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On measure solutions of backward stochastic differential equations

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Abstract

We consider backward stochastic differential equations (BSDEs) with nonlinear generators typically of quadratic growth in the control variable. A measure solution of such a BSDE will be understood as a probability measure under which the generator is seen as vanishing, so that the classical solution can be reconstructed by a combination of the operations of conditioning and using martingale representations. For the case where the terminal condition is bounded and the generator fulfills the usual continuity and boundedness conditions, we show that measure solutions with equivalent measures just reinterpret classical ones. For the case of terminal conditions that have only exponentially bounded moments, we discuss a series of examples which show that in the case of non-uniqueness, classical solutions that fail to be measure solutions can coexist with different measure solutions.

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0. Introduction

The generally accepted natural framework for the most efficient formulation of pricing and hedging contingent claims on complete financial markets, for instance in the classical

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Merton–Scholes problem, is given by martingale theory, more precisely by the elegant notion of martingale measures. Martingale measures represent a view of the world in which price dynamics do not have inherent trends. From the perspective of this world, pricing a claim amounts to taking expectations, while hedging boils down to pure conditioning and using martingale representation.

At first glance, hedging a claim is, however, a problem calling upon stochastic control: it consists in choosing strategies to steer the portfolio into a terminal random endowment that the portfolio holder has to ensure. Solving stochastic backward equations (BSDEs) is a technique tailor-made for this purpose. This powerful tool has been introduced to stochastic control theory by Bismut [1]. Its mathematical treatment in terms of stochastic analysis was initiated by Pardoux and Peng [2], and its particular significance for the field of utility maximization in financial stochastics clarified in El Karoui, Peng and Quenez [3] (see El Karoui and Barrieu [15] in the context of risk measures). To fix ideas, we restrict our attention to a Wiener space probabilistic environment. In this framework, a BSDE with terminal variable ξ at time horizon T and generator f is solved by a pair of processes (Y, Z) on the interval $[0, T]$ satisfying

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}W_s + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s, \quad t \in [0, T]. \tag{1}$$

In the case of vanishing generator, the solution just requires an application of the martingale representation theorem in the Wiener filtration, and Z will be given as the stochastic integrand in the representation, to which we will refer as the *control process* in the sequel. The classical approach of existence and uniqueness for BSDEs involves *a priori* inequalities as a basic ingredient, by which unique solutions are constructed via fixed point arguments, just as in the case of forward stochastic differential equations.

In this paper we are looking for a notion in the context of BSDEs that plays the role of the martingale measure in the context of hedging claims. Our main interest is directed to BSDEs of the type (1) with generators that are non-Lipschitzian, and depend on the control variable z quadratically, typically $f(s, y, z) = z^2 b(s, z)$, $s \in [0, T]$, $z \in \mathbb{R}$, with a bounded function b. These generators were given a thorough treatment in Kobylanski [4], Briand and Hu [5], and Lepeltier and San Martin [6]. While [4,6] consider existence and uniqueness questions for bounded terminal variables ξ, [5] goes to the limit of possible terminal variables by considering ξ for which $\exp(\gamma|\xi|)$ has finite expectation for some $\gamma > 2\|b\|_\infty$. All these papers employ different methods of approach following the classical pattern of arguments mentioned above. In contrast to this, our work will investigate an alternative notion of solution of BSDEs, the generators of which fulfill similar conditions. Using the analogy with martingale measures in hedging which effectively eliminate drifts in price dynamics (see for example [17]), we shall look for probability measures under which the generator of a given BSDE is seen as vanishing. Given such a measure $\mathbb{Q}$ which we call the *measure solution* of the BSDE and supposing that $\mathbb{Q} \sim \mathbb{P}$, the processes Y and Z are the results of projection and representation respectively, i.e. $Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = Y_0 + \int_0^\cdot Z_s \mathrm{d}\widetilde{W}_s$, where $\widetilde{W}$ is a Wiener process under $\mathbb{Q}$. The first main finding of the paper roughly states that provided the terminal variable ξ is bounded, all classical solutions can be interpreted as measure solutions. More precisely, we show that if the generator satisfies the usual continuity and quadratic boundedness conditions, classical solutions (Y, Z) exist if and only if measure solutions with $\mathbb{Q} \sim \mathbb{P}$ exist. So existence theorems obtained in the papers quoted are recovered in a more elegant and concise way in terms of measure solutions. We do not touch uniqueness questions in general. Of course, determining a measure $\mathbb{Q}$ under which the generator vanishes amounts to doing a Girsanov change of probability that eliminates it. We therefore have to look at the BSDEs in the form

$$Y_t = \xi - \int_t^T Z_s \left[\mathrm{d}W_s - \frac{f(s, Y_s, Z_s)}{Z_s} \mathrm{d}s \right], \quad t \in [0, T], \tag{2}$$

define $g(s, y, z) = \frac{f(s,y,z)}{z}$, and study the measure

$$\mathbb{Q} = \exp\left(M - \frac{1}{2} \langle M \rangle \right) \cdot \mathbb{P}$$

for the martingale $M = \int_0^{\cdot} g(s, Y_s, Z_s) \mathrm{d}W_s$. One of the fundamental problems that took some effort to solve consists in showing that $\mathbb{Q}$ is a probability measure. Here one has to dig essentially deeper than Novikov's or Kazamaki's criteria allow. We successfully employed a criterion which is based on the explosion properties of the quadratic variation $\langle M \rangle$, which we learnt from a conversation with M. Yor, and which has been latent in the literature for a while; see Liptser, Shiryaev [7], or the more recent paper by Wong, Heyde [8]. This criterion allows a simple treatment of the problem of existence of measure solutions in the case of bounded terminal variable, and a still elegant and efficient one in the borderline case of exponentially integrable terminal variable considered by Briand and Hu [5]. If ξ is only exponentially bounded, things turn essentially more complex immediately. Specializing to a very simple generator, we find a wealth of different situations looking confusing at first sight. Just to quote three basic scenarios exhibited in a series of examples of different types: in the first type we obtain one solution which is a measure solution at the same time; in the second one we find two different solutions both of which are measure solutions; in the third one we encounter two solutions one of which is a measure solution, while the other one is not. We even combine these basic examples to develop a scenario in which there exists a continuum of measure solutions, and another one in which a continuum of non-measure solutions is given.

Here is an outline of the presentation of our material. Throughout we consider BSDEs possessing generators with quadratic nonlinearity in z. In a first section we discuss the case of bounded terminal variable ξ, and show that if the generator satisfies continuity and quadratic boundedness conditions, classical solutions (Y, Z) exist if and only if measure solutions with $\mathbb{Q} \sim \mathbb{P}$ exist. Things become essentially more complex in the second section, where we pass to exponentially integrable terminal variables. Taking the simple generator $f(s, z) = \alpha z^2, s \in [0, t], z \in \mathbb{R}$, with some $\alpha \in \mathbb{R}$, a wealth of different scenarios arises in which in the case of non-uniqueness in particular solutions can be measure solutions, while different ones fail to have this property. In the final section we construct measure solutions from first principles without using strong solutions in our algorithm, for generators which are Lipschitz continuous with time dependent and random constants. By iterating the successive applications of martingale representation and Girsanov change of measure with respect to drifts obtained from the martingale representation density of the previous step we obtain a sequence of probability measures which can be seen to be tight in the weak topology, and thus have accumulation points which yield measure solutions.

1. Measure solutions: Definition and first examples

In this section we first recall some basic definitions concerning BSDEs. We then introduce and exemplify the notion of a measure solution by looking at a special class of BSDEs.

Throughout let T be a non-negative real, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space, and $(W_t)_{0\le t\le T}$ a one-dimensional Brownian motion, whose natural filtration, augmented by $\mathcal{N}$, is denoted by $(\mathcal{F}_t)_{0\le t\le T}$, where

$$\mathcal{N} = \{A \subset \Omega, \exists G \in \mathcal{F}, A \subset G \text{ and } \mathbb{P}(G) = 0\}.$$

Let ξ be an $\mathcal{F}_T$-measurable random variable, and let $f : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for all $z \in \mathbb{R}$ the mapping $f(\cdot, \cdot, z)$ is predictable. A classical solution of the BSDE with *terminal condition* ξ and *generator* f is defined to be a pair of predictable processes (Y, Z) such that almost surely we have $\int_0^T Z_s^2 \mathrm{d}s < \infty$, $\int_0^T |f(s, Y_s, Z_s)|\mathrm{d}s < \infty$, and for all $t \in [0, T]$,

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}W_s + \int_t^T f(s, Z_s)\mathrm{d}s. \tag{3}$$

The solution processes (Y, Z) are often shown to satisfy some integrability properties and to belong to the following function spaces. For $p \ge 1$ let $\mathcal{H}^p$ denote the set of all $\mathbb{R}$-valued predictable processes ζ such that $E \int_0^1 |\zeta_t|^p \mathrm{d}t < \infty$, and by $\mathcal{S}^\infty$ we denote the set of all essentially bounded $\mathbb{R}$-valued predictable processes.

If ξ is square integrable and f satisfies a Lipschitz condition, then it is known that there exists a unique pair $(Y, Z) \in \mathcal{H}^2 \otimes \mathcal{H}^2$ solving (3). Recall that the solution process Y_t has a nice representation as a conditional expectation with respect to a new probability measure if f is a linear function of the form

$$f(s, z) = b_s z, \tag{4}$$

where b is a predictable and bounded process. More precisely, if $D_t = \exp(\int_0^t b_s \mathrm{d}W_s - \frac{1}{2}\int_0^t b_s^2 \mathrm{d}s)$, and $\mathbb{Q}$ is the probability measure with density $\mathbb{Q} = D_T \cdot \mathbb{P}$, then

$$Y_t = \mathbb{E}^{\mathbb{Q}}[\xi|\mathcal{F}_t]. \tag{5}$$

In the following we will discuss whether Y still can be written as a conditional expectation of ξ if f does not have a representation as in (4) with b bounded, but satisfies only a quadratic growth condition in z. We aim at finding sufficient conditions guaranteeing that the process Y_t of a classical solution of a quadratic BSDE has a representation as a conditional expectation of ξ with respect to a new probability measure. For this purpose we consider the class of generators $f : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, satisfying, for some constant $c \in \mathbb{R}_+$,

Assumption (H1):

(i) $f(s, z) = f(\cdot, s, z)$ is adapted for any $z \in \mathbb{R}$,
(ii) $g(s, z) = \frac{f(s,z)}{z}$, $z \in \mathbb{R}$, is continuous in z, for all $s \in [0, T]$,
(iii) $|f(s, z)| \le c(1 + z^2)$ for any $s \in [0, T]$, $z \in \mathbb{R}$,
(iv) there exists $\varepsilon > 0$ and a predictable process $(\psi_s)_{s\ge 0}$ such that $\int_0^\cdot \psi_s \mathrm{d}W_s$ is a BMO-martingale and for every $|z| \le \varepsilon$, $|g(s, z)| \le \psi_s$.

Let ξ be an $\mathcal{F}_T$-measurable random variable. We introduce for BSDEs with generators satisfying (H1) our concept of measure solutions.

Definition 1.1. A triplet $(Y, Z, \mathbb{Q})$ is called *measure solution* of the BSDE (3) if $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{F})$, (Y, Z) a pair of $(\mathcal{F}_t)$-predictable stochastic processes such that

$\int_0^T Z_s^2 \mathrm{d}s < \infty$, $\mathbb{Q}$-a.s., and the following conditions are satisfied:

$$\widetilde{W} = W - \int_0^{\cdot} g(s, Z_s)\mathrm{d}s \quad \text{is a } \mathbb{Q}\text{-Brownian motion,}$$
$$\xi \in L^1(\Omega, \mathcal{F}, \mathbb{Q}),$$
$$Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t) = \xi - \int_t^T Z_s \mathrm{d}\widetilde{W}_s, \quad t \in [0, T].$$

It is known from the literature that if the terminal condition ξ is bounded and the generator f satisfies Assumption (H1), then the BSDE (3) has a classical solution (Y, Z) (see for example Kobylanski [4]). We show that in this case there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution.

Theorem 1.1. *Assume that ξ is bounded, and that f satisfies Assumption* (H1). *Then for every classical solution (Y, Z) there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of* (3).

Proof. Let (Y, Z) be a classical solution of (3). From Kobylanski [4], $Y \in L^\infty(\Omega \times [0, T])$. Moreover the martingale $Z.W$ is a BMO ($\mathbb{P}$)-martingale (see for example Barrieu et al. [9]).

We shall prove that under this condition also a measure solution exists. For this purpose, we define

$$M = \int_0^{\cdot} g(s, Z_s)\mathrm{d}W_s. \tag{6}$$

It is clear that all we have to establish is that the measure

$$\mathbb{Q} = V_T \cdot \mathbb{P},$$

with

$$V = \exp\left(M - \frac{1}{2}\langle M\rangle\right)$$

leads to a probability measure equivalent to $\mathbb{P}$. Note that due to Assumption (H1)

$$|g(s, z)| \leq C(\psi_s + |z|), \quad \mathrm{s} \in [0, T], z \in \mathbb{R}^d,$$

for some $C > 0$.

Therefore M is also a BMO ($\mathbb{P}$)-martingale, and from Kazamaki [10] $\mathbb{Q}$ is a probability equivalent to $\mathbb{P}$.

Under $\mathbb{Q}$, by definition,

$$W^{\mathbb{Q}} = W - \int_0^{\cdot} g(s, Z_s)\, \mathrm{d}s$$

is a Brownian motion, and our BSDE may be written as

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}W_s^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_t)$$

for $t \in [0, T]$. This shows that $(Y, Z, \mathbb{Q})$ is a measure solution. $\square$

It is straightforward to see that every measure solution gives rise to a classical solution. Consequently, under the assumptions of Theorem 1.1, measure solutions exist if and if only classical solutions exist. More precisely, we obtain the following.

Corollary 1.1. *Assume that ξ is bounded, and that f satisfies Assumption* (H1). *Then (Y, Z) is a classical solution if and only if there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of (3).*

We remark that the previous results can be extended to the case where W is a d-dimensional Brownian motion. Let $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be a generator for which there exists a constant $c \in \mathbb{R}_+$ such that

$$|f(s, z)| \le c(1 + |z|^2), \quad s \in [0, T], z \in \mathbb{R}^d, \tag{7}$$

and assume that $g : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a function that is continuous in z and satisfies

$$\langle z, g(s, z)\rangle = f(s, z), \quad \text{for all } z \in \mathbb{R}^d \text{ and } s \in [0, T]. \tag{8}$$

If ξ is bounded and $\mathcal{F}_T$-measurable, then one can show with arguments similar to those used in the preceding proof that, starting from a classical solution (Y, Z), there exists a probability measure $\mathbb{Q}$ such that $W - \int_0^\cdot g(s, Z_s)\,\mathrm{d}s$ is a $\mathbb{Q}$-Brownian motion, and $Y_t = E^{\mathbb{Q}}(\xi|\mathcal{F}_t)$.

Notice that the relation (8) may be satisfied by more than one continuous g, and consequently there may exist more than one measure solution in the multidimensional case. For example, let $d = 2$, $f(s, z) = z_1 z_2$, and observe that $|f(z)| \le \frac{1}{2}|z|^2$. For any $a \in (0, \infty)$ let $g_a(z) = (az_1, \frac{1}{a}z_2)$. Then, we have $\langle z, g_a(s, z)\rangle = f(s, z)$, and thus there exists more than one measure solution for a BSDE with generator f and a bounded terminal condition ξ.

In the following sections we shall discuss quadratic BSDEs with terminal conditions that are not bounded. As is known from the literature (see for example Briand and Hu [5,11]), this case is by far more complex. For example, here, even if the generators are smooth, solutions are no longer unique. We shall exhibit examples below which complement the result shown in Briand and Hu [11], according to which uniqueness is guaranteed for the case where the generator of the BSDE possesses additional convexity properties, and the terminal variable possesses exponential moments of all orders. This fact underlines that also variations in the generator affect questions of existence and uniqueness of solutions a lot. For this reason, and also to keep better oriented on a windy track with many bifurcations, in the next section we shall choose a simpler generator, and assume that our generator is given by

$$f(s, z) = \alpha z^2.$$

2. Measure and non-measure solutions of quadratic BSDEs with unbounded terminal condition

In this section we will study in more detail the BSDEs with generator of the form

$$f(z) = \alpha z^2.$$

We shall further assume without loss of generality that $\alpha > 0$. This can always be obtained in our BSDE by changing the signs of ξ, and the solution pair (Y, Z).

Nonetheless, it turns out that positive and negative terminal variables need a separate treatment. We will first show (see Section 2.1) the existence of measure solutions for terminal conditions ξ bounded from below. Note that with a linear shift of Y we may assume that $\xi \geq 0$. We shall further work under exponential integrability assumptions in the spirit of Briand and Hu [5]. According to this paper, exponential integrability of the terminal variable of the form

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{9}$$

for some $\gamma > 2\alpha$ is sufficient for the existence of a solution. Let us first exhibit an example to show that one cannot go essentially beyond this condition without losing solvability.

Example. Let $T = 1$, and let $\alpha = \frac{1}{2}$. Let us first consider

$$\xi = \frac{W_1^2}{2}.$$

It is immediately clear from the fact that W_1 possesses the standard normal density that $\mathbb{E}\exp(2\alpha|\xi|) = \infty$, and hence of course also for $\gamma > 2\alpha$ (9) is not satisfied. To find a solution (Y, Z) of (3) on any interval $[t, 1]$ with $t > 0$ define

$$Z_s = \frac{W_s}{s}, \quad s > 0,$$

and set for completeness $Z_0 = 0$. Let $t > 0$ and use the product formula for Itô integrals to deduce

$$\begin{aligned}\int_t^1 Z_s \mathrm{d}W_s &= \frac{1}{2}\frac{W_s^2}{s}\bigg|_t^1 + \frac{1}{2}\int_t^1 \frac{W_s^2}{s^2}\mathrm{d}s \\ &= \xi - \frac{1}{2}\frac{W_t^2}{t} + \frac{1}{2}\int_t^1 Z_s^2 \mathrm{d}s.\end{aligned} \tag{10}$$

This means that, if we set for convenience again $Y_0 = 0$, the pair of processes $(Y_s, Z_s) = (\frac{1}{2}\frac{W_s^2}{s}, \frac{W_s}{s})$, $s \in [0, 1]$, solves the BSDE (3) on $[t, 1]$ for any $t > 0$. Of course, the definition of Y_0 is totally inconsistent with the BSDE. Worse than that, Z is not square integrable on $[0, 1]$, as is well known from the path behavior of Brownian motion. Hence (Y, Z) is not a solution of (3). To put it more strictly, there is no classical solution of (3) on $[0, 1]$, since, due to local Lipschitz conditions, any such solution would have to coincide with (Y, Z) on any interval $[t, 1]$ with $t > 0$.

According to Jeulin and Yor [12], transformations of this type are related to a phenomenon that they call *appauvrissement de filtrations*. In fact, when $\frac{1}{2}$ is replaced with a parameter λ, they show that the natural filtration of the transformed process gets poorer than that of the Wiener process, iff $\lambda > \frac{1}{2}$. Hence in the case that we are interested in the Wiener filtration is preserved.

Let us now reduce the factor of W_1^2 in the definition of ξ a bit, to show that solutions exist in this setting. For $k \in \mathbb{N}$, let

$$\xi_k = \frac{W_1^2}{2(1 + 1/k)},$$

and consider the BSDE (3) with the generator f chosen above, and terminal condition ξ_k. In this setting, we clearly have

$$\mathbb{E}\exp(\gamma\xi_k) < \infty \quad \text{for } 2\alpha \leq \gamma < 2\alpha(1 + 1/k).$$

This shows that the condition of Briand and Hu [5] is satisfied. It is not hard to construct the solutions of the corresponding BSDEs explicitly, in the same way as above. In fact, for $k \in \mathbb{N}$ we may define $f_k(t) = \frac{1}{k} + t$, $t \in [0, 1]$, and set

$$Z_t^k = \frac{W_t}{f_k(t)}, \quad t \in [0, 1].$$

We may then repeat the product formula for the Itô integrals argument used above to obtain for $t \geq 0$

$$\begin{aligned}\int_t^1 Z_s^k \mathrm{d}W_s &= \frac{1}{2} \frac{W_s^2}{f_k(s)}\bigg|_t^1 + \frac{1}{2} \int_t^1 \frac{W_s^2 f_k'(s)}{f_k(s)^2} \mathrm{d}s \\ &= \frac{1}{2} \frac{W_1^2}{f_k(1)} - \frac{1}{2} \frac{W_t^2}{f_k(t)} + \frac{1}{2} \int_t^1 (Z_s^k)^2 \mathrm{d}s. \end{aligned} \tag{11}$$

Hence we set

$$Y_t^k = \frac{1}{2} \frac{W_t^2}{f_k(t)}, \quad t \in [0, 1],$$

to identify the pair of processes (Y^k, Z^k) as a solution of the BSDE

$$Y_t^k = \xi_k - \int_t^1 Z_s^k \mathrm{d}W_s + \frac{1}{2} \int_t^1 (Z_s^k)^2 \mathrm{d}s, \quad t \in [0, 1]. \tag{12}$$

We do not know at this moment whether (3) possesses more solutions. □

2.1. *Exponentially integrable lower bounded terminal variable*

For under the exponential integrability assumption $\mathbb{E}(\exp(2\alpha\xi)) < \infty$, we will now derive measure solutions from given classical solutions. Leaving the difficult question of uniqueness aside for a moment, we remark that with our simple generator, we obtain an explicit solution given by the formula

$$Y_t = \frac{1}{2\alpha} \ln M_t - \frac{1}{2\alpha} \ln M_0, \quad Z_t = \frac{1}{2\alpha} \frac{H_t}{M_t}, \tag{13}$$

where

$$M_t = \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t) = M_0 + \int_0^t H_s \mathrm{d}W_s, \quad t \in [0, T].$$

In the sequel, we shall work with this explicit solution. In the following lemma, we prove integrability properties for the square norm of Z which will be crucial for stating the martingale property of M and other related processes later.

Lemma 2.1. *For any $p \geq 1$ we have*

$$\mathbb{E}\left(\left[\int_0^T Z_s^2 \mathrm{d}s\right]^p\right) < \infty.$$

In particular, $\int_0^\cdot Z_s \mathrm{d}W_s$ is a uniformly integrable martingale.

Proof. Let $t \in [0, T]$. By Itô's formula, applied to N

$$\frac{1}{2\alpha}[\ln M_t - \ln M_0] = \frac{1}{2\alpha}\left[\int_0^t \frac{H_s}{M_s}\mathrm{d}W_s - \frac{1}{2}\int_0^t \left(\frac{H_s}{M_s}\right)^2 \mathrm{d}s\right] = \int_0^t Z_s \mathrm{d}W_s - \alpha \int_0^t Z_s^2 \mathrm{d}s.$$

Hence

$$\alpha \int_0^t Z_s^2 \mathrm{d}s = -\frac{1}{2\alpha}[\ln M_t - \ln M_0] + \int_0^t Z_s \mathrm{d}W_s. \tag{14}$$

By concavity of the ln and Jensen's inequality

$$\ln M_t = \ln \mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t) \geq \mathbb{E}(2\alpha\xi|\mathcal{F}_t).$$

Using this in (14), we obtain

$$\alpha \int_0^t Z_s^2 \mathrm{d}s \leq -\mathbb{E}(\xi|\mathcal{F}_t) + \frac{1}{2\alpha}\ln M_0 + \int_0^t Z_s \mathrm{d}W_s.$$

Taking p-norms in this inequality and using the inequality of Burkholder, Davis and Gundy for the stochastic integral, we obtain with universal constants c_1, c_2, c_3

$$\begin{aligned}\mathbb{E}\left(\left[\int_0^t Z_s^2 \mathrm{d}s\right]^p\right) &\leq c_1 \left[\mathbb{E}\left(|\mathbb{E}(\xi|\mathcal{F}_t)|^p\right) + |\ln M_0|^p + \mathbb{E}\left(\left[\int_0^t Z_s^2 \mathrm{d}s\right]^{\frac{p}{2}}\right)\right] \\ &\leq c_2 \left[\mathbb{E}(|\xi|^p) + |\ln M_0|^p + \mathbb{E}\left(\left[\int_0^t Z_s^2 \mathrm{d}s\right]^{\frac{p}{2}}\right)\right].\end{aligned}$$

By a standard argument this entails

$$\mathbb{E}\left(\left[\int_0^t Z_s^2 \mathrm{d}s\right]^p\right) \leq c_3[\mathbb{E}(|\xi|^p) + |\ln M_0|^p + 1],$$

and finishes the proof. □

We shall now prove that (Y, Z) gives rise to a measure solution.

Theorem 2.1. *Assume that (Y, Z) are defined as in (13). Then there exists a probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that $(Y, Z, \mathbb{Q})$ is a measure solution of (3).*

Proof. Let

$$S = \int_0^\cdot Z_s \, \mathrm{d}W_s.$$

Due to Lemma 2.1, we know that S is a uniformly integrable martingale. We may write

$$\begin{aligned}\alpha S - \frac{1}{2}\alpha^2\langle S\rangle &= \alpha\left[\int_0^\cdot Z_s \mathrm{d}W_s - \alpha\int_0^\cdot Z_s^2 \mathrm{d}s\right] + \int_0^\cdot \left(\alpha^2 Z_s^2 - \frac{1}{2}\alpha^2 Z_s^2\right) \mathrm{d}s \\ &= \alpha(Y - Y_0) + \frac{1}{2}\alpha^2 \int_0^\cdot Z_s^2 \mathrm{d}s.\end{aligned} \tag{15}$$

Now define stopping times $\tau_n = T \wedge \inf\{t \geq 0 : \langle S\rangle_t \geq n\}$. For any $n \in \mathbb{N}$ we have

$$\mathbb{E}\exp\left(\alpha S_{\tau_n} - \frac{1}{2}\alpha^2\langle S\rangle_{\tau_n}\right) = 1,$$

and consequently Fatou's lemma implies

$$\mathbb{E}\exp\left(\alpha[\xi - Y_0] + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 \mathrm{d}s\right) \leq \liminf_{n\to\infty} \mathbb{E}\exp\left(\alpha S_{\tau_n} - \frac{1}{2}\alpha^2 \langle S\rangle_{\tau_n}\right) = 1. \tag{16}$$

Using this and the positivity of the terminal variable ξ, we can now obtain the exponential integrability property

$$\mathbb{E}\exp\left[\frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 \,\mathrm{d}s\right] < \infty. \tag{17}$$

We shall now use (15) together with (16) to prove the exponential integrability of $\frac{1}{2}\alpha S_T$. In fact, we have

$$\frac{1}{2}\alpha S_T = \frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2 \,\mathrm{d}s.$$

Hence we obtain

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha S_T\right) < \infty, \tag{18}$$

and together with the uniform integrability of the martingale S, proved in Lemma 2.1, this enables us to apply the criterion of Kazamaki (see Revuz and Yor [13], p. 332). Hence we have proved the existence of a measure solution to our BSDE (3). □

As a by-product of our main result, we obtain the exponential integrability of the quadratic variation of S.

Corollary 2.1. *Under the conditions of Theorem 2.1 we have*

$$\mathbb{E}\exp\left(\frac{1}{2}\alpha^2 \int_0^T Z_s^2 \,\mathrm{d}s\right) < \infty.$$

Proof. This follows immediately from (17) and the lower boundedness of ξ. □

2.2. A quadratic BSDE with two solutions

Let us now come back to the question of uniqueness of solutions, and their measure solution property. Briand and Hu [5] prove the existence of solutions (Y, Z) in the usual sense, given that (9) is satisfied. In a setting with more general generators, the nonlinear z-part being bounded by αz^2, they provide pathwise upper and lower bounds for Y, given by the known explicit solution for this generator $(\frac{1}{2\alpha}\log\mathbb{E}(\exp(2\alpha\xi)|\mathcal{F}_t)_{t\in[0,T]})$ used above, and its negative counterpart $(-\frac{1}{2\alpha}\log\mathbb{E}(\exp(-2\alpha\xi)|\mathcal{F}_t)_{t\in[0,T]})$. In a more recent paper, Briand and Hu [11] also provide a uniqueness result for the same setting, which is satisfied under the stronger integrability hypothesis

$$\mathbb{E}(\exp(\gamma|\xi|)) < \infty \tag{19}$$

for all $\gamma > 0$ and a convexity assumption concerning the generator. Let us start our discussion of uniqueness and the measure solution property by giving some examples.

For $b > 0$, let $\tau_b = \inf\{t \geq 0 : W_t \leq bt - 1\}$. We first consider a BSDE with random time horizon τ_b. Let the generator be further specified by $\alpha = \frac{1}{2}$. Let $\xi = 2a(b-a)\tau_b - 2a$, where

$a > 0$. It will become clear along the way why this choice of terminal variable is made. In the first place, it is motivated by the striking simplicity of the solutions that we shall construct. We shall in fact give two explicit solutions of the BSDE

$$Y_{t\wedge\tau_b} = \xi - \int_t^{\tau_b} Z_s \mathrm{d}W_s + \int_t^{\tau_b} \frac{1}{2} Z_s^2 \mathrm{d}s. \tag{20}$$

Appropriate choices of a and b allow for terminal variables that are bounded below as well as bounded above. The fact that the time horizon is random is not crucial. Indeed, by using a time change, any solution of Eq. (20) can be transformed into a solution of a BSDE with the same generator and with time horizon 1. To this end consider the time change $\rho(t) = \frac{t}{1+t}$, $t \in [0, \infty]$, and observe that the inverse of ρ is given by $\rho^{-1}(t) = \frac{t}{1-t}$, $t \in [0, 1]$. Let $h(t) = \frac{1}{1-t}$ for all $t \in [0, 1]$. Then the process defined by

$$\tilde{W}_t = \int_0^t h^{-1}(s)\mathrm{d}(W_{\rho^{-1}(s)}), \quad t \in [0, 1], \tag{21}$$

is a Brownian motion on [0, 1]. Note that $W_t = \int_0^{\rho(t)} h(s)\mathrm{d}\tilde{W}_s$ (and this is how we have to define W, if $\tilde{W}$ is given). Moreover, the stopping time

$$\hat{\tau}_b = \inf\left\{ t \geq 0 : \int_0^t h(s)\mathrm{d}\tilde{W}_s \leq \frac{t}{1-t} - 1 \right\}$$

is equal to $\rho(\tau_b)$. We can now define a time changed analogue of Eq. (20) with time horizon 1.

Lemma 2.2. *Let (Y_t, Z_t) be a solution of the BSDE (20), and let $\hat{\xi} = 2a(b-a)\frac{\hat{\tau}_b}{1-\hat{\tau}_b} - 2a$. Then $(y_t, z_t) = (Y_{\rho^{-1}(t)}, h(t)Z_{\rho^{-1}(t)})$ is a solution of the BSDE*

$$y_t = \hat{\xi} - \int_t^1 z_s \mathrm{d}\tilde{W}_s + \int_t^1 \frac{1}{2} z_s^2 \mathrm{d}s. \tag{22}$$

Proof. Since stochastic integration and continuous time changes can be interchanged (see Proposition 1.5, Chapter V in [13]), we have

$$\begin{aligned} y_t &= Y_{\rho^{-1}(t)} = \int_0^{\rho^{-1}(t)} Z_s \mathrm{d}W_s - \frac{1}{2}\int_0^{\rho^{-1}(t)} Z_s^2 \mathrm{d}s \\ &= \int_0^t Z_{\rho^{-1}(s)} \mathrm{d}W_{\rho^{-1}(s)} - \frac{1}{2}\int_0^t Z_{\rho^{-1}(s)}^2 \mathrm{d}\rho^{-1}(s) \\ &= \int_0^t Z_{\rho^{-1}(s)} h(s)\mathrm{d}\tilde{W}_s - \frac{1}{2}\int_0^t Z_{\rho^{-1}(s)}^2 h^2(s)\mathrm{d}s, \end{aligned}$$

and hence the result. $\square$

Let us first assess exponential integrability properties of ξ. For this, let $\gamma > 0$ be arbitrary. Then we have

$$\mathbb{E}\mathrm{e}^{\gamma|\xi|} = \mathbb{E}\mathrm{e}^{\gamma|2a(b-a)\tau_b - 2a|} \leq \mathrm{e}^{2a\gamma}\mathbb{E}\mathrm{e}^{\gamma 2a|b-a|\tau_b}.$$

Define the auxiliary stopping time

$$\sigma_b = \inf\{t \geq 0 : W_t \leq t - b\}.$$

It is well known and proved by the scaling properties of Brownian motion that the laws of τ_b and $\frac{\sigma_b}{b^2}$ are identical (see Revuz and Yor [13]). Moreover, the Laplace transform of σ_b is equally well known. According to Revuz and Yor [13] we therefore have for $\lambda > 0$

$$E(\exp(-\lambda\tau_b)) = E\left(\exp\left(-\frac{\lambda}{b^2}\sigma_b\right)\right) = \exp\left(-b\left[\sqrt{1+\frac{2\lambda}{b^2}}-1\right]\right). \tag{23}$$

Moreover, it is seen by analytic continuation arguments that this formula is even valid for $\lambda \geq -\frac{b^2}{2}$. Now choose $\lambda = -2a|b-a|\gamma$. Then the inequality

$$-2a|b-a|\gamma \geq -\frac{1}{2}b^2$$

amounts to

$$\gamma \leq \frac{b^2}{4a|b-a|}. \tag{24}$$

This in turn means that we have exponential integrability of orders bounded by $\frac{b^2}{4a|b-a|}$; in particular we may reach arbitrarily high orders by choosing a and b sufficiently close. But no combination of a and b allows exponential integrability of all orders. In the light of Briand and Hu [11] this means that the entire field of pairs of positive a and b promises multiple solutions, and this is precisely what we will exhibit.

2.2.1. *The first solution*

It is clear from the definition that the pair (Y_t, Z_t), defined by $Y_t = 2aW_{t\wedge\tau_b} - 2a^2(\tau_b \wedge t)$ and $Z = 2a1_{[0,\tau_b]}$, is a solution of (20). To answer the question of whether this defines a measure solution, we have to investigate

$$\begin{aligned}\mathbb{E}\exp\left[\int_0^{\tau_b}\frac{1}{2}Z_s\mathrm{d}W_s - \frac{1}{8}\int_0^{\tau_b}Z_s^2\mathrm{d}s\right] &= \mathbb{E}\exp\left[aW_{\tau_b} - \frac{a^2}{2}\tau_b\right]\\ &= \mathbb{E}\left(\exp\left(a\left(b-\frac{a}{2}\right)\tau_b - a\right)\right).\end{aligned}$$

Due to (23) we have

$$\begin{aligned}\mathbb{E}\left(\exp\left(a\left(b-\frac{a}{2}\right)\tau_b - a\right)\right) &= \exp\left(-b\left[\sqrt{1-\frac{2}{b^2}a\left(b-\frac{a}{2}\right)}-1\right]-a\right)\\ &= \exp\left(-b\left[\left|1-\frac{a}{b}\right|-1\right]-a\right),\end{aligned}$$

and the latter equals 1 in the case $b \geq a$ and $\exp(2(b-a)) < 1$ in the case $a > b$. This simply means that our first solution is a measure solution of (22) provided $b \geq a$, and it fails to be one in the case $a > b$. We will show that this first solution does not necessarily correspond to the particular solution discussed at the beginning of the section.

2.2.2. *The second solution*

We show now that the BSDE (20) with the same terminal variable as above possesses a second solution. By Lemma 2.2 there exists a second solution of (22) as well. Once this is shown, for any possible degree γ of exponential integrability we will have exhibited a negative random variable satisfying $\mathbb{E}(\exp(\gamma|\xi|)) < \infty$ for which (20) possesses at least two solutions. This in turn will underline that the Briand and Hu [11] uniqueness result, valid under (19), cannot be improved by much. Note that the solution that we will exhibit is again of the explicit form (13) encountered earlier. Let $M_t = \mathbb{E}[e^{\xi}|\mathcal{F}_t]$ for all $t \geq 0$. Due to the martingale representation property there exists a process H such that $M_t = M_0 + \int_0^t H_s \mathrm{d}W_s$. We know that $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is a solution of (20). We will show below that

$$\ln M_{\tau_b\wedge t} = 2b - 4a + 2(b-a)W_{\tau_b\wedge t} - 2(b-a)^2(\tau_b \wedge t), \quad \text{if } 2a > b, \tag{25}$$

$$\ln M_{\tau_b} = 2aW_{\tau_b\wedge t} - 2a^2\tau_b \wedge t, \quad \text{if } 2a \leq b. \tag{26}$$

This implies that the solution $(\ln M_{\tau_b\wedge t}, \frac{H_{\tau_b\wedge t}}{M_{\tau_b\wedge t}})$ is different from the solution $(2aW_{\tau_b\wedge t} - 2a^2(\tau_b \wedge t), 2a)$ obtained above in the case $2a > b$. Hence by Lemma 2.2 we obtain a second solution of (22) in this case.

First note that

$$\begin{aligned} M_t &= e^{-2a}\mathbb{E}[e^{2a(b-a)\tau_b}|\mathcal{F}_t] \\ &= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\leq t\}} + e^{2a(b-a)t}\mathbb{E}[e^{2a(b-a)[\tau_b-t]}|\mathcal{F}_t]1_{\{\tau_b>t\}}\right). \end{aligned} \tag{27}$$

Let $\sigma_b(x,t) = \inf\{s \geq 0 : W_{s+t} - W_t \leq b(s+t) - 1 - x\}$ and observe that on the set $\{\tau_b > t\}$ we have $\tau_b - t = \sigma_b(W_t, t)$. Therefore, by using again our knowledge on the Laplace transforms of $\sigma(x,t)$ (see [13]), we get

$$\begin{aligned} \mathbb{E}[e^{2a(b-a)[\tau_b-t]}|\mathcal{F}_t]1_{\{\tau_b>t\}} &= \mathbb{E}[e^{2a(b-a)\sigma_b(x,t)}]|_{x=W_t}1_{\{\tau_b>t\}} \\ &= e^{-b(1+W_t-bt)[\sqrt{1-\frac{4a(1-a)}{b^2}}-1]}1_{\{\tau_b>t\}} \\ &= e^{-b(1+W_t-bt)[|1-\frac{2a}{b}|-1]}1_{\{\tau_b>t\}}. \end{aligned}$$

Consequently,

$$\begin{aligned} M_t &= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\leq t\}} + e^{2a(b-a)t}e^{-b(W_t+1-bt)[|1-\frac{2a}{b}|-1]}1_{\{\tau_b>t\}}\right) \\ &= e^{2a((1-a)(\tau_b\wedge t)-1)}1_{\{\tau_b\leq t\}} + e^{-2(b-a)(W_t+1-bt)}1_{\{\tau_b>t\}}. \end{aligned}$$

Hence in the case $2a > b$

$$\begin{aligned} \ln M_{\tau_b\wedge t} &= 2a((b-a)(\tau_b\wedge t) - 1) - 2(a-b)(W_{\tau_b\wedge t} + 1 - (\tau_b\wedge t)) \\ &= -4a + 2b + [-2b + 4a - 2a^2](\tau_b\wedge t) - 2(a-b)W_{\tau_b\wedge t} \\ &= 2b - 4a + 2(b-a)W_{\tau_b\wedge t} - 2[b-a]^2(\tau_b\wedge t). \end{aligned}$$

This confirms the first Eq. (25). Let finally $2a \leq b$. Then we have

$$\begin{aligned} M_t &= e^{-2a}\left(e^{2a(b-a)\tau_b}1_{\{\tau_b\leq t\}} + e^{2a(b-a)t}e^{2a(W_t+1-bt)}1_{\{\tau_b>t\}}\right) \\ &= e^{2a((b-a)(\tau_b\wedge t))+2a(W_{\tau_b\wedge t}+1-b\tau_b\wedge t)} \\ &= e^{2aW_{\tau_b\wedge t}-2a^2\tau_b\wedge t}. \end{aligned}$$

Hence in this case

$$\ln M_{\tau_b\wedge t} = 2aW_{\tau_b\wedge t} - 2a^2\tau_b \wedge t.$$

Note that in the case $2a \le b$ we recover the solution already obtained as the first solution.

Let us finally show that this second solution is in fact a measure solution for any possible combination of parameters.

Lemma 2.3. $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{\tau_b \wedge t}})$ *can be extended to a measure solution of* (20), *and hence provides a measure solution of* (22).

Proof. For the first solution in the case $a \le b$, which is identical to the one considered in the case $2a \le b$, we have already established the measure solution property. Let us therefore consider the case $2a > b$. Note that for all t, $M_{t \wedge \tau_b} = \mathrm{e}^{2b-4a} + \int_0^{t \wedge \tau_b} H_s \mathrm{d}W_s$. Itô's formula applied to $\mathrm{e}^{2(b-a)W_{\tau_b \wedge t} - 2[b-a]^2(\tau_b \wedge t)}$ yields

$$H_{s \wedge \tau_b} = 2(b-a)\mathrm{e}^{2(b-a)W_{\tau_b \wedge t} - 2[b-a]^2(\tau_b \wedge t)}.$$

As a consequence, we have

$$Z_{s \wedge \tau_b} = \frac{H_{s \wedge \tau_b}}{M_{s \wedge \tau_b}} = 2(b-a)1_{[0,\tau_b]}(s),$$

and therefore

$$\begin{aligned}
\mathcal{E}\Big(\frac{1}{2}\int Z \mathrm{d}W\Big)_{\tau_b} &= \mathrm{e}^{(b-a)W_{\tau_b} - \frac{1}{2}(b-a)^2\tau_b} \\
&= \mathrm{e}^{(b-a)(b\tau_b - 1) - \frac{1}{2}(b-a)^2\tau_b} \\
&= \mathrm{e}^{(a-b)}\mathrm{e}^{\frac{1}{2}(b-a)(b+a)\tau_b}.
\end{aligned}$$

Again the explicit representation of the Laplace transform in (23) gives

$$\mathbb{E}\mathcal{E}\Big(\frac{1}{2}\int Z \mathrm{d}W\Big)_{\tau_b} = \mathrm{e}^{(a-b)}\mathbb{E}\mathrm{e}^{-\frac{1}{2}(b-a)(b+a)\tau_b} = \mathrm{e}^{(a-b)}\mathrm{e}^{-b(\sqrt{1-(1-\frac{a^2}{b^2})}-1)} = 1.$$

This implies the claimed result that our second solution $(\ln M_{\tau_b \wedge t}, \frac{H_{\tau_b \wedge t}}{M_{b \wedge t}})$ is a measure solution of (20). □

Remark. 1. We can summarize the findings of our investigations of the examples by stating that there are three basic scenarios: (a) for $b \ge 2a$ we obtained one solution which is a measure solution at the same time; (b) in the range $2a > b \ge a$ we found two different solutions both of which are measure solutions; (c) if $a > b$ we finally encountered two solutions one of which is a measure solution, while the other one is not.

2. Note that our examples exhibiting solutions of (20) that are not measure solutions are all for negative terminal variables ξ. Positive terminal variables arise in scenarios (a) or (b), and therefore only produce multiple measure solutions.

2.2.3. A continuum of solutions

Let us now combine the first and second solutions to obtain a continuum of solutions of our BSDE (20). To do this, we have to consider a still somewhat more general class of stopping times. For $c \in \mathbb{R}$, let

$$\rho_c = \inf\{t \ge 0 : W_t \le t - c\}.$$

We investigate the terminal variables

$$\xi = 2a(a-1)\rho_c + d$$

with further constants $a \neq 0, d \in \mathbb{R}$. Note first that the integrability properties of ξ are the same as those obtained before for $b = 1$. According to the preceding paragraphs, our BSDE (20) possesses the following two solutions:

$$Z^1 = 2a1_{[0,\rho_c]}, \quad Y^1 = d_1 + 2aW_{\rho_c\wedge\cdot} - 2a^2\rho_c \wedge \cdot, \tag{28}$$

$$Z^2 = 2(1-a)1_{[0,\rho_c]}, \quad Y^2 = d_2 + 2(1-a)W_{\rho_c\wedge\cdot} - 2(1-a)^2\rho_c \wedge \cdot, \tag{29}$$

with $d_1 = -2ac$ and $d_2 = -2(a-1)c$ respectively. Let us now take $c = 1$ and combine the two solutions to obtain a continuum of new ones. To do this, we start with the equation

$$\rho_1 = \rho_c + \rho_{1-c} \circ \theta_{\rho_c},$$

where θ_t is the shift on Wiener space defined by

$$\theta_t(\omega) = W_{t+\cdot}(\omega) - W_t(\omega),$$

and $c \in]0, 1[$. It describes the first time for reaching the line with slope 1 that cuts the vertical at level -1, by decomposition with the intermediate time for reaching the line with slope 1 cutting the vertical at $-c$. We mix the two solutions on the two resulting stochastic intervals; more precisely we put for $c \in]0, 1[$, $l \in \mathbb{R}$,

$$\begin{aligned} Z^c &= 2a1_{[0,\rho_c]} + 2(1-a)1_{[\rho_c,\rho_1]}, \\ Y^c &= l + 2aW_{\rho_c\wedge\cdot} - 2a^2\rho_c \wedge \cdot + 2(1-a)[W_{\rho_1\wedge\cdot} - W_{\rho_c\wedge\cdot}] \\ &\quad - 2(1-a)^2[\rho_1 \wedge \cdot - \rho_c \wedge \cdot]. \end{aligned} \tag{30}$$

Since we have

$$\begin{aligned} Y^c_{\rho_1} &= l + 2aW_{\rho_c} - 2a^2\rho_c + 2(1-a)[W_{\rho_1} - W_{\rho_c}] - 2(1-a)^2[\rho_1 - \rho_c] \\ &= l + 2a(1-a)\rho_1 - 2ac - 2(1-a)(1-c), \end{aligned}$$

we have to set

$$l - 2ac - 2(1-a)(1-c) = d$$

in order to obtain a solution of (20) with $c = 1$. According to the treatment of the first and second solutions, the constructed mixture is a measure solution if and only if both components of the mixture are. This is the case for $2a(1-a) > 0$, whereas for $2a(1-a) < 0$ we obtain a continuum of solutions that are not measure solutions.

Remark. 1. This time, we may summarize our results by saying that there are two scenarios: (a) for $2a(1-a) > 0$ there is a continuum of measure solutions of (20), while for $2a(1-a) < 0$ a continuum of non-measure solutions is obtained.

2. Note that the initial conditions of our solutions continuum vary in a convex way between $-2a$ and $-2(1-a)$ as c varies in $]0, 1[$, spanning the whole interval.

We shall now point out that the measure solution property of the second solution in the case $a > b$ exhibited in the example above is not a coincidence. In fact, it will turn out that also for negative exponentially integrable ξ, solutions given by (13) provide measure solutions. To prove this, we will reverse the sign of ξ by looking at our BSDE from the perspective of an equivalent measure.

2.3. *Exponentially integrable upper bounded terminal variable*

Sticking with the positivity of α in the generator

$$f(s, z) = \alpha z^2, \quad s \in [0, T], z \in \mathbb{R}$$

we shall now consider terminal variables ξ that fulfill the exponential integrability condition (9), but are bounded above by a constant. Again, by a constant shift of the solution component Y, we can assume that the upper bound is 0, i.e. $\xi \leq 0$. So fix a non-positive terminal variable ξ satisfying (9) for some $\gamma > 2\alpha$, and denote by (Y, Z) the pair of processes given by the explicit representation of (13) solving our BSDE according to Briand and Hu [5]. With respect to the following probability measure, ξ will effectively change its sign, so we can hook up to the previous discussion. Recall $S = \int_0^\cdot Z_s \mathrm{d}W_s$.

Lemma 2.4. *Let $V = \exp(2\alpha S - 2\alpha^2 \langle S \rangle)$. Then V is a martingale of class (D), and consequently*

$$R = V_T \cdot \mathbb{P}$$

is a probability measure equivalent to $\mathbb{P}$. Moreover,

$$W^R = W - 2\alpha \int_0^\cdot Z_s \mathrm{d}s$$

is a Brownian motion under R.

Proof. By (3), we may write

$$2\alpha[Y - Y_0] = 2\alpha S - 2\alpha^2 \langle S \rangle,$$

and hence also

$$2\alpha[\xi - Y_0] = 2\alpha S_T - 2\alpha^2 \langle S \rangle_T.$$

According to Briand and Hu [5], Theorem 2, there exists $\delta > 2\alpha$ such that

$$\mathbb{E}(\sup_{t\in[0,T]} \exp(\delta |Y_t|)) < \infty, \tag{31}$$

and therefore $\beta > 1$ with the property

$$\mathbb{E}(\sup_{t\in[0,T]} V_t^\beta) < \infty. \tag{32}$$

This clearly implies that V is a martingale of class (D), and consequently R is a probability measure. Finally, Girsanov's theorem implies that W^R is a Brownian motion under R. □

Now consider our BSDE under the perspective of the measure R with respect to the Brownian motion W^R. We may write

$$Y = \xi - \int_\cdot^T Z_s \mathrm{d}W_s + \alpha \int_\cdot^T Z_s^2 \mathrm{d}s = \xi - \int_\cdot^T Z_s \mathrm{d}W_s^R - \alpha \int_\cdot^T Z_s^2 \mathrm{d}s. \tag{33}$$

But this just means that by switching signs in (Y, Z), we may return, under the new measure R, to our old BSDE with ξ replaced with $-\xi$. So our measure change puts us back into the framework of the previous subsection, and we may resume our arguments there by setting

$$S^R = -\int_0^{\cdot} Z_s \mathrm{d}W_s^R.$$

We need an analogue of Lemma 2.1 to guarantee that R is a uniformly integrable martingale.

Lemma 2.5. *For any $p \geq 1$ we have*

$$\mathbb{E}^R\left(\left[\int_0^T Z_s^2 \mathrm{d}s\right]^p\right) < \infty.$$

In particular, S^R is a uniformly integrable martingale under R.

Proof. By definition of R, we have for any $p > 1$

$$\mathbb{E}^R\left(\left[\int_0^T Z_s^2 \mathrm{d}s\right]^p\right) = \mathbb{E}\left(\exp(2\alpha[\xi - Y_0])\left[\int_0^T Z_s^2 \mathrm{d}s\right]^p\right).$$

Now since $\xi \leq 0$, the density $\exp(2\alpha[\xi - Y_0])$ is bounded above. Therefore the asserted moment finiteness follows from Lemma 2.1. □

We are in a position to prove the main result of this subsection.

Theorem 2.2. *Assume that f satisfies $f(s, z) = \alpha z^2$, $z \in \mathbb{R}$, $s \in [0, T]$, and that ξ is bounded above and satisfies (9). Then there is a measure solution of (3) with a measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$.*

Proof. We may assume $\xi \leq 0$. Let us first show that, in analogy to the proof of Theorem 2.1,

$$V^R = \exp\left(\alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle\right)$$

is a uniformly integrable martingale under R, using Kazamaki's criterion. For this purpose, let

$$\tau_n^R = \inf\{t \geq 0 : \langle S^R \rangle_t \geq n\} \wedge T, \quad n \in \mathbb{N}.$$

Then, due to $\langle S \rangle = \langle S^R \rangle$, we deduce for all $n \in \mathbb{N}$ that $\tau_n = \tau_n^R$. Since $\tau_n^R \to T$ as $n \to \infty$, even with $\tau_n^R = T$ for all but finitely many n, Fatou's lemma allows us to deduce

$$\mathbb{E}^R(V_T) \leq \liminf_{n \to \infty} \mathbb{E}^R(V_{\tau_n^R}^R) \leq 1. \tag{34}$$

Moreover, by the form of the BSDE translated to W^R under R,

$$\begin{aligned}
\alpha S^R - \frac{1}{2}\alpha^2 \langle S^R \rangle &= \alpha\left[-\int_0^{\cdot} Z_s \mathrm{d}W_s^R - \frac{1}{2}\alpha \int_0^{\cdot} Z_s^2 \mathrm{d}s\right] \\
&= \alpha\left[-\int_0^{\cdot} Z_s \mathrm{d}W_s^R - \alpha \int_0^{\cdot} Z_s^2 \mathrm{d}s\right] + \frac{1}{2}\alpha^2 \int_0^{\cdot} Z_s^2 \mathrm{d}s \\
&= \alpha[-Y + Y_0] + \frac{1}{2}\alpha^2 \int_0^{\cdot} Z_s^2 \mathrm{d}s.
\end{aligned}$$

Using the negativity of ξ and the identity just derived, we get the integrability property

$$\mathbb{E}^R \exp\left[\frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, \mathrm{d}s\right] < \infty. \tag{35}$$

Using this and the positivity of the terminal variable ξ, we can now obtain the exponential integrability property

$$\mathbb{E} \exp\left[\frac{1}{2}\alpha(\xi - Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, \mathrm{d}s\right] < \infty. \tag{36}$$

Again, we may now use (35) together with (34) to prove the exponential integrability of $\frac{1}{2}\alpha S_T^R$. In fact, from the BSDE viewed with W^R under R we have

$$\frac{1}{2}\alpha S_T^R = \frac{1}{2}\alpha(-\xi + Y_0) + \frac{1}{2}\alpha^2 \int_0^T Z_s^2\, \mathrm{d}s.$$

Hence we obtain

$$\mathbb{E}^R \exp\left(\frac{1}{2}\alpha S_T^R\right) < \infty. \tag{37}$$

Now appeal to the uniform integrability of the martingale S^R under R, proved in Lemma 2.5, to see that the criterion of Kazamaki (see Revuz and Yor [13], p. 332) may be applied. Hence V_R is a uniformly integrable martingale under R.

We have to show that this implies uniform integrability of

$$V = \exp\left(\alpha S - \frac{1}{2}\alpha^2 \langle S\rangle\right)$$

under $\mathbb{P}$. To see this, note that

$$\begin{aligned}\exp\left(\alpha S - \frac{1}{2}\alpha^2\langle S\rangle\right) &= \exp(2\alpha S - 2\alpha^2\langle S\rangle)\cdot \exp\left(-\alpha S + \frac{3}{2}\alpha^2\langle S\rangle\right)\\ &= \exp(2\alpha S - 2\alpha^2\langle S\rangle)\cdot \exp\left(\alpha S^R - \frac{1}{2}\alpha^2\langle S^R\rangle\right).\end{aligned}$$

Hence for $n \in \mathbb{N}$

$$\mathbb{E}(V_{\tau_n} 1_{\{\tau_n < T\}}) = \mathbb{E}^R(V^R_{\tau_n^R} 1_{\{\tau_n^R < T\}}), \tag{38}$$

and the latter expression tends to 0 as $n \to \infty$ by the first part of the proof. Hence the uniform integrability of V under $\mathbb{P}$ follows from the explosion criterion

$$\mathbb{Q}^n(\tau_n < T) \to 0 \quad (n \to \infty). \tag{39}$$

In fact let

$$\mathbb{Q}^n = V_T \cdot \mathbb{P}|_{\mathcal{F}_{\tau_n}}$$

be the measure change defined locally on $\mathcal{F}_{\tau_n}$. We know that $\mathbb{Q}^n$ is a probability measure equivalent to $\mathbb{P}$, and the Radon–Nikodym density of $\mathbb{Q}^n$ with respect to $\mathbb{P}$ on $\mathcal{F}_{\tau_n}$ is given by

$$V_{\tau_n} = \exp\left(\alpha S_{\tau_n} - \frac{1}{2}\langle \alpha S\rangle_{\tau_n}\right).$$

Namely, (39) implies

$$\lim_n \mathbb{E}(V_T 1_{\{\tau_n = T\}}) = \lim_n [\mathbb{E}(V_{\tau_n}) - \mathbb{E}(V_{\tau_n} 1_{\{\tau_n < T\}})] = 1 - \lim_n \mathbb{Q}^n(\tau_n < T) = 1. \tag{40}$$

On the other hand, dominated converges yields $E(V_T) = \lim_n \mathbb{E}(V_T 1_{\{\tau_n = T\}})$, and hence that V induces a probability measure. We remark that the criterion (39) can be found in [7], and appears also as Lemma 1.5 in [10].

This completes the proof. □

Remark. The results of the preceding two subsections clearly call for similar ones for our BSDE with exponentially integrable terminal variable that are not bounded. Due to the nonlinearity of the generator of the BSDE, it seems impossible to derive such properties by combining the results of Theorems 2.1 and 2.2.

3. The existence of measure solutions in the Lipschitz case

We shall now construct measure solutions from first principles. In particular, we shall not assume any knowledge about strong solutions. We shall give a complete self-contained construction for measure solutions with Lipschitz continuous generator for which the Lipschitz constant may be time dependent. Our construction provides the measure solution through an algorithm which just iterates the procedures of projecting the terminal variable by a given measure. The conditions on the generator are less restrictive than in El Karoui and Huang [16]. The martingale representation theorem with respect to the measure $\mathbb{Q}^n$ in step n will produce a control process Z^n which is then plugged into the generator of the BSDE. The resulting drift is taken off by applying Girsanov's theorem which produces a new measure $\mathbb{Q}^{n+1}$ with which we continue along the lines just sketched in step $n+1$. The sequence $(\mathbb{Q}^n)_{n \in \mathbf{N}}$ thus produced has to be shown to possess at least an accumulation point in the weak topology. This is seen by a simple argument using the Lipschitz and boundedness properties. The extension to a continuous or quadratic generator and bounded terminal condition is straightforward from this perspective, and uses monotone approximations following the scheme in [6]. But this result is already contained in the results of [4] and Theorem 1.1. Hence we do not write the details here. The extension of our intrinsic construction of measure solutions to unbounded terminal conditions is left for future research.

In order to obtain a self-contained theory that is not using any knowledge on classical solutions, we first construct measure solutions in a setting for which they have been studied mostly: for generators that increase at most linearly and possess Lipschitz properties with time dependent and random Lipschitz constants. More formally, in this section we consider the following class of generators. Let

$$f : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$$

satisfy the **Assumption (H2)**: for some $\gamma \geq 1$ and some non-negative process ϕ:

1. $\xi \in L^\gamma(\Omega)$;
2. $f(s, z) = f(\cdot, s, z)$ is adapted for any $z \in \mathbb{R}$;
3. $\mathbb{E}\left(\int_0^T |f(s, 0)|^\gamma \mathrm{d}s\right) < \infty$;
4. the set $\{s \in [0, T], f(s, .)$ is not continuous$\}$ is of Lebesgue measure zero;
5. $|f(s, z) - f(s, z')| \leq \phi_s |z - z'|$ for all $s \in [0, T]$, $(z, z') \in \mathbb{R}^2$.

We shall assume in the following that $f(s,0) = 0$ for all $s \in [0,T]$. This can be done without loss of generality, since we may replace ξ with the γ-integrable random variable

$$\tilde{\xi} = \xi + \int_0^T f(s,0)\mathrm{d}s.$$

Now we define the function $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ by the requirement that for all $s \in [0,T]$, $z \in \mathbb{R}$:

$$\begin{aligned} g(s,z) &= \frac{f(s,z)}{z}, \quad \text{if } z \neq 0, \\ &= 0, \quad \text{if } z = 0. \end{aligned}$$

Therefore we have defined the function g with values in $\mathbb{R}$ and g is bounded by the process ϕ.

The process ϕ verifies either

$$\exists \kappa > 1, \quad \mathbb{E}\left[\exp\left(\frac{\kappa}{2}\int_0^T \phi_r^2 \mathrm{d}r\right)\right] < +\infty \tag{41}$$

or

$$\text{the martingale } \left(L_t = \int_0^t \phi_r \mathrm{d}W_r\right)_{t\in[0,T]} \text{ is BMO.} \tag{42}$$

We denote by $\|L\|$ the BMO_2-norm of L. From Theorem 2.2 in [10], (42) implies (41), with $1/\kappa = 2\|L\|^2$. Remark that (41) is a stronger Novikov condition. From these assumptions (see [10], Theorem 2.3), we know that for $0 \le t \le T$,

$$\mathcal{E}(\phi W)_t = \exp\left(\int_0^t \phi_r \mathrm{d}W_r - \frac{1}{2}\int_0^t \phi_r^2 \mathrm{d}r\right)$$

is a uniformly integrable martingale.

We define the process Φ by

$$\forall t \in [0,T], \quad \Phi_t = \int_0^t \phi_s^2 \mathrm{d}s,$$

and **Assumption (H3)** holds: there exist two constants $\alpha > \Psi$ and $\delta > \Psi$ such that

$$\mathbb{E}(\mathrm{e}^{\alpha \Phi_T} |\xi|^\delta) < +\infty. \tag{43}$$

The constant $\Psi > 1$ is given for (41) by

$$\Psi(\kappa) = \Psi_{(41)}(\kappa) = 1 + 4\frac{\sqrt{\kappa}}{(\sqrt{\kappa}-1)^2} = \left(1 + \frac{2\sqrt{\kappa}+1}{\kappa}\right)\frac{\kappa}{(\sqrt{\kappa}-1)^2},$$

and for (42) by

$$\Psi(\|L\|) = \Psi_{(42)}(\|L\|) = \left(1 + \frac{\|L\|}{2}\right)\frac{\theta^{-1}(\|L\|)}{\theta^{-1}(\|L\|) - 1}.$$

The function $\theta :]1,+\infty[\to \mathbb{R}_+^*$ is the continuous decreasing function given by

$$\forall q \in]1,+\infty[, \theta(q) = \left\{1 + \frac{1}{q^2}\ln\frac{2q-1}{2(q-1)}\right\}^{\frac{1}{2}} - 1.$$

We can check that $\Psi_{(42)} :]0, +\infty[\to]1, +\infty[$ is an increasing function such that $\Psi(0) = 1$ and $\Psi(\infty) = \infty$.

Remark 3.1. If f is a Lipschitz function:

$$|f(t, z) - f(t, z')| \leq K|z - z'|,$$

then ϕ is the constant K. Then (41) is satisfied for all $\kappa > 1$, and (43) holds if $\gamma > 1$.

The solution algorithm for our BSDE (3)

$$Y_t = \xi + \int_t^T f(s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s$$

is based on a recursively defined change of measure. Let $\mathbb{Q}^0 = \mathbb{P}$, and $W^0 = W$, the coordinate process which is a Wiener process under $\mathbb{Q}^0$. Set

$$Y^1 = \mathbb{E}(\xi|\mathcal{F}_\cdot) = \mathbb{E}(\xi) + \int_0^\cdot Z_s^1 \mathrm{d}W_s^0,$$

and

$$\mathbb{Q}^1 = \exp\left(\int_0^T g(s, Z_s^1)\mathrm{d}W_s - \frac{1}{2}\int_0^T g(s, Z_s^1)^2 \mathrm{d}s\right) \cdot \mathbb{P} = R_T^1 \cdot \mathbb{P}.$$

Then

$$W^1 = W - \int_0^\cdot g(s, Z_s^1)\mathrm{d}s$$

is a Wiener process under $\mathbb{Q}^1$. Indeed under (41), the Novikov condition is satisfied, and under (42), the martingale

$$M_t^1 = \int_0^t g(s, Z_s^1)\mathrm{d}W_s$$

is BMO. Now since $(\mathbb{Q}^1, \mathbb{Q}^0)$ is a Girsanov pair, it is well known that the predictable representation property is inherited from the Brownian motion W^0 to the Brownian motion W^1. See for example Revuz and Yor [13], p. 335. Hence there exists a pair (Y^2, Z^2) of processes such that for all $t \in [0, T]$

$$Y_t^2 = \mathbb{E}^{\mathbb{Q}^1}(\xi|\mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}^1}(\xi) + \int_0^t Z_s^2 \mathrm{d}W_s^1.$$

Assume that $\mathbb{Q}^n$ is recursively defined, along with the Brownian motion

$$W^n = W - \int_0^\cdot g(s, Z_s^n)\mathrm{d}s$$

under $\mathbb{Q}^n$. Then Revuz and Yor [13] may be applied to obtain two processes (Y^{n+1}, Z^{n+1}) such that

$$Y^{n+1} = \mathbb{E}^{\mathbb{Q}^n}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^n(\xi|\mathcal{F}_\cdot) = \mathbb{E}^n(\xi) + \int_0^\cdot Z_s^{n+1} \mathrm{d}W_s^n.$$

Now set

$$\mathbb{Q}^{n+1} = \exp\left[\int_0^T g(s, Z_s^{n+1})\mathrm{d}W_s - \int_0^T g(s, Z_s^{n+1})^2 \mathrm{d}s\right] \cdot \mathbb{P} = R_T^{n+1} \cdot \mathbb{P}$$

to complete the recursion step. Then from our assumptions on ϕ, and from the boundedness of g by ϕ, the sequence of probability measures $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ is well defined and consists of measures equivalent with P. It is not hard to show tightness for this sequence.

Proposition 3.1. *Under (41) or (42), the sequence $(\mathbb{Q}^n)_{n\in\mathbb{N}}$ is tight.*

Proof. In this proof, $\mathbb{E}^n$ denotes the expectation under $\mathbb{Q}^n$. For $0 \le s \le t \le T$, $n \in \mathbb{N}$, we have, recalling that W is the coordinate process on the canonical space,

$$\begin{aligned}
\mathbb{E}^n\left(|W_t - W_s|^4\right) &\le \mathbb{E}^n\left(|W_t^n - W_s^n + \int_s^t g(u, Z_u^n)\mathrm{d}u|^4\right) \\
&\le C\left[\mathbb{E}^n\left(|W_t^n - W_s^n|^4\right) + \mathbb{E}^n\left(\left|\int_s^t g(u, Z_u^n)\mathrm{d}u\right|^4\right)\right] \\
&\le C|t-s|^2 + C|t-s|^2\mathbb{E}^n\left(\int_s^t g(u, Z_u^n)^2\mathrm{d}u\right)^2 \\
&\le C|t-s|^2 + C|t-s|^2\mathbb{E}^n\left(\int_s^t \phi_u^2\mathrm{d}u\right)^2 \\
&\le C|t-s|^2 + C|t-s|^2\mathbb{E}\left[\left(\int_s^t \phi_u^2\mathrm{d}u\right)^2 R_T^n\right] \\
&\le C|t-s|^2\left\{1 + \left[\mathbb{E}\left(\int_s^t \phi_u^2\mathrm{d}u\right)^{2p}\right]^{1/p}\left[\mathbb{E}(R_T^n)^q\right]^{1/q}\right\},
\end{aligned} \tag{44}$$

from the Hölder inequality with $p > 1$ and $p^{-1} + q^{-1} = 1$.

Suppose that ϕ satisfies the assumption (41). From the Novikov condition applied to the martingale

$$M_t^n = \int_0^t g(u, Z_u^n)\mathrm{d}W_u,$$

we know that $\mathcal{E}(M^n)$ is a uniformly integrable martingale under $\mathbb{P}$. Moreover if $C \le \kappa$

$$\mathbb{E}\left[\exp\left(\frac{\sqrt{C}}{2}M_T^n\right)\right] \le \mathbb{E}\left[\exp\left(\frac{C}{2}\langle M^n\rangle_T\right)\right]^{1/2} \le \mathbb{E}\left[\exp\left(\frac{\kappa}{2}\langle L\rangle_T\right)\right]^{1/2} < +\infty.$$

From Theorem 1.5 in [10], we deduce that if $p > p^*$ with

$$\frac{\sqrt{p^*}}{\sqrt{p^*}-1} = \sqrt{\kappa} \iff p^* = \frac{\kappa}{(\sqrt{\kappa}-1)^2},$$

then for $q < q^*$

$$\mathbb{E}\left[\mathcal{E}\left(g(., Z^n)W\right)_T^q\right] = \mathbb{E}(R_T^n)^q \le C. \tag{45}$$

Now if ϕ verifies the assumption (42), the martingale M^n is also BMO, and the BMO-norm of M^n is smaller than the BMO-norm of L. Therefore from Theorem 3.1 in [10] (or more precisely from the proof of this result), we deduce that there exists $q > 1$ and C s.t.

$$\mathbb{E}\left[\mathcal{E}\left(g(., Y^n, Z^n)W\right)_T^q\right] = \mathbb{E}(R_T^n)^q \leq C. \tag{46}$$

The constant q must satisfy the following inequality: $q < q^*$ with

$$\|L\| = \theta(q^*) \iff q^* = \theta^{-1}(\|L\|) \iff p^* = \frac{\theta^{-1}(\|L\|)}{\theta^{-1}(\|L\|) - 1}.$$

Moreover, from the John–Nirenberg inequality (see [10], Theorem 2.2),

$$\mathbb{E}\left[\exp\left(\frac{1}{4\|L\|_{BMO_2}^2}\int_0^T \phi_u^2 \mathrm{d}u\right)\right] \leq 2 \Longrightarrow \mathbb{E}\left(\int_s^t \phi_u^2 \mathrm{d}u\right)^{2p} < +\infty.$$

Finally from (44)

$$\mathbb{E}^n\left(|W_t - W_s|^4\right) \leq C|t-s|^2.$$

Hence by a well known criterion (see for example Kallenberg [14], p. 261), tightness follows. □

In a second step, we shall now establish the boundedness in L^2 of the control sequence $(Z^n)_{n\in\mathbb{N}}$ obtained by the algorithm. Before this, let us give some estimates.

Lemma 3.1. *If $\delta > \Psi$ and (43) holds, there exist two constants $\beta > 0$ and $p > 1$ such that*

$$\forall n \in \mathbb{N}, \quad \mathbb{E}^{n-1}\left(\mathrm{e}^{\beta\Phi_T}|\xi|^p\right) < +\infty. \tag{47}$$

Proof. In the proof of Proposition 3.1, we already see that there exists $q^* > 1$ such that for every $1 < r < q^*$, and for every n, $\mathbb{E}\left(R_T^{n-1}\right)^r \leq C_r < +\infty$. Thus using Hölder's inequality

$$\mathbb{E}^{n-1}\left(\mathrm{e}^{\beta\Phi_T}|\xi|^p\right) \leq \left[\mathbb{E}\left(\mathrm{e}^{s\beta\Phi_T}|\xi|^{sp}\right)\right]^{1/s} \times \left[\mathbb{E}\left(R_T^{n-1}\right)^r\right]^{1/r} \leq C_r\left[\mathbb{E}\left(\mathrm{e}^{s\beta\Phi_T}|\xi|^{sp}\right)\right]^{1/s}.$$

From (43), $\delta > \Psi$ implies that $\delta > p^* = (1-1/q^*)^{-1}$. Hence for $r < q^*$, $\delta/s > 1$ and we can find $p > 1$ such that $sp < \delta$. Then choosing β sufficiently small, $s\beta < \alpha$ and the conclusion follows. □

From Lemma 3.1, we deduce:

Lemma 3.2. *There exists a constant C such that for every $n \in \mathbb{N}$,*

$$\mathbb{E}^{n-1}\left[\sup_{t\in[0,T]}(\mathrm{e}^{\beta\Phi_t}|Y_t^n|^p) + \left(\int_0^T \mathrm{e}^{\beta\Phi_t}|Z_t^n|^2\mathrm{d}t\right)^{p/2}\right] \\ \leq C\mathbb{E}^{n-1}\left[\exp\left(\beta\Phi_T \max\left(\frac{p}{2}, 1\right)\right)|\xi|^p\right].$$

Proof. Recall that for every n, $Y_t^n = \mathbb{E}^{n-1}(\xi|\mathcal{F}_t) = \xi - \int_t^T Z_s^n \mathrm{d}W_s^{n-1}$. Therefore

$$\mathrm{e}^{(\beta/p)\Phi_t}|Y_t^n| \leq \mathbb{E}^{n-1}(\mathrm{e}^{(\beta/p)\Phi_t}|\xi||\mathcal{F}_t) \leq \mathbb{E}^{n-1}(\mathrm{e}^{(\beta/p)\Phi_T}|\xi||\mathcal{F}_t).$$

Using Doob's inequality we deduce

$$\mathbb{E}^{n-1} \sup_{t\in[0,T]} (\mathrm{e}^{\beta \Phi_t}|Y_t^n|^p) \leq C_p \mathbb{E}^{n-1}(\mathrm{e}^{\beta \Phi_T}|\xi|^p).$$

Now we have

$$\int_t^T \mathrm{e}^{\beta \Phi_s/2} Z_s^n \mathrm{d}W_s^{n-1} = \mathrm{e}^{\beta \Phi_T/2}\xi - \mathrm{e}^{\beta \Phi_t/2} Y_t^n - (\beta/2)\int_t^T \mathrm{e}^{\beta \Phi_s/2} Y_s^n \phi_s^2 \mathrm{d}s.$$

Using the Burkholder–Davis–Gundy inequality and the previous estimate on Y^n

$$\mathbb{E}^{n-1}\left[\left(\int_0^T \mathrm{e}^{\beta \Phi_t}|Z_t^n|^2 \mathrm{d}t\right)^{p/2}\right] \leq C\mathbb{E}^{n-1}(\mathrm{e}^{\beta \Phi_T p/2}|\xi|^p). \quad \square$$

Proposition 3.2. *Under Assumption* (H2)*, if* $\delta > \Psi$ *and if* (43) *holds, there exists* $\beta > 0$ *and* $p > 1$ *such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} \mathrm{e}^{\beta \Phi_t}(Y_t^n)^p\right) \quad \text{and} \quad \mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]$$

are bounded sequences.

Proof. We give just the proof for the sequence $\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p}{2}}\right]$. For the other sequence, the sketch is the same. Define for $n \in \mathbb{N}$

$$R^n = R_T^n = \exp\left(\int_0^T g(s, Z_s^n)\mathrm{d}W_s - \frac{1}{2}\int_0^T g(s, Z_s^n)^2 \mathrm{d}s\right).$$

Then for $p > 1$ and $\varepsilon > 0$

$$\begin{aligned}
\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p}{2}} &= \mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p}{2}} (R^{n-1})^{\frac{1}{1+\varepsilon}} (R^{n-1})^{-\frac{1}{1+\varepsilon}}\right] \\
&\leq \left[\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p(1+\varepsilon)}{2}} R^{n-1}\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\
&= \left[\mathbb{E}^{n-1}\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p(1+\varepsilon)}{2}}\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}}.
\end{aligned}$$

With Lemmas 3.2 and 3.1we obtain

$$\mathbb{E}^{n-1}\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p(1+\varepsilon)}{2}} \leq C\mathbb{E}^{n-1}\left(\mathrm{e}^{\beta \Phi_T \frac{p(1+\varepsilon)}{2}} |\xi|^{p(1+\varepsilon)}\right).$$

Thus for some $\eta > 0$

$$\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta \Phi_s}(Z_s^n)^2 \mathrm{d}s\right)^{\frac{p}{2}} \leq C\left[\mathbb{E}\left(\mathrm{e}^{\beta \Phi_T \frac{p(1+\varepsilon)}{2}} |\xi|^{p(1+\varepsilon)} R^{n-1}\right)\right]^{\frac{1}{1+\varepsilon}} \left[\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\le C\left\{\mathbb{E}e^{\beta\frac{p(1+\varepsilon)}{2}(1+\eta)\Phi_T}|\xi|^{p(1+\varepsilon)(1+\eta)}\right\}^{\frac{1}{(1+\varepsilon)(1+\eta)}}$$

$$\times\left\{\mathbb{E}(R^{n-1})^{\frac{1+\eta}{\eta}}\right\}^{\frac{\eta}{(1+\varepsilon)(1+\eta)}}\left\{\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right\}^{\frac{\varepsilon}{1+\varepsilon}}. \tag{48}$$

From the conditions (41) or (42), we can prove that there exist $\eta > 0$ and $\varepsilon > 0$ s.t.

$$\sup_{n\in\mathbb{N}}\left\{\mathbb{E}(R^{n-1})^{\frac{1+\eta}{\eta}}\right\}^{\frac{\eta}{(1+\varepsilon)(1+\eta)}}\left\{\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}}\right\}^{\frac{\varepsilon}{1+\varepsilon}} < +\infty.$$

First assume that (41) holds. Then

$$\begin{aligned}(R^{n-1})^{-\frac{1}{\varepsilon}} &= \exp\left[-\frac{1}{\varepsilon}\int_0^T g(s,Z_s^{n-1})\mathrm{d}W_s + \frac{1}{2\varepsilon}\int_0^T g(s,Z_s^{n-1})^2\mathrm{d}s\right]\\ &= \exp\left[\int_0^T\left(-\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)\mathrm{d}W_s - \frac{1}{2}\int_0^T\left(\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)^2\mathrm{d}s\right]\\ &\quad\times\exp\left[\frac{1}{2\varepsilon^2}(1+\varepsilon)\int_0^T g(s,Z_s^{n-1})^2\mathrm{d}s\right].\end{aligned} \tag{49}$$

Now if

$$\Gamma^{n-1,\varepsilon} = -\int_0^T\frac{g(u,Z_u^{n-1})}{\varepsilon}\mathrm{d}W_u,$$

we have for $C > 1$

$$\mathbb{E}\left[\exp\left(\frac{\sqrt{C}}{2}\Gamma^{n-1,\varepsilon}\right)\right] \le \mathbb{E}\left[\exp\left(\frac{C}{2}\langle\Gamma^{n-1,\varepsilon}\rangle\right)\right]^{1/2} \le \mathbb{E}\left[\exp\left(\frac{C}{2\varepsilon^2}\langle L\rangle_T\right)\right]^{1/2} < +\infty,$$

when $C/\varepsilon^2 = \kappa$. Thus

$$\mathbb{E}\left[\exp\left[\int_0^T\left(-\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)\mathrm{d}W_s - \frac{1}{2}\int_0^T\left(\frac{g(s,Z_s^{n-1})}{\varepsilon}\right)^2\mathrm{d}s\right]\right]^q < +\infty$$

when $1/q + 1/p = 1$ and

$$\frac{\sqrt{p}}{\sqrt{p}-1} = C = \varepsilon\sqrt{\kappa} \iff p = \frac{\kappa\varepsilon^2}{\left(\varepsilon\sqrt{\kappa}-1\right)^2}.$$

And we have

$$\mathbb{E}\exp\left[\frac{p}{2\varepsilon^2}(1+\varepsilon)\int_0^T g(s,Z_s^{n-1})^2\mathrm{d}s\right] \le \mathbb{E}\exp\left[\frac{p(1+\varepsilon)}{2\varepsilon^2}\int_0^T\phi_s^2\mathrm{d}s\right] < +\infty,$$

if

$$\frac{p(1+\varepsilon)}{\varepsilon^2} \le \kappa \iff \varepsilon \ge \frac{1+2\sqrt{\kappa}}{\kappa} \iff 1+\varepsilon = \frac{\kappa+2\sqrt{\kappa}+1}{\kappa}.$$

From (49) and with Hölder's inequality we deduce that $\mathbb{E}R_{n-1}^{-\frac{1}{\varepsilon}} \le C$. With (45) we already know that there exists η s.t. $\mathbb{E}R_{n-1}^{\frac{1+\eta}{\eta}} \le C$. We have to take $\sqrt{1+\eta} = \sqrt{p^*} = \frac{\sqrt{\kappa}}{\sqrt{\kappa}-1}$.

Assume that (42) holds. Then we already know (46): there exists $\eta > 0$ such that $\mathbb{E}R_{n-1}^{\frac{1+\eta}{\eta}} \leq C$, if η satisfies

$$\|L\| < \theta\left(\frac{1+\eta}{\eta}\right).$$

We use Theorem 2.4 in [10] in order to prove that $\mathbb{E}(R^{n-1})^{-\frac{1}{\varepsilon}} \leq C$. We must choose ε s.t.

$$\|L\| \leq \sqrt{2}\left(\sqrt{1+\varepsilon}-1\right).$$

The two constants η and ε depend on the constant κ in (41) or the BMO-norm $\|L\|$ in (42). Coming back to (48) we deduce that

$$\mathbb{E}\left(\int_0^T e^{\beta\Phi_s}\left(Z_s^n\right)^2 ds\right)^{\frac{p}{2}} \leq C\left\{\mathbb{E}e^{\beta\frac{p(1+\varepsilon)}{2}(1+\eta)\Phi_T}|\xi|^{p(1+\varepsilon)(1+\eta)}\right\}^{\frac{1}{(1+\varepsilon)(1+\eta)}}.$$

Remark now that $(1+\varepsilon)(1+\eta) = \Psi$. Thereby from assumption (43), if $\delta > \Psi$, the desired boundedness follows for some $p > 1$ such that $\delta \geq p\Psi$ and by choosing $\beta > 0$ such that $\alpha \geq \beta p\Psi/2$. □

Proposition 3.3. *The sequence $(Z_n)_{n\in\mathbb{N}}$ converges in $L^2([0,T]\times\mathbb{P})$.*

Proof. Applying Itô's formula we have

$$\begin{aligned}
e^{\beta\Phi_t}|Y_t^{n+1}-Y_t^n|^2 + \int_t^T e^{\beta\Phi_u}|Z_u^{n+1}-Z_u^n|^2 du &= -\beta\int_t^T \phi_u^2 e^{\beta\Phi_u}|Y_u^{n+1}-Y_u^n|^2 du \\
&\quad -2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(-Z_u^{n+1}g(u,Z_u^n)+Z_u^n g(u,Z_u^{n-1}))du \\
&\quad -2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(Z_u^{n+1}-Z_u^n)dW_u \\
&= -\beta\int_t^T \phi_u^2 e^{\beta\Phi_u}|Y_u^{n+1}-Y_u^n|^2 du + 2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(Z_u^{n+1}-Z_u^n)g(u,Z_u^n)du \\
&\quad +2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(f(u,Z_u^n)-f(u,Z_u^{n-1}))du \\
&\quad +2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)g(u,Z_u^{n-1})(Z_u^{n-1}-Z_u^n)du \\
&\quad -2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(Z_u^{n+1}-Z_u^n)dW_u.
\end{aligned}$$

Recall that g is bounded by the process ϕ. Hence with some positive constants ε and η

$$\begin{aligned}
\int_t^T e^{\beta\Phi_u}|Z_u^{n+1}-Z_u^n|^2 du &\leq \int_t^T \left(\frac{1}{\varepsilon}+2\frac{1}{\eta}-\beta\right)\phi_u^2 e^{\beta\Phi_u}|Y_u^{n+1}-Y_u^n|^2 du \\
&\quad +\varepsilon\int_t^T e^{\beta\Phi_u}|Z_u^{n+1}-Z_u^n|^2 du + 2\eta\int_t^T e^{\beta\Phi_u}|Z_u^n-Z_u^{n-1}|^2 du \\
&\quad -2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1}-Y_u^n)(Z_u^{n+1}-Z_u^n)dW_u.
\end{aligned}$$

Choosing β such that

$$\frac{1}{\varepsilon} + 2\frac{1}{\eta} = \beta, \tag{50}$$

we have

$$(1-\varepsilon)\int_0^T e^{\beta\Phi_u}|Z_u^{n+1} - Z_u^n|^2 du \le 2\eta \int_0^T e^{\beta\Phi_u}|Z_u^n - Z_u^{n-1}|^2 du$$
$$-2\int_t^T e^{\beta\Phi_u}(Y_u^{n+1} - Y_u^n)(Z_u^{n+1} - Z_u^n) dW_u.$$

If $\alpha > 4.5\Psi$, then we can choose $\beta > 9$ such that $\alpha \ge \beta\Psi/2$ (see the end of the proof of Proposition 3.2) and ε and η such that (50) holds with $2\eta/(1-\varepsilon) < 1$. Since the conclusion of Proposition 3.2 holds, the local martingale in the previous expression is a true martingale. Hence taking the expectation we obtain

$$\mathbb{E}\int_0^T e^{\beta\Phi_u}|Z_u^{n+1} - Z_u^n|^2 du \le \frac{2\eta}{1-\varepsilon}\mathbb{E}\int_0^T e^{\beta\Phi_u}|Z_u^n - Z_u^{n-1}|^2 du.$$

Therefore the sequence $(Z_n)_{n\in\mathbb{N}}$ converges in $L^2([0,T]\times\mathbb{P})$. □

Lemma 3.3. *There exists a subsequence of Z^n (still denoted as Z^n) which converges $\mathbb{P}\otimes\lambda$-a.e. to some process Z.*

Lemma 3.4. *The sequence R_T^n converges also $\mathbb{P}$-a.s. to*

$$R_T = \exp\left(\int_0^T g(s, Z_s) dW_s - \frac{1}{2}\int_0^T (g(s, Z_s))^2 ds\right).$$

Proof. We may w.l.o.g. assume that $g(s, .)$ is continuous for all $s \in [0, T]$. The rest follows from Lemma 3.3. □

Equipped with these results, we are now in a position to state our existence theorem.

Theorem 3.1. *Suppose Assumption* (H1) *holds. There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and an adapted process Z such that $\mathbb{E}\int_0^T |Z_s|^2 ds < \infty$ such that, setting*

$$R_T = \exp\left(\int_0^T g(s, Z_s) dW_s - \frac{1}{2}\int_0^T g(s, Z_s)^2 ds\right), \qquad W^{\mathbb{Q}} = W - \int_0^{\cdot} g(s, Z_s) ds,$$

we have

$$\mathbb{Q} = R_T \cdot \mathbb{P},$$

and such that the pair (Y, Z) defined by

$$Y = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^{\mathbb{Q}}(\xi) + \int_0^{\cdot} Z_s dW_s^{\mathbb{Q}}$$

solves the BSDE (3).

Proof. Using Theorem 3.1, choose a probability measure $\mathbb{Q}$ and another subsequence of the corresponding subsequence of $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ which converges weakly to $\mathbb{Q}$. We denote this subsequence again by $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ and the corresponding subsequence of controls by $(Z^n)_{n\in\mathbb{N}}$. We have

$$\mathbb{Q} = R_T \cdot \mathbb{P}.$$

Moreover for all $n \in \mathbb{N}$,

$$Y_t^n = \mathbb{E}^{n-1}(\xi) + \int_0^t Z_s^n \mathrm{d}W_s^n = \mathbb{E}^{n-1}(\xi) + \int_0^t Z_s^n \mathrm{d}W_s - \int_0^t Z_s^n g(s, Z_s^{n-1})\mathrm{d}s.$$

The only thing that we have to prove is that the sequence $Y_0^n = \mathbb{E}^n(\xi)$ also converges. But $Y_0^n = \mathbb{E}^n(\xi) = \mathbb{E}(\xi R^n)$, and ξ belongs to L^γ; R^n also belongs to some L^p space with $1/p + 1/\gamma = 1$ if and only if

$$\gamma \geq \frac{\kappa}{(\sqrt{\kappa} - 1)^2}.$$

But it is true since $\gamma \geq \Psi(\kappa)$. Taking a subsequence if necessary, we deduce that Y_0^n converges to $\mathbb{E}^{\mathbb{Q}}(\xi)$.

Hence we obtain

$$Y_t = \mathbb{E}^{\mathbb{Q}}(\xi|\mathcal{F}_\cdot) = \mathbb{E}^{\mathbb{Q}}(\xi) + \int_0^t Z_s \mathrm{d}W_s^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a $\mathbb{Q}$-Brownian motion. Finally (Y, Z) solves the BSDE (3). □

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