and propagated through ages by all forward-looking decisions: the expectations of aging lead the terminal condition just before the retirement age to feed back on all younger ages. Using the expression of the job surplus $S_i(\epsilon)$, it is possible to analyze the impact of these expectations on the set of productivity profitable at each age. For $S_i(\epsilon) > 0$, we have:

$$\begin{aligned}\epsilon \;=\;& b-\phi(e_i)-\lambda\left[\int_{R_i}^{1} S_i(x)dG(x)-S_i(\epsilon)\right]+\gamma e_i p(\theta)\left[\int_{R_i}^{1} S_i(x)dG(x)-S_i(R_i)\right]\\ &\underbrace{-[\lambda-\gamma e_i p(\theta)]\left[\beta\int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)-\int_{R_i}^{1} S_i(x)dG(x)\right]-(1-\lambda)\left[\beta S_{i+1}(\epsilon)-S_i(\epsilon)\right]}_{Expectations\ of\ aging}\end{aligned}$$

There are two effects related to aging in the previous equation[5]:

- The variation in the asset value $\beta S_{i+1}(\epsilon)-S_i(\epsilon)$ is the loss associated with the capitalization effect: aging reduces the horizon and then the value of any particular draw $\epsilon$. In order to compensate for this loss, the current productivity must be higher.
- The term $\beta\int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)-\int_{R_i}^{1} S_i(x)dG(x)$ is the loss/gain associated with the dynamics of the expected value of the new opportunities. This dynamics is the combination of the capitalization effect on any profitable productivity and of a selection effect on the set of new opportunities: aging by reducing the horizon is a loss through the capitalization effect($S_{i+1}(x)-S_i(x)<0$), but aging can also lead to be more or less strict in the selection of the new opportunities, either inside or outside the firm, according to the age-dynamics of the productivity thresholds. This selection effect is more stringent when the age-dynamics of reservation productivity is increasing ($R_i < R_{i+1}$), reinforcing the capitalization effect to make the expected value of new opportunities (the expected job surplus) age-decreasing. On the other hand, it is less stringent for age-decreasing reservation productivity ($R_i > R_{i+1}$), which can compensate for the capitalization effect.

The expectations of aging explain why the solution for the job surplus is highly dependent on the age-dynamics of the productivity thresholds. Solving for the age-dynamics of job creations and job destructions implies to postulate the existence of a given age-profile for the productivity thresholds in order to derive its implications for the job surplus and search effort and finally to characterize the conditions under which this age-profile exists.

[5]Beyond the role of the current outside opportunity, the first two components are linked to the valuation of the new draw relatively to an initial condition. For a given age $i$, $\int_{R_i}^{1} S_i(x)dG(x)-S_i(\epsilon)$ is the surplus associated to a new draw "inside" the firm. For a given age $i$, $\int_{R_i}^{1} S_i(x)dG(x)-S_i(R_i)$ is the surplus associated to a new draw "outside" the firm.

### 4.2 Age-increasing productivity thresholds: the low persistence case

We first consider the case with age-increasing productivity thresholds.

**Proposition 1.** *If $R_i < R_{i+1}$, $\forall i$, then the job surplus, the expected job surplus and the productivity reservation are solution to:*

$$S_i(\epsilon) = \sum_{j=0}^{T-1-i} \beta^j (1-\lambda)^j \max\{\epsilon - R_{i+j}; 0\} \quad (14)$$

$$ES_{i+1} \equiv \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x) = \sum_{j=0}^{T-1-i} \beta^j (1-\lambda)^j I(R_{i+j}) \quad (15)$$

$$R_i = b - \phi(e_i) - [\lambda - \gamma e_i p(\theta)]\beta ES_{i+1} \quad (16)$$

*with the terminal conditions $R_{T-1} = b$ and $e_{T-1} = 0$.*

*Proof.* Straightforward using equations (10) and (11) and noticing that $I(R_{i+j}) \equiv \int_{R_{i+j}}^{1}(x - R_{i+j})dG(x) = \int_{R_{i+j}}^{1}(1 - G(x))dx$. □

When the age-profile of the productivity thresholds is increasing, this implies that the current reservation productivity may become unprofitable at the next age. This is unambiguously the case for the marginal job productivity: $S_{i+1}(R_i) = 0$. The job surplus is lower for the older workers due to this more stringent selection effect. This reinforces the capitalization effect, due to a summation over a shorter horizon, to make the older workers' surplus lower.

**Proposition 2.** *If $R_i < R_{i+1}$, then $e_i > e_{i+1}$.*

*Proof.* From equation (15), we deduce that $ES_i - ES_{i+1} = [I(R_i) - I(R_{i+1})] + \beta(1-\lambda)(ES_{i+1} - ES_{i+2})$ with $I(R_{i+j}) = \int_{R_{i+j}}^{1}(1 - G(x))dx$. As $ES_{T-2} > ES_{T-1}$, it is then straightforward that $ES_i \geq ES_{i+1}, \forall i$. Then, using equation (12), we obtain $e_i > e_{i+1}$, given that $\phi''(\cdot) > 0$. □

As the expected job surplus is age-decreasing when the productivity thresholds are age-increasing, the return on searching for the unemployed workers is lower for older workers: the search effort is then age-decreasing. As the retirement age gets closer, the return on job-search investments decreases because the horizon over which workers can recoup their investment is reduced on the one hand and because less jobs are profitable due to a more severe selection effect on the other hand. This contributes to generating age-decreasing hiring rates. From Proposition 2, it is straightforward that both job creation and job destruction are age-increasing when $R_i < R_{i+1}$.

**Proposition 3.** *If $\lambda \geq \gamma p(\theta) e_i$, $R_i < R_{i+1}$, $\forall i < T-1$. A sufficient condition ensuring $R_i < R_{i+1}$ is then $\lambda \geq \gamma$.*

*Proof.* Consistently with equations (12) and (16), the solution for the productivity reservation can be written as follows:

$$\gamma p(\theta)R_i = \gamma p(\theta)b - \underbrace{[\gamma p(\theta)\phi(e_i) + (\lambda - \gamma p(\theta)e_i)\phi'(e_i)]}_{\equiv \Upsilon(e_i)}$$

$R_i$ is then age-increasing if and only if

$$\begin{array}{rcccl} \gamma p(\theta)(R_{i+1} - R_i) & = & \Upsilon(e_i) - \Upsilon(e_{i+1}) & \geq & 0 \qquad \forall i < T-2 \\ \gamma p(\theta)(R_{T-1} - R_{T-2}) & = & \Upsilon(e_{T-2}) & \geq & 0 \end{array}$$

We have $R_i \leq R_{i+1}$, $\forall i < T-2$, implying that $e_i \geq e_{i+1}$ (Proposition 2), if and only if $\Upsilon'(e_i) \geq 0$. As $\Upsilon'(e_i) = (\lambda - \gamma p(\theta)e_i)\phi''(e_i)$, the condition $\lambda \geq \gamma p(\theta)e_i$, $\forall i$, must be satisfied. On the other hand, for $i = T-2$, the terminal restriction is given by $\Upsilon(e_{T-2}) \geq 0 \Leftrightarrow \lambda \geq \left(\frac{\omega(e_{T-2})-1}{\omega(e_{T-2})}\right)\gamma p(\theta)e_{T-2}$ which is a less restrictive condition than $\lambda \geq \gamma p(\theta)e_{T-2}$. This is why, $\forall i < T-1$, $\lambda \geq \gamma p(\theta)e_i$ implies that $R_i < R_{i+1}$. Finally, because $e_i p(\theta) < 1$, a sufficient condition for $R_i < R_{i+1}$ is then $\lambda \geq \gamma$ □

Proposition 3 states that older workers face more job separations, that is $R_i < R_{i+1}$, when the persistence of the productivity level is low enough to make the value of future opportunities unambiguously better inside the firm than outside. This is the case because the arrival rate of new draw inside the firm is higher than the arrival rate of job offers. Because the horizon of older workers is shorter, firms invest less in labor-hoarding activities at the end of the life cycle.

## 4.3 Age-decreasing productivity thresholds: the high persistence case

Let us now turn to the case with continuous age-decreasing productivity thresholds.

**Proposition 4.** *If $R_i > R_{i+1}$, $\forall i$, then the job surplus, the expected job surplus and the productivity reservation are solution to:*

$$S_i(\epsilon) = P_i(T)\max\{\epsilon - R_i; 0\} \quad \text{with } P_i(T) = \textstyle\sum_{j=0}^{T-1-i} \beta^j (1-\lambda)^j \tag{17}$$

$$ES_{i+1} = \int_{R_{i+1}}^{1} S_i(x)dG(x) = P_{i+1}(T)I(R_{i+1}) \tag{18}$$

$$R_i = b - \phi(e_i) - [\lambda - \gamma e_i p(\theta)]\beta ES_{i+1} - (1-\lambda)\beta S_{i+1}(R_i) \tag{19}$$

*with the terminal conditions $R_{T-1} = b$ and $e_{T-1} = 0$.*

*Proof.* Straightforward using equations (10) and (11) and noticing that $I(R_{i+1}) \equiv \int_{R_{i+1}}^1 (x - R_{i+j})dG(x) = \int_{R_{i+1}}^1 (1 - G(x))dx$. □

When the reservation productivity profile is age-decreasing, a job opened at a given age remains profitable at older ages until retirement as long as the level of productivity is unchanged. This implies that the capitalization effect depends on the degree of persistence as in MP, but also, more interestingly, on the horizon of the workers until retirement through the variable $P_i(T)$. Younger workers yield a higher job surplus for any given level of productivity since $P_i(T) > P_{i+1}(T)$. However, when the productivity thresholds are age-decreasing, the selection effect is more stringent for younger workers: less jobs are profitable, although the distance to retirement is longer. This is why the expected job surplus is not necessarily lower for older workers, as the capitalization effect and the selection effect go in the opposite direction, as can be deduced from equation (18):

$$ES_i - ES_{i+1} = \underbrace{[P_i(T) - P_{i+1}(T)]I(R_i)}_{capitalization\ age-dynamics} - \underbrace{P_{i+1}(T)\int_{R_{i+1}}^{R_i}[1 - G(x)]dx}_{selection\ age-dynamics} \tag{20}$$

Given that $P_i(T) > P_{i+1}(T)$ and $R_i > R_{i+1}$, $ES_i \lesseqgtr ES_{i+1}$. The expected surplus is age-decreasing when the effect of the capitalization age-dynamics dominates the effect of the selection age-dynamics.

**Proposition 5.** *If $R_i > R_{i+1}$, then $e_i \lesseqgtr e_{i+1}$. If the effect of the capitalization age-dynamics dominates the effect of the selection age-dynamics, then $e_i > e_{i+1}$.*

*Proof.* Using equation (12) and (20), we then deduce that $e_i \lesseqgtr e_{i+1}$, given that $\phi''(\cdot) > 0$. If the effect of the capitalization age-dynamics dominates that of the selection age-dynamics, i.e. $[P_i(T) - P_{i+1}(T)]I(R_i) > P_{i+1}(T)\int_{R_{i+1}}^{R_i}[1 - G(x)]$, then $ES_i > ES_{i+1}$ and $e_i > e_{i+1}$. □

The search effort of older workers is lower when the expected return on searching for a job is age-decreasing. When $R_{i+1} < R_i$, it is the case only when the effect of the capitalization age-dynamics overcompensate for the effect of the selection age-dynamics. If $R_{i+1} < R_i$, the job destructions are age-decreasing, whereas the age-dynamics of the job creations is indeterminate.

**Proposition 6.** *A necessary condition ensuring $R_i > R_{i+1}$, $\forall i < T - 1$, is $\lambda < \gamma$. An age-decreasing dynamics for $e_i$ is a sufficient condition to ensure that $R_i > R_{i+1}$, $\forall i < T - 1$, provided that the terminal condition $R_{T-2} > R_{T-1}$ holds.*

*Proof.* Combining equations (12) and(19) leads to:

$$\gamma p(\theta)R_i = \gamma p(\theta)b - \Upsilon(e_i) + \gamma p(\theta)\beta(1 - \lambda)P_{i+1}(T)(R_{i+1} - R_i)$$

Then, $R_i$ is age-decreasing if and only if

$$\begin{array}{rcllr} R_{i+1} - R_i &=& \frac{\Upsilon(e_i)-\Upsilon(e_{i+1})+\gamma p(\theta)\beta(1-\lambda)P_{i+2}(T)(R_{i+2}-R_{i+1})}{\gamma p(\theta)[1+\beta(1-\lambda)P_{i+1}(T)]} &<& 0 \qquad \forall i < T-2 \\ R_{T-1} - R_{T-2} &=& \frac{\Upsilon(e_{T-2})}{\gamma p(\theta)[1+\beta(1-\lambda)]} &<& 0 \end{array}$$

If the terminal restriction is satisfied ($R_{T-1} - R_{T-2} < 0 \Leftrightarrow \Upsilon(e_{T-2}) < 0$), by backward induction, it is sufficient to determine the restriction which ensures $R_{i+1} - R_i < 0$, given that $R_{i+2} - R_{i+1}$ is negative. The terminal condition is: $\lambda < \left(\frac{\omega(e_{T-2})-1}{\omega(e_{T-2})}\right)\gamma p(\theta)e_{T-2}$, which is more restrictive than the condition $\lambda < \gamma p(\theta)e_{T-2}$, with $e_{T-2} = \phi'^{-1}(\gamma p(\theta)\beta\int_b^1(1-G(x))dx)$. If $e_i$ is age-decreasing and $\Upsilon' < 0$, i.e. $\lambda < \gamma p(\theta)e_i, \forall i < T-2$, then $\Upsilon(e_i) - \Upsilon(e_{i+1}) < 0$ and $R_{i+1} - R_i < 0, \forall i < T-2$. Consistently with the age-decreasing dynamics of $e_i$, the condition $\lambda < \gamma p(\theta)e_i$ is the most restrictive for $i = T-2$, implying that the terminal condition is enough to ensure that $R_{i+1} < R_i$. A necessary condition for $R_{i+1} < R_i$ is then $\lambda < \gamma$. $\square$

Proposition 6 shows that a high degree of persistence ($\lambda < \gamma$ at least) implies that the search sector gives more chances than the firm to have new opportunities. Let us emphasize that $\lambda < \gamma$ is only a necessary condition for continuously age-decreasing productivity thresholds. Indeed, the condition $\lambda < \gamma p(\theta)e_i,\ \forall i < T-2$ can be more restrictive for low values of the search intensity, especially for older ages if the search intensity is age-decreasing. Ceteris paribus, when the persistence is particularly high, the shorter horizon of older workers may lead to less separations on their jobs. However, this conclusion is valid only if the search value is lower for older workers. This is the case if the capitalization effect dominates the selection effect, i.e. the search effort is age-decreasing. There are indeed two opposite forces induced by the age-decreasing productivity thresholds which could lead to a self-inconsistency. Firstly, if the reservation productivity is lower and lower as workers age, the selection process is less and less stringent at the end of the working life, and the new opportunities in the search sector are easier and easier to reach for older workers. This dynamic selection effect, ceteris paribus, implies that the older workers are more fired, which is not internally consistent. Secondly, as the horizon over which the next reservation productivity will be capitalized is longer in the case of younger workers, the age-profile of the search value may be increasing, the capitalization effect compensating for the dynamic selection effect. In this case, the search value of older workers is lower whereas there are less job separations for older workers, i.e. $R_i$ is age-decreasing. In a nutshell, when the persistence of the shocks is high, workers accept a lower wage at the end of the working life if aging is viewed as a reduction in the value of outside opportunities, which is consistent with the domination of the capitalization effect.

### 4.4 Discussion

**Model predictions.** In the two previous sections, the age-profile of job creation and separation have been characterized. Whatever the case considered, the horizon effect is at the origin of heterogeneous flows across ages independently of any other factors. The shorter horizon of the older workers unambiguously leads to a lower capitalization effect over any profitable jobs. This explains why the job separation rates may be higher and the job creation rate weaker when retirement is imminent. The shorter horizon also reduces the number of new opportunities in the case of bad shocks. If the persistence of the shock is weak, this feature reinforces the implications of the capitalization effect. However, if the persistence is high, searching for a new job is a better strategy than labor hoarding, especially for younger workers with a longer horizon. In that case, there are less separations on the older worker jobs. The degree of persistence relative to the bargaining power of workers then appear a crucial feature determining the age-profile of job separation.

However, the dynamics of the search intensity over the life cycle is also key for that of the separation rate. Proposition 6 naturally predicts that the age-profile of the separation rate is hardly age-decreasing at the end of the life-cycle: as the search intensity is low for these ages, an age-increasing dynamics is consistent with almost any values of the persistent level. This prediction implies that the life-cycle matching model may help to explain why countries experience a drop in their employment rate at the end of the working life in Figure 1 and why it occurs at different ages. The important dimension of the model is indeed the retirement age. Only the distance between the current age and the retirement age matters according to a horizon effect. The biological age does not matter in itself. These results suggest that the observed low employment rate of near-to-retirement people cannot be considered as a reason for not postponing the age of retirement. The reasoning is completely reversed: retirement postponement is actually likely to increase the employment rate of these workers, thereby contradicting the widespread view that the low employment rate of older workers makes any extension of the retirement age pointless[6].

Can the age-dynamics of the job separation rates be continuously increasing over the life cycle? Proposition 3 presents the maximal shock duration able to ensure an age-increasing separation rate: $\lambda$ must higher than $\gamma$. For instance, for a realistic value of $\gamma$ equal to 0.5, the expected duration of a shock must be lower than 2 years. Such values of persistence are not necessarily unrealistic. Indeed, it is assumed that the shocks uniformly hit the workers, whatever their age, in order to focus on the horizon effect. In that sense, these shocks drive worker flows, and not job flows as in Mortensen and Pissarides [1994]. The persistence of these job-worker pair shocks is lower than the persistence of the job-technology pair shocks. Indeed, there exists some empirical evidence of a low persistence of the shocks that hit job-worker pairs[7]. For instance, in Ljungqvist and Sargent [2008], the earnings are the product of the wage by unit of human capital by the level of the human capital. In this context, they consider that the average time between wage draws is 1.9 years within a skilled occupation, whereas the average duration for a skilled occupation is 1.435 years.

However, it is certainly more relevant to believe that the reservation productivity displays a U-shaped age-profile. If $\lambda < \gamma$, a continuous age-increasing profile for the productivity thresholds is no longer ensured. In that case, the age-profile of the search effort becomes determinant.

[6]See Hairault, Langot and Sopraseuth [2010] for such policy experiments.

[7]The average duration of a job for a given worker is lower than the average duration of an equipment in a firm.

As the younger workers are likely to intensively search for a job, the reservation productivity age-dynamics may be U-shaped.

**Proposition 7.** *A U-shaped reservation productivity age-dynamics is such that $R_i > R_{i+1}$ for $i < \widehat{i} - 1$ and $R_i < R_{i+1}$ for $i \geq \widehat{i} - 1$. The age $\widehat{i}$ is defined by the condition $\lambda = \gamma p(\theta) e_{\widehat{i}}$. A sufficient condition for the existence of this equilibrium is $\lambda < \gamma p(\theta) e_{\widehat{i-j}}$ and $\lambda > \gamma p(\theta) e_{\widehat{i+j}}$ for $j > 0$, provided that the search effort $e_i$ is age-decreasing.*

*Proof.* See Appendix C. □

The intuition of Proposition 7 is straightforward[8]. Whereas the reservation productivity is age-increasing for older workers until retirement, the age-profile can be inverted below a given age $\hat{i}$ if the search value dominates the labor hoarding value when considering younger workers with higher search intensity. The age $\widehat{i}$ is defined such that workers are indifferent between labor hoarding and search strategies. This is the case when the search intensity is such that $\lambda = \gamma p(\theta) e_{\widehat{i}}$. Below this age ($i < \widehat{i}$), searching for a job dominates labor hoarding ($\lambda < \gamma p(\theta) e_i$) and the productivity thresholds are age-decreasing. Above this age ($i > \widehat{i}$), the opposite is true ($\lambda > \gamma p(\theta) e_i$) and the productivity thresholds are age-increasing. Over the life-cycle, the age-dynamics of the reservation productivity is then U-shaped when the search intensity is high enough to invert the age-increasing profile featuring older ages. In that case, the job creations are unambiguously age-decreasing after $\widehat{i}$, whereas this age-profile is indeterminate before $\widehat{i}$ when considering younger workers, as the result of the two opposite dynamics on $R_i$ and $e_i$.

**What are the facts?** Menzio, Telyukova and Visschers [2010] have recently shown on US data that both the flows from employment to unemployment and the flows from unemployment to employment are age-decreasing. More precisely, the job creation rate is fairly flat over the life cycle, with a large decline after 50, whereas the age-profile of the job separation rate has a more pronounced downward slope over the life-cycle. These results on job separations are partially at odds with those shown in a previous paper (Chéron, Hairault and Langot [2009]), which are characterized by a U-shaped profile over the life cycle on US and French data (Labor Force Survey): the job separation rate would increase at the end of the working life before retirement. On the other hand, there is no discrepancy on the age-profile for the job creations. The antagonist result on the separation flows at the end of the working life relies on the difference in the pool of unemployed workers considered. Even more than for the other ages, the unemployment occupation definition is too restrictive when based on the self-declaration of looking for a job[9].

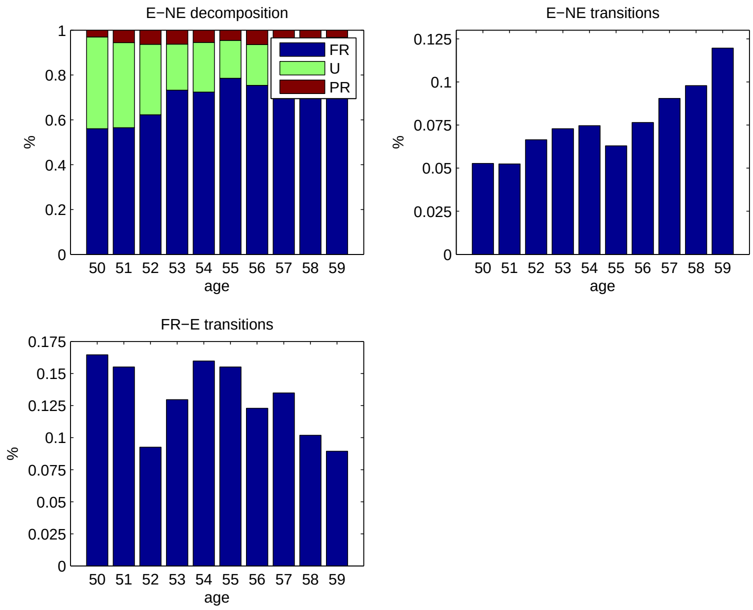
[8]Though intuitive, Proposition 7 implies some complexity to be proved due to the inversion of the age-dynamics after a given age $\widehat{i}$. See Appendix C.

[9]The seminal Blanchard and Diamond [1989],[1990] works suggest to be less restrictive concerning the unemployment definition: one half of the new matches does not come from this set of unemployed people, showing that workers not reporting "looking for a job" search because some of them obtain a job.

We believe that this is the specificity of older worker labor market to generate a specific unemployment status characterized by a so low search intensity that unemployed people claim to be retired, as the probability to become employed again is very weak. This is in line with our model predictions. They are actually unemployed workers with a particularly low search intensity, and not actually retired people as they claimed to be.

Using the HRS data set allows us to illustrate this point by focusing on the older worker status. In the HSR survey, there is a clear distinction between retirement and not in the labor force status, and also between partially retired and fully retired. The former people are looking for a job, and then can unambiguously be associated with unemployed workers, whereas the latter workers declare not searching. For people between 50 and 62 (the early eligibility age to Social Security), the "fully retired" position becomes dominant among the not-employed people. More particularly, it is striking to see on Figure 2 that the structure of job separations are highly biased toward fully retirement. As people age, the share of the retirement status grows as that of the unemployed falls. The proportion of those leaving work to become partly retired remains stable at a low level. For instance, 56% of people of age 50 leaving employment declare to be fully retired. This proportion jumps to 85% around 60, when workers are closer to the age at which it is possible to withdraw the SS benefits.

Figure 2: The full retirement status into question




Source: HRS data for 1995 (authors' calculation). E-NE decomposition is the structure of the separations. E-NE transitions refer to separations from employment to (unemployment+fully retirement). FR-E transitions are job findings from fully retired people.

Are these workers still participating despite their claim to be fully retired? We believe that they may be considered as unemployed workers with a particularly low search intensity. Figure 2 shows the probability of working for individuals “retired" at different age. What pops up is that although fully retired individuals self-declare not looking for a job, and despite certainly their low search effort, there is a significant fraction of them who works in the future. Once the “fully retired" workers are included in the pool of workers searching for a job, Figure 2 shows that the evolution of separation rates from employment to unemployment is dramatically changed[10]: it increases almost continuously from age 50 to age 60. In that case, our model can simultaneously replicate the age-decline of the job creation and the age-increase of the separation rate when the retirement age is approaching[11].

This interpretation of the fully retired status for the workers in their fifties is debatable and a more comprehensive empirical analysis would be needed to conclude[12]. We acknowledge that retirement and unemployment with low search effort are hardly distinguishable. However, it is beyond doubt that the end of the working life intrinsically generates this mixed status between retirement and unemployment. From our point of view, a key point is that the job separations toward this mixed status are strongly affected by the distance to retirement. In other words, an increase in the "real" retirement age would eliminate most of these separations and increase the search intensity for a given age[13], as the distance to retirement would be higher. Section 6, by making the retirement age endogenous, will help us to shed more light on these interactions between the decisions underlying retirement and labor market flows.

# 5 Optimal allocation and age-policy

The previous sections highlighted economic rationales sustaining age heterogeneity of job separations and job creations. It remains to show that this behavior is at odds with social optimality. The existence of an additional externality, namely an intergenerational externality, could lead to reconsidering the efficiency result obtained in an infinite horizon when the Hosios condition holds. Specific policies, in particular designed by age, should then be implemented to restore the social optimality.

## 5.1 The efficient allocation

The problem of the planner is to determine the optimal allocation of each worker between the production and the search sectors, and the optimal investment in the search sector. The per-

[10]Leaving aside this population leads to an age-decreasing separation rate as shown by Menzio, Telyukova and Visschers [2010].

[11]Menzio, Telyukova and Visschers [2010] explain the age-decreasing separation dynamics they observe by taking into account the higher human capital of older workers in order to compensate for the horizon effect.

[12]This is beyond the scope of this paper to provide with such empirical analysis.

[13]See Hairault, Langot and Sopraseuth [2010] for such counterfactual experiments.

unemployed worker social value in the search sector and the per-employed worker social value in the good sector are respectively given by:

$$
\begin{aligned}
Y_i^s &= b - \phi(e_i^\star) - c e_i^\star \theta^\star + e_i^\star p(\theta^\star)\beta \int_0^1 Y_{i+1}(x)dG(x) + (1 - e_i^\star p(\theta^\star))\beta Y_{i+1}^s && (21)\\
Y_i(\epsilon) &= \max\left\{\epsilon + \lambda\beta \int_0^1 Y_{i+1}(x)dG(x) + (1-\lambda)\beta Y_{i+1}(\epsilon); Y_i^s\right\} && (22)
\end{aligned}
$$

where $ce_i^\star\theta^\star$ represents the total cost of vacancies in efficiency units per-unemployed worker. The planner's decisions $R_i^\star$, $e_i^\star$ and $\theta^\star$ are solution to:

$$
\max_{\{e_i^\star, R_i^\star, \theta^\star\}} \sum_{i=1}^{T-1} u_i^\star Y_i^s \quad s.t. \quad Y_i(R_i^\star) = Y_i^s \quad \forall i \in [2, T-1]
$$

At each age, the social allocation between the two sectors is governed by the arbitration condition: $Y_i(R_i^\star) = Y_i^s$. On the other hand, the non-segmented search process implies choosing the labor market tightness that maximizes the social value of search per-unemployed worker averaged over age groups. The job surplus for the planer is defined by $S_i^\star(\epsilon) \equiv Y_i(\epsilon) - Y_i^s$:

$$
S_i^\star(\epsilon) = \max\left\{\epsilon - b + \phi(e_i^\star) + ce_i^\star\theta^\star + [\lambda - e_i^\star p(\theta^\star)]\beta \int_0^1 S_{i+1}^\star(x)dG(x) + (1-\lambda)\beta S_{i+1}^\star(\epsilon); 0\right\}
$$

**Definition 2.** *Let* $\eta(\theta^\star) = 1 - \frac{\theta^\star p'(\theta^\star)}{p(\theta^\star)}$*, the maximum value of steady-state social welfare is reached when* $\theta^\star$*,* $\{R_i^\star\}_{i=2}^{T-1}$ *and* $\{e_i^\star\}_{i=1}^{T-1}$ *solve:*

$$
\begin{aligned}
\frac{c}{q(\theta^\star)} &= [1-\eta(\theta^\star)]\beta \sum_{i=1}^{T-1} \left( \frac{e_i^\star u_i^\star}{\widetilde{u}^\star} \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x) \right) && (23)\\
S_i^\star(\epsilon) &= \max\{\epsilon - R_i^\star + (1-\lambda)\beta[S_{i+1}^\star(\epsilon) - S_{i+1}^\star(R_i^\star)]; 0\} && (24)\\
R_i^\star &= b - \phi(e_i^\star) - ce_i^\star\theta^\star - [\lambda - e_i^\star p(\theta^\star)]\beta \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x) - (1-\lambda)\beta S_{i+1}^\star(R_i^\star) && (25)\\
\phi'(e_i^\star) &= -c\theta^\star + p(\theta^\star)\beta \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x) && (26)\\
u_{i+1}^\star &= u_i^\star\left[1 - e_i^\star p(\theta^\star)(1 - G(R_{i+1}^\star))\right] + \lambda G(R_{i+1}^\star)(1-u_i^\star) && (27)
\end{aligned}
$$

*with the terminal conditions* $e_{T-1}^\star = 0$ *and* $R_{T-1}^\star = b$*.*

The equation (23) is similar to the equation (9) obtained in the decentralized equilibrium, on condition that the worker share of employment surplus ($\gamma$) is now replaced by the elasticity relative to the unemployment in the matching function ($\eta(\theta^\star)$). The equation (25) shows that a job is destroyed when the expected profit from the marginal job (the current product plus the option value for expected productivity shocks) fails to cover the social return on unemployed worker. For the social planner, the return on an additional age $i$ unemployed worker is reduced by the cost of vacancy per age $i$ "efficient" unemployed worker, which is equal to $c\theta e_i$. Finally, the equation (26) shows that the marginal cost of the search effort is equalized to the social return on the search process for unemployed workers. The first return on this activity is the surplus associated with the transition from unemployment to employment for a worker aged $i$ ($p(\theta^\star)\beta \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x)$). Moreover, the planner takes into account the fact that more effort devoted to the search from a worker aged $i$ reduces the likelihood for the other workers of finding a job. In order to keep the job opportunities per unemployed workers constant, the planner must raise the number of vacancies: this cost is measured by $c\theta^\star$ in the equation (26). Hence, this implies that an unemployed worker must search only if the expected value of the transition from unemployment to employment is higher than this congestion cost. More formally, we have:

$$e_i^\star \geq 0 \quad \text{if and only if} \quad p(\theta^\star)\beta \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x) \geq c\theta^\star \qquad (C)$$

Let us note that the condition $(C)$ is always satisfied in the MP model[14]: the search costs are lower than the expected hiring value for an unemployed worker. In our model, whereas the search costs are shared by all the unemployed workers, the expected hiring value for a specific age can be lower than its average value. The efficient allocation can then be such that the optimal search effort is nil before the retirement age if the condition $(C)$ does not hold. In this case, the arbitration of the social planner is simply to reallocate these unemployed workers to the home production sector. In the following, we assume for simplicity, and without loss of generality, that the condition $(C)$ is always satisfied.

**Proposition 8.** *If $\lambda \geq p(\theta^\star)e_i^\star$, $\forall i < T-1$, then $R_i^\star < R_{i+1}^\star$ implying that $e_i^\star > e_{i+1}^\star$ and $\tau_i^\star > \tau_{i+1}^\star$.*

*Proof.* See Appendix A. □

Proposition 8 restates at the social optimum the condition under which the job separations are age-increasing and the implication of this dynamics for the search intensity over the life cycle. This proposition emphasizes that higher (lower) job destruction (creation) rates for older workers are not only an equilibrium outcome but also an efficient age-pattern of labor market flows when persistence is low enough. Because of their shorter horizon, older workers must be fired more and hired less. On the other hand, for the same reasons as in the market equilibrium allocation, it is optimal to have an age-increasing separation dynamics when the persistence is high enough (Proposition 9). Let us emphasize that the individual hiring value $\tau_i^\star$, defined by the counterpart of the equation (8) at the social optimum, can still be age-decreasing when the search intensity $e_i$ is age-decreasing, i.e. when the capitalization effect dominates the selection effect.

[14]Indeed, in the MP model with homogenous agents, the free entry condition is simply $\frac{c}{q(\theta)} = (1-\eta(\theta))\int_R^1 [1-G(x)]dx \Leftrightarrow c\theta = (1-\eta(\theta))p(\theta)\int_R^1[1-G(x)]dx$. For $\eta(\theta) \in ]0;1[$, we deduce that in the MP model $c\theta < p(\theta)\int_R^1[1-G(x)]dx$.

**Proposition 9.** *A necessary condition ensuring $R^\star_{i+1} < R^\star_i$, $\forall i$, is $\lambda < p(\theta^\star)e^\star_i$, $\forall i$. An age-decreasing dynamics for $e^\star_i$ is a sufficient condition to ensure $R^\star_i > R^\star_{i+1}$, $\forall i$, provided that the terminal condition $R^\star_{T-2} > R^\star_{T-1}$ holds. Under this sufficient condition, $\tau^\star_i > \tau^\star_{i+1}$.*

*Proof.* See Appendix B. □

Although the optimal and equilibrium labor market flows share the same age-profile, this does not imply that their equilibrium levels are consistent with the efficient allocation.

## 5.2 The inefficiency of the labor market equilibrium: the intergenerational externality

Traditionally, the equilibrium unemployment framework is known to generate search externalities which take the decentralized equilibrium away from the efficient allocation. However, when the elasticity relative to vacancies in the matching function is equal to the bargaining power of workers (Hosios condition), social optimality can be reached. Does this result still hold when life-cycle features are considered?

It is well-known since Davis [1995] that a single matching function when there is some heterogeneity across workers or jobs can create some additional inefficiencies. Traditionally, it is assumed that there are heterogenous jobs with different productivity, whereas workers are homogenous[15]. This is exactly the symmetric case to ours where the jobs are identical and the workers differs across ages. Beyond this difference, it must be emphasized that there is a life-cycle specificity of the efficiency gap created by the worker heterogeneity when firms are not allowed to direct their search. The specificity of this intergenerational externality implies that the job creation is intrinsically not efficient. To easily unveil this specificity, let us assume that the search effort is constant over the age and normalized to unity ($e_i = 1$, $\forall i$) and that the destruction rate is exogenous, denoted by $s$. Then, the equilibrium and the first best allocations are defined as follows:

Equilibrium

$$\begin{aligned}
S_i &= y - b - \tfrac{1}{1-\gamma}c\theta\tau_i + c\theta\tau_i + \beta(1-s)S_{i+1} \\
\tfrac{c}{q(\theta)} &= \beta(1-\gamma)\textstyle\sum_{i=1}^{T-1}\left(\tfrac{u_i}{\widetilde{u}}S_{i+1}\right) \\
u_{i+1} &= u_i\left[1-p(\theta)\right] + s(1-u_i) \\
\tau_i &= \frac{S_{i+1}}{\sum_{i=1}^{T-1}\left(\frac{u_i}{\widetilde{u}}S_{i+1}\right)}
\end{aligned}$$

First best

$$\begin{aligned}
S^\star_i &= y - b - \tfrac{1}{1-\eta(\theta^\star)}c\theta^\star\tau^\star_i + c\theta^\star + \beta(1-s)S^\star_{i+1} \\
\tfrac{c}{q(\theta^\star)} &= \beta[1-\eta(\theta^\star)]\textstyle\sum_{i=1}^{T-1}\left(\tfrac{u^\star_i}{\widetilde{u}^\star}S^\star_{i+1}\right) \\
u^\star_{i+1} &= u^\star_i\left[1-p(\theta^\star)\right] + s(1-u^\star_i) \\
\tau^\star_i &= \frac{S^\star_{i+1}}{\sum_{i=1}^{T-1}\left(\frac{u^\star_i}{\widetilde{u}^\star}S^\star_{i+1}\right)}
\end{aligned}$$

The Hosios condition $\gamma = \eta(\theta^\star)$ allows the market and the planner to agree on the search returns of an age-$i$ worker ($\frac{1}{1-\gamma}c\theta\tau_i = \frac{1}{1-\eta(\theta^\star)}c\theta^\star\tau^\star_i$). But they still disagree on the search costs ($c\theta\tau_i$ for

[15] See for example Acemoglu [2001], Bertola and Caballero [1994] or Ljunqvist and Sargent [2000].

the market versus $c\theta^\star$ for the planner), and then on the job surplus. Indeed, the ex-ante search costs are sunk for the market, and only the ex-post search costs saved by an age-specific hiring is accounted for through the wage bargaining process. On the contrary, for the planner, only the ex-ante search costs, independent of the worker age, matter. As each individual hiring value differs from the average hiring value ($\tau_i \neq 1$), the social search costs are not properly internalized at the equilibrium. If the matched worker is young (old), the market over(under)-estimates the search costs.

Nevertheless, these age-specific gaps could lead to an efficient vacancy decision, as this latter decision depends on the average surplus over all unemployed workers. Given the expressions of the age-specific job surplus in the simplified model, this is the case if and only if:

$$\sum_{i=1}^{T-1} \frac{u_i}{\widetilde{u}} \left[ \tau_i + \sum_{j=1}^{T-i-1} \tau_{i+j} \beta^j (1-s)^j \right] c\theta \quad = \quad \sum_{i=1}^{T-1} \frac{u_i^\star}{\widetilde{u}^\star} \left[ 1 + \sum_{j=1}^{T-i-1} \beta^j (1-s)^j \right] c\theta^\star \tag{28}$$

As long as workers are heterogenous in terms of distance to retirement, the condition (28) is never verified and the job creation is not socially efficient. As the job duration differs across ages, the distortion does not vanish because the summation over ages is truncated by the retirement of older workers. This is why the condition (28) is satisfied when the jobs last only one period ($s \to 1$). This result also holds in a model with some job persistence but without any heterogeneity in the expected duration of the match, typically in an infinite-life environment (see Appendix E). In our framework, only the initial distortions at the hiring stage of all the jobs compensate each other, because, by definition of $\tau_i$, we have $\sum_{i=1}^{T-1} \frac{u_i}{\widetilde{u}} \tau_i = 1$: the overestimation of the search costs induced by the hirings of younger workers is fully compensated by the underestimation of these costs induced by the hirings of older workers. This is no longer the case over all the job duration, the future of younger workers being not compensated for. Moreover, as this future is aging, the average job surplus is underestimated at the equilibrium, due to the age-decreasing profile of $\tau_i$. There are lower incentives to post vacancies at the market equilibrium than for the planer ($\theta < \theta^\star$). In a nutshell, the intergenerational externality is specific, not only because it derives from an endogenous heterogeneity due to the distance to retirement, but also because it implies an inefficient job creation decision *per se*.

The comparison between the equilibrium and the first best allocations in the simplified model shows that they differ in terms of the valuation of the job surplus. This difference identically exists in the complete model with endogenous separation and search effort. As these latter decisions are based on the individual job surplus, there are necessarily distorted, even in the case of a non-distorted job creation[16]. To show this result, it is more convenient to rewrite equations

[16] In the particular case of purely idiosyncratic shocks ($\lambda = 1$), as jobs last only one period, the vacancy decision is not distorted *per se*.

(25) and (26) as follows:

$$\begin{aligned} R_i^\star + \lambda\beta \int_{R_{i+1}^\star}^1 S_{i+1}^\star(x)dG(x) + (1-\lambda)\beta S_{i+1}^\star(R_i^\star) &= b - \phi(e_i^\star) + \frac{1}{1-\eta(\theta^\star)} c e_i^\star \theta^\star \tau_i^\star - c e_i^\star \theta^\star && (29) \\ \phi'(e_i^\star) &= \frac{1}{1-\eta(\theta^\star)} c\theta^\star \tau_i^\star - c\theta^\star && (30) \end{aligned}$$

and to compare them to their equilibrium counterparts:

$$\begin{aligned} R_i + \lambda\beta \int_{R_{i+1}}^1 S_{i+1}(x)dG(x) + (1-\lambda)\beta S_{i+1}(R_i) &= b - \phi(e_i) + \frac{1}{1-\gamma} c\theta e_i \tau_i - c\theta e_i \tau_i && (31) \\ \phi'(e_i) &= \frac{1}{1-\gamma} c\theta\tau_i - c\theta\tau_i && (32) \end{aligned}$$

As the evaluation of the search costs is not the same for the market ($c\theta\tau_i$ in equations (31) and (32)) than for the planner ($c\theta^\star$ in equations (29) and (30)), the Hosios condition no longer achieves efficiency of the separation and search decisions. When $\tau_i < 1$, for the older workers, the search cost is under-estimated: the reservation productivity and the search effort are then both too high at the equilibrium. The opposite is true for younger workers. It is worth to reemphasizing this efficiency gap by considering the equilibrium wage:

$$w_i(\epsilon) = \gamma\left(\epsilon + c\theta e_i\right) + (1-\gamma)\left(b - \phi(e_i)\right) + \gamma c\theta e_i\left(\tau_i - 1\right) \tag{33}$$

whereas the optimal wage would be:

$$w_i^\star(\epsilon) = \eta(\theta^\star)\left(\epsilon + c\theta^\star e_i^\star\right) + (1-\eta(\theta^\star))\left(b - \phi(e_i^\star)\right) + c\theta^\star e_i^\star\left(\tau_i^\star - 1\right) \tag{34}$$

If $\gamma = \eta(\theta^\star)$, the gap between the equilibrium wage and the optimal wage is:

$$w_i^\star(\epsilon) - w_i(\epsilon) = (1-\gamma)c\theta e_i(\tau_i - 1)$$

For the younger (older) workers $\tau_i^\star > 1$ ($\tau_i^\star < 1$), implementing the optimal wage would lead to a higher (lower) wage. As vacancy investments are made in function of the representative worker, when the contact is a younger (older) worker, the job surplus is higher (lower) than expected. The optimal evaluation of this ex post gain (loss) is $c\theta^\star e_i^\star\left(\tau_i - 1\right)$ (see equation (34)). At the equilibrium, only a fraction $\gamma$ of this value is captured by the worker (see equation (33)), leading to a wrong evaluation of the individual search value. Should the bargaining power of workers $\gamma$ be equal to one, this inefficiency would be eliminated. This result is in line with Davis [2001]: there exists a tension between the condition for an efficient mix of jobs and the condition for an efficient total of jobs.

### 5.3 The optimal policy

Labor market policies designed by age may allow firms and workers to internalize the intergenerational externality. In order to specifically deal with this externality, we assume that the Hosios condition is satisfied ($\eta(\theta^\star) = \gamma$). To restore optimality, we propose introducing age-specific taxes or subsidies. As different decisions are distorted, from the job creation to the job separation by passing through the search effort, different instruments may be necessary. First, we assume that the search effort is exogenous in order to show that it is possible to restore optimality on the job creation and separation through a unique tax on filled jobs; secondly, we reintroduce an endogenous search effort: we then show that a unique tax on the search effort is enough to restore optimality on all decisions.

### 5.3.1 The exogenous search effort and employment policy

When the search effort is exogenous, the age-specific search externality cannot be partially compensated by the age-specific search intensity: an old worker has the same probability of meeting a firm as a young worker. The equilibrium search value can be influenced only through a change in the surplus of a filled job. Let us consider an age-specific tax/subsidy ($a_i$) on jobs once they are filled. This policy aims at eliminating the gap between the equilibrium and optimal search values, or equivalently between the equilibrium and the optimal reservation productivity, can be derived from the comparison of the equations (35) and (36):

$$\underbrace{R_i + \lambda\beta \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x) + (1-\lambda)\beta S_{i+1}(R_i)}_{\substack{\text{Lowest worker value inside} \\ \text{the production sector}}} = \underbrace{b - a_i + \frac{1}{1-\gamma} c\theta\tau_i - c\theta\tau_i}_{\substack{\text{Worker value outside} \\ \text{the production sector}}} \tag{35}$$

$$\overbrace{R_i^\star + \lambda\beta \int_{R_{i+1}^\star}^{1} S_{i+1}^\star(x)dG(x) + (1-\lambda)\beta S_{i+1}^\star(R_i^\star)} = \overbrace{b + \frac{1}{1-\eta(\theta^\star)} c\theta^\star\tau_i^\star} - c\theta^\star \tag{36}$$

**Proposition 10.** *An optimal age-sequence for employment subsidies, denoted* $\{a_i\}_{i=1}^{T-1}$ *solves* $a_i = c\theta^\star\,(1-\tau_i^\star)$*, where* $\theta^\star$ *and* $R_i^\star$ *are defined by the efficient allocation.*

*Proof.* Straightforward by comparing equations (35) and (36). $\square$

It is noticeable that the optimal policy rule is independent of the degree of persistence, as the relative hiring value $\tau_i^\star$ is always age-decreasing, according to Propositions 8 and 9 .

**Corollary 1.** *Whatever the level of persistence* $\lambda$*, the optimal age-dynamics of employment subsidies are increasing with age,* $a_i^\star < a_{i+1}^\star \quad \forall i \in [0, T-1]$*, and there exists a threshold age* $\tilde{i}$ *such that* $a_i \leq 0 \quad \forall i \in [0, \tilde{i}]$ *and* $a_i \geq 0 \quad \forall i \in [\tilde{i}, T-1]$.

*Proof.* Whatever the level of productivity threshold age-dynamics, the expected surplus is age-decreasing under the conditions of Propositions 8 and 9. We have necessarily $\tau_i^\star > \tau_{i+1}^\star$ implying $a_i^\star < a_{i+1}^\star$. Because $\tau_1^\star > 1$ and $\tau_{T-1}^\star < 1$, there exists $a_{\tilde{i}}^\star = c\theta^\star(1-\tau_{\tilde{i}}^\star) = 0$ $\square$

This implies subsidizing the employment of older workers more, and even taxing the employment of younger workers (for $i \leq \tilde{i}$). By equalizing the equilibrium job surplus with its optimal counterpart at each age ($S_i(x) = S_i^\star(x)$), the expected surplus over all the unemployed workers then coincide too, leading to efficient job creations. The investment in vacancies taken without any information concerning the future match is now consistent with the planner decision, as there is a compensatory age-dependent subsidy/tax once the job is filled by a worker characterized by a given distance to retirement.

### 5.3.2 Endogenous search effort and unemployment policy

Let us consider the general case with endogenous search effort. Actually, it is fairly intuitive that correcting for this decision at the equilibrium, as it implies to modify the value of the search cost internalized by the unemployed workers, can also benefit to the job separation and job creation decisions. Let us consider a subsidy conditional on the level of the search effort, $s_i e_i$. The instantaneous utility of an unemployed worker is then $b - \phi(e_i) + s_i e_i$. Let us consider again the employment subsidy $a_i$. In this case, the equilibrium conditions are:

$$R_i + \lambda\beta \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x) + (1-\lambda)\beta S_{i+1}(R_i) = b - a_i - \phi(e_i) + s_i e_i + \frac{1}{1-\gamma}c\theta e_i \tau_i - c\theta e_i \tau_i \quad (37)$$

$$\phi'(e_i) = s_i + \frac{1}{1-\gamma}c\theta\tau_i - c\theta\tau_i \quad (38)$$

whereas the optimal allocation satisfies equations (29) and (30). The equation (38) shows that the marginal cost of the search effort is equal to its marginal return net of the subsidy $s_i$.

**Proposition 11.** *An optimal age-sequence for employment and search effort subsidies, denoted* $\{a_i^\star, s_i^\star\}_{i=1}^{T-1}$ *solves* $a_i^\star = 0$ *and* $s_i^\star = -c\theta^\star(1 - \tau_i^\star)$.

Proposition 11 shows that only one instrument is needed to reach the optimal allocation. The reason is intuitive: the search policy intervenes upstream from the employment policy, at the roots of the inefficiency centered on the search value. Restoring the optimality of the search effort decision implies that the search value coincides with its optimal level, which then makes the separation and the vacancy decisions optimal.

**Corollary 2.** *Whatever the level of persistence, the optimal age-dynamics of search effort subsidies is characterized by* $s_i^\star > s_{i+1}^\star \;\; \forall i \in [0, T-1]$, *and there exists a threshold age* $\tilde{i}$ *such that* $s_i \geq 0 \;\; \forall i \in [0, \tilde{i}]$ *and* $s_i \leq 0 \;\; \forall i \in [\tilde{i}, T-1]$.

*Proof.* See the proof of Corollary 1. □

This suggests that it is optimal to tax the search effort of the older workers (and to subsidize the younger worker effort). It is consistent with the result that their wages are too high. It is then optimal to discourage the older workers from searching for jobs. This result gives some theoretical arguments for the existence of pre-retirement schemes in some European countries. In France, for instance, older unemployed workers aged more than 57 receive generous benefits owing to the fact that they do not have to search for a job. It must be emphasized that our result validates this policy and its particular age threshold only because these workers are near to retirement (at 60 in France).

On the other hand, the size of the tax/subsidy correcting for the efficiency gap gives some insights on the quantitative importance of the intergenerational externality. The job surplus is close to zero for older workers ($\tau_o^\star = 0$), implying that the tax is of the same magnitude as the search costs ($s_o = -c\theta^\star$). It is possible from sort of back to envelope calculation to determine the subsidy that must be given to the youngest workers: let us assume that the population consists of three age groups of the same size and that the expected surplus of each population is proportional to their horizon, leading to $\tau_y^\star = 2$ for the younger workers, the prime-age worker being representative of the average surplus[17]. Then, the optimal policy for younger workers is $s_y = c\theta^\star$. The correction induced by the policy is then of the same order of the labor market frictions, which implies that the magnitude of the intergenerational externality is quite significant in the search approach[18].

# 6 Endogenous retirement ages

Until now, the retirement age was exogenous. In this section, we propose to make the extensive margin of the labor market participation endogenous through the retirement decision. We then show that our main results are robust to the introduction of this additional margin.

## 6.1 Accounting for an age-increasing retirement value

In usual models on retirement choices, households accumulates financial or social security assets allowing them to quit the labor market before an exogenous death $D$. After the retirement age, they use these assets to smooth their consumption paths. In our model, without any saving behavior, it seems to be heroic to derive a rational behavior for the retirement age[19], as long

[17]Let us notice these numbers depends on the population structure, which gives the relative scarcity of the younger workers. More younger workers in the population would imply that they are closer to the average group. This would lead to a subsidy lower than proportional to the search costs whereas the tax of older workers would remain unchanged.

[18]Given the optimal choices of the unemployed workers, implying $e_y > 0$ and $e_o = 0$, the total subsidies perceived by the unemployed workers are $s_y^\star e_y^\star = 2\left(\frac{\gamma}{1-\gamma}\right)(c\theta^\star)^2$ and the tax paid by the older workers are nil. Using the Pissarides [2009] values ($c = .356$, $\theta = .72$, $\gamma = \eta = .5$ and $b = .7$), we then deduce that $s_y e_y = 0.1314$, corresponding to a 18.77% increase in the unemployment incomes ($b + s_y e_y$).

[19]See, for example, Hairault, Langot and Sopraseuth [2008] for a model with endogenous retirement choices, endogenous saving and accumulation of social security wealth.

as no exogenous heterogeneity across ages is considered, for instance an increasing disutility of working[20]. A simplifying assumption consists of introducing exogenous earnings $\psi(i)$, assumed to increase with retirement age ($\psi'(\cdot) > 0$ and $\psi''(\cdot) < 0$). Once workers are retired, and we assume that retirement is an absorbing state, these earnings are constant. These exogenous earnings can be considered as a reduced form allowing the matching model to easily capture the age-increasing earnings generated by accumulated assets, either on the financial market or through the social security system. Accounting for a retirement value leads to modifying the value functions as follows:

$$\mathcal{W}_i(\epsilon) = \max\left\{w_i(\epsilon) + \beta\left[\lambda\int_0^1 \mathcal{W}_{i+1}(x)dG(x) + (1-\lambda)\mathcal{W}_{i+1}(\epsilon)\right]; \mathcal{B}_i\right\} \tag{39}$$

$$\mathcal{B}_i = \max_{e_i}\left\{b - \phi(e_i) + \beta\left[e_i p(\theta)\int_0^1 \mathcal{W}_{i+1}(x)dG(x) + (1 - e_i p(\theta))\mathcal{B}_{i+1}\right]; \mathcal{R}_i\right\} \tag{40}$$

$$\mathcal{R}_i = \sum_{j=0}^{D-i}\beta^j(b + \psi(i)) \tag{41}$$

where $\mathcal{B}_i$ the non-employment value and $\mathcal{R}_i$ the retirement value. Employed and unemployed workers have now the outside opportunity to retire when the retirement value $\mathcal{R}$ dominates the job search value.

### 6.2 Retirement decisions

The retirement occupation is chosen when the worker has no longer incentives to participate either as employed or as unemployed. Firstly, there is a maximal retirement age in the economy above which no jobs are opened: all workers are then retired after this age.

**Proposition 12.** *If the function $\psi(\cdot)$ is such that $\psi(T^e) = 1 - b$ for $T^e < D$, there exists a maximum age of retirement $T^e$ for the employed workers. From this age on, the job surplus is never again positive, even for the highest productivity level. The retirement decision $T^e$ is optimal.*

*Proof.* If $\psi(T^e) = 1-b$, as, by assumption, $\psi(\cdot)$ is increasing with age, we have $\psi(D) > 1-b$, and then $S_D(1) = \max\{1 - b - \psi(D); 0\} = 0$. This leads to $S_{D-1}(1) = \max\{1 - b - \psi(D-1); 0\} = 0$ because $\int_0^1 S_D(x)dG(x) = 0$. By backward induction, we find that all job surpluses are nil above $T^e$. $T^e$ is such that $\psi(T^e) = 1 - b$ implying that $S_{T^e-1}(1) = 1 - b - \psi(T^e) > 0$. Identically, for the planner, we have $S^\star_{T^e}(1) = 0$ and $S^\star_{T^e+j}(1) < 0\ \ \forall j > 0$, implying $\psi(T^{e\star}) = 1 - b$ and then $T^e = T^{e\star}$. □

At age $T^e$, the workers (even) at the highest level of productivity choose to retire. There are no more jobs profitable occupied by older workers after $T^e$, and unemployed workers are then

[20] In that case, it would create another age heterogeneity across workers, which would be at odds with our objective to focus on the endogenous heterogeneity created by the finite-life hypothesis.

obviously also retired. Proposition 12 shows that the retirement age is independent of the labor market equilibrium. The retirement age $T^e$ is simply a decreasing function of the outside market opportunities $b$. This also explains why the planner chooses the same optimal retirement age as the agents in the decentralized economy.

Before this age $T^e$, there can be flows from employment to retirement if the unemployment value is dominated by the retirement value, but also flows between employment and unemployment as long as unemployed workers choose to participate. The key age to determinate is then that of retirement for unemployed workers, which happens when the unemployment value converges to the retirement value.

**Proposition 13.** *All unemployed workers retire at $T^u$ before $T^e$. Even if the Hosios condition is satisfied, the intergenerational externality implies that this retirement age is not optimal.*

*Proof.* The equilibrium retirement age for the unemployed worker, $T^u$, solves $\mathcal{R}_{T^u} = \mathcal{U}_{T^u}$, implying, as $\mathcal{R}_{T^u+1} > \mathcal{U}_{T^u+1}$, that $\psi(T^u) = -\phi(e_{T^u}) + e_{T^u} p(\theta)\gamma\beta \int_0^1 S_{T^u+1}(x)dG(x)$. $T^u$ is strictly inferior to $T^e$. Indeed, we know that there is no job surplus after $T^e$. $T^u = T^e$ would imply that $\psi(T^u) = -\phi(e_{T^u})$, which has no solution. Moreover, the retirement decision is not optimal as the equilibrium decision solves $\psi(T^u) = -\phi(e_{T^u}) + \frac{1}{1-\gamma} c\theta e_{T^u}\tau_{T^u} - c\theta e_{T^u}\tau_{T^u}$ whereas the optimal retirement age is given by $\psi(T^{u\star}) = -\phi(e^\star_{T^{u\star}}) + \frac{1}{1-\eta(\theta^\star)} c\theta^\star e^\star_{T^{u\star}}\tau^\star_{T^{u\star}} - c\theta^\star e^\star_{T^{u\star}}$. Even if $\gamma = \eta(\theta^\star)$, the comparison between these two equations shows that $T^u \neq T^{u\star}$ because $c\theta e_{T^u}\tau_{T^u} \neq c\theta^\star e^\star_{T^{u\star}}$. □

Once search frictions are considered, it is straightforward that it is optimal that unemployed workers retire before the employed workers. However, the existence of the intergenerational externality makes the retirement age of the unemployed workers suboptimal. At the end of the life cycle, the older workers under-estimate the search costs, which leads them to retire too late, compared to the first best allocation.

**Proposition 14.** *After the retirement age $T^u$, job separations are optimal retirement decisions and are age-increasing.*

*Proof.* See Appendix D.1 □

Proposition 14 shows that the job separations after $T^u$ are taken with respect to the retirement position. These job separations are optimal because the outside option after $T^u$ only depends on the same exogenous function $\psi(.)$ as that of the planner. These separations are retirement decisions. Our framework is enough to deliver a continuous distribution of the retirement age. There is a minimal retirement age, given by the first age at which the unemployed workers choose to become retired. After this minimal age, the continuous separation process feeds the retirement pool until the employed workers even with the maximal productivity level choose to retire at $T_e$.

These job separations greatly differ from those before $T^u$ as they are optimal. They are not affected by the intergenerational externality, which matters only when the greater outside opportunity of employment is unemployment. This result is intuitive. The intergenerational externality arises from the hiring process: if there are no longer workers in the search process, there are no externalities, neither the aggregate search externality nor the intergenerational one.

**Proposition 15.** *Before $T^u$, the age-dynamics of job creations and separations are qualitatively the same as in the benchmark model with an exogenous retirement age $T$.*

*Proof.* See the appendix D.2 and D.3. □

Proposition 15 sheds some light on the previous results in the exogenous retirement case. The retirement age of the unemployed workers is crucial as it closes the search market. After this age, only some (optimal) separations from employment to retirement happen. On the other hand, before $T_u$, job separations and job creations are featured by the horizon effect and distorted by the intergenerational externality. It is worth emphasizing that the search intensity decreases as far as the retirement age $T_u$ is approaching, but these workers are unemployed, and not yet retired. The endogenous retirement case reinforces the idea that the existence of these somehow discouraged unemployed workers is intrinsic to the proximity to the retirement age.

# 7 Conclusion

Because the horizon of older workers is shorter, firms and workers invest less in job-search and labor-hoarding activities at the end of the life cycle: hiring (separation) rate decreases (increases) with age. This result shows that the normal retirement age is the key factor which governs the employment rate of older workers. Countries with a low retirement age would also suffer from a depressed employment rate for older workers at a relatively early age. This may explain why countries with a retirement age of around 60 such as France and Belgium also have a lower employment rate for workers aged between 55 and 59 than those with a retirement age of 65, such as Sweden and the United States.

Age-policies are necessary to reach the optimal level of output when the search process is not age-directed. At equilibrium, due to an intergenerational externality, there are indeed not enough (too many) job destructions for younger (older) workers. This is why it is optimal to subsidize the employment of the older workers. Concerning the labor supply, we show that it could be optimal to discourage the search effort of older workers. For a given retirement age, these results give some rationale for the age policies implemented in some European countries, such as pre-retirement for unemployed workers.

It must be emphasized that all these results are independent of the age-profile of human capital.

Traditionally, this issue is at the heart of the life cycle approach, especially when the aim is to provide a quantitative analysis of the age-profile of wages. This paper leaves aside this dimension to focus on the basics of the life-cycle equilibrium unemployment theory. On the one hand, in the case of specific human capital, as it is not transferable to a new job, it is quite intuitive that the horizon effect is still the only dimension that determines the relative hiring value across ages. On the other hand, the expected hiring value of older workers can be higher than that of younger workers if their higher level of general human capital dominates their shorter horizon. In that case, the age-dependent policy could be reversed. But this case is quite unrealistic, as the age-profile of general human capital accumulation is typically considered as hump-shaped with a peak at about age forty (Kotlikoff and Gokhale [1992]). We believe that our normative results are robust to the introduction of human capital[21]. However, we acknowledge that taking into account human capital could be useful for a more quantitative analysis of the life cycle unemployment framework. This is left for further research.

[21]In Cheron, Hairault and Langot [2008], all these results are explicitly shown. Moreover, we emphasize that considering costly investments, and not human capital accumulation resulting from an exogenous learning-by-doing process, would lead to reinforcing the horizon effect on the employment flows, by decreasing investments in the working life.

# A Proof of Proposition 8

If $R_i^\star \leq R_{i+1}^\star$, $\forall i$, then the solutions for the surplus, the productivity reservation and the search effort are:

$$\begin{aligned} S_i^*(\epsilon) &= \sum_{j=0}^{T-1-i} \beta^j (1-\lambda)^j \max\{\epsilon - R_{i+j}^\star; 0\} && (42) \\ R_i^\star &= b - \phi(e_i^\star) - c e_i^\star \theta^\star - [\lambda - e_i^\star p(\theta^\star)]\beta E S_{i+1}^\star && (43) \\ \phi'(e_i^\star) &= -c\theta^\star + p(\theta^\star)\beta E S_{i+1}^\star && (44) \end{aligned}$$

Let us denote $I(y) = \int_y^1 (x-y)dG(x) = \int_y^1 (1-G(x))dx$, with $I'(x) < 0$, and $ES_i^\star = \int_0^1 S_i^*(x)dG(x) = \sum_{j=0}^{T-1-i} \beta^j (1-\lambda)^j I(R_{i+j}^\star)$.

Let us start by showing that $R_{i+1}^\star \geq R_i^\star$ implies $e_{i+1}^\star \leq e_i^\star$. As $ES_i^\star - ES_{i+1}^\star = [I(R_i^\star) - I(R_{i+1}^\star)] + \beta(1-\lambda)(ES_{i+1}^\star - ES_{i+2}^\star)$, and given that $I'(x) < 0$ and $ES_{T-2}^\star > ES_{T-1}^\star$, it is then straightforward that $ES_i^\star \geq ES_{i+1}^\star, \forall i$. Then, if the condition $(C)$ is satisfied and using equation (44), we obtain $e_{i+1}^\star \leq e_i^\star$, given that $\phi(\cdot) > 0$, $\phi'(\cdot) > 0$ and $\phi''(\cdot) > 0$. If the age-decreasing pattern of $ES_i^\star$ implies that there exists $\bar{i}$ such that $p(\theta^\star)\beta ES_{\bar{i}+1}^\star < c\theta < p(\theta^\star)\beta ES_{\bar{i}}^\star$ then $e_i^\star = 0$, $\forall i \geq \bar{i}$.

It remains to show that $R_i^\star \leq R_{i+1}^\star$, $\forall i < T-1$ if $\lambda \geq p(\theta^\star)e_i^\star$. Consistently with equations (43) and (44), the solution for the productivity reservation can be written as follows:

$$p(\theta^\star)R_i^\star = p(\theta^\star)b - \lambda c\theta^\star - \underbrace{[p(\theta^\star)\phi(e_i^\star) + (\lambda - p(\theta^\star)e_i^\star)\phi'(e_i^\star)]}_{\equiv \mathcal{Z}(e_i)}$$

If the condition $(C)$ is satisfied, $R_i^\star$ is then age-increasing if and only if

$$\begin{aligned} p(\theta^\star)(R_{i+1}^\star - R_i^\star) &= \mathcal{Z}(e_i^\star) - \mathcal{Z}(e_{i+1}^\star) &\geq 0 \qquad \forall i < T-2 \\ p(\theta^\star)(R_{T-1}^\star - R_{T-2}^\star) &= \lambda c\theta^\star + \mathcal{Z}(e_{T-2}^\star) &\geq 0 \end{aligned}$$

We have $R_i^\star \leq R_{i+1}^\star$, $\forall i < T-2$, implying that $e_i^\star \geq e_{i+1}^\star$, if and only if $\mathcal{Z}'(e_i^\star) \geq 0$. As $\mathcal{Z}'(e_i) = (\lambda - p(\theta^\star)e_i^\star)\phi''(e_i^\star) \geq 0$, the condition $\lambda \geq p(\theta^\star)e_i^\star$, $\forall i$, must be satisfied. On the other hand, for $i = T-2$, the terminal restriction is given by $\lambda c\theta^\star + \mathcal{Z}(e_{T-2}^\star) \geq 0 \Leftrightarrow \lambda \geq \left(\frac{\omega(e_{T-2}^\star)-1}{\omega(e_{T-2}^\star)\left(1+\frac{c\theta^\star}{\phi'(e_{T-2}^\star)}\right)}\right) p(\theta^\star)e_{T-2}^\star$ where $\omega(e_{T-2}^\star)$ is the elasticity of the search cost function at the age $T-2$. This inequality is a less restrictive condition than $\lambda \geq p(\theta^\star)e_{T-2}^\star$. This is why, $\forall i < T-1$, $\lambda \geq p(\theta^\star)e_i^\star$ implies that $R_i^\star \leq R_{i+1}^\star$.

# B Proof of Proposition 9

If the $R^\star_i > R^\star_{i+1}$, $\forall i$, then the solutions for the surplus, the reservation productivity and the search effort are:

$$
\begin{aligned}
S^*_i(\epsilon) &= P_i(T)\max\{\epsilon - R^\star_i; 0\} \quad \text{with } P_i(T) = \textstyle\sum_{j=0}^{T-1-i}\beta^j(1-\lambda)^j && (45)\\
R^\star_i &= b - ce^\star_i\theta^\star - \phi(e^\star_i) - [\lambda - e^\star_i p(\theta)]\beta ES^\star_{i+1} - (1-\lambda)\beta P_{i+1}(T)(R^\star_i - R^\star_{i+1}) && (46)\\
\phi'(e^\star_i) &= -c\theta^\star + p(\theta^\star)\beta ES^\star_{i+1} && (47)
\end{aligned}
$$

where $ES^\star_i = \int_0^1 S^*_i(x)dG(x) = P_i(T)I(R^\star_i)$, with $I(y) = \int_y^1(1-G(x))dx$ and then $I'(x) < 0$.

Let us start by showing that $R^\star_{i+1} < R^\star_i$ implies $e^\star_{i+1} \leq e^\star_i$ only if the capitalization effect dominates the selection effect. As $ES^\star_i - ES^\star_{i+1} = [P_i(T) - P_{i+1}(T)]I(R^\star_i) - P_{i+1}(T)\int_{R^\star_{i+1}}^{R^\star_i}[1 - G(x)]dx$, given that $[P_i(T) - P_{i+1}(T)] > 0$ and $R^\star_{i+1} < R^\star_i$, $ES^\star_i \lesseqgtr ES^\star_{i+1}$. Using equation (47), we then deduce that $e^\star_i \lesseqgtr e^\star_{i+1}$, given that $\phi(\cdot) > 0$, $\phi'(\cdot) > 0$ and $\phi''(\cdot) > 0$. If the capitalization effect dominates the selection effect, i.e. $[P_i(T) - P_{i+1}(T)]I(R^\star_i) > P_{i+1}(T)\int_{R^\star_{i+1}}^{R^\star_i}[1-G(x)]$, then $ES^\star_i > ES^\star_{i+1}$ and $e^\star_i > e^\star_{i+1}$.

The equations (46) and (47) lead to:

$$p(\theta)R_i = p(\theta)b - \lambda c\theta^\star - \mathcal{Z}(e^\star_i) + p(\theta^\star)\beta(1-\lambda)P_{i+1}(T)(R^\star_{i+1} - R^\star_i)$$

If the condition $(C)$ is satisfied, $R^\star_i$ is age-decreasing if and only if

$$
\begin{aligned}
R^\star_{i+1} - R^\star_i &= \tfrac{\mathcal{Z}(e^\star_i) - \mathcal{Z}(e^\star_{i+1}) + p(\theta^\star)\beta(1-\lambda)P_{i+2}(T)(R^\star_{i+2} - R^\star_{i+1})}{p(\theta^\star)[1+\beta(1-\lambda)P_{i+1}(T)]} &< 0 \qquad \forall i < T-2\\
R^\star_{T-1} - R^\star_{T-2} &= \tfrac{\lambda c\theta^\star + \mathcal{Z}(e^\star_{T-2})}{p(\theta^\star)[1+\beta(1-\lambda)]} &< 0
\end{aligned}
$$

If the terminal restriction is satisfied ($R^\star_{T-1} - R^\star_{T-2} < 0 \Leftrightarrow \lambda c\theta^\star + \mathcal{Z}(e^\star_{T-2}) < 0$), by backward induction, it is sufficient to determine the restriction which ensures $R^\star_{i+1} - R^\star_i < 0$, given that $R^\star_{i+2} - R^\star_{i+1}$ is negative. The terminal condition is $\lambda < \frac{\omega(e^\star_{T-2})-1}{\omega(e^\star_{T-2})\left(1+\frac{c\theta^\star}{\phi'(e^\star_{T-2})}\right)}p(\theta^\star)e^\star_{T-2}$, which is more restrictive than the condition $\lambda < p(\theta)e^\star_{T-2}$. If $e^\star_i$ is age-decreasing and $\mathcal{Z}'(e_i) < 0$, i.e. $\lambda < p(\theta^\star)e^\star_i$, $\forall i < T-2$, then $\mathcal{Z}(e^\star_i) - \mathcal{Z}(e^\star_{i+1}) < 0$ and $R^\star_{i+1} - R^\star_i < 0$, $\forall i < T-2$. Consistently with the age-decreasing dynamics of $e^\star_i$, the condition $\lambda < p(\theta^\star)e^\star_i$ is the most restrictive for $e^\star_{T-2}$, implying that the terminal condition is enough to ensure that $R^\star_{i+1} < R^\star_i$.

# C The U-shaped reservation productivity dynamics

## C.1 The match surplus

If there exists an age $\widehat{\imath}$ such that $R_i > R_{i+1}$ for $i < \widehat{\imath}$ and $R_i \leq R_{i+1}$ for $i \geq \widehat{\imath}$, the job surplus is given by:

$$S_i(\epsilon) = \begin{cases} \max\left\{(\epsilon - R_i) + \beta(1-\lambda)\left[S_{i+1}(\epsilon) - S_{i+1}(R_i)\right]; 0\right\} & \text{for} \quad i < \widehat{\imath} \\ \max\left\{(\epsilon - R_i) + \beta(1-\lambda)S_{i+1}(\epsilon); 0\right\} & \text{for} \quad i \geq \widehat{\imath} \end{cases}$$

Forward iterations lead to the following expressions of the surplus, for $i \in [\widehat{\imath}; T-1]$

$$S_i(\epsilon) = \sum_{j=0}^{T-1-i} [\beta(1-\lambda)]^j \max\left\{\epsilon - R_{i+j}; 0\right\}$$

whereas for $i < \widehat{\imath}$, we have

$$\begin{aligned} S_i(\epsilon) &= \max\left\{\left(\sum_{j=0}^{\widehat{\imath}-i-1} [\beta(1-\lambda)]^j\right)(\epsilon - R_i) + [\beta(1-\lambda)]^{\widehat{\imath}-i}\left[S_{\widehat{\imath}}(\epsilon) - S_{\widehat{\imath}}(R_i)\right]; 0\right\} \\ &= P_i(\widehat{\imath}) \max\left\{\epsilon - R_i; 0\right\} + \mathcal{V}_i(\widehat{\imath}, \epsilon) \end{aligned} \tag{48}$$

The U-shaped dynamics for the $R_i$ implies the presence of the additional term $\mathcal{V}_i(\widehat{\imath}, \epsilon)$ when considering the age-decreasing dynamics before $\widehat{\imath}$. On the other hand, after $\widehat{\imath}$, the age-increasing dynamics is unchanged.

## C.2 The age-dynamics of the reservation productivity after the age $\widehat{\imath}$

For $i \geq \widehat{\imath}$ we have

$$R_{i+1} - R_i = \frac{-\Upsilon(e_{i+1}) + \Upsilon(e_i)}{\gamma p(\theta)}$$

Because $e_i$ is age-decreasing (see Proposition 2), and if $\Upsilon'(e_i) > 0$, i.e. $\lambda > \gamma p(\theta) e_i, \forall i \geq \widehat{\imath}$, then $-\Upsilon(e_{i+1}) + \Upsilon(e_i) > 0$ which ensures that $R_{i+1} - R_i > 0, \forall i \geq \widehat{\imath}$.

## C.3 The age-dynamics of the reservation productivity before the age $\widehat{\imath}$

The dynamics of the reservation productivity is given by:

$$\begin{aligned} R_{i+1} - R_i &= \frac{-\Upsilon(e_{i+1}) + \Upsilon(e_i)}{\gamma p(\theta)[1 + \beta(1-\lambda)P_i(\widehat{\imath})]} \\ &+ \frac{\gamma p(\theta)(1-\lambda)\beta[P_{i+2}(\widehat{\imath})(R_{i+2} - R_{i+1}) + (\mathcal{V}_{i+1}(\widehat{\imath}, R_i) - \mathcal{V}_{i+2}(\widehat{\imath}, R_{i+1}))]}{\gamma p(\theta)[1 + \beta(1-\lambda)P_{i+1}(\widehat{\imath})]} \\ &= \frac{-\Upsilon(e_{i+1}) + \Upsilon(e_i)}{\gamma p(\theta)[1 + \beta(1-\lambda)P_i(\widehat{\imath})]} \\ &+ \Phi_{1,i}(R_{i+2} - R_{i+1}) + \Phi_{2,i}(\mathcal{V}_{i+1}(\widehat{\imath}, R_i) - \mathcal{V}_{i+2}(\widehat{\imath}, R_{i+1})) \end{aligned} \tag{49}$$

with $\Phi_{j,i} > 0$ for $i = 1, 2$. In the case where the reservation productivities are continuously age-decreasing, a sufficient condition to ensure the existence of this sequence is $-\Upsilon(e_{i+1}) + \Upsilon(e_i) < 0$. In the U-shaped case, it is necessary to solve the continuation values after $\widehat{i}$, ie. $\mathcal{V}_{i+1}(\widehat{i}, R_i)$ and $\mathcal{V}_{i+2}(\widehat{i}, R_{i+1})$, in order to determine the sufficient condition ensuring that $R_{i+1} < R_i$. Using the definition of $\mathcal{V}_{i+1}(\widehat{i}, R_i)$, we obtain:

$$\begin{aligned}
\mathcal{V}_{i+2}(\widehat{i}, R_{i+1}) &= [\beta(1-\lambda)]^{\widehat{i}-i-2} \left[S_{\widehat{i}}(R_{i+1}) - S_{\widehat{i}}(R_{i+2})\right] \\
\mathcal{V}_{i+1}(\widehat{i}, R_i) &= [\beta(1-\lambda)]^{\widehat{i}-i-1} \left[S_{\widehat{i}}(R_i) - S_{\widehat{i}}(R_{i+1})\right] \\
\Rightarrow \mathcal{V}_{i+1}(\widehat{i}, R_i) - \mathcal{V}_{i+2}(\widehat{i}, R_{i+1}) &= [\beta(1-\lambda)]^{\widehat{i}-i-2} \left\{[\beta(1-\lambda) - 1](S_{\widehat{i}}(R_i) - S_{\widehat{i}}(R_{i+1})) + \Delta\right\} \\
\text{with } \Delta &= \left[S_{\widehat{i}}(R_i) - S_{\widehat{i}}(R_{i+1})\right] - \left[S_{\widehat{i}}(R_{i+1}) - S_{\widehat{i}}(R_{i+2})\right]
\end{aligned}$$

where the first term is unambiguously negative, whereas the sign of $\Delta$ is a priori indeterminate. Using the definition of $S_i(\epsilon)$ for $i > \widehat{i}$, we have:

$$\begin{aligned}
S_{\widehat{i}}(R_{i+1}) - S_{\widehat{i}}(R_{i+2}) &= \sum_{j=0}^{T-\widehat{i}-2} [\beta(1-\lambda)]^j \left(\max\{R_{i+1} - R_{\widehat{i}+j}; 0\} - \max\{R_{i+2} - R_{\widehat{i}+j}; 0\}\right) \\
S_{\widehat{i}}(R_i) - S_{\widehat{i}}(R_{i+1}) &= \sum_{j=0}^{T-\widehat{i}-1} [\beta(1-\lambda)]^j \left(\max\{R_i - R_{\widehat{i}+j}; 0\} - \max\{R_{i+1} - R_{\widehat{i}+j}; 0\}\right)
\end{aligned}$$

There exists some ages $\widehat{i}+p$, $\widehat{i}+n$ and $\widehat{i}+m$ from which the marginal jobs, respectively $R_i$, $R_{i+1}$ and $R_{i+2}$, are closed after $\widehat{i}$. The associated productivity thresholds are such that $\max\{R_i - R_{\widehat{i}+p}; 0\} = 0$, $\max\{R_{i+1} - R_{\widehat{i}+n}; 0\} = 0$ and $\max\{R_{i+2} - R_{\widehat{i}+m}; 0\} = 0$. As $R_i > R_{i+1} > R_{i+2}$, we have $p > n > m$. From the previous expressions, we deduce that:

$$\begin{aligned}
S_{\widehat{i}}(R_{i+1}) - S_{\widehat{i}}(R_{i+2}) &= \left(\sum_{j=0}^{\widehat{i}+m-1} [\beta(1-\lambda)]^j\right)(R_{i+1} - R_{i+2}) \\
&\quad + [\beta(1-\lambda)]^m \sum_{j=0}^{n-m} [\beta(1-\lambda)]^j (R_{i+1} - R_{\widehat{i}+m+j})
\end{aligned}$$

$$\begin{aligned}
S_{\widehat{i}}(R_i) - S_{\widehat{i}}(R_{i+1}) &= \left(\sum_{j=0}^{\widehat{i}+n-1} [\beta(1-\lambda)]^j\right)(R_i - R_{i+1}) \\
&\quad + [\beta(1-\lambda)]^n \sum_{j=0}^{p-n} [\beta(1-\lambda)]^j (R_i - R_{\widehat{i}+n+j})
\end{aligned}$$

The surplus gap between two ages comes from the initial difference of the productivity thresholds, which persists after $\widehat{i}$, plus the additional advantage of having a longer job duration for the younger worker. Thus, we deduce that:

$$\begin{aligned}\Delta &= \left(\sum_{j=0}^{\widehat{\imath}+n-1}[\beta(1-\lambda)]^j\right)(R_i - R_{i+1}) - \left(\sum_{j=0}^{\widehat{\imath}+m-1}[\beta(1-\lambda)]^j\right)(R_{i+1} - R_{i+2}) \\ &\quad \underbrace{+[\beta(1-\lambda)]^m\left(\sum_{j=0}^{p-n}[\beta(1-\lambda)]^{j+n-m}(R_i - R_{\widehat{\imath}+n+j}) - \sum_{j=0}^{n-m}[\beta(1-\lambda)]^j(R_{i+1} - R_{\widehat{\imath}+m+j})\right)}_{\Omega}\end{aligned}$$

$\Omega$ gives the relative value associated to the additional durations of the different jobs ($p-n$ versus $n-m$). If $p-n \equiv \tau \leq n-m$, and using the notation $\tau + \theta = n-m$, we have[22]:

$$\Omega = \widetilde{\Omega} - [\beta(1-\lambda)]^{m+\tau+1}\sum_{j=0}^{\theta-1}[\beta(1-\lambda)]^j(R_{i+1} - R_{\widehat{\imath}+m+\tau+1+j})$$

with

$$\begin{aligned}\widetilde{\Omega} &= [\beta(1-\lambda)]^m\left\{\left(\sum_{j=0}^{\tau}[\beta(1-\lambda)]^j\right)(R_i - R_{i+1}) - \sum_{j=0}^{\tau}[\beta(1-\lambda)]^j(R_{\widehat{\imath}+n+j} - R_{\widehat{\imath}+m+j})\right. \\ &\quad \left. +[1-[\beta(1-\lambda)]^\tau]\left[-R_i\left(\sum_{j=0}^{\tau}[\beta(1-\lambda)]^j\right) + \sum_{j=0}^{\tau}[\beta(1-\lambda)]^j R_{\widehat{\imath}+n+j}\right]\right\}\end{aligned}$$

Finally, it can be deduced the following expression:

$$\mathcal{V}_{i+1}(\widehat{\imath}, R_i) - \mathcal{V}_{i+2}(\widehat{\imath}, R_{i+1}) = -\Psi_{1,i+1} + \Psi_{2,i+1}(R_i - R_{i+1}) - \Psi_{3,i+1}(R_{i+1} - R_{i+2}) \quad (50)$$

where $\Psi_{j,i+1} > 0$ for $j = 1, 2, 3$. Using equation (50), we then obtain:

$$\begin{aligned}R_{i+1} - R_i &= \frac{-\Upsilon(e_{i+1}) + \Upsilon(e_i)}{\left(\gamma p(\theta)[1 + \beta(1-\lambda)P_i(\widehat{\imath})]\right)(1 + \Phi_{2,i}\Psi_{2,i+1})} \\ &\quad + \frac{-\Phi_{2,i}\Psi_{1,i+1}}{1 + \Phi_{2,i}\Psi_{2,i+1}} + \frac{\Phi_{1,i} + \Phi_{2,i}\Psi_{3,i+1}}{1 + \Phi_{2,i}\Psi_{2,i+1}}(R_{i+2} - R_{i+1})\end{aligned}$$

where $\Phi_{x,y} > 0$, $\forall x, y$. A sufficient condition ensuring that $R_{i+1} - R_i < 0$ is that $-\Upsilon(e_{i+1}) + \Upsilon(e_i) < 0$. If $e_i$ is age-decreasing and $\Upsilon'(e_i) < 0$, i.e. $\lambda < \gamma p(\theta)e_i, \forall i < \widehat{\imath} - 1$, then $-\Upsilon(e_{i+1}) + \Upsilon(e_i) < 0$ and $R_{i+1} - R_i < 0, \forall i < \widehat{\imath} - 1$, provided that the terminal condition $R_{\widehat{\imath}-1} > R_{\widehat{\imath}}$ holds.

[22]The condition $p - n \leq n - m$ simply indicates that the net returns of the search activity must be concave. After the age $\widehat{\imath}$, the gap between two successive reservation productivities is given by $\gamma p(\theta)(R_{i+1} - R_i) = \Upsilon(e_i) - \Upsilon(e_{i+1}) \geq 0$, $\forall i < T - 2$, where $\Upsilon(e_i) \equiv \gamma p(\theta)\phi(e_i) + (\lambda - \gamma p(\theta)e_i)\phi'(e_i)$, $\Upsilon'(e_i) = (\lambda - \gamma p(\theta)e_i)\phi''(e_i)$ and $\Upsilon''(e_i) = \left[\lambda - \gamma p(\theta)e_i\left(1 + \frac{1}{\mu(e_i)}\right)\right]\phi'''(e_i)$ with $\mu(e_i) = \phi'''(e_i)\frac{e_i}{\phi''(e_i)}$. Thus, with $e_i > e_{i+1}$, we have $R_{\widehat{\imath}+j+2} - R_{\widehat{\imath}+j+1} \geq R_{\widehat{\imath}+j+1} - R_{\widehat{\imath}+j}$, $\forall j \geq 0$, if $\Upsilon''(e_i) \leq 0$. A sufficient conditions is $\Phi'''(e_i) < 0$, but if $\Phi'''(e_i) > 0$, we can have $\Upsilon''(e_i) \leq 0$ if $\mu(e_i) \leq \frac{e_i p(\theta)}{1 - e_i p(\theta)}$. Intuitively, these two conditions suggest that the gap between two successive reservation productivities is age-increasing or constant if the net return of the search activity is increasing and concave. This convexity in the $R_i$ dynamics after $\widehat{\imath}$ is a necessary condition to ensure that $p - n \leq n - m$.

### C.4 The terminal condition

At the age $\widehat{\imath}$, the age-dynamics of the reservation productivity changes. Indeed, for $i = \widehat{\imath} - 1$, we have $\gamma p(\theta) R_{\widehat{\imath}-1} = \gamma p(\theta) b - \Upsilon(e_{\widehat{\imath}-1}) - \gamma p(\theta)(1-\lambda)\beta S_{\widehat{\imath}}(R_{\widehat{\imath}-1})$, and for $i \geq \widehat{\imath}$, we have $\gamma p(\theta) R_i = \gamma p(\theta) b - \Upsilon(e_i)$. We then deduce that:

$$
\begin{aligned}
R_{\widehat{\imath}} - R_{\widehat{\imath}-1} &= \frac{\Upsilon(e_{\widehat{\imath}-1}) - \Upsilon(e_{\widehat{\imath}}) + \gamma p(\theta)(1-\lambda)\beta S_{\widehat{\imath}}(R_{\widehat{\imath}-1})}{\gamma p(\theta)} \\
R_{\widehat{\imath}+1} - R_{\widehat{\imath}} &= \frac{\Upsilon(e_{\widehat{\imath}}) - \Upsilon(e_{\widehat{\imath}+1})}{\gamma p(\theta)}
\end{aligned}
$$

The first equation shows that the gap between $R_{\widehat{\imath}}$ and $R_{\widehat{\imath}-1}$ depends, through $S_{\widehat{\imath}}(R_{\widehat{\imath}-1})$, on the sequence of the future reservation productivity. Nevertheless, if we have $R_{\widehat{\imath}+1} > R_{\widehat{\imath}-1}$, then the solution for $S_{\widehat{\imath}}(R_{\widehat{\imath}-1})$ is simply $R_{\widehat{\imath}-1} - R_{\widehat{\imath}}$. Indeed, a first order approximation around $e_i = e_{\widehat{\imath}}$ leads to $\Upsilon(e_{\widehat{\imath}}) - \Upsilon(e_{\widehat{\imath}+1}) \approx \gamma p(\theta)(e_{\widehat{\imath}+1} - e_{\widehat{\imath}})^2 \phi''(e_{\widehat{\imath}})$ and $\Upsilon(e_{\widehat{\imath}-1}) - \Upsilon(e_{\widehat{\imath}}) \approx -\gamma p(\theta)(e_{\widehat{\imath}-1} - e_{\widehat{\imath}})^2 \phi''(e_{\widehat{\imath}})$. This implies that:

$$
\begin{aligned}
R_{\widehat{\imath}} - R_{\widehat{\imath}-1} &\approx \frac{-\gamma(e_{\widehat{\imath}-1} - e_{\widehat{\imath}})^2 p(\theta)\phi''(e_{\widehat{\imath}}) + \gamma p(\theta)(1-\lambda)\beta S_{\widehat{\imath}}(R_{\widehat{\imath}-1})}{\gamma p(\theta)} \\
R_{\widehat{\imath}+1} - R_{\widehat{\imath}} &\approx \frac{\gamma(e_{\widehat{\imath}+1} - e_{\widehat{\imath}})^2 p(\theta)\phi''(e_{\widehat{\imath}})}{\gamma p(\theta)}
\end{aligned}
$$

Then, for $e_{\widehat{\imath}-1} = e_{\widehat{\imath}} + \delta$ and $e_{\widehat{\imath}+1} = e_{\widehat{\imath}} - \delta$, we obtain $R_{\widehat{\imath}+1} - R_{\widehat{\imath}-1} \approx \frac{\gamma p(\theta)(1-\lambda)\beta S_{\widehat{\imath}}(R_{\widehat{\imath}-1})}{\gamma p(\theta)} > 0$ and then $S_{\widehat{\imath}}(R_{\widehat{\imath}-1}) = R_{\widehat{\imath}-1} - R_{\widehat{\imath}}$. Then, we deduce that:

$$
\begin{aligned}
R_{\widehat{\imath}} - R_{\widehat{\imath}-1} &\approx \frac{-\gamma(e_{\widehat{\imath}-1} - e_{\widehat{\imath}})^2 p(\theta)\phi''(e_{\widehat{\imath}})}{\gamma p(\theta)[1 + \gamma p(\theta)\beta(1-\lambda)]} < 0 \\
R_{\widehat{\imath}+1} - R_{\widehat{\imath}} &\approx \frac{\gamma(e_{\widehat{\imath}+1} - e_{\widehat{\imath}})^2 p(\theta)\phi''(e_{\widehat{\imath}})}{\gamma p(\theta)} > 0
\end{aligned}
$$

showing that $R_{\widehat{\imath}-1} > R_{\widehat{\imath}}$ when $\lambda - \gamma e_{\widehat{\imath}-1} p(\theta) < 0$ and $R_{\widehat{\imath}} < R_{\widehat{\imath}+1}$ when $\lambda - \gamma e_{\widehat{\imath}+1} p(\theta) > 0$.

## D Solution when retirement age is endogenous

### D.1 After the age $T^u$

For $i \geq T^u$, if $R_i \leq R_{i+1}$, the agents choose between employment and retirement occupations. Then, the surplus is given by

$$
\begin{aligned}
S_i(\epsilon) &= \max\left\{\epsilon - b - \psi(i) + \beta\left(\lambda \int_0^1 S_{i+1}(x)dG(x) + (1-\lambda)S_{i+1}(\epsilon)\right); 0\right\} \\
&= = \max\left\{\epsilon - R_i + \beta(1-\lambda)S_{i+1}(\epsilon); 0\right\} = \sum_{j=0}^{T^e - i} [\beta(1-\lambda)]^j \max\left\{\epsilon - R_{i+j}; 0\right\}
\end{aligned}
$$

After $T^u$, the decision rule concerning the separation becomes $\mathcal{W}_i(\widetilde{R}_i) = \mathcal{R}_i \Leftrightarrow S_i(\widetilde{R}_i) = 0$ for $i \in ]T^u; T^e[$. Because the outside opportunity is the identical for the planner and for the market, the job surplus at the equilibrium corresponds to its optimal value. Separations are then optimal. It is possible to show that they are unambiguously age-increasing. The dynamics of the reservation productivity is given by:

$$\widetilde{R}_i = b + \psi(i) - \lambda\beta \int_{\widetilde{R}_{i+1}}^{1} S_{i+1}(x)dG(x) \tag{51}$$

with the terminal conditions $\widetilde{R}_{T^e-1} = b + \psi(T^e - 1)$ and $1 = b + \psi(T^e)$. Then, we deduce that the gap between $\widetilde{R}_i$ and $\widetilde{R}_{i+1}$ evolves as follows:

$$\widetilde{R}_{i+1} - \widetilde{R}_i = \underbrace{\psi(i+1) - \psi(i)}_{+} + \lambda\beta \underbrace{\left[ \int_{\widetilde{R}_{i+1}}^{1} S_{i+1}(x)dG(x) - \int_{\widetilde{R}_{i+2}}^{1} S_{i+2}(x)dG(x) \right]}_{+}$$

with the terminal condition $\widetilde{R}_{T^e-1} - \widetilde{R}_{T^e-2} = \psi(T^e-1) - \psi(T^e-2) + \lambda\beta \int_{\widetilde{R}_{T^e-1}}^{1} S_{T^e-1}(x)dG(x)$. As $\psi(T^e-1) - \psi(T^e-2) > 0$, we have $\widetilde{R}_{T^e-1} > \widetilde{R}_{T^e-2}$ and then $\widetilde{R}_{i+1} > \widetilde{R}_i$, for $i \in [T^u; T^e - 1]$.

### D.2 Before the age $T^u$: the case of age-increasing reservation productivity

If $R_i \leq R_{i+1}$, for $i < T^u$, we have $S_{i+1}(R_i) = 0$ and then, $\forall i \leq T^e$,

$$S_i(\epsilon) = \max\{\epsilon - R_i + \beta(1-\lambda)S_{i+1}(\epsilon); 0\} = \sum_{j=0}^{T^e-i} [\beta(1-\lambda)]^j \max\{\epsilon - R_{i+j}; 0\}$$

For $i < T^u$, the reservation productivity is:

$$R_i = b - \phi(e_i) - [\lambda - \gamma e_i p(\theta)]\beta \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)$$

Using the FOC on the search effort, implying $\frac{\phi'(e_i)}{\gamma p(\theta)} = \beta \int_{R_{i+1}}^{1} S_{i+1}(x)dG(x)$, we have:

$$\gamma p(\theta)(R_{i+1} - R_i) = \Upsilon(e_i) - \Upsilon(e_{i+1})$$

where $\Upsilon(e_j) = \gamma p(\theta)\phi(e_j) + [\lambda - \gamma e_j p(\theta)]\phi'(e_j)$, for $j = i, i+1$. We have $R_i \leq R_{i+1}$, implying that $e_i$ is age-decreasing (see proposition 2) if $\Upsilon'(e_j) \geq 0$. As $\Upsilon'(e_j) = (\lambda - \gamma p(\theta)e_j)\phi''(e_j)$, the condition $\lambda \geq \gamma p(\theta)e_{i+1}$, $\forall i < T^u$, must be satisfied. Nevertheless, because $e_i p(\theta) < 1$, a sufficient condition for $R_i \leq R_{i+1}$ is then $\lambda \geq \gamma$. By comparison with the case where retirement age is exogenous and homogenous among workers (see Proposition 3), we obtain the same sufficient condition. Note that, unlike the case with exogenous retirement, the search effort just before the retirement age of the unemployed workers is not zero. Then, the terminal condition, which gives the gap between $R_{T^u}$ and $R_{T^u-1}$, is the same as the condition giving the gap between $R_i$ and $R_{i+1}$. Finally, using the definition of $T^u$, we have $R_{T^u} = \widetilde{R}_{T^u}$, ensuring that the sequence of productivity reservation is continuous.

### D.3 Before the age $T^u$: the case of age-decreasing reservation productivity

If $R_i > R_{i+1}$, the match surplus is for $i < T^u$:

$$\begin{aligned} S_i(\epsilon) &= \max\left\{\epsilon - b + \phi(e_i) + \beta\left(\lambda\int_0^1 S_{i+1}(x)dG(x) + (1-\lambda)S_{i+1}(\epsilon)\right)\right. \\ &\quad \left. -\gamma e_i p(\theta)\beta\int_0^1 S_{i+1}(x)dG(x); 0\right\} \\ &= \max\left\{\left(\sum_{j=0}^{T^u-i-1}[\beta(1-\lambda)]^j\right)(\epsilon - R_i) + [\beta(1-\lambda)]^{T^u-i}\left[S_{T^u}(\epsilon) - S_{T^u}(R_{T^u-1})\right]; 0\right\} \\ &= P_i(T^u)\max\{\epsilon - R_i; 0\} + \mathcal{V}_i(T^u,\epsilon) \end{aligned}$$

This case with age-decreasing reservation productivity is equivalent to the U-shaped case before $\widehat{i}$, as can be shown by comparing the last equation with equation (48) (Appendix C), when $\widehat{i}$ is replaced by $T^u$. It can be derived that a sufficient condition ensuring that $R_{i+1} - R_i < 0$ is $-\Upsilon(e_{i+1}) + \Upsilon(e_i) < 0$. If $e_i$ is age-decreasing and $\Upsilon'(e_i) < 0$, i.e. $\lambda < \gamma p(\theta)e_i$, $\forall i < T^u - 1$, then $-\Upsilon(e_{i+1}) + \Upsilon(e_i) < 0$, ensuring $R_{i+1} - R_i < 0, \forall i < T^u - 1$. As in the case with exogenous retirement age $T$, $\lambda < \gamma p(\theta)e_i$ is a sufficient condition, and $\lambda \leq \gamma$ a necessary condition, for $R_i > R_{i+1}, \forall i < T^u - 1$, provided that the terminal condition $R_{T^u} - R_{T^u-1} < 0$ holds.

At the age $T^u$, there is change in the law of motion of the reservation productivity, implying:

$$\begin{aligned} \widetilde{R}_{T^u+1} - \widetilde{R}_{T^u} &= \psi(T^u+1) - \psi(T^u) + \lambda\beta\left[\int_{\widetilde{R}_{T^u+1}}^1 S_{T^u+1}(x)dG(x) - \int_{\widetilde{R}_{T^u+2}}^1 S_{T^u+2}(x)dG(x)\right] \\ R_{T^u} - R_{T^u-1} &= \frac{-\Upsilon(e_{T^u}) + \Upsilon(e_{T^u-1}) + \gamma p(\theta)(1-\lambda)\beta S_{T^u}(R_{T^u-1})}{\gamma p(\theta)} \end{aligned}$$

The optimal retirement age for the unemployed workers $T^u$ is such that $\psi(T^u) = -\phi(e_{T^u}) + e_{T^u}p(\theta)\gamma\beta\int_{\widetilde{R}_{T^u+1}}^1 S_{T^u+1}(x)dG(x)$, implying that

$$\begin{aligned} \widetilde{R}_{T^u+1} - R_{T^u-1} &= \psi(T^u+1) + \underbrace{\phi(e_{T^u-1}) - e_{T^u-1}p(\theta)\gamma\beta\int_{\widetilde{R}_{T^u}}^1 S_{T^u}(x)dG(x)}_{\phi(e_{T^u-1}) - e_{T^u-1}\phi'(e_{T^u-1})<0} \\ &\quad \underbrace{-\beta\left[\begin{array}{l} \lambda\int_{\widetilde{R}_{T^u+2}}^1 S_{T^u+2}(x)dG(x) \\ -\left(\lambda\int_{\widetilde{R}_{T^u}}^1 S_{T^u}(x)dG(x) + (1-\lambda)S_{T^u}(R_{T^u-1})\right) \end{array}\right]}_{\lambda(ES_{T^u+2} - ES_{T^u}) - (1-\lambda)S_{T^u}(R_{T^u-1})<0} \end{aligned}$$

Under the assumption that the capitalization effect dominates the selection effect, implying $ES_{T^u+2} < ES_{T^u}$, a sufficient condition for $\widetilde{R}_{T^u+1} - \widetilde{R}_{T^u-1} > 0$ is that $\psi(T^u+1) > -\phi(e_{T^u-1}) + e_{T^u-1}\phi'(e_{T^u-1})$. Using the definition of the retirement age of the unemployed workers, this restriction is always satisfied if the function $\psi(\cdot)$ is sufficiently increasing, ie. such that $\psi'(T^u) >$

$e_{T^u}\phi''(e_{T^u})(e_{T^u-1} - e_{T^u})$. Then, under this restriction, we deduce that $S_{T^u}(R_{T^u-1}) = R_{T^u-1} - R_{T^u}$, implying that

$$R_{T^u} - R_{T^u-1} \quad = \quad \frac{-\Upsilon(e_{T^u}) + \Upsilon(e_{T^u-1})}{\gamma p(\theta)[1+\beta(1-\lambda)]}$$

Then, if $e_i$ is age-decreasing and $\Upsilon(e_i) < 0$, ie. $\lambda < \gamma p(\theta) e_{T^u}$, then $-\Upsilon(e_{T^u}) + \Upsilon(e_{T^u-1}) < 0$, ensuring that $R_{T^u} - R_{T^u-1} < 0$. Compared with the case where the retirement age is exogenous (see Proposition 6), this terminal condition leads to a less restrictive condition on $\lambda$. Finally, let us notice that the definition of $T^u$ implies that $R_{T^u} = \widetilde{R}_{T^u}$, ensuring the continuity of the reservation productivity sequence.

# E When persistent heterogeneity does lead to additional inefficiency

Let assume that there productivity heterogeneity across workers ($y(j),\ j = 1, ..., N$) in an infinite-life environment along the line of Davis [1995] would lead to an efficient job creation.

$$\begin{array}{rclcrcl}
 & & \text{Equilibrium} & & & & \text{First best} \\
S_j & = & \frac{y(j)-b-\frac{1}{1-\gamma}c\theta\tau_j + c\theta\tau_j}{1-\beta(1-s)} & & S_j^\star & = & \frac{y(j)-b-\frac{1}{1-\eta(\theta^\star)}c\theta^\star\tau_j^\star + c\theta^\star}{1-\beta(1-s)} \\
\frac{c}{q(\theta)} & = & \beta(1-\gamma)\sum_{j=1}^N\left(\frac{u_j}{\widetilde{u}}S_j\right) & & \frac{c}{q(\theta^\star)} & = & \beta[1-\eta(\theta^\star)]\sum_{j=1}^N\left(\frac{u_j^\star}{\widetilde{u}^\star}S_j^\star\right) \\
u_j & = & \frac{s}{s+p(\theta)} & & u_j^\star & = & \frac{s}{s+p(\theta^\star)} \\
\tau_j & = & \frac{S_j}{\sum_{j=1}^N\left(\frac{u_j}{\widetilde{u}}S_j\right)} & & \tau_j^\star & = & \frac{S_j^\star}{\sum_{j=1}^N\left(\frac{u_j^\star}{\widetilde{u}^\star}S_j^\star\right)}
\end{array}$$

In this case, all jobs have the same expected duration. The condition of equivalence between the equilibrium and the first best allocations becomes: $\sum_{j=1}^N \frac{u_j}{\widetilde{u}}\tau_j = 1$, which is obviously verified, given the definition of $\tau_j$.