Pressure and shear horizontal guided waves excitation: Nonuniform, time-periodic source distribution of finite extent on the boundaries

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There is a class of waveguides problems in which energy is provided by a source distribution of finite extent on the surface of the guide and for which usually the behavior of the acoustic field created is not analytically correlated with the source strength. The present paper aims at providing an analytical model describing the field which can be obtained from such nonuniformly distributed, time-periodic source creating either pressure waves in fluid-filled waveguides or Shear Horizontal waves in isotropic solid plates. This model involves convolution products between appropriate Green's functions and the source stress. A relevant example is finally displayed. © *2011 American Institute of Physics*. [doi:10.1063/1.3580772]

The nature of acoustic fields within fluid-filled or isotropic solid, finite, or infinite waveguides is a topic of persisting importance in fundamental acoustics and in its practical applications. There is a broad class of waveguide problems in which the energy is provided by a source distribution of finite extent on the surfaces of the guides and for which the acoustic field created is a modal field, either a pressure field or a Shear Horizontal (SH)-wave, depending on the set-up considered. In the literature, specific examples are addressed in the frame of analytical models suitable for describing the modes coupling due to scattering on small one-dimensional irregularities of the surface of both fluid-filled waveguides (acoustic pressure waves) and isotropic solid waveguides (SH waves inside a plate) (see Refs. 1–8, and references contained therein). For these interior problems, the analytic procedure whereby one expresses specific acoustic fields, created by such source distributions on the surface of finite waveguides, as an integral over the surface of the source, is relatively unexplored (relevant formulation for infinite fluid-filled waveguide was provided by Doak).[9] The present paper deals with such analytical formulation for pressure and SH waves (or torsional waves in a cylinder), which involves convolution products between appropriate Green's functions and the source stress acting on the boundaries of the medium, upstream the domain of interest, where the bounding surfaces of the waveguide are natural surfaces in a coordinate system. A relevant example, starting from the behavior of a nonuniformly distributed, time-periodic source of finite extent set on the surface of fluid or solid infinite waveguide is displayed. It shows how an appropriate SH or pressure wave can be created downstream the source, respectively, in a fluid-filled or solid waveguide.

The two or three-dimensional rectangular $(x, \mathbf{w}=x, y, z)$ or circular $(x, \mathbf{w}=x, r, \theta)$ structure of semi-infinite $[x \in (-\ell, \infty)]$ or finite $[x \in (-\ell, L)]$ extent under consideration is schematically shown in Fig. 1. It is either a fluid-filled waveguide (acoustic pressure waves) or an isotropic solid waveguide (SH waves inside a plate, torsional waves in a cylinder). An acoustic source of finite extent, which is set on the surface S_0 of coordinates $\mathbf{w}=\mathbf{w}_0$, namely, x_0, y_0 or r_0, θ_0 (or acting inside a negligible skin thickness) of the waveguide in the domain D_1 $[x \in (-\ell, 0)]$ upstream the domain of interest (domain $x>0$ labeled D_2), creates an appropriate acoustic field at the entrance of the finite $[x \in (0, L)]$ or semi-infinite $x \in (0, \infty)$ waveguide.[7] This source is assumed to be a nonuniformly distributed, time-periodic source [it depends on the coordinate x (the axis of the waveguide) and on the coordinates of the surface $\mathbf{w}_0=(x_0, y_0)$ or (r_0, θ_0)]. It is worth noting that the waveguide considered $[x \in (-\ell, L)]$ is equivalent to a waveguide which is symmetrical with respect to the abscissa $x=-\ell$.

The source is described by its normal velocity inside the fluid-filled waveguide or the tangential stress acting on the surface S_0 (contour Γ_0, coordinate $\mathbf{w}_0$) of the solid waveguide. In both situations, these parameters depend on the first derivative of the acoustic field (pressure field in fluid-filled waveguide or displacement field in solid waveguide) with respect to the spatial coordinates. Therefore, the boundary conditions on the surfaces of the waveguides in the domain D_2 considered are Neumann boundary conditions, namely the normal velocity field vanishes in the fluid-filled waveguide and the tangential component of the stress vanish in the solid waveguide. Besides, owing to the symmetrical geometry of the structure with respect to the coordinate $x=-\ell$, the first derivative with respect to x of the x- (axial) and $\mathbf{w}$-coordinates of the velocity fields, and the x-component of

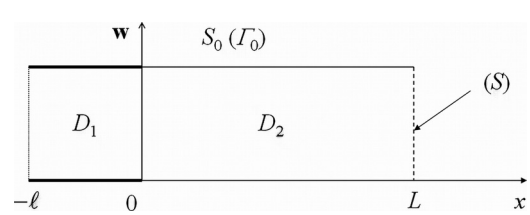

FIG. 1. Sketch of a waveguide of axis x with lateral surfaces denoted S_0 (contour Γ_0) and section denoted (S), with a source distribution on the surface in the interval $x \in (-\ell, 0)$, assumed to be symmetrical with respect to the abscissa $x=-\ell$ (or rigid walled at $x=-\ell$).

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the velocity field vanish at the coordinate $x=-\ell$.

The acoustic fields expressed analytically below are the pressure fields $p_2(x,\mathbf{w})$ (fluid-filled waveguides) or the y-component of the displacement field $U_{2y}(x,z)$ (SH wave), in the domain of interest D_2 [$x \in (0,L)$, L finite, or infinite], both denoted below $\phi_2(x,\mathbf{w})$ [$\phi_1(x,\mathbf{w})$ in the domain D_1]. Actually, the field in the domain D_2 depends on both the field in the domain D_1 and the interface conditions between both domains. The formalism used, in both domains, lies on the classical integral formalism, assuming appropriate Green's functions.

The problem addressed herein, in the domains D_i, $i=1,2$, is governed by the following set of equations, including the propagation equation and the boundary conditions, which is written as

$$(\partial_{xx}^2+\partial_{\mathbf{ww}}^2+k^2)\phi_i(x,\mathbf{w})=0, \text{ domain } D_i, \tag{1a}$$

$$\partial_{\mathbf{n}_0}\phi_i(x,\mathbf{w})=\sigma(x,\mathbf{w}), \text{ on the lateral surfaces } S_0, \tag{1b}$$

with $\sigma(x,\mathbf{w})=-i\omega\rho_0 V(x,\mathbf{w})$ (fluid) or $\sigma(x,\mathbf{w})=(1/\mu)T_{zy}(x,\mathbf{w})$ (over the surface of the source, and $\sigma(x,\mathbf{w})=0$ elsewhere, where $V(x,\mathbf{w})$ denotes the velocity of the source (normal to the fluid waveguide, outwardly directed) and $(1/\mu)T_{zy}(x,\mathbf{w})$ the y-component of the stress (solid plate), ω,k,ρ_0,μ, and $\partial_{\mathbf{n}_0}$ being, respectively, the angular frequency, the adiabatic wavenumber, the density (fluid), the second Lamé coefficient (solid), and the normal derivative (outwardly directed).

The integral formulation of the problem stated above (1a) and (1b), in the domain D_1, can be written as follows:

$$\begin{aligned}\phi_1(x,\mathbf{w})=&\int_{-\ell}^{0}dx_0\oint_{\Gamma_0}d\Gamma_0 G_1(x,\mathbf{w};x_0,\mathbf{w}_0)\partial_{\mathbf{n}_0}\phi_1(x_0,\mathbf{w}_0)\\&+\iint_{(S)}[G_1(x,\mathbf{w};0,\mathbf{w}_0)\partial_{x_0}\phi_1(0,\mathbf{w}_0)\\&-\phi_1(0,\mathbf{w}_0)\partial_{x_0}G_1(x,\mathbf{w};0,\mathbf{w}_0)]d\mathbf{w}_0,\end{aligned} \tag{2}$$

[the first term in the right hand side lying on the lateral surface of the domain D_1 and the second one on the section (S) of the waveguide at $x=0$], where the Green's function $G_1(x,\mathbf{w};x_0,\mathbf{w}_0)$ is given by the eigenfunction expansion

$$G_1(x,\mathbf{w};0,\mathbf{w}_0)=\sum_{n=0}^{\infty}g_n(x,x_0)\psi_n(\mathbf{w}_0)\psi_n(\mathbf{w}), \tag{3a}$$

the orthogonal, normalized eigenfunctions $\psi_n(\mathbf{w})$ being solutions of the homogeneous Helmholtz equation with the Neumann boundary conditions at the outer surface of the waveguide, with the requirement of symmetry of the Green's function at the abscissa $x=-\ell$, this being fulfilled by

$$\begin{aligned}g_n(x,x_0)=&\{\exp(-ik_{xn}|x-x_0|)+\exp[-ik_{xn}(x+x_0\\&+2\ell)]\}/(2ik_{xn}),\end{aligned} \tag{3b}$$

with $k_n^2=k_{xn}^2+k_{\mathbf{w}n}^2$, k_n denoting the eigenvalues (the subscript n standing for two integers).

In the domain D_2, the Green's function chosen is required to satisfy to the Neumann boundary condition at the outer surface of the waveguide and to satisfy a given boundary condition at the end $x=L$ of the domain (R being a known reflection coefficient accounting for the input properties of the device loading the end of the waveguide), leading to

$$G_2(x,\mathbf{w};x_0,\mathbf{w}_0)=\sum_{n=0}h_n(x,x_0)\psi_n(\mathbf{w}_0)\psi_n(\mathbf{w}), \tag{4a}$$

$$\begin{aligned}h_n(x,x_0)=&\{\exp(-ik_{xn}|x-x_0|)+R\exp(ik_{xn}[(x+x_0)\\&-2L])\}/(2ik_{xn}).\end{aligned} \tag{4b}$$

Given this Green's function, the solution of the problem addressed here in the domain D_2 is readily expressed as follows:

$$\begin{aligned}\phi_2(x,\mathbf{w})=&\sum_n\psi_n(\mathbf{w})\{[\partial_{x_0}h_n(x,x_0=0)]\iint_{(S)}\phi_2(x_0\\&=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0-h_n(x,0)\iint_{(S)}\partial_{x_0}\phi_2(x_0\\&=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0\}.\end{aligned} \tag{5}$$

Finally, we must express the continuity conditions at the interface $x=0$ between both domains D_1 and D_2 mentioned above. They are given by

$$\phi_2(0,\mathbf{w})=\phi_1(0,\mathbf{w}) \text{ and } \partial_x\phi_2(0,\mathbf{w})=\partial_x\phi_1(0,\mathbf{w}). \tag{6}$$

Therefore, invoking the inner product of ϕ_1 by ψ_n [Eqs. (2) and (3a)], and using Eq. (6), the integral in the right hand side of Eq. (5) can be expressed as

$$\begin{aligned}\iint_{(S)}\phi_2(x_0=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0=&\iint_{(S)}\phi_1(x_0\\&=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0=[F_n(0)\\&+g_n(0,0)\iint_{(S)}\partial_{x_0}\phi_2(x_0\\&=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0]/[1+\partial_{x_0}g_n(0,0)],\end{aligned} \tag{7}$$

with

$$F_n(x)=\int_{-\ell}^{0}dx_0\oint_{\Gamma_0}d\Gamma_0 g_n(x,x_0)\psi_n(\mathbf{w}_0)\sigma(x_0,\mathbf{w}_0). \tag{8}$$

Then reporting this result in Eq. (5), multiplying by the eigenfunction ψ_n and integrating over the section of the waveguide, and finally derivating with respect to x, yields the relevant expression for the last integral in Eq. (7), at $x=0$:

$$\begin{aligned}\iint_{(S)}\partial_x\phi_2(x=0,\mathbf{w})\psi_n(\mathbf{w})d\mathbf{w}=&\frac{\partial_x\partial_{x_0}h_n(0,0)}{1+\partial_{x_0}g_n(0,0)}[F_n(0)\\&+g_n(0,0)\iint_{(S)}\partial_{x_0}\phi_2(x_0=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0]\\&-\partial_x h_n(x=0,x_0=0)\iint_{(S)}\partial_{x_0}\phi_2(x_0\\&=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0,\end{aligned} \tag{9a}$$

namely, noting that the integral over the surface (S) in the left hand side of this equation is nothing else that those in the right hand side,

$$\iint_{(S)} \partial_{x_0}\phi_2(x_0=0,\mathbf{w}_0)\psi_n(\mathbf{w}_0)d\mathbf{w}_0 = \frac{\partial_x\partial_{x_0}h_n(0,0)}{[1+\partial_{x_0}g_n(0,0)][1+\partial_x h_n(0,0)]-g_n(0,0)\partial_x\partial_{x_0}h_n(0,0)} \times F_n(0). \quad (9b)$$

Finally, reporting this result in Eq. (7) then in Eq. (5), and invoking expressions (3b) and (4b) for the Green's coefficients $g_n(x,x_0)$ and $h_n(x,x_0)$, the solution in the domain D_2 takes the following form:

$$\phi_2(x,\mathbf{w}) = \sum_n \left\{ \frac{F_n(0)}{1+R\exp(-ik_{xn}2L)[1-2\exp(-ik_{xn}\ell)\cos(k_{xn}\ell)]} \times \{\exp(-ik_{xn}x)+R\exp[ik_{xn}(x-2L)]\}\psi_n(\mathbf{w}) \right\}. \quad (10)$$

Note that reporting results obtained above in Eq. (2), accounting for Eqs. (3a) and (3b), leads straightforwardly to the solution in the domain D_1.

The simple example presented hereinafter involves both an infinite rectangular or cylindrical fluid-filled waveguide or a two-dimensional infinite solid waveguide (SH wave in a plate) and a source activity given by the harmonic factor $\sigma(x,\mathbf{w})$ of the source distribution set on the lateral wall in the interval $x\in(-\ell,0)$, assuming the symmetry with respect to $x=-\ell$, such as $\sigma(x,\mathbf{w})=\sigma_0\cos k_{xm}(x+\ell)$, where k_{xm} is linked to the eigenvalue labeled "m" (m being in this example a given integer) and where $k_{xm}\ell$ is assumed to be greater than π.

For a plate of thickness L_z (in the z-direction), the factor (8) at $x=0$ which appears in the solution (10), for a source distribution $\sigma(x,\mathbf{w})$ chosen on the wall set at $z=0$, takes the following form:

$$F_n(0) = (-1)^{m+n+1} i\frac{\sigma_0\ell}{2k_{xm}}\sqrt{\frac{2-\delta_{n0}}{L_z}}\exp(-ik_{xn}\ell)[\mathrm{sinc}(k_{xm}+k_{xn})\ell+\mathrm{sinc}(k_{xm}-k_{xn})\ell], \quad (11)$$

(sinc denoting the cardinal sine) showing that when $m\neq n$ ($k_{xm}\ell$ and $k_{xn}\ell$ being greater than π) the amplitude of the mode n created by the source is very small (the term in the bracket is much lower than one), unlike when $m=n$ the amplitude of this mode, is important. Then the field created is given mainly by this mode. In order to illustrate that result,

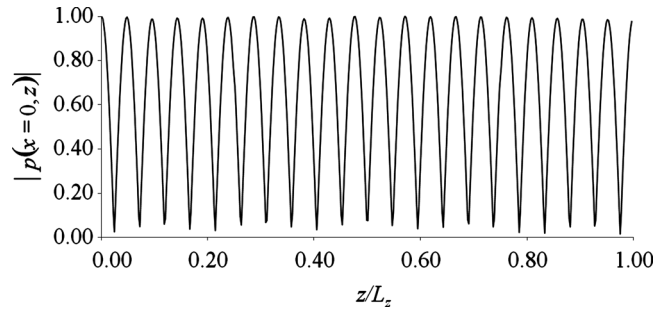

FIG. 2. Shapes of the amplitudes (normalized to one) of the fields created by sinusoidal sources for the antisymmetrical mode $m=10$ ($k_{x10}\ell=10\pi$), when considering 21 propagative modes in the calculus.

Fig. 2 show the shapes of the amplitudes (normalized to one) of the fields created by sinusoidal sources for the antisymmetrical mode $m=10$ ($k_{x10}\ell=10\pi$), when considering 21 propagative modes in the calculus.

In conclusion, it should be noted that this analytical model describing the field obtained from a time-periodic source nonuniformly distributed on the wall of a waveguide, which accounts for the discontinuity between the part of the boundary of the waveguide containing this external source distribution and the other part assumed to be motionless, would be effective in many applications, even though the discrepancies between theoretical and experimental strengths of the source distribution could be not so small as to be negligible.

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