Hysteresis in response of nonlinear bistable interface to continuously varying acoustic loading

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Abstract

In many experimental situations it is an equation of the forced relaxator and not of the forced oscillator that describes a variation in the acoustic field of the interface width (i.e. of a characteristic distance between the surfaces composing the interface). The developed theory predicts that some types of the nonlinear relaxators (depending on the structure of the nonlinear interaction force between the surfaces) exhibit hysteresis in their response to continuous acoustic loading of first increasing and then decreasing amplitude. Nonlinear (unharmonic) variation of the interface width starts at threshold amplitude of the incident sinusoidal acoustic wave, which is higher than threshold amplitude for returning to sinusoidal motion. This dynamic hysteresis (and accompanying it bistability) are possible, in particular, if the dependence of the effective interaction force on the interface width admits two quasi-equilibrium positions of the interface (bistable interface) or if the force itself is hysteretic (hysteretic interface). These theoretical predictions are relevant to some recent experimental observations on the interaction of powerful ultrasonic fields with cracks.

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1. Introduction

Starting from the pioneer experiments on harmonic generation at unbonded interfaces and fatigue cracks [1,2] and first theoretical modeling of the observations [3] there is a continuous interest to various nonlinear phenomena in the interaction of the acoustic waves with interfaces [4–16] and layers [17,18]. Both theoretical [4–8] and experimental [6,7,9] methods were developed for the nonlinear acoustic diagnostics of adhesive/diffusion bonds between the interfaces. Significant progress was achieved in the investigation of the nonlinear contact phenomena at unbonded interfaces [10–13]. Acoustic demodulation (rectification) was observed in reflection from a solid–solid interface [10,11]. Two types of popping (clapping) nonlinearity both associated with intermittent contact between the two surfaces (initiated by the acoustic wave) were identified [12,13]. In the first regime the acoustic wave induces temporary separation of the surfaces initially in contact, in the second acoustic wave induces clapping of the surfaces initially (i.e. in the absence of the acoustic field) separated by a gap. These two types of the nonlinear vibrations exhibited a threshold behavior as a function of the acoustic wave amplitude and were separated by a regime of instability, in which a chaotic process evolves through multiple bifurcations of the acoustic period [11–13]. Theoretical predictions were also obtained relative to a possibility of the nonlinear acoustic diagnostics of the contact between rough surfaces [14–16] and of the acoustically thin nonlinear layers [17,18].

Quite recently a new class of nonlinear acoustic phenomena has been observed for the surface acoustic bursts interacting with cracked defects in solids [19]. Acoustic wave impact on a crack is shown to exhibit amplitude hysteresis and storage for parametric and nonlinear acoustic effects. Both hysteresis and memory are attributed in Ref. [19] to thermoelastic effects. Thermoelastic mechanism is also proposed for the explanation of logarithmic slow dynamics and memory in bulk elastic wave interactions with individual cracks [20].
We have reported recently [21] an observation of the self-induced hysteresis for the nonlinear bulk acoustic wave in cracked material. Hysteresis in the self-action of the monochromatic pump wave, in the excitation of its superharmonics and subharmonics is observed when the amplitude of the cw pump wave first continuously increases and then decreases. The proposed theoretical explanation [21] attributes the phenomenon to hysteresis in transition from a nonclapping regime to a regime of clapping intermediate contacts in the acoustically induced oscillation of cracks. It is suggested that the hysteresis might be caused not only by thermoelastic effects but also might be an intrinsic property of the nonlinear oscillations with impacts. Actually the hysteresis phenomenon is well documented in the investigations of the impact oscillators [22–28], so the mechanism proposed in Ref. [21] is possible. However under some circumstances the contact interface behaves in the acoustic field rather as a nonlinear relaxator than as a nonlinear oscillator. Here we present a detailed theoretical model which predicts the hysteresis phenomenon in the response to the cw acoustic loading of the nonlinear bistable relaxator. In Section 3 the forces (composing the interface) by the surfaces (normal to the direction of the acoustic waves in Fig. 1) is considered as being delta-localized in the plane $x = 0$ normal to the direction of the acoustic wave propagation (normal to $x$ axis). In the following we denote the initial position of the left and right surfaces respectively. It is assumed that the surfaces interact due to the existence of the intermediate contacts, due to inter-molecular (inter-atomic) forces, etc. This interaction is characterized by an average force per unit area of the surface $F(y)$, which depends on the separation $y = u(0 + 0) − u(0 − 0)$ between the surfaces (see the inset in Fig. 1). Here we use the notation $u$ for the mechanical displacement in the material.

Acoustic field in the system can be decomposed into the plane longitudinal acoustic waves propagating in the opposite directions. We use, in the notations, the indices $−/+, −/−, +/+,$ and $+/−$, where the first part of the index refers to the region of the wave propagation (“−” for $x < 0$ and “+” for $x > 0$), while the second part indicates the direction of the wave propagation (“+” and “−” for the positive and negative direction along the $x$ axis, respectively). The solution of the one-dimensional linear acoustic wave equation has the form

\[
\begin{align*}
    u_{−/+} &= e^{−zx}f_{−/+}(t − x/c), & u_{−/−} &= e^{zx}f_{−/−}(t + x/c), \\
    u_{+/+} &= e^{−zx}f_{+/+}(t − x/c), & u_{+/−} &= e^{zx}f_{+/−}(t + x/c).
\end{align*}
\]

(1)

Here $t$ is the time variable, $z$ denotes the coefficient of sound attenuation, $c$ is the sound velocity, and $f$ are still arbitrary functions to be determined from the boundary conditions. We model the boundary conditions at the emitter and the receiver by

\[
\begin{align*}
    u_{−/+} + u_{−/−} &= u_{−/−} + H/c, & x &= −H, \\
    u_{+/−} − Ru_{+/+} &= 0, & x &= h,
\end{align*}
\]

(2)

2. Forced relaxator as a model for the acoustic wave interaction with an interface

In the classical experimental geometry the interface under evaluation is placed between the emitting and receiving acoustic transducers [1,9,12,21]. The one-dimensional schematic presentation of the experiments carried with longitudinal acoustic waves is given in Fig. 1. We assume for definiteness that the emitting transducer is located in the plane $x = −H$ and the receiving transducer is located in the plane $x = h$. A solid/solid interface is considered being composed of two interacting surfaces of the similar material separated by a microscopic/nanoscopic distance, which can be of the order of particle displacement in the acoustic wave. In Fig. 1 the interface is presented as being delta-localized in the plane $x = 0$ normal to the direction of the acoustic wave propagation (normal to $x$ axis). In the following we denote the initial position of the left and right surfaces (composing the interface) by $x = 0 − 0$ and $x = 0 + 0$, respectively. It is assumed that the surfaces interact due to the existence of the intermediate contacts, due to inter-molecular (inter-atomic) forces, etc. This interaction is characterized by an average force per unit area of the surface $F(y)$, which depends on the separation $y = u(0 + 0) − u(0 − 0)$ between the surfaces (see the inset in Fig. 1). Here we use the notation $u$ for the mechanical displacement in the material.

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    u_{+/+} &= e^{−zx}f_{+/+}(t − x/c), & u_{+/−} &= e^{zx}f_{+/−}(t + x/c).
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(1)

Here $t$ is the time variable, $z$ denotes the coefficient of sound attenuation, $c$ is the sound velocity, and $f$ are still arbitrary functions to be determined from the boundary conditions. We model the boundary conditions at the emitter and the receiver by

\[
\begin{align*}
    u_{−/+} + u_{−/−} &= u_{−/−} + H/c, & x &= −H, \\
    u_{+/−} − Ru_{+/+} &= 0, & x &= h,
\end{align*}
\]

(2)
where \( u_{-H} \) denotes the displacement of the boundary \( x = -H \) initiated by the emitting transducer, and \( R \) is the wave reflection coefficient at the boundary \( x = h \). With the help of Eq. (1) we rewrite Eq. (2) in the form

\[
f_{-/+}(t + H/c) + e^{-2\lambda t}f_{-/+}(t - H/c) = e^{-\lambda t}u_{-H}(t + H/c),
\]

\[
f_{-/+}(t + h/c) = Re^{-\lambda t}f(t - h/c).
\]

(3)

(4)

The conditions for the acoustical stresses \( \sigma \) at the surfaces \( x = 0 \pm 0 \) are

\[
\sigma(x = 0 \pm 0) = -F[u(0 + 0) - u(0 - 0)] - \sigma_0 = -F_{eff}[u(0 + 0) - u(0 - 0)].
\]

(5)

Here \( \sigma_0 \) is the mechanical stress due to a possible static preloading that can exist in the absence of the acoustic excitation, and the notation \( F_{eff} \) is introduced for an effective force (combining the interaction between the surfaces and the preloading). Taking into account the Hooke’s law (\( \sigma = \rho c^2(2\mu/hc) \), where \( \rho \) denotes the density of the material) and decomposing the total displacement into the sum of the contributions from the counter-propagating waves (1), the conditions in Eq. (5) are reduced to

\[
\rho c^2 \left( x + \frac{1}{c} \frac{\partial}{\partial t} \right) [f_{+/+}(t) - f_{+/+}(t)] = F_{eff}[f_{+/+}(t) + f_{-/+}(t) - f_{-/+}(t) - f_{-/+}(t)],
\]

\[
\rho c^2 \left( x + \frac{1}{c} \frac{\partial}{\partial t} \right) [f_{-/+}(t) - f_{+/+}(t)] = F_{eff}[f_{+/+}(t) + f_{+/+}(t) - f_{+/+}(t) - f_{+/+}(t)].
\]

(6)

(7)

Note that the equality of the right-hand-sides in Eqs. (6) and (7) (providing the continuity of the stress across the interface) leads to the following relation between the different components of the acoustic field at the interface

\[
f_{+/+}(t) - f_{+/+}(t) = f_{-/+}(t) - f_{+/+}(t).
\]

(8)

We will characterize the state of the interface by its “width” \( y \)

\[
y = u(0 + 0) - u(0 - 0) = f_{+/+}(t) + f_{+/+}(t) - f_{+/+}(t) - f_{+/+}(t).
\]

(9)

Using Eqs. (8) and (9) the relation between the components of the acoustic field at the opposite sides of the interface can be expressed in the form

\[
f_{+/+}(t) = f_{+/+}(t) - y(t)/2,
\]

\[
f_{+/+}(t) = f_{+/+}(t) - y(t)/2.
\]

(10)

By substituting Eq. (4) into Eq. (6), and also by substituting first Eq. (10) and then Eq. (4) into Eq. (3) we get a system of two equations for the interface width \( y \) and the function \( f_{+/+} \)

\[
(\pi c + \frac{\partial}{\partial t}) \left[ Re^{-2\lambda t}f_{+/+}(t - \frac{2H}{c}) - f_{+/+}(t) \right] + \frac{1}{\rho c}F_{eff}(y) = 0,
\]

\[
(\pi c + \frac{\partial}{\partial t}) \left[ Re^{-2\lambda t}f_{+/+}(t - \frac{2(H + h)}{c}) + \frac{1}{\rho c}F_{eff}(y) \right] = e^{-\lambda t}u_{-H}(t).
\]

(11)

(12)

The system (11) and (12) of the delay differential equations [29] describes the acoustic waves in the solid state resonator (\( -H \leq x \leq h \)) composed of two linear parts (\( -H \leq x \leq 0 \) and \( 0 \leq x \leq h \)) interacting through an effective force \( F_{eff} \), which is, in general, nonlinear. It should be mentioned that in the propagative acoustic waves (assumed everywhere in the following) the attenuation length \( \lambda^{-1} \) by definition exceeds significantly the acoustic wavelength. Because of this in the operator \( (\pi c + \partial/\partial t) \), in Eq. (11) and everywhere before, the first part is negligible in comparison with the second one. The latter is estimated by a characteristic cyclic frequency of the acoustic wave \( (\partial/\partial t) \propto \omega \). The system (11) and (12) can be significantly simplified and reduced to a single equation for the interface width \( y \) in a few limiting cases important for the applications. Here we describe only two of them.

The first case corresponds to the experimental situation in Ref. [12,13], where the dimensions \( h \) and \( H \) of the system were significantly smaller than the acoustic wavelength \( \lambda (\lambda \approx 10 \text{ m at frequency } 300 \text{ Hz in steel}). In this case, on the one hand, the attenuation can be neglected completely in Eqs. (11) and (12) because \( x(H + h) \ll 2\pi(H + h)/\lambda \ll 1 \). On the other hand the delay in the arguments of the functions is so small in comparison with characteristic time scale \( (\propto 1/\omega) \) of the functions variation that only first few terms of Taylor’s expansion are sufficient for the approximation of the functions. As a result the system (11) and (12) takes the form

\[
(\pi c + \frac{\partial}{\partial t}) \left[ (R - 1)f_{+/+}(t) - \frac{2h}{c}R \frac{\partial f_{+/+}(t)}{\partial t} \right] + \frac{1}{\rho c}F_{eff}(y) = 0,
\]

\[
(R + 1)f_{+/+}(t) - \frac{2(H + h)}{c}R \frac{\partial f_{+/+}(t)}{\partial t} - y(t) = u_{-H}(t).
\]

(13)

We are still keeping the terms with higher order derivatives in Eq. (13) just to account for some particular experimental situations where the leading order terms (those proportional to \( (R - 1) \) and \( (R + 1) \)) might be small due to \( R \approx 1 \) or \( R \approx -1 \). The first situation (with \( R \approx 1 \)) is rather typical for the experiments with resonant bars, when the detection of sound at \( x = h \) is realized with the help of very light accelerometers.
For \( |1 - R| \ll 2(h/c)\omega \propto 4\pi(h/\lambda) \) the equations in (13) take the form
\[
- \left( 2(h/c)\partial^2 f_{\pm}(\lambda)/\partial \lambda^2 + (1/pc)F_{\text{eff}}(\lambda) \right) = 0,
\]
\[
2f_{\pm} - y = u_{\pm}(\lambda).
\]

They can be combined in the equation of forced oscillations for the interface width
\[
\frac{\partial^2 y}{\partial t^2} - \frac{1}{\rho c^2} F_{\text{eff}}(y) = -\frac{\partial^2 u_{\pm}(\lambda)}{\partial t^2}.
\]  
(14)

The second situation (with \( R \approx -1 \)) corresponds to rigid boundary conditions at \( x = h \). For \( |1 + R| \ll 2[(h + h)/c]\omega \propto 4\pi(H + h)/\lambda \) the equations in (13) take the form
\[
- 2\partial f_{\pm} / \partial t + (1/pc)F_{\text{eff}}(y) = 0,
\]
\[
[(H + h)/c]2f_{\pm} / \partial t - y = u_{\pm}(\lambda).
\]

They can be combined in the algebraic equation
\[
y - \frac{H + h}{\rho c^2} F_{\text{eff}}(y) = -u_{\pm}(\lambda),
\]
describing a quasistatic variation of the interface width instantaneously following the displacement of the boundary \( x = -H \) (the equation of force balance). Apart of these limiting cases (i.e. \( R \approx \pm 1 \)) and, in particular, in all the situations where the receiving transducer in the plane \( x = h \) is acoustically matched to the material in order to diminish the reflected wave, the leading order approximation for the system (13) is
\[
(R - 1)f_{\pm}/\partial t + (1/pc)F_{\text{eff}}(y) = 0,
\]
\[
(R + 1)f_{\pm} - y = u_{\pm}(\lambda).
\]

These equations are combined in the equation of a forced relaxator for the interface width
\[
\frac{\partial y}{\partial t} - \frac{1}{\rho c} \frac{1}{(1 - R)} F_{\text{eff}}(y) = -\frac{\partial u_{\pm}(\lambda)}{\partial t}.
\]  
(15)

The dynamics is over damped with no inertial effects giving oscillations around equilibrium.

The second case, admitting the simplification of the system (11) and (12) of the delay differential equations, corresponds to the experimental situation in Ref. [21], where the attenuation of the acoustic waves was strong enough \((xH \gg 1, \, z_h \geq 1)\) to neglect all the waves reflected from the boundaries at \( x = -H \) and \( x = h \). The Eqs. (11) and (12) take the form
\[
- \partial f_{\pm}/\partial t + (1/pc)F_{\text{eff}}(y) = 0,
\]
\[
f_{\pm} - y/2 = e^{-\alpha_H}u_{\pm}(\lambda),
\]
and can be combined in the equation of a forced nonlinear relaxator
\[
\frac{\partial y}{\partial t} - \frac{2}{\rho c} F_{\text{eff}}(y) = -2e^{-\alpha_H} \frac{\partial u_{\pm}(\lambda)}{\partial t}.
\]  
(16)

In Eq. (16) \( \exp(-2\alpha_H)u_{\pm}(\lambda) \) is the displacement in the acoustic wave incident on the surface located at \( x = 0 \), and can be combined in the equation of a forced nonlinear relaxator
\[
\frac{\partial y}{\partial t} - \frac{2}{\rho c} F_{\text{eff}}(y) = -2e^{-\alpha_H} \frac{\partial u_{\pm}(\lambda)}{\partial t}.
\]  
(17)

Clearly the Eq. (15) can be reduced to the same form (17) by an appropriate change of the notations. Because of this in the rest part of the paper we will analyze Eq. (17). Note that we kept the coefficient “2” in the right-hand-side (r.h.s.) of Eq. (17), because it has a clear physical sense providing the doubling of the displacement, when a wave arrives at a mechanically free surface.

The equation of a forced nonlinear relaxator appears also in the theory of a pulse response of a nonlinear layer, when the layer is acoustically thin [17]. Together with the simple analysis undertaken just above this leads to the conclusion that the acoustically-induced motion of the interface in many experimental situations is a forced relaxation process rather than an oscillation process. Consequently it might be inappropriate in some cases to use the results obtained in the theory of the nonlinear oscillations (relative to the stochastization of the oscillations and relative to the hysteresis phenomena, for example) [22–28,32–34] for the interpretation of the acoustic wave interaction with unbonded interfaces. Because of this the development of a theory of hysteresis in the forced motion of the nonlinear relaxator looks topical.

3. Two-well potential as a model for an effective interaction between surfaces composing an interface

In the models existing in nonlinear acoustics for the description of an unbonded interface [3,14,15], it is assumed that a single possible equilibrium width of the interface is controlled by \( F_{\text{eff}} \) that is by the superposition of the external forces applied to the sample (i.e., the static pressure \( p = -\sigma_0 \) applied at \( x = -H \) and \( x = h \) in Fig. 1, for example [1,3,12]) and the repulsion force \( F \) due to intermediate contacts or due to complete dynamic contact between the surfaces \( x = 0 \). The effective force in this case is a monotonous decreasing function of the interface width of the type schematically presented in Fig. 2(a). In Fig. 2(a) \( y_0 \) denotes the equilibrium separation distance between the surfaces. The vertical line at \( y = 0 \) takes into account that the mutual penetration of the surfaces \( x = 0 \) and \( x = 0 \) (that is the region \( y < 0 \) is forbidden. In the case when the intermediate contacts between the surfaces are approximated by Hertzian contacts of a height \( \delta \) (which is significantly smaller than their curvature \( R \)) the force \( F \) in the region \( 0 \leq y \leq \delta \) can be modeled by \( F = F_H \propto (\delta - y)^{3/2} \). It is common to model the real distribution of the effective force by a piece-wise linear (Fig. 2(b)) or by the distribution of the impact type (Fig. 2(c)).
forces $F_{\text{im}}$ acting between plane surfaces is presented in schematic form in Fig. 3(a). The equilibrium width $y_\text{c}$ in Fig. 3(a) is of the order of the interatomic distance. However the attractive force can be effective at significantly larger separations between the surfaces [37,38,40], though usually significantly shorter than the scale of the surface roughness $\delta$ ($y_\text{c} < y_{\text{min}} \ll \delta$). It should be mentioned that intermolecular force is far from being a single possible attractive force between the surfaces [37,39,40]. For example capillary forces due to the presence of liquids at the interface can contribute among the others [37–40]. In this situation the application of a sufficiently soft external loading in the case of sufficiently soft intermediate contacts $0 < F_{\text{mmax}}^\text{eff} + \sigma_0 < -F_{\text{min}}^\text{eff}$ leads to the distribution of the effective force $F_{\text{eff}} = F + \sigma_0$ with two possible positions of stable equilibrium (Fig. 3(b)). The qualitative piece-wise linear approximation for the effective force providing two possible equilibrium widths of the interface is presented in Fig. 3(c). The higher slope of the force in the region $0 < y < y_{\text{imp}}$ accounts for the shorter scales and higher amplitudes of the attractive forces in the considered regime in comparison with long-scale distribution of $F_{\text{H}} + \sigma_0$.

The existence of the physical realizations of the dependence of the effective force on the separation distance of the type presented in Fig. 3(b) and (c) is confirmed by multiple experiments in atomic force microscopy (see, for example, Ref. [37–39] and the references therein). With the help of the force curve in Fig. 3(c) the behavior of a tip of an atomic force microscope (AFM) with the varying quasistatic loading can be described as follows. Let the initial equilibrium position be $y = y_0$. We start to

![Fig. 2. Qualitative models of the large-scale effective interaction force $F_{\text{eff}}(y)$: an interaction $F_{\text{eff}}(y) \propto F_{\text{H}} + \sigma_0$ through the Hertzian contacts at the distance $y < \delta$ (a); an approximation by a piece-wise linear dependence (b); a piece-wise linear approximation with an instantaneous impact (c); the Richardson’s [3] model (d). The preloading stress is denoted by $\sigma_0 < 0$.](image)

![Fig. 3. Qualitative models for the effective force $F_{\text{eff}}(y)$ between the surfaces composing the interface: a “textbook” short-scale intermolecular force $F_{\text{im}}$ (a); an effective force $F_{\text{eff}}(y) \propto F_{\text{mm}} + F_{\text{H}} + \sigma_0$ (b); a piece-wise linear approximation for the effective force (c); a distribution of the interaction potential for the case of a bistable interface (d).](image)
diminish \( \sigma_0 < 0 \) (by introducing the variation \( \Delta \sigma_0 < 0 \)). This provides the shift of the total curve \( F_{\text{eff}} = F + \sigma_0 \) downward, and the displacement of the equilibrium position to the left. When \( \Delta \sigma_0 = -F_0 \) the separation width \( y = y_0 \) becomes equal to \( y_{\text{imp}} \) (see the notation in Fig. 3(c)), the equilibrium becomes unstable and the separation distance diminishes by a jump to the value \( y = y_c \). Note that the position of \( y_c \) also shifts to the left when \( \Delta \sigma_0 < 0 \) is applied. This jump corresponds to the jump of the tip to contact observed in AFM [37–39]. If now one wants to separate the surfaces it is not enough just to remove \( \Delta \sigma_0 < 0 \), but it will be necessary to apply \( \Delta \sigma_0 = F_c > 0 \) the equilibrium position \( y = y_c \) coincides with \( y_{\text{imp}} \) and the separation distance will increase by a jump. Thus the effective force curve in Fig. 3(c) correctly takes into account the basic features of the mechanical hysteresis observed in the quasistatic behavior of the AFM. However it should be clearly stated here that the so-called adhesion hysteresis [37] due to possible dependence of some type of the attractive forces on the sign of the separation gap variation is not taken into account by the single-valued dependence of \( F + \sigma_0 \) on \( y \) in Fig. 3(c). The qualitative conditions \( 0 < F_{\text{H}}^{\text{max}} + \sigma_0 < -F_{\text{min}} \) proposed above for the bistable interface correspond to the known fact in AFM: the weaker cantilever, the greater the mechanical hysteresis effects [35]. In fact our Fig. 3(c) corresponds to Fig. 5.3 in Ref. [35].

The basic idea that stimulated the present investigation can be formulated as follows. Because the complicated effective forces has been observed in the interaction of the surface with the individual contact (tip), then there is no reason to reject the possibility that a complex regime of this type might be also realized in the interaction of a group of contacts with the surface (that is in the interaction between the rough surfaces). Similar situations might be also expected for the interaction between the surfaces of the crack, though in the latter case an initial equilibrium width of the crack \( y_0 \) ("open" state) is due to a rather complicated stress distribution around the crack. Neither external loading nor the surface roughness are necessary to keep the crack in this equilibrium position. Clearly the effective force keeping the crack in the open position can be modeled by a distribution of the type presented in Fig. 2(c). Moreover the dependence of the effective force on the separation \( y \) between the sides of the crack can be estimated by \( F_{\text{eff}} \propto E(y_0/D) [(y_0 - y)/y_0] \). Here \( E \) is the Young modulus of the material, and \( D \) is the characteristic length of the crack (or the diameter of the crack). The estimate takes into account that a penny-like crack is \( (D/y_0) \gg 1 \) times softer than the surrounding material. Consequently when the attractive forces are included in the modeling then (in the case of sufficiently soft \( (D/y_0) \gg 1 \) cracks) the total effective force might be modeled by the curve in Fig. 3(c).

The next sections will be devoted to the analysis of the nonlinear bistable relaxator driven by the incident acoustic wave. The dependence of the effective force on the interface width will be modeled by the piecewise linear function in Fig. 3(c), which corresponds to the two-level effective potential in Fig. 3(d). In line with terminology existing in the other branches of physics the considered relaxator is a two-level system. The mechanical hysteresis in the response of the system to quasistatic loading has been demonstrated just above. In the following we will analyze the response of the two-level (bistable) relaxator to the periodic acoustic excitation of the continuously varying amplitude.

4. Interaction of the bistable relaxator with low frequency acoustic wave

For the quantitative mathematical analysis of the response of a bistable relaxator to the action of an acoustic field we introduce the following parameterization of the effective force in Eq. (17) (Fig. 3(c))

\[
F_{\text{eff}} = \begin{cases} -k_0(y - y_0), & y_{\text{imp}} < y; \\ -k_c(y - y_c), & y < y_{\text{imp}}. \end{cases}
\]

Here \( y_{0, c} \) and \( k_{0, c} \) denote the equilibrium width of the interface (crack) and its rigidity per unit area in the open ("o" index) and closed ("c" index) states, respectively. We will call the impact width \( y_{\text{imp}} \) the separation distance between the surfaces, where the force abruptly changes from repulsive to attractive (see Fig. 3(c)). We will assume that the product \( k_c y_c \) is sufficiently large to avoid mutual penetration of the interacting surfaces \( x = 0 + 0 \) and \( x = 0 - 0 \). With the use of Eq. (18) we rewrite Eq. (17) for the sinusoidal incident acoustic wave with the amplitude \( u_{\text{ac}} \) and frequency \( \omega \left( u_t = u_{\text{ac}} \sin(\omega t) \right) \) in the form valid in both regions (i.e. for \( y > y_{\text{imp}} \) and \( y < y_{\text{imp}} \))

\[
\frac{\partial}{\partial \theta} (y - y_{0,c}) + \omega_{0,c} (y - y_{0,c}) = -2u_{\text{ac}} \omega \cos(\omega t).
\]

Here \( \omega_{0,c} = 2k_{0,c} / (pc) \) are the characteristic relaxation frequencies of the interface (crack) in two states of equilibrium. It is convenient to introduce for the analysis of Eq. (19) the nondimensional time variable \( \theta = \omega t \) and the nondimensional relaxation parameters \( R_{0,c} = \omega_{0,c} / \omega \)

\[
\frac{\partial}{\partial \theta} (y - y_{0,c}) + R_{0,c} (y - y_{0,c}) = -2u_{\text{ac}} \cos \theta.
\]

We identify in this section the regime of the low-frequency (LF) acoustic action as the regime where the acoustic frequency \( \omega \) is significantly lower than both relaxation frequencies. In this regime \( R_{0,c} \gg 1 \) in Eq. (20) and the crack width follows the variation of the acoustic field quasistatically.
\[ y - y_{0,c} \approx -\frac{2u_{ac}}{R_{ac}} \cos \theta, \]  
(21)

unless there are instantaneous jumps when the relaxation process changes the attractive equilibrium state. For the accepted model of the effective force the scenario depends on the initial state of the interface.

If initially the interface is open \((y = y_{0})\) and the amplitude of the acoustic wave starts to increase (from zero value), then for the amplitudes

\[ u_{ac} < R_{o}(y_{0} - y_{imp})/2 \equiv u_{o}^{(LF)} \]

(22)

the width of the crack varies harmonically (in accordance with Eq. (21)) relative to the constant average \(y = y_{0}\) (Fig. 4, trajectory (1)). The forces in the region \(y_{imp} \leq y\) cannot compensate the acoustic action with the displacement amplitude exceeding \(u_{o}^{(LF)}\). Because of this, when the crack width \(y\) reaches with increasing \(u_{ac}\) the impact width \(y_{imp}\), the crack width instantaneously diminishes (to the value denoted by \(y'\) in Fig. 4). We will assume in the following for definiteness that \(F_{c} = k_{c}(y_{imp} - y_{c}) > k_{o}(y_{0} - y_{imp}) = F_{o}\). Then the variation of the crack width, when the acoustic amplitude satisfies the inequality

\[ u_{o}^{(LF)} \equiv R_{o}(y_{0} - y_{imp})/2 < u_{ac} < R_{c}(y_{imp} - y_{c})/2 \equiv u_{c}^{(LF)}, \]

(23)

will be a harmonic oscillation relative to the new constant average position \(y = y_{c}\) with a lower amplitude than before (Fig. 4, trajectory (2)). The amplitude diminishes because in Eq. (21) for the assumed \(k_{c} > k_{o}\) the relaxation in the closed state is faster than in the open one \((R_{c} > R_{o})\). Further increase in the acoustic wave amplitude will lead under the condition \(u_{ac} > u_{c}^{(LF)}\) to the strongly unharmonic (nonlinear) periodic motion (clapping), where the system spends part of each period in the region \(y \leq y_{imp}\) in the field of the attractive center in \(y = y_{c}\) and the other part in the region \(y_{imp} \leq y\) near the attractive center in \(y = y_{0}\) (Fig. 4, trajectory (3)). The period-averaged crack width \(y\) starts to depend continuously on the acoustic wave amplitude. We do not give here the analytical solution for the crack width in this regime (it can be readily obtained, if necessary, by combining the solutions in Eq. (21) valid in different regions). For the current analysis of the low-frequency limiting regime it is important that, when the amplitude of the acoustic wave starts to decrease, the strongly nonlinear clapping regime disappears at the same acoustic amplitude \(u_{ac} = u_{c}^{(LF)}\), at which it has started (without any hysteresis). The second important point is that continuous diminishing of \(u_{ac}\) to zero value finally leaves the interface in the closed position \(y = y_{c}\). The LF acoustic wave of first increasing (exceeding the critical amplitude \(u_{o}^{(LF)}\) and then decreasing amplitude induces the closing of the interface (crack).

If the acoustic action starts on the initially closed interface \((y = y_{c})\), then the transition to clapping also starts for \(u_{ac} > u_{c}^{(LF)}\) and also following the harmonic motion around \(y = y_{c}\). However subsequent diminishing of the incident wave amplitude will finally leave the crack in the same closed state. The return to the harmonic oscillation from the clapping regime again takes place without any hysteresis.

It can be concluded that it is the difference in the amplitude of the effective forces in the impact plane \((F_{o} < F_{c})\) that makes the difference between the scenarios starting from \(y_{0}\) and \(y_{c}\) in the case of the LF acoustic action. Finally it should be noted that in the LF regime due to the inequality \(R_{ac} \gg 1\) the amplitude of the acoustic wave necessary to initiate the interface clapping significantly exceeds the initial interface (crack) width \(u_{ac} = u_{c}^{(LF)} \gg [(y_{imp} - y_{c}), (y_{0} - y_{imp})]\).

5. Interaction of the bistable relaxator with a high frequency acoustic wave

We identify in this section the regime of the high frequency (HF) acoustic action as the regime where the acoustic frequency \(\omega\) is significantly higher than both

![Fig. 4](image-url)
relaxation frequencies $\omega_{ac}$. In this regime $R_{ac} \ll 1$ in Eq. (20) and in the leading approximation the variation in the crack width follows the displacement in the acoustic field: $y \approx c' - 2u_i(\theta)$. Clearly for the periodic acoustic action the integration constant $c'$ in the above solution has a meaning of the average interface width

$$y \approx \langle y \rangle - 2u_i(\theta). \quad (24)$$

It is the average interface width $\langle y \rangle$ that contains in the HF regime the information on the interaction between the surfaces $x = 0 - 0$ and $x = 0 + 0$ composing the interface. In this regime the periodic acoustic action on the interface manifests itself in the variation of the average interface width. In other words the nonlinearity of the interface acts as a diode (or a rectifier). The similar high-frequency regime in the AFM (when the interaction between the tip and the surface is modulated by an ultrasonic wave incident on the surface or by HF vibrations of the cantilever) is known as the ultrasonic force mode (UFM) [41–43]. It is worth mentioning that vibrations of the cantilever) is known as the ultrasonic relaxation frequencies $\omega_{ac}$. In this regime $R_{ac} \ll 1$ in Eq. (20) and in the leading approximation the variation in the crack width follows the displacement in the acoustic field: $y \approx c' - 2u_i(\theta)$. Clearly for the periodic acoustic action the integration constant $c'$ in the above solution has a meaning of the average interface width $y \approx \langle y \rangle - 2u_i(\theta)$. \quad (24)

In the case of the forced oscillator (Eq. (13)) (relevant to the description of the UFM) the average separation between the tip and the sample can be found from the condition [41]

$$\langle F_{ef}(y) \rangle = \langle F_{ef}(\langle y \rangle - 2u_i(\theta)) \rangle = 0, \quad (25)$$

that follows from the averaging of Eq. (13) over an acoustic period. Obviously the same condition Eq. (25), that the average effective force should be zero, is valid for the acoustic relaxator in Eq. (16) as well. Consequently the dependence of the average interface width $\langle y \rangle$ on the amplitude of the acoustic wave can be evaluated by substitution of the solution (24) in the condition (25) for a particular model of the effective force.

In principle the description of the HF regime (as well as in the case of an arbitrary frequency [44]) can be obtained for the sinusoidal acoustic loading. However the critical parameters for the transition between qualitatively different types in the behavior of the nonlinear relaxator even in the asymptotic HF regime, might be found only approximately as the solutions of some transcendental algebraic equations. In order to establish simple (in mathematical sense) and clear (in physical sense) boundaries between different types of HF behavior we found useful to approximate the sinusoidal acoustic action by the stepwise linear

$$u_i = \begin{cases} 4u_{ac}\theta, & -1/4 \leq \theta \leq 1/4, \\ -4u_{ac}(\theta - 1/2), & 1/4 \leq \theta \leq 3/4. \end{cases} \quad (26)$$

In Eq. (26) $\theta$ is the normalized time variable ($\theta = t/T$, where $T$ is the acoustic period). Using the solution (24), we find from the condition $y = y_{imp}$ that in the clapping regime the transition from the region $y_{imp} < y$ (we call it open region) to the region $y < y_{imp}$ (closed region) takes place at the time

$$\theta_1 = (\langle y \rangle - y_{imp})/8u_{ac}. \quad (27)$$

The relaxator returns back to the open region at the time

$$\theta_2 = 1/2 - \theta_1. \quad (28)$$

Using the properties of the symmetry and periodicity of the solution (26), and also the solutions (27) and (28) for the transition times we derived the description of the average effective force

$$\langle F_{ef}(y) \rangle \approx \frac{1}{2}(k_i - k_o) \left\{ \langle y \rangle^2 - 2\langle y \rangle k_c(2u_{ac} + y_i) + k_c(2u_{ac} - y_o) \right\} \frac{k_o - k_c}{k_c - k_o}$$

$$+ k_c[(2u_{ac} + y_i)^2 - (y_{imp} - y_o)^2] - k_o[(2u_{ac} - y_o)^2 - (y_o - y_{imp})^2] \right\}$$

$$= \frac{F^2}{k} (\langle y \rangle - y) \quad (29)$$

For further analysis it is convenient to introduce the following nondimensional parameters: $k = k_i/k_o$ is the ratio of the crack stiffness in the open and closed states, $F = F_o/F_o = k(y_{imp} - y_c)/(y_0 - y_{imp})$ is the ratio of the attractive to the repulsive forces in the impact plane $y = y_{imp}$ (see Fig. 3(c)), $E = E_o/E_o = k(y_{imp} - y_c)/(y_0 - y_{imp})^2 = F^2/k$ is the ratio of the separations of the potential energy minima in $y = y_c$ and $y = y_o$ from the local energy maximum (potential barrier) in $y = y_{imp}$ (see Fig. 3(d)). Clearly the parameter $E$ controls which state (i.e. open or closed) is absolutely stable if the presence of the thermal fluctuations in the system is admitted. In the case $E > 1$ ($E < 1$) the closed state of the interface is stable (metastable) while the open state is metastable (stable). The situation corresponding to the case $E < 1$ is qualitatively presented in Fig. 3(d). For the normalization of the distances we use the scale $y_0 - y_{imp} \equiv \Delta_0^2/2, U = 2u_{ac}/(y_0 - y_{imp}), \langle Y \rangle = \langle y \rangle/(y_o - y_{imp}), \Delta_o = y_0 - y_{imp}$. In these notations, if the system before the acoustic action is in the open state, then the impact takes place first when $U = 1$. If the system is initially in the closed state then the first impact takes place when $U = (E/F) \equiv \Delta_0^2/E_o$. The solution of the quadratic equation in Eqs. (25) and (29) for the average interface width can be presented in the form

$$\langle Y \rangle - Y_o = \frac{F^2(U - \frac{\theta}{2}) + E(U - 1) - (F^2 - E) \pm \sqrt{E^2(F - 1)^2 + 4F^2E(U - \frac{\theta}{2})(U - 1)}}{F^2 - E}. \quad (30)$$
For the existence of a real-valued solution for the interface width \( \langle Y \rangle \) the polynomial under the square root in Eq. (30) should be nonnegative. In fact it is a quadratic polynomial in the acoustic amplitude \( U \). The trivial analysis demonstrates that this polynomial is nonnegative for all possible values of \( U \) under the condition

\[
(E - 1)(k - 1) \geq 0.
\]  
(31)

Under the opposite condition

\[
(E - 1)(k - 1) < 0
\]  
(32)

there is a forbidden gap in the values of \( U \) where the real-valued solution (30) does not exist.

The solution in Eq. (30) is multi-valued, composed of two branches discriminated by the sign (\( \pm \)) in front of the square root. The condition of stability for the variation of the average crack width can be formulated as

\[
\left( \frac{\partial (F^c)}{\partial \langle Y \rangle} \right) < 0.
\]

If we adopt the convention that the square root in Eq. (30) has a positive real part, then the analysis of the last inequality for the average force (29) leads to the following conclusion. The lower (upper) branch of the solution is stable in the case \( k > 1 \) (\( k < 1 \)), respectively.

It can be verified that one branch of the solution in Eq. (30) passes through the point \( A(\langle Y \rangle = Y_o, U = 1) \), which is the point of the first impact in the case when the acoustic action of the increasing amplitude is applied to an initially open interface. The other branch passes through the point \( B(\langle Y \rangle = Y_c = Y_o - 1 - (E/F), U = (E/F)) \) corresponding to the point of the first impact when the interface is initially closed. The condition that the impact point \( A(\langle Y \rangle = Y_o, U = 1) \) belongs to the stable branch can be formulated as

\[
(F - 1) < 0.
\]  
(33)

The requirement for the impact point \( B(\langle Y \rangle = Y_c, U = (E/F)) \) to belong to the stable branch of the solution (30) is just the opposite case

\[
(F - 1) > 0.
\]  
(34)

In the following we concentrate on the detailed analysis of the nonlinear relaxator characterized by the inequalities \( k > 1, F > 1 \). As it was discussed in Section 6 the relaxator, satisfying these requirements, is expected to be the closest model for the physical reality of interest. The description of the nonlinear relaxator behavior for the other domains of the parameters \( k \) and \( F \) can be obtained similarly if necessary. In the considered case \( k > 1, F > 1 \) the stable branch of the solution (30) is the lower one. In accordance with (33) and (34) the impact point \( A \) is unstable and the impact point \( B \) is stable. Two types of the relaxator behavior are possible depending on the value of the parameter \( E \).

In the case \( E > 1 \) (where the stable position of the interface in the presence of the thermal fluctuations is the closed position) the solution (30) exists in accordance with Eq. (31) for all values of \( U \). In Fig. 5(a) and (b) we provide a graphical illustration of the interface width variation in this case. The curves marked by (+) and (−) in Fig. 5 correspond to the unstable (upper) and stable (lower) branches of the solution (30), respectively. If the initial state is the closed one then the increase in acoustic amplitude \( U \) does not lead to variation of the interface width until the first impact takes place. This qualitatively corresponds in Fig. 5(a) to the motion of the relaxator along the trajectory \( CB(\langle Y \rangle = Y_c, U < (E/F)) \). The clapping starts at point \( B \) when \( U = (E/F) \). The subsequent increase in the wave amplitude moves the system along the stable branch (−). With diminishing amplitude the system returns back to the initial state.

If the initial state is the open one then the increase of \( U \) does not lead to the variation of \( \langle Y \rangle \) until the impact takes place in the point \( A \). This is qualitatively illustrated in Fig. 5(a) by the motion of the relaxator (i.e. the interface width) along the horizontal trajectory \( DA(\langle Y \rangle = Y_o, U < 1) \). When \( U = 1 \) the clapping starts and the interface average width diminishes by a jump from the unstable (+) to the stable (−) branch of the solution. The subsequent variation of \( U \) causes the variation of \( \langle Y \rangle \) along the lower branch. The diminishing of the acoustic action will first lead the relaxator to the impact point \( B \) and then to the nonclapping regime described by the \( \langle Y \rangle \) motion along the straight line \( BC \). Consequently the acoustic action (similar to the thermal fluctuations) can transfer in the considered case the system from the metastable to the stable state. Under the condition \( E < F \) assumed in Fig. 5(a) the clapping starts at higher wave amplitudes \( (U = 1) \) than it stops \( (U = E/F < 1) \). There is a hysteresis in the switching-on and the switching-off the clapping regime.

Fig. 5(b) illustrates the same case \( E > 1 \) but under the condition \( E > F \). In this situation the nonclapping interface motion around the initial position of the unstable equilibrium at \( \langle Y \rangle = Y_o \) is transferred by the impact in point \( A \) into the nonclapping motion around the position of the stable equilibrium at \( \langle Y \rangle = Y_c \) (see Fig. 5(b)). This situation resembles the LF regime described earlier in Section 3. The subsequent variation in the wave amplitude can either induce clapping (if \( U > E/F > 1 \)) or lead the system to the final stop in the closed state even without intermediate clapping regime (if \( U < E/F \)).

In the case \( E < 1 \) (where the stable position of the interface in the presence of the thermal fluctuations is the open position) the real-valued solution (30) does not exist (in accordance with Eq. (32)) for some values of \( U \). In Fig. 5(c) corresponding to this case the left and the right curves presenting the solution (30) are separated by a gap \( (U_c^{(c)} < U < U_o^{(c)}) \). Note that in the considered case \( E < 1, F > 1 \) the inequality \( E/F < 1 \) holds and, consequently, we do not need two figures to illustrate the case \( E < 1 \) as it was in the previous case \( E > 1 \). From
the analysis of Fig. 5(c) one can arrive to the following conclusions. If the relaxator is initially in the open (stable) state then the impact (at $U = 1$) leads to an abrupt decrease of the interface average width accompanied by the transition to clapping regime. The return to nonclapping regime with diminishing wave amplitude but it takes place at lower acoustic amplitudes ($U < U^{(cr)}_0$). Thus the hysteresis phenomenon is predicted.

If the relaxator is initially in the closed (metastable) state then the impact at $U = E/F < 1$ is followed first by the clapping regime (the stable trajectory $BG$ in Fig. 5(c)). From the latter regime the relaxator has still opportunity to return to the initial metastable position. However if the increasing wave amplitude overcomes the critical value $U^{(cr)}_c$ (see Fig. 5(c)) the abrupt transition to the regime of the nonclapping motion around the stable (open) position takes place and the relaxator loses opportunity to return to the initial position.

The above presented analysis underlines two important features of the high-frequency regime. First, the acoustic action inducing the interface clapping switches in the most of the cases the interface from the metastable state (in the local minimum of the potential) to the stable state (in the absolute minimum of the potential). Second, under some conditions the clapping regime stops at lower amplitude of the acoustic action than those when it starts (with hysteresis). The predicted hysteresis phenomenon can contribute to the hysteresis in the interaction of the powerful acoustic wave with cracks observed experimentally [21]. The hysteresis of the predicted type (when the tip of the AFM starts tapping the surface at higher amplitudes than it stops tapping) has been recently observed also in UFM [43], though it has been attributed to the hysteresis in the adhesion forces rather than to the existence of two equilibrium positions in the system composed of the mechanical oscillator and the elastic surface.

In order to understand which of the predicted phenomena survives at intermediate frequencies of the acoustic action we have analyzed numerically the analytical solution of Eq. (19) obtained for the sinusoidal action and an arbitrary frequency. These results, demonstrating when the dynamic hysteresis survives, will be presented elsewhere [44].

To close this section we give the estimates indicating that the realization of the HF regime is possible in practice. As the simplest example let us consider that the roughness of the interface might be modeled by the identical spherical contacts of curvature radius $r$, separated by a distance $\geq r$. If initially the contacts are unloaded and the interface width diminishes by $\partial Y$ this will lead to the interaction Hertzian force [45].
The characteristic dimension \( a \) of the strained part of the Hertzian contact is of the order of the contact radius \( a \propto 1/\tau_a \).
Consequently \( \omega_a \propto \sqrt{r/(\delta y)} \), and the inequality \( \omega_a < \omega < \omega_o \) (necessary for the description of the HF regime for the open crack by Eq. (19) might be verified if \( n\pi \delta y^2/(\delta y/r) \ll 1 \). When the distance between the contacts exceeds \( r \) the first multiplier in the latter inequality is estimated by \( (n\pi)^2 \ll 1 \). We conclude that the HF regime is possible if the interface width variation is sufficiently smaller in comparison with the curvature of the contacts \( (\delta y/r) \ll 1 \). Clearly for sufficiently small \( (n\pi \delta y^2)/(\delta y/r) \) a more general inequality \( \omega_a < \omega_c < \omega < \omega_o \) (necessary for the description of the HF regime in both open and closed regions of crack width variation by Eq. (19)) can be also satisfied.

6. Discussion and conclusions

The developed theory predicts that the variation of the interface width under the action of an incident acoustic wave can exhibit hysteresis. The manifestations of the theoretical predicted hysteresis are similar to the recent experimental observations [19,21]: the transition to highly nonlinear regime of crack interaction with the acoustic field of first increasing and then decreasing amplitude takes place at higher threshold amplitudes than the inverse transition. The theory relates this hysteretic behavior to the specific dependence of the interaction force (between the surfaces composing the interface) on the interface width (on the nanoscopic/microscopic separation between the surfaces). The case of the bistable interface, where the interaction force allows two different positions of equilibrium (closed and open interfaces) has been analyzed in details. The interaction force is single-valued in this case. However, in general, similar hysteretic behavior might be related to adhesion hysteresis, when the interaction force is itself hysteretic (see the recent UFM AFM experiments [43], for example). The dependence of the effective force on the interface width in the latter case is multi-valued and controlled also by the sign of the interface width variation (i.e. the force is different for the processes of the interface opening or closing). Surely, it is also possible that in reality the interface is hysteretic and bistable simultaneously and that the thermoelastic effects [19,20] also contribute to dynamic hysteresis in acoustic fields.

To discriminate between the different possible mechanisms the measurements of the variation of the signal at fundamental frequency reflected by (or transmitted through) the interface as well as the measurements of the higher harmonics radiated by the interface with the incident wave amplitude, might be useful [3,46,47]. The theoretical evaluation of the frequency spectrum of the reflected and transmitted acoustic wave as a function of the incident wave amplitude might be considered as one of the perspectives for future research.

Finally it is worth mentioning that, as in many other physical situations, the hysteretic behavior manifests itself simultaneously with bistability (see Ref. [48] for an acoustic example). In fact if there is a hysteretic behavior of the acoustic relaxator (see, for example, the interval \( U_{61}^{\delta y} \leq U \leq 1 \) in Fig. 5(c)) then clearly in the same region the relaxator can move differently in the same acoustic field depending on the prehistory of its periodic loading. In the case illustrated in Fig. 5(c) in the same acoustic field with an amplitude in the interval \( U_{61}^{\delta y} \leq U \leq 1 \) the motion of the interface can be either harmonic or highly unharmonic (with the impacts). In other words the interface exhibits bistable behavior in the acoustic field. Consequently the developed theory is not only a mathematical demonstration of the hysteresis in dynamic response of a nonlinear relaxator, but also a demonstration of a bistable nonlinear relaxator.

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References