Improvement of the computation of Fourier integrals using the complex plane: Application to acoustic fields

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This letter aims at showing the interest in using an integration path in the complex plane in order to calculate Fourier integrals, instead of using a real path, as needed for the fast Fourier transform algorithm as follows: fastness, accuracy, avoiding of singularities, or even aliasing phenomenon. The method is applied to the calculation of the acoustic field emitted by a transducer or a loudspeaker; for the same number of sampling points, the calculation of the spatial Fourier integrals is more accurate using a complex path than using a real path. © *2010 American Institute of Physics*. [doi:10.1063/1.3298359]

Optic or acoustic fields can be obtained using the well-known principle of decomposition into monochromatic plane waves (angular spectrum decomposition). Thus, in a fluid medium (sound speed c_0 and density ρ_0), the complex acoustic pressure $\hat{p}(x,z;t)=\hat{P}(x,z)\exp(-i\omega t)$ (where the "hat" means that the quantity may be complex and where ω is the angular frequency of the waves) may be calculated in the half-space $z\geq 0$ (see Fig. 1) by means of a continuous superposition of plane or evanescent waves[1–4]

$$\hat{P}(x,z) = \int_{-\infty}^{+\infty} \hat{A}(k_x)\exp[i(k_x x + \hat{k}_z z)]\mathrm{d}k_x. \tag{1}$$

Each wave has its own wave number vector $\mathbf{k}_0=\omega/c_0\mathbf{n}=k_x\mathbf{e}_x+\hat{k}_z\mathbf{e}_z$, where $\mathbf{n}$ is the propagation direction vector of the wave. The component $\hat{k}_z$ of the wave vector is real or purely imaginary valued and depends on k_x via the usual dispersion equation

$$k_x^2 + \hat{k}_z^2 = \|\mathbf{k}_0\|^2 = k_0^2 = (\omega/c_0)^2, \tag{2}$$

where the choice of the determination of this function $\hat{k}_z(k_x)$ is crucial.

Indeed, the uniqueness of the solution (1) of the wave propagation equation in the fluid medium is ensured by a radiation (or decreasing) condition when $z\rightarrow+\infty$, for each plane or evanescent wave constituting the beam, by choosing the appropriate value for $\hat{k}_z$, through the dispersion Eq. (2) as follows:

- for $|k_x|\leq k_0$, $\hat{k}_z$ is real. The constitutive wave is a progressive plane wave which propagates toward $z>0$ if the positive real determination is chosen for $\hat{k}_z$. (a)
- for $|k_x|>k_0$, $\hat{k}_z$ is purely imaginary. The constitutive wave is evanescent in the direction Oz, and decreases exponentially toward $z>0$ if the positive imaginary determination is chosen. (b)

More specifically, the function $\hat{k}_z(\hat{k}_x)$ must be specified in the complex plane $\hat{k}_x$. The conditions [(a) and (b)] on $\hat{k}_z$ lead to select the relevant branch (the principal branch) of the multivalued function $\hat{k}_z(\hat{k}_x)$. This principal branch corresponds to the condition $\Re e(\hat{k}_z)>0$ where $\Re e(\hat{X})$ and $\Im m(\hat{X})$ denote the real and imaginary parts of $\hat{X}$, respectively. The holomorphy domain of this branch is limited by the two following cuts on the real axis: $-\infty<k_x<-k_0$ and $k_0<k_x<+\infty$, on which $\Re e(\hat{k}_z)=0$ (Fig. 2). The real integral path in k_x of the Fourier transform (1), has to be specified as following, when the principal branch $\hat{k}_z(k_x)$ is introduced in this integral:

$$\text{-upper edge of the cut } -\infty<k_x<-k_0\ [\Im m(\hat{k}_z)>0], \tag{3a}$$

$$\text{-segment } -k_0<k_x<k_0\ [\Re e(\hat{k}_z)>0], \tag{3b}$$

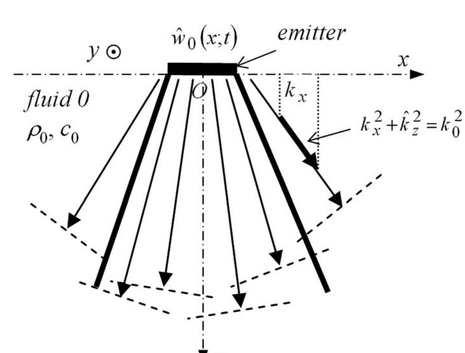

FIG. 1. Schematic diagram of the plane waves which constitute the acoustic bounded beam created by an emitter in a fluid medium.

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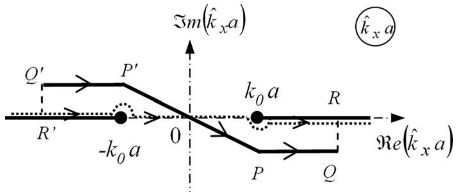

FIG. 2. Complex integration paths in the complex plane of $\hat{k}_x a$.

-lower edge of the cut $k_0 < k_x < +\infty$ $[\Im m(\hat{k}_z) > 0]$, (3c)

which corresponds to the dotted path on Fig. 2.

Once this choice has been done, by using the classical relation between particle velocity and acoustic pressure,[5] the complex amplitude $\hat{A}(k_x)$ is determined as a function of the component k_x from the emission data of the normal velocity $\hat{W}_0(x)$ on the plane $z=0$ (membrane of a loudspeaker or the front face of a transducer).

Then, the pressure field $\hat{P}(x,z)$ can be calculated, for any given $z \geq 0$, using the Fourier integral (1), in the form[6]

$$\frac{\hat{P}(x,z)}{\rho_0 c_0 W_0} = k_0 a \int_{-\infty}^{+\infty} \frac{\hat{G}(k_x)}{\hat{k}_z} \exp[i(k_x x + \hat{k}_z z)] \mathrm{d}k_x, \tag{4}$$

with $\hat{G}(k_x) = \dfrac{1}{\pi} \dfrac{\sin(k_x a)}{k_x a}$ for a crenel (piston mode), (5a)

where $\hat{W}_0(x)$ is a constant W_0 on the interval $-a \leq x \leq a$ and is zero outside, and

$$\hat{G}(k_x) = \frac{1}{2\sqrt{\pi}} \exp\left[-\left(\frac{k_x a}{2}\right)^2\right] \quad \text{for a Gaussian emitter with nominal diameter } 2a. \tag{5b}$$

The computation of this Fourier integral could classically be done using a discrete Fourier transform, which permits to use a fast Fourier transform (FFT) algorithm. This implies that the parameter k_x be real [and the real path integration is that specified in Eq. (3)]. In addition, this technique is restrictive insofar as it imposes a strict association between the sampling points of the integration variable and the points of the physical space where the values of the pressure field are calculated. The number of these points will have to be equal to the number of the integration points and will have to be regularly distributed on a line parallel to the emitting line.

On the other hand, the matrix computation languages now available (MATLAB® or SCILAB software for instance) permit to calculate very fastly integrals of the form (1), using a simple classical trapezoid method. Here, the constraint on mapping points in the physical space is avoided. Moreover, using the Cauchy integral theorem, the path integration in k_x may be modified in the plane of the complex variable $\hat{k}_x$. This is the numerical technique we propose in this letter, in order to calculate spatial Fourier integrals of the form of Eq. (4).

Figure 3 present cartographies in the spatial plane (O,x,z) of the acoustic field generated by an emitter, using a

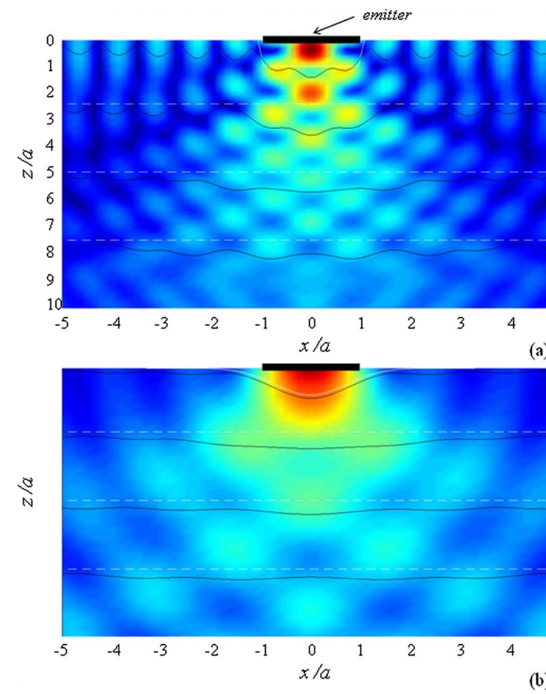

FIG. 3. (Color online) Cartographies of the acoustic pressure (modulus) field in a fluid with four cuts at $z=0$, $z_e=2.5a$, $z_e=5a$ and $z_e=7.5a$, using a real path for the calculation of Fourier integral (1). (a) Piston mode, $k_0 a=4$ $(-5 \leq k_x a \leq 5$, 30 sampling points for $k_x)$. (b) Gaussian mode, $k_0 a=2$ $(-4 \leq k_x a \leq 4$, 50 sampling points for $k_x)$.

real path for computing the Fourier integrals when the normal velocity $\hat{W}_0(x)$ is a crenel [piston mode, see Fig. 3(a)] or is a Gaussian [Fig. 3(b)]. For the chosen sampling in k_x (only 30 or 50 points) the fields are not well described and present many irregularities. At least, a 500 points sampling is necessary to obtain a good resolution, for each point $M(x,z)$ of the physical space. Also, the points $k_x = \pm k_0$ of the real axis, for which $\hat{k}_z=0$, are (integrable) singularities for the Fourier integral; therefore, the sampling values of k_x must avoid these points, and a great number of points is necessary in order to obtain a suitable approximated value of the integral in the neighboring of these singularities.

Instead of using a real path for the integration in k_x, it is also possible to use a complex path, according to Cauchy integral theorem, since the function to be integrated is analytic.

The study of the behavior at infinity of the function $\hat{G}(k_x)$ [see Eqs. (4) and (5)] and of the propagation exponential permits to perform a possible distortion of the integral path at infinity. Following the classical theorems in complex analysis,[7] the path may be moved away at infinity, parallel to the real axis, above this axis in the region $\Re e(\hat{k}_x) < 0$ and below it in the region $\Re e(\hat{k}_x) > 0$.

Practically, the real integration path is substituted by the finite path $Q'P'PQ$ (Fig. 2). This change of path is justified if the integral values on the segment QR and on the half-straight line $[R,+\infty[$ are negligible with respect to the value of the whole integral. As the trapezoid method makes it very easy to calculate the acoustic field at only one point of the physical space, the path $Q'P'PQ$ can be optimized by studying the values of the integrals for some benchmark points of the physical space. For instance, in the case of a piston mode

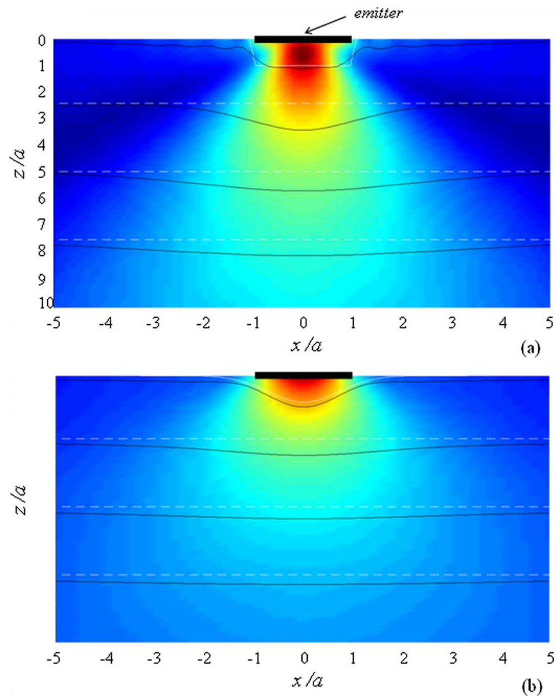

FIG. 4. (Color online) Cartographies of the acoustic pressure field (modulus) in a fluid with four cuts at $z=0$, $z_e=2.5a$, $z_e=5a$, and $z_e=7.5a$, using a complex path $P'P$ for the calculation of Fourier integral (1). (a) Piston mode, $k_0a=4$ (k_xa on the line $[-5+0.2i,5-0.2i]$, 30 sampling points for k_x). (b) Gaussian mode, $k_0a=2$ (k_xa on the line $[-4+0.2i,4-0.2i]$, 50 sampling points for k_x).

emitter ($k_0a=4$), for a benchmark point located on the acoustic axis $10a$ far from the emitter, the modulus of the reduced pressure $\hat{P}(x,z)/(\rho_0 c_0 W_0)$ [left hand member of Eq. (4)] is equal to 0.5033, whereas the values of the integrals on QR and on $[R,+\infty[$ (abscissa of R located at $k_xa=4.5$) are respectively equal to 6×10^{-12} and to 7×10^{-12} (integration limited to $k_xa=50$). Considering these results, it is even justified to merge points P and Q (and P' and Q'), and thus to restrict the integration to only one segment $P'P$. The study for benchmark points closer to the emitter show that the abscissa of R has to be slightly increase, (increasing influence of evanescent waves). A good compromise, adopted here for the drawing of the cartography [Fig. 4(a)], is to choose the segment $[-5+0.2i,5-0.2i]$. The number of the sampling points can be reduced to 50 or even to 30 in the case of this cartography. The choice of the distance $\Im m(k_xa)=0.2$ is guided by the caring about both being far enough from singular points $k_xa=\pm4$, and being close enough to the real axis, in order to minimize the value of the integral on QR.

As an example, Fig. 4 present cartographies in the spatial plane (O,x,z) of the acoustic field, in the same conditions as those of Fig. 3 (for the same number of points for the sampling of k_x), but using such a complex integration path. As it can be shown comparing Figs. 3 and 4, using such an integration path hugely improves the accuracy of the calculus of integral (1), without increasing the number of points for the sampling of k_x.

This method saves a significant amount of time. As a rough guide, on a personal computer (Intel T7500, 2.20 GHz, 1.99 GB RAM), the integration along a real path (1000 sampling points for k_x) takes 1.25 s for a mapping of 250 000 points in physical space, while, for a comparable accuracy, the integration along a complex path (40 sampling points for k_x are sufficient) takes only 0.1 s.

It is possible to extend this method to the computation of three-dimensional fields, for which double spatial Fourier integrals have to be calculated. Cauchy's theory is still valid, but now, integration surfaces (dimension 2) have to be distorted in a four-dimensional space. However, as the distortions from the real axis are small, the geometrical complexity is reduced. Some encouraging results have been obtained in this sense.

As a conclusion, this complex integration method, which is fast and accurate, is interesting when the number of the calculation points in the physical space becomes large, like for drawing cartographies such as those of Fig. 4, or when the beam interacts with a fluid or solid multilayered structure.[6] Moreover, the fact to be no longer dependent on the constraints imposed by the FFT algorithm becomes particularly useful in the case of oblique incidence on a structure.

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