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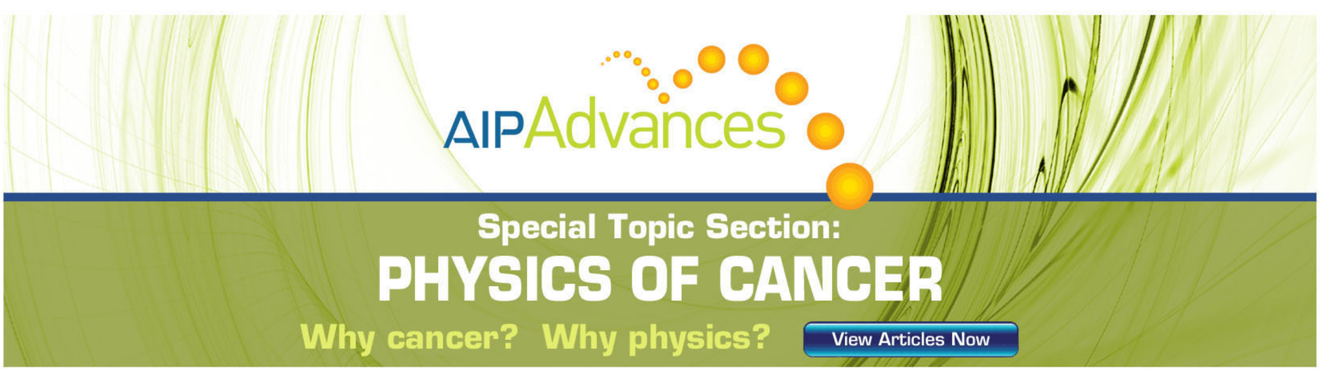

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Stochastic simulation of the high-frequency wave propagation in a random medium

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A stochastic model is proposed to simulate the propagation of an acoustic wave in a random medium characterized by weak velocity fluctuations. After the acoustic wave propagation through a random velocity field, the propagation field becomes itself a random field. In the developed stochastic approach, the wave field in such random medium is modeled by the combination of the wave field in a mean homogeneous medium and of fluctuation corrections. These corrections are provided by a random field generator whose inputs are the statistical moments of the travel times. Using this stochastic modeling, the propagation of both an incident plane wave and an acoustic realistic beam generated by a real transducer in a random velocity field is calculated and the corresponding simulations are validated by comparison with those obtained with a deterministic model based on geometrical optics. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4748274]

I. INTRODUCTION

Wave propagation phenomena are involved in a large amount of scientific fields and often associated with important industrial issues. When the media of propagation are not homogeneous, their physical characteristics (such as bulk density, temperature, elastic stiffness, and particle motion) may vary randomly in space but with a spatial correlation. A wave propagating through such media undergoes some stochastic spatiotemporal fluctuations (the so-called inhomogeneities) of one or several structural medium parameters which depend on the nature and environment of the propagation medium. The spatial correlation of these inhomogeneities can be characterized by a correlation function (defined in Sec. II A) and the characteristic length. This characteristic length gives information on the average homogeneity size. Different ratio of characteristic length to the wavelength lead to three different propagation regimes: low frequency, medium frequency, and high frequency. In this paper, we are only interested in high frequency problems involving the wave propagation through media containing large scale random inhomogeneities.

A medium composed of random inhomogeneities is often too complicated to be described deterministically; it is common to model the real medium by a random medium using a stochastic process. For one sample generation of random velocity medium, the wave propagation in such medium can be calculated using a deterministic propagation model as the method of geometrical optics,[1,2] the method of smooth perturbation,[3,4] and the numerical methods.[5–7] Often these deterministic methods require significant computation time.

Therefore, the stochastic analysis based on rigorous mathematical and statistical developments is adopted here to reveal the nature and degree of influence of the velocity variations on the wave propagation. Numerous papers dealing with stochastic simulation have been published in the view of various applications: oceanography, materials science, hydrology, and geophysics. Among these papers, various techniques of stochastic simulation have been developed including matrix decomposition techniques,[8,9] moving average,[10] nearest neighbor,[11] spectral methods,[12,13] and turning bands.[14] More recently, the analysis of stochastic systems with input parameters variations has been the subject of extensive researches in the past two decades in stochastic mechanics.[15–17] Numbers of stochastic analysis have also been carried out in the topics of seismic ground motions[18–20] whose spatial variation is an important factor that should be carefully considered in the seismic design of buried lifelines such as tunnels and pipelines.

One of the techniques that have been widely applied in engineering problems for simulating stochastic seismic waves is the method of integral spectral decomposition.[21] The application of this method to the wave propagation in random media consists in representing the random elastic modulus and mass density as Fourier integrals using their spectral densities called spectral representations. Substituting the spectral representations of the random material parameters into the wave equation, a stochastic wave can finally be obtained by solving that wave equation. But this method is not quite efficient in practice because the resolution of such stochastic inhomogeneous wave equation requires the complex one of an integral characteristics equation for the wave number.[22]

In this paper, we propose a new fast simulation method to address the problem of the wave propagation in a weakly inhomogeneous random medium. This modeling problem is a challenging complex one of practical interest since it can be encountered in different fields of physics and can deal

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with numerous industrial applications. The new proposed modeling considers that the wave field in a random medium is the combination of a homogenous field and a fluctuation contribution. The homogeneous field is the wave propagation field in an average homogeneous medium, which is simple to calculate. The fluctuation contribution is due to the random velocity variations and can be generated by a random field generator whose inputs are the statistical moments of the wave field in the studying random velocity medium. According to the theoretical studies of Chernov,[23] Tatarskii,[24] LaCasce,[25] and Rytov,[26] the statistics of the random wave field are strongly related to the statistical moments of the random medium. Using the perturbation method,[26,27] the analytical expressions of the statistical moments of wave field are determined. The spectral method of Shinozuka[12] is used as the random field generator owing to its simplicity and its efficiency. Using this new method, we can simulate efficiently the fluctuations part of the wave field which is related to the random properties of the medium velocity. This developed model owns the advantages to deal with the wave propagation in a three dimensional space and to lead to a very short computation time.

The developed stochastic simulation is theoretically described in the case of acoustic scalar waves but this concept can be easily extended to other kinds of propagating waves, in elastodynamics, electromagnetism, and optics. We devote Sec. II of this paper to the use of the classical determinist method based on geometrical optics in order to calculate the ray tracing and travel times of an incident plane wave in a Gaussian random field. In Sec. III, the statistical moments of travel times in random medium are first studied using the ray perturbation method. For the wave propagation in a random medium, it enables to obtain the analytical solutions for the average and the correlation functions of travel times for an incident plane wave. These analytical expressions are the main inputs of the stochastic model described in Sec. III B. And finally, in Sec. IV, the propagation field simulations provided by the deterministic method and the stochastic model are compared in a realistic application: the prediction of the acoustic beam generated by a real transducer.

II. SIMULATING THE PROPAGATION OF AN ACOUSTIC WAVE VIA A DETERMINISTIC MODEL

The method of geometrical optics is of great importance in wave-field analysis. It is a simple and attractive model to deal with a wide range of wave phenomena. It works well as long as the wavelength is small compared to the characteristic length of the heterogeneity. This method is applied here to calculate the ray-tracing of a plane wave in a random velocity medium.

A. Geometrical optics equations in a inhomogeneous fluid medium

The following is a brief derivation of the geometrical optics equations for the simplest case of the scalar monochromatic wave propagating through an inhomogeneous medium. We consider a monochromatic wave whose wave field $u(\mathbf{r})$ at a point $\mathbf{r}$ can be solved by the inhomogeneous Helmholtz equation[1]

$$\Delta u(\mathbf{r}) + k_0^2 n^2(\mathbf{r}) u(\mathbf{r}) = 0, \tag{1}$$

k_0 denotes the modulus of the average acoustic wave number vector $\mathbf{k}$. $n(\mathbf{r})$ is the spatially varying refractive index of the medium. This inhomogeneous Helmholtz equation [Eq. (1)] can describe under certain conditions waves propagation in different physical domains.[1]

For acoustic waves, the acoustic Helmholtz equation [Eq. (1)] is derived from fundamental laws[28,29] (Euler and mass conservation hydrodynamic equations, the thermodynamic state equation connecting pressure, density, and entropy) for a non-viscous fluid. Several main assumptions were then done to obtain the Helmholtz equation [Eq. (1)]. By neglecting the thermal conductivity, the sound propagation is assumed to be an adiabatic process. The material density is supposed to be spatially constant. The material properties are also independent of time. The fluid is assumed to be non-moving at rest and contains no source. Note that more general derivations than equation [Eq. (1)] for the acoustic wave equation can be found in the books of Brekhovskikh and Godin[28] and Bruneau.[29] In practice, it can be shown that the propagation of monochromatic acoustic waves in weakly turbulent fields can be accurately described by equation [Eq. (1)] if the acoustic wave length is less than a typical length scale of the turbulent field depending on its Prandtl number.[30] The equation [Eq. (1)] governs acoustic waves by setting

$$k_0 = \frac{\omega}{c_0} \quad \text{and} \quad n(\mathbf{r}) = \frac{c_0}{c(\mathbf{r})}, \tag{2}$$

where ω is the angular frequency, $c(\mathbf{r})$ represents the local velocity in the fluid medium whose average value is c_0. We are only interested in this paper in the influence of the medium velocity variation on the propagation field in the case of weakly inhomogeneous media.

As previously said, the perturbation field $\epsilon(\mathbf{r})$ defined as follows refers to small fluctuations ($|\epsilon(\mathbf{r})| \ll 1$) defined by

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2}[1 + \epsilon(\mathbf{r})], \tag{3}$$

where $c(\mathbf{r})$ represents the local velocity in the fluid medium whose average value is c_0. We consider a smooth inhomogeneous medium and suppose that $\epsilon(\mathbf{r})$ varies only slightly over the wavelength λ and has a zero average and a variance σ_ϵ^2

$$\sigma_\epsilon^2 = \mathrm{E}[\epsilon^2(\mathbf{r})], \tag{4}$$

where E[..] denotes the mathematical expectation. As $\epsilon(\mathbf{r})$ is a set of random values in space, the dependency between these values can be described by a function of the distance separating them in space using the covariance function $C_\epsilon(\mathbf{h})$. For a reference point $\mathbf{r}_1$, the relation between the value on this point and that on another point $\mathbf{r}_2$ can be quantified by

$$C_\epsilon(\mathbf{h}) = \mathrm{E}[\epsilon(\mathbf{r}_1)\epsilon(\mathbf{r}_2)] = \mathrm{E}[\epsilon(\mathbf{r}_1)\epsilon(\mathbf{r}_1+\mathbf{h})], \tag{5}$$

where $\mathbf{h} = \mathbf{r}_2 - \mathbf{r}_1$ is the vector connecting the two points $\mathbf{r}_1$ and $\mathbf{r}_2$. For an isotropic homogeneous random medium, the covariance function C_ϵ does not depend on the $\mathbf{h}$ direction, but only on the vector modulus $h = |\mathbf{h}|$. By normalizing the covariance function $C_\epsilon(\mathbf{h})$ by the variance of random field $\epsilon(\mathbf{r})$, one can obtain its correlation coefficient

$$R_\epsilon(h) = \frac{\mathrm{E}[\epsilon(\mathbf{r})\epsilon(\mathbf{r}+h)]}{\sigma_\epsilon^2}, \tag{6}$$

which varies between -1 and 1 and is used to indicate the random field correlation structure. For example, for a Gaussian random field, the correlation coefficient is given by

$$R_\epsilon(h) = e^{-(h^2/l_\epsilon^2)}, \tag{7}$$

where l_ϵ is called the characteristic length of this random field, and it is also an indicator of the distance inside which two random variables can be correlated.

The inhomogeneous propagation medium is assumed to be weakly and also smoothly variable. Since $\epsilon(\mathbf{r})$ is supposed to vary only slightly over the wavelength λ, we can assume that the field u at each point can be locally approximated by a plane wave

$$u(\mathbf{r},\omega) \simeq U(\mathbf{r})e^{-\mathrm{i}\omega T(\mathbf{r})}, \tag{8}$$

where $U(\mathbf{r})$ denotes the amplitude, $T(\mathbf{r})$ is the local travel time and ω is the pulsation. The smaller the ratio of the wavelength to the characteristic scale of inhomogeneity l_ϵ is, the better the high frequency approximation Eq. (8) is. Substituting the solution Eq. (8) into the wave equation [Eq. (1)] leads to two equations independent of frequency, the eikonal equation and the transport equation[1]

$$(\nabla T)^2 = c^{-2} = c_0^{-2}[1+\epsilon], \tag{9a}$$

$$2\nabla U \cdot \nabla T + U\nabla^2 T = 0, \tag{9b}$$

where

$$\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

By solving the eikonal equation [Eq. (9a)], the rays trajectories and the times of flight are obtained, whereas the amplitude is given by the transport equation [Eq. (9b)]. In our study, we are only interested in the time of flight $T(\mathbf{r})$ of the propagation field. Different mathematical and numerical approaches have been proposed to solve the eikonal equation and implemented for various applications; the most popular one is called the characteristic method.[31]

According to the characteristic method, the eikonal equation can be transformed into a system of ordinary differential equations system[2] including the derivatives with respect to time of both the spatial location $\mathbf{r}$ of a point along the ray and the slowness vector $\mathbf{s}(\mathbf{r}) = \nabla T(\mathbf{r})$

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}T} = c^2(\mathbf{r})\mathbf{s}(\mathbf{r}), \tag{10a}$$

$$\frac{\mathrm{d}\mathbf{s}(\mathbf{r})}{\mathrm{d}T} = -\frac{\nabla c(\mathbf{r})}{c(\mathbf{r})}. \tag{10b}$$

The two differential equations are known as the kinematic ray tracing system, where d T is the time step along the ray, $c(\mathbf{r})$ denotes the velocity at point $\mathbf{r}$ and $\nabla c(\mathbf{r})$ its spatial derivative. The slowness vector can also be defined as

$$\mathbf{s}(\mathbf{r}) = \frac{\mathbf{t}}{c(\mathbf{r})}, \tag{11}$$

where $\mathbf{t}$ is the vector tangent to the ray and normal to the phase front $T = const$.

The numerical procedures for the solution of a system of ordinary differential equations of the first order with specified initial conditions (i.e., the acoustic source position $\mathbf{r}_0$ and the initial slowness vector $\mathbf{s}_0$) are well known. Once the solution of system Eqs. (10a) and (10b) is determined by the Runge-Kutta method, the ray path $\mathbf{r}_0 \to \mathbf{r}'$ ($\mathbf{r}'$ is a current point along the ray) can be obtained. Such a procedure is usually called ray tracing. Travel time at the current point $T(\mathbf{r}')$ can be simply computed along the ray path $\mathbf{r}_0 \to \mathbf{r}'$ as follows:

$$T(\mathbf{r}') = \int_{\mathbf{r}_0}^{\mathbf{r}'} \mathrm{d}T. \tag{12}$$

B. Ray tracing and travel time in random media

Setting the Cartesian coordinate system in which $\mathbf{r} = (x, y, z)$, we assume a plane wave propagating along the x direction with unit amplitude: e^{ikx} coming from the region $x < 0$, where $c(\mathbf{r}) = c_0$, and penetrating the random region $x > 0$, where the velocity $c(\mathbf{r})$ is defined by Eq. (3). In Fig. 1, an example of simulated Gaussian random velocity map is represented in a two dimensional area $[x/l_\epsilon, y/l_\epsilon]$; in this realization

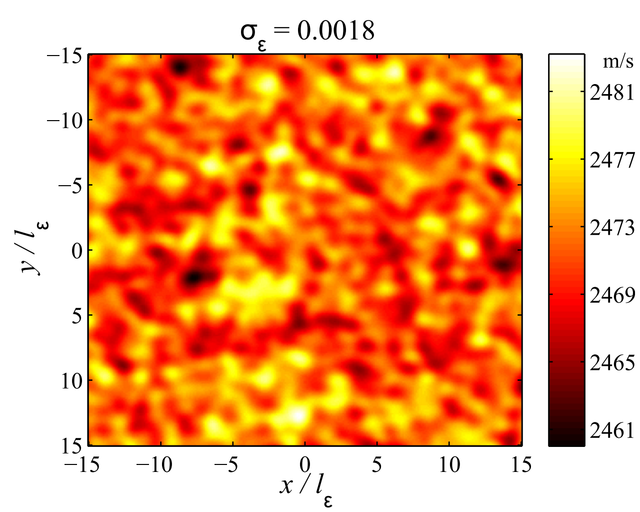

FIG. 1. Realization of a Gaussian 2D random field, for a characteristic length $l_\epsilon = 0.1$ m. Velocity magnitude, represented by the color code, varies from 2461 m/s to 2481 m/s.

of velocity map, the modeled velocity is the sound celerity which varies from 2461 m/s to 2481 m/s and its standard deviation is $\sigma_\epsilon = 1.8 \times 10^{-3}$. Figs. 2 and 3 show for an incident plane wave two examples of ray tracing in two different media with a Gaussian random velocity field whose standard deviations are respectively $\sigma_\epsilon = 1.8 \times 10^{-3}$ and $\sigma_\epsilon = 3.5 \times 10^{-3}$, whose characteristic lengths l_ϵ are 0.1 m. 80 rays, which are initially linear trajectories, are launched with a regular step of 0.03 m in the transversal direction. The rays are plotted in a plane (x, y) and in terms of the non-dimensional variables x/l_ϵ and y/l_ϵ. The ray method allows to model ray bending in inhomogeneous media: modeled rays are curved owing to the medium velocity inhomogeneities. The rays are randomly distorted by the velocity fluctuations and this phenomenon increases for a more important standard deviation σ_ϵ as observed by comparing the two figures. The rays disturbance increases with the propagation distance increase for a given characteristic length l_ϵ as shown by the figures; on the contrary, at a fixed position, smaller the characteristic length l_ϵ is, more important the disturbance of rays is. Caustics (due to multi-pathing) and shadow areas (in which no ray penetrates) may occur at large distances from the source as shown in Figs. 2 and 3; see also Ref. 32 for an analysis of the caustics occurrence in turbulent media. The ray method is no longer valid to calculate the field amplitude at caustics where it is predicted infinite. On the other hand, as explained later, the occurrence of caustics does not alter the validity of the stochastic model developed in Sec. III.

C. Fluctuation of travel times in random media

In order to find out how the travel times can be disturbed by the random velocity field, a screen is positioned perpendicularly to the initial propagation direction and then the travel times are recorded when the rays reach the screen. Screens have been set in six positions from $x/l_\epsilon = 10$ to $x/l_\epsilon = 60$. For each given normalized distance x/l_ϵ, the time

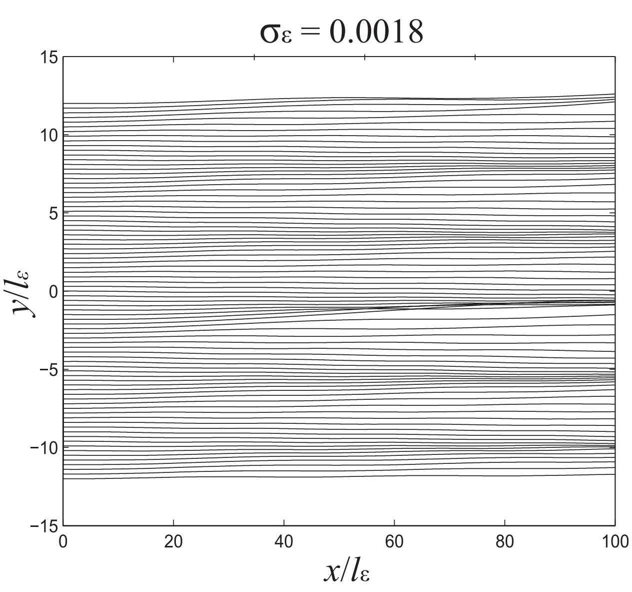

FIG. 2. Ray tracing through a Gaussian random field with $\sigma_\epsilon = 1.8 \times 10^{-3}$, $x/l_\epsilon = 0.1$m.

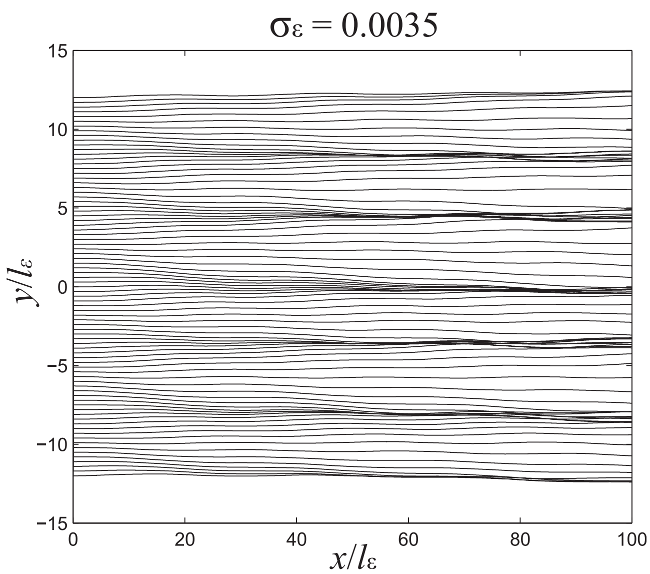

FIG. 3. Ray tracing through a Gaussian random field with $\sigma_\epsilon = 3.5 \times 10^{-3}$, $x/l_\epsilon = 0.1$ m.

fluctuations $T'(y) = T(y) - \mathrm{E}[T(y)]$ are shown in Fig. 4 with the following Gaussian random medium: $\sigma_\epsilon = 1.8 \times 10^{-3}$ and $l_\epsilon = 0.1$m. As to the results shown in Fig. 4, after propagation in a random medium (whose average velocity c_0 is 2472 m/s), the travel time field becomes itself a random process, whose values vary from $-0.5\,\mu$s to $0.5\,\mu$s for a propagation distance shorter than 6 m. The results also show that the time fluctuations are faster when the distance x/l_ϵ increases. There are transversal correlations of travel time between neighboring rays since the curves are continuous. The dependencies of the travel time fluctuations between two neighboring x/l_ϵ positions can be also observed and are related to the existence of a time longitudinal correlation.

In fact, the geometrical optics method is time consuming while it is used to calculate the wave propagation for a long distance in an inhomogeneous medium. And for the wave propagation in a random medium, a single result is not representative; hence one needs to generate a large number of random velocity fields in which ray tracings are carried out. Therefore, a deterministic model is not convenient for providing a fast simulation of the propagation in a random medium. That is why we propose in the following a stochastic model. This model does not need as input a deterministic velocity field but just the statistical properties of the random field.

III. SIMULATING THE PROPAGATION OF AN ACOUSTIC WAVE IN RANDOM MEDIA VIA A STOCHASTIC MODEL

A new fast technique to simulate the propagation of an acoustic wave in a random medium is presented in this section. First, the wave propagation in a mean homogeneous medium is calculated using simple geometrical optics, and the velocity of the homogeneous medium is chosen as the average value c_0 of the real random medium. We assume that the fluctuations of velocity field are quite weak $(\sigma_\epsilon \ll 1)$.

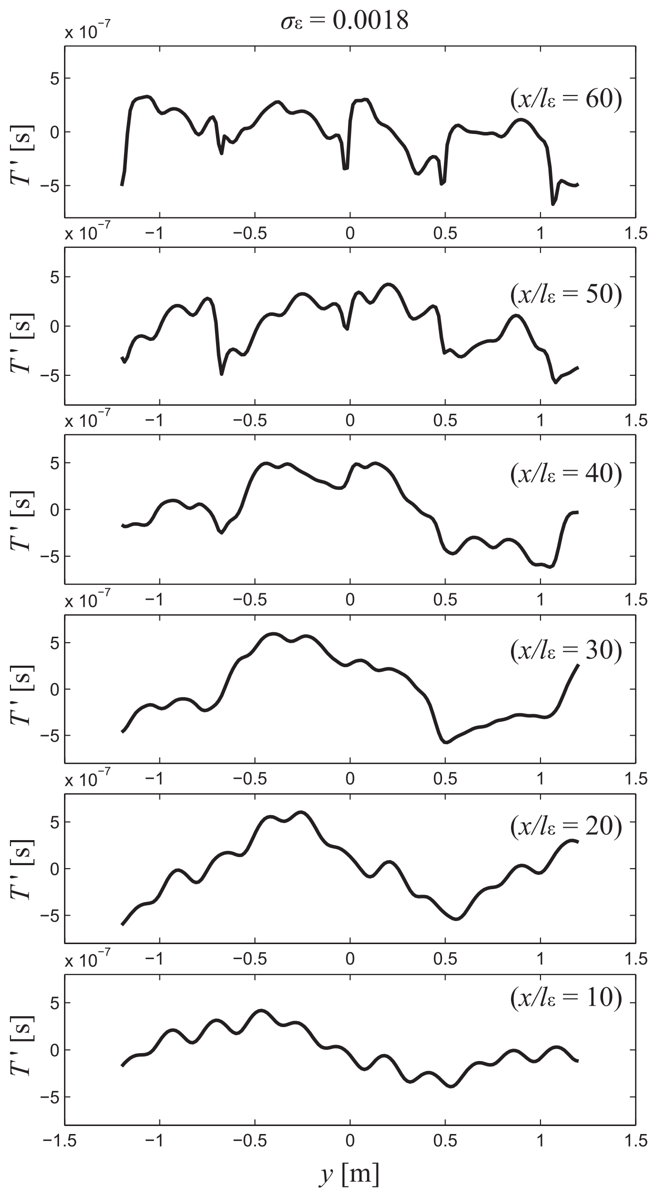

FIG. 4. Travel time fluctuations for different propagation distances, calculated by the geometrical optics method.

The developed stochastic propagation model consists in keeping the amplitudes modeled by geometrical optics in the mean homogeneous velocity medium and in only modifying the corresponding travel times by a set of travel time corrections provided by a random field generator. The principle of this stochastic propagation model is summed up by the following formula:

$$T(\mathbf{r}) = \mathrm{E}[T(\mathbf{r})] + T'(\mathbf{r}), \tag{13}$$

where $T(\mathbf{r})$ is the total travel time, $\mathrm{E}[T(\mathbf{r})]$ the time average in the mean homogeneous medium, and $T'(\mathbf{r})$ is the random fluctuation. These time corrections are randomly generated but they must respect the statistical moments (average, variance, correlation functions) of the travel times in a random medium. In fact, these moments are the inputs of the random field generator and are function of the statistical properties of the random medium.

A. Inputs of the stochastic model: statistical moments of the travel time fluctuations

The theoretical relationships between the statistical moments of travel times and the statistical moments of the random velocity field are established by using the perturbation method.[26,27] This theory shows that the average and variance of travel times obtained after propagation of an incident plane wave in a random medium vary with the propagation distance x. An approximation of the average is

$$\mathrm{E}[T(x)] = \frac{x}{c_0} - \sqrt{\pi}\frac{\sigma_\epsilon^2}{8c_0}\frac{x^2}{l_\epsilon} + O(x^3\epsilon^2), \tag{14}$$

and the variance is also approximated as

$$\sigma_T^2(x) = \frac{\sqrt{\pi}}{4}\frac{\sigma_\epsilon^2}{c_0^2}l_\epsilon^2\frac{x}{l_\epsilon} + \frac{\pi}{32}\frac{\sigma_\epsilon^2}{c_0^2}l_\epsilon^2\left(\frac{x}{l_\epsilon}\right)^4 + O(x^9\epsilon^4). \tag{15}$$

To model travel time correlations similar to those modeled in Fig. 4 using geometrical optics, we use as inputs of the stochastic model the correlation coefficient [Eq. (6)]. For a plane wave, the correlation coefficient in the "transversal" plane (see Fig. 5) is different from that in the propagation ("longitudinal") direction. In the transversal plane, the travel time correlation coefficient is the same as that of the velocity field, so the corresponding transversal correlation function is also an isotropic function, given by

$$C_T(\rho) = \sigma_T^2(x)R_\epsilon(\rho), \tag{16}$$

where $\sigma_T^2(x)$ is the travel time variance defined in Eq. (15), $R_\epsilon(\rho)$ is the velocity correlation coefficient given by Eq. (6), and ρ is the distance between two different points $\mathbf{r}_{\perp 1}, \mathbf{r}_{\perp 2}$ in the transversal plane (see Fig. 5): $\rho = \|\mathbf{r}_{\perp 2} - \mathbf{r}_{\perp 1}\|$. So the characteristic length in the transversal plane is the same as that of the velocity medium: $l_\perp = l_\epsilon$, since their correlation coefficients are the same $R_\perp(\rho) = R_\epsilon(\rho)$, which is shown in Fig. 6.

The longitudinal correlation function, which characterizes the travel time correlation along the propagation direction, is not an isotropic function but depends on the direction

$$C_L(x_1, x_2) = \frac{x_<}{4c_0^2}\sigma_\epsilon^2\sqrt{(\pi)}l_\epsilon, \tag{17}$$

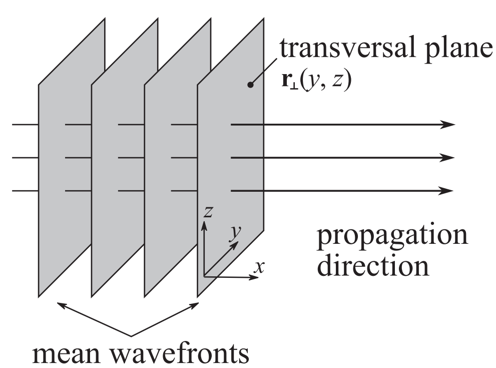

FIG. 5. Definitions of the "transversal plane" and of the "longitudinal" propagation direction.

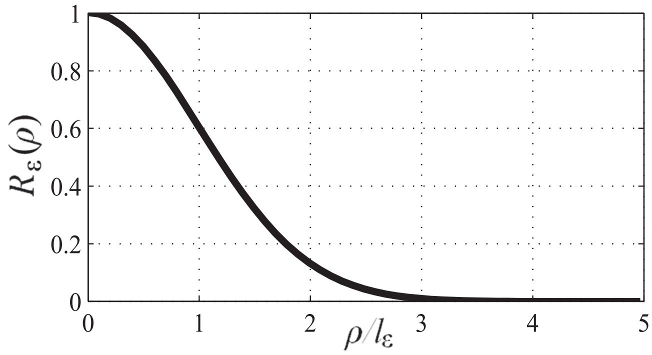

FIG. 6. Transversal correlation coefficient of travel time.

where x_1 and x_2 are two propagation distances and

$$x_< = \min(x_1, x_2). \tag{18}$$

The longitudinal correlation coefficient is obtained by

$$R_L(x_1, x_2) = \frac{C_L(x_1, x_2)}{\sigma_T(x_1)\sigma_T(x_2)}, \tag{19}$$

hence

$$R_L(x_1, x_2) = \frac{x_<}{\sqrt{x_1 x_2}} = \begin{cases} \sqrt{x_2/x_1} & \text{for } x_2 < x_1 \\ \sqrt{x_1/x_2} & \text{for } x_2 > x_1 \end{cases}. \tag{20}$$

For a given propagation distance x_1, the evolution of versus x_2 is presented in Fig. 7.

Once the analytical expressions of travel times statistical moments are obtained, we can use them as inputs of a random field generator to generate a set of travel time corrections.

B. Principle of the phase aberration for the stochastic simulation of the propagation in random media

In this section, we are going to be interested in the stochastic simulation of the propagation in random media using a random generator. Importing the travel time statistical moments in the random generator, the fluctuation parts of travel times in the whole space can be obtained, which enables to simulate the influence of the random velocity. The random field generator presented here is based on the spectral representation method,[12] simple to implement and very efficient. First, a two-dimensional random time field in the transversal plane of a plane wave propagation is generated. In this transversal plane, the travel time field is isotropic as the velocity field. The algorithm will then be extended to the three-dimensional case.

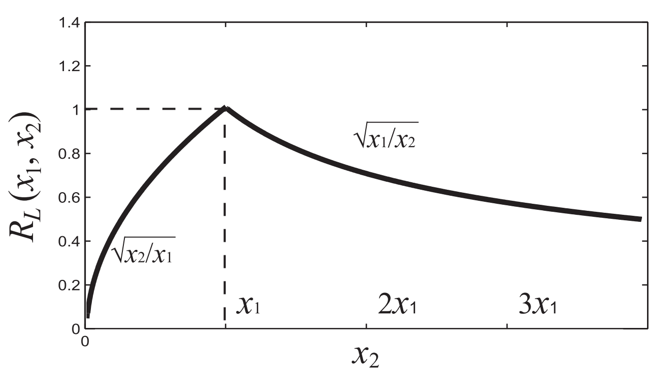

FIG. 7. Longitudinal correlation coefficient R_L of travel time versus x_2.

1. Spectral representation of a two-dimensional process

During incident plane wave propagation into a random medium, a transversal two-dimensional screen $\mathbf{r}_\perp(y, z)$ (see Fig. 5) is set in order to measure the arrival time of the wave on every point of this screen. By locating this screen at different positions (propagation distances), a two-dimensional travel time field $T(\mathbf{r}_\perp)$ on the screen $\mathbf{r}_\perp$ can be obtained and we consider the fluctuation part $T'(\mathbf{r}_\perp)$ of the travel time [defined in Eq. (13)] as a two-dimensional isotropic random process. In fact, by using the formula proposed by Shinozuka and Jan,[12] the random time field $T'(\mathbf{r}_\perp)$ can be simulated by a series of cosine functions weighted by functions A_n

$$T'(\mathbf{r}_\perp) = \sqrt{2}\sum_{n=1}^{N} A_n \cos(\xi_n \cdot \mathbf{r}_\perp + \varphi_n), \tag{21}$$

where $\xi_n = (\xi_{n1}, \xi_{n2})$ is the wave vector, and

$$A_n = \sqrt{S_{T_\perp}(\xi_n)\Delta\xi_n}, n = 1, 2, \cdots, N, \tag{22}$$

where $S_{T_\perp}(\xi)$ is the so-called spectral density function of the two-dimensional random process, and $\Delta\xi_n$ denotes the wave vector step. In Eq. (21), each cosine function can be considered as a real Fourier mode corresponding to a simple plane wave (see Fig. 8) characterized by its wave vector ξ_n (whose angle with respect to the y-axis is θ_n), its phase φ_n and its amplitude A_n. We chose to define the modulus of each mode $\xi_n = |\xi_n|$ linearly distributed in the range $[\xi_{\min}, \xi_{\max}]$ with the step $\Delta\xi$

$$\xi_n = \xi_{\min} + (n-1)\Delta\xi, \tag{23}$$

with $\Delta\xi = (\xi_{\max} - \xi_{\min})/(N-1)$ and $\xi_{\max} = -\xi_{\min}$. We define here that $\Delta\xi_n = \Delta\xi$.

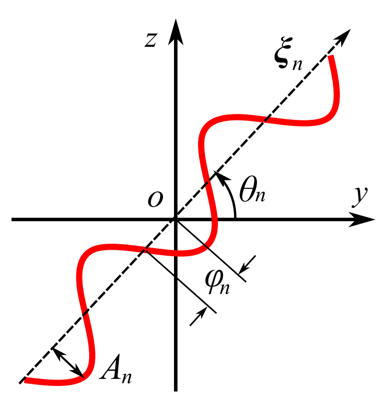

FIG. 8. Single Fourier mode characterized by its wave vector ξ_n whose direction is specified by θ_n, its amplitude A_n, and its phase φ_n.

To generate a two-dimensional homogeneous isotropic random field, the spatial distribution of those Fourier modes should be uniform. Therefore, the direction θ_n of each mode should be uniformly and randomly distributed from 0 to 2π; its probability density function is

$$\Pr[0 \le \theta_n \le 2\pi] = \frac{1}{2\pi}, \tag{24}$$

and the distribution for φ_n is also homogeneous from 0 to π. The spectral density function of the two-dimensional random process is given by the Fourier transform of the correlation function $C_T(\rho)$ defined by Eq. (16)

$$S_T(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\mathrm{i}\rho\cdot\xi} C_T(\rho)\,\mathrm{d}\rho, \tag{25}$$

with $\rho = |\mathbf{r}_{\perp 2} - \mathbf{r}_{\perp 1}|$ is the distance between two points in the transversal plane. For a Gaussian random velocity field, the coefficient $R_\epsilon(\rho)$ of the correlation function $C_T(\rho)$ takes the form of Eq. (6) and the equation becomes

$$S_T(\xi) = \frac{\sigma_T^2}{2\pi}\int_{-\infty}^{\infty} e^{-(\rho/l_\epsilon)^2} e^{-\mathrm{i}\rho\cdot\xi}\,\mathrm{d}\rho = \frac{\sigma_T^2 l_\epsilon^2}{4\pi} e^{-\frac{\xi^2 l_\epsilon^2}{4}}, \tag{26}$$

which is also a Gaussian function. Since the modes directions are uniformly distributed in space [Eq. (24)], the spectral density function is isotropic, then we have

$$S_T(\xi) = S_T(\xi), \tag{27}$$

where $\xi = |\xi|$. The normalized spectral density function of a Gaussian random field $S_T(\xi)/\sigma_T^2$ is shown in Fig. 9. Applying the random generator corresponding to Eq. (21), a simulation of travel time fluctuation on a two-dimensional transversal plane located at $x = 30 l_\epsilon$ is shown in Fig. 10. The continuity of travel times which are modeled are consistent with that obtained with the deterministic simulation (see Fig. 4). There are two major types of errors introduced by the use of this stochastic simulation. The first is due to the estimation of the statistics of the underlying random field from a limited data set. Here we suppose that an incident plane wave propagates in an ideal random velocity field (e.g., the Gaussian random field) for which the analytical expression of the statistics exists. The second type of errors is called "within simulation error"[14] and these errors are caused by the choice of the frequency step $\Delta\xi$ and the definition of the spectral range $[\xi_{\min}, \xi_{\max}]$. Numerically, the spectral range can be defined as the boundaries of the spectral density function of the two-dimensional process shown in Fig. 9. To study the spectral range influence on the accuracy of the stochastic simulation, we can compare the theoretical two-dimensional covariance function given by Eq. (16) with the simulated covariance function. In the used configuration, $\xi_{\max}$ varies from $2/l_\epsilon$ to $8/l_\epsilon$; the frequency step is fixed for $\Delta\xi = 0.16/l_\epsilon$. The results are illustrated in Fig. 11: the spectral range of the converged results matches well the spectral range of the spectral density function (see Fig. 9) that means $|\xi_{\max}| \ge 6/l_\epsilon$. In fact, outside the spectral range $S_T(\xi_n) = 0$ (see Fig. 9), then $A_n = 0$ (according to Eq. (22) the mode n has no contribution to the series Eq. (21)). There is no need to take into account the modes outside the spectral range. We chose to fix the spectral range as follows: $-6/l_\epsilon = \xi_{\min} < \xi_n < \xi_{\max} = 6/l_\epsilon$. Once the spectral range is fixed, the influence of the step $\Delta\xi$ on the simulation accuracy can be examined. Fig. 12 shows that for $\Delta\xi = 2.47/l_\epsilon$ (the corresponding modes number is $N = 6$) the accuracy is very poor; for $\Delta\xi = 1.65/l_\epsilon$ ($N = 9$) the accuracy is satisfying and for $\Delta\xi = 0.70/l_\epsilon$ ($N = 18$), the accuracy is excellent. So we have chosen $\Delta\xi = 0.70/l_\epsilon$.

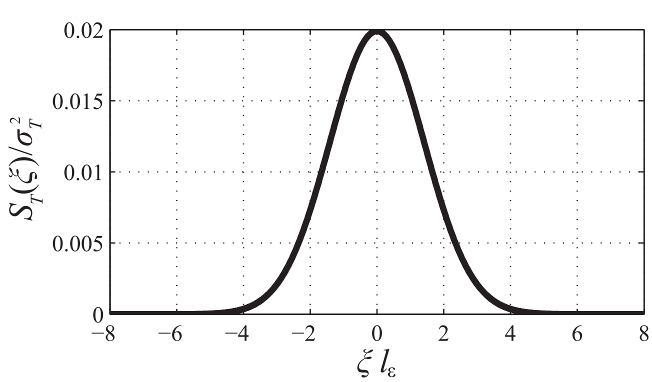

FIG. 9. Normalized spectral density function $S_T(\xi)/\sigma_T^2$ of a Gaussian process.

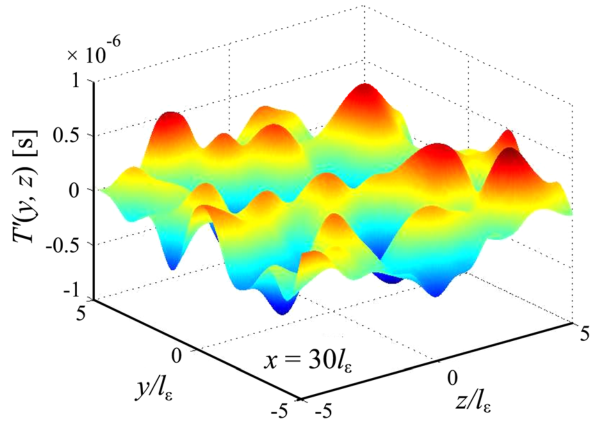

FIG. 10. Realization of travel time fluctuation on a screen for $x = 30\,l_\epsilon$, $l_\epsilon = 0.1$ m, and $\sigma_\epsilon = 0.0018$.

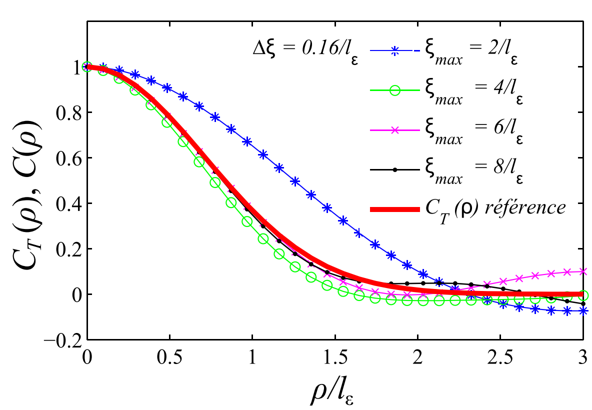

FIG. 11. Comparison between the theoretical (red line) and simulated covariance functions obtained for $\xi_{\max} = 2/l_\epsilon$ (blue), $4/l_\epsilon$ (green), $6/l_\epsilon$ (purple), $8/l_\epsilon$ (black), and $\Delta\xi = 0.16/l_\epsilon$.

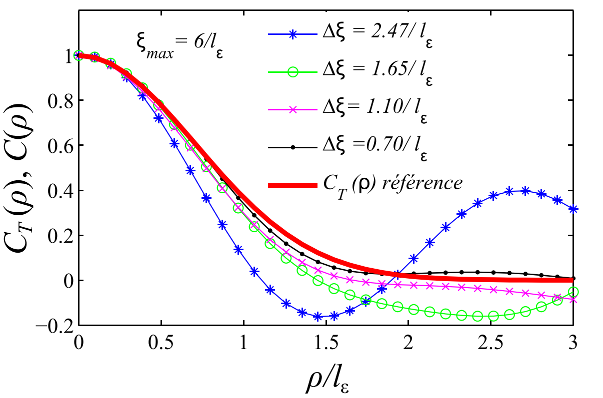

FIG. 12. Comparison between the theoretical (red line) and simulated covariance functions obtained for $\Delta\xi = 2.47/l_\epsilon$ (blue), $1.65/l_\epsilon$ (green), $1.10/l_\epsilon$ (purple), $0.70/l_\epsilon$ (black), and $\xi_{\max} = 6/l_\epsilon$.

2. Three-dimensional random process simulation

To simulate the travel time of an incident plane wave in a three dimensional space ($T'(\mathbf{r}_\perp) \to T'(\mathbf{r})$), the spatial correlation of travel times between two neighboring points lying on the longitudinal direction should be taken into account. According to Eq. (15), the travel time variance increases with the propagation distance x and in the longitudinal direction, the time covariance function is different from that in the transversal plane. Consequently, the three dimensional simulation cannot be generated by a simple extension of the two dimensional process but the modeling of a three dimensional non homogeneous anisotropic random field is needed. The Fourier transform Eq. (26) takes the new form

$$S_T(\xi) = \frac{1}{(2\pi)^2}\int\int_{\mathbb{R}^2} e^{-\mathrm{i}\mathbf{r}\cdot\xi} C_T(\mathbf{r})\,\mathrm{d}\rho\,\mathrm{d}x, \tag{28}$$

where $\mathbf{r}(\rho, x)$ denotes the three dimensional coordinates and where the correlation function $C(\mathbf{r})$ is no longer isotropic since the travel time correlation in the transversal plane is different from that in the propagation direction. We need to take into account the longitudinal correlation function $C_L(x_1, x_2)$ into the three dimensional random field simulation. But $C_L(x_1, x_2)$ is a non symmetrical function and there is no analytical expression of its spectral density function. Thus, an approximation is done, which consists in replacing it by a symmetrical Gaussian function and in choosing an approximated longitudinal characteristic length $l_\parallel$

$$C_L(x_1, x_2) \approx C_L^*(h) = \sigma_T^2 e^{-h^2/l_\parallel^2}, \tag{29}$$

where $h = |x_2 - x_1|$. Then the total correlation function is written as

$$C(\mathbf{r}) = C_T(\rho) C_L^*(h). \tag{30}$$

By applying one dimensional Fourier transform of Eq. (29), the spectral density function in the longitudinal direction is given by

$$S_L(\xi) = \frac{\sigma_T^2 l_\parallel}{\sqrt{\pi}} e^{\frac{(\xi l_\parallel)^2}{4}}, \tag{31}$$

which is also a Gaussian function. Hence the total spectral density function can be obtained by combining the transversal spectral density function and the longitudinal one

$$S(\xi) = S_T(\xi_1, \xi_2) S_L(\xi_3) = \frac{|\xi| l_\epsilon^2 l_\parallel}{2\sqrt{\pi}} e^{\frac{(\xi_1 l_\parallel)^2 + (\xi_2 l_\epsilon)^2 + (\xi_3 l_\epsilon)^2}{4}}, \tag{32}$$

with the projections of ξ on the three axis defined by

$$\begin{cases} \xi_1 = |\xi| \cos\beta \\ \xi_2 = |\xi| \cos\alpha \sin\beta, \\ \xi_3 = |\xi| \sin\alpha \sin\beta \end{cases} \tag{33}$$

where $\alpha \in [0, 2\pi[$ denotes the azimuth angle and $\beta \in [0, \pi]$ the polar angle (see Fig. 13).

Then the three-dimensional travel time fluctuations are modeled by

$$S(\xi) = S_T(\xi_1, \xi_2) S_L(\xi_3) = \frac{\sigma_T^2 l_\epsilon^2 l_\parallel}{8(\pi)^{3/2}} e^{\frac{(\xi_1 l_\parallel)^2 + (\xi_2 l_\epsilon)^2 + (\xi_3 l_\epsilon)^2}{4}}. \tag{34}$$

The variance of the time fluctuation increases with the propagation distance x and the three-dimensional phase coherence is insured by the correlation function [Eq. (30)]. The time fluctuations calculated by the stochastic model with Eq. (30) on screens at six positions from $x/l_\epsilon = 10$ to $x/l_\epsilon = 60$ (as done using the geometrical optics in Fig. 4) are shown in Fig. 14. For the results at one given position, the curve smoothness represents the transversal correlation of time fluctuations in the transversal plane, which is consistent with the results shown in Fig. 4. Then we calculate the variances of time fluctuations at each position and compare in Table I with those calculated from the deterministic model results (shown in Fig. 4) and the theoretical values given by Eq. (15). As the theoretical prediction, the time fluctuations calculated by the stochastic model increase with the propagation distance, and the variance growths of these three models are consistent. Finally, the correlation between curves of two successive positions can be observed in Fig. 14, as already seen with the deterministic results shown in Fig. 4. This consistency comes from the account of the longitudinal correlation in the stochastic modeling [Eq. (34)].

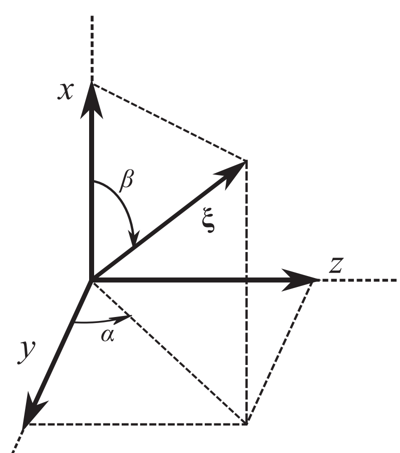

FIG. 13. Wave vector geometry of a single Fourier mode.

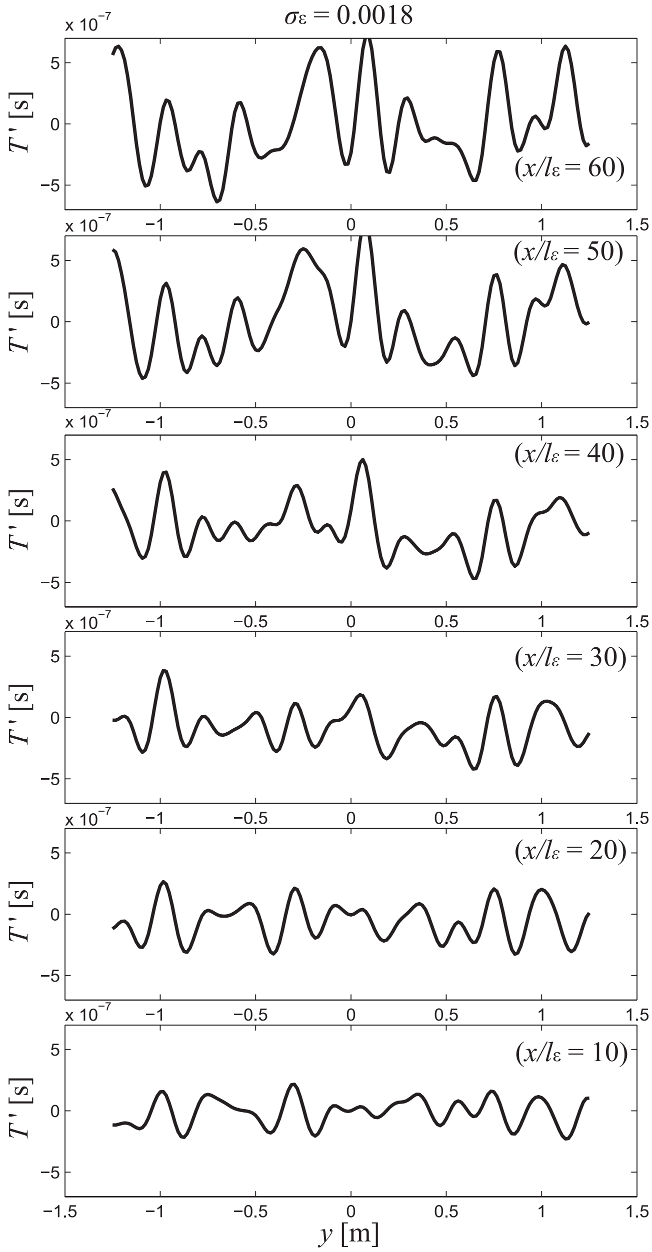

FIG. 14. Travel time fluctuations for different propagation distances, calculated by the stochastic simulation. The standard deviations of the random velocity field is $\sigma_\epsilon = 0.0018$, the transversal characteristic length is $l_\perp = 0.01m\,(l_\perp = l_\epsilon)$, and the longitudinal characteristic length is $l_\parallel = 3m$.

We have presented here the principle of the stochastic model and its application to an incident plane wave in a random medium. The statistical behaviors of the travel time fluctuations calculated by the stochastic model by taking into account the theoretical time variance and correlation functions are consistent with the results given by the deterministic model.

It has to be noticed that caustics will not call into question the validity of this stochastic model. In the mean homogeneous velocity medium used for the amplitude calculation, caustics will not occur, contrary to an inhomogeneous medium, if we assume an unbounded medium including no large scattering object able to focus rays by reflection or refraction. The presence of caustics does also not interfere

TABLE I. Comparison of variances of time fluctuations ($\times 10^{-13}$s^2) calculated on screens at different positions from $x/l_\epsilon = 10$ to $x/l_\epsilon = 60$, knowing that the travel time averages on the screens vary from 0.40×10^{-3}s to 2.40×10^{-3}s. Theoretical values given by Eq. (15); variances of deterministic model calculated from the time fluctuations shown in Fig. 4; variances of stochastic model calculated from the time fluctuations shown in Fig. 14.

Position x/l_ϵ	10	20	30	40	50	60
Theoretical model	0.90	1.84	2.90	4.25	6.60	8.63
Deterministic model	0.42	0.97	1.18	3.06	6.82	7.95
Stochastic model	0.65	1.36	2.06	2.98	7.03	9.52

with the statistical results of the propagation field in terms of travel times (analytical expressions Eqs. (14) and (15)). The used correlation functions are Gaussian ensuring a spatial smoothing of the modeled travel times.

This stochastic model is applied in Sec. IV to a practical case, the realistic acoustic beam generated by a transducer.

IV. APPLICATION: SIMULATION OF THE ACOUSTIC WAVE BEAM EMITTED BY A PLANE TRANSDUCER

Ultrasonic techniques are widely used notably in nondestructive evaluation (NDE) in the aim of characterizing defects[33] or in telemetry for locating immersed objects.[34] Modeling is crucial to ensure the ability of the used echographical methods. The simulation of a whole echography mainly requires two modelling steps, one dedicated to transducer radiation and one to scattering from flaws in solids[33,35] or from immersed targets.[34] The acoustic/elastodynamic beam radiated by a transducer in a fluid or in an isotropic, anisotropic and/or heterogeneous solid specimen has been modeled by Gengembre and Lhémery[36,37] using geometrical optics. Using both the deterministic model (geometrical optics[1,36]) and the stochastic model, the acoustic beam propagation can be simulated in our studied random velocity field. We will see that the stochastic model is much faster than geometrical optics. The results calculated by geometrical optics are taken as references; by comparing them with the results obtained with the stochastic model, the stochastic model will be validated.

The acoustic wave beam generated by a transducer and then propagating in a fluid can be modeled by the radiation of a distribution of particle velocity sources at points $\mathbf{r}_0(y', z')$ over the surface P_0. The Rayleigh integral describes the acoustic scalar field potential $\varphi_v(\mathbf{r}, t)$ at the considered field observation point $\mathbf{r}$ due to the sources $v(t)$ located on the radiating surface P_0 (see Fig. 15)

$$\varphi_v(\mathbf{r}, t) = v(t) \otimes \int\int_{\mathbf{r}_0 \in P_0} \frac{\delta(t - T(\mathbf{r}))}{2\pi|\mathbf{r} - \mathbf{r}_0|} d\mathbf{r}_0, \tag{35}$$

where $v(t)$ is the uniform particle velocity on the surface P_0, $\otimes$ represents the convolution, $|\mathbf{r} - \mathbf{r}_0| = [x^2 + (y - y')^2 + (z - z')^2]^{1/2}$ denotes the acoustic path between a source point $\mathbf{r}_0(0, y', z')$ and an observation point $\mathbf{r}(x, y, z)$ and this path gives rise to a delay $T(\mathbf{r}) = |\mathbf{r} - \mathbf{r}_0|/c_0$ in a homogeneous fluid medium of sound velocity c_0. Then the acoustic beam field is modeled as the sum of contributions due to

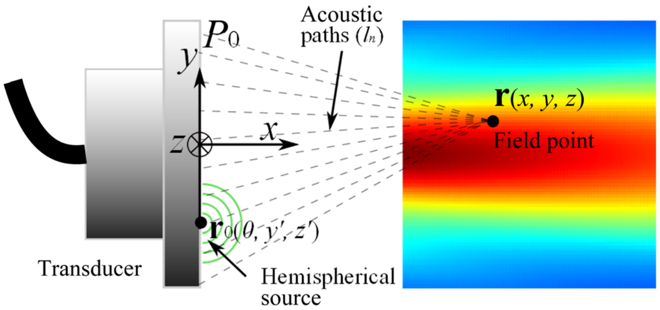

FIG. 15. Transducer of radiating surface P_0 and acoustic beam prediction; each contribution on the transducer surface is a hemispherical source which is linked to the field point by acoustic paths l_n.

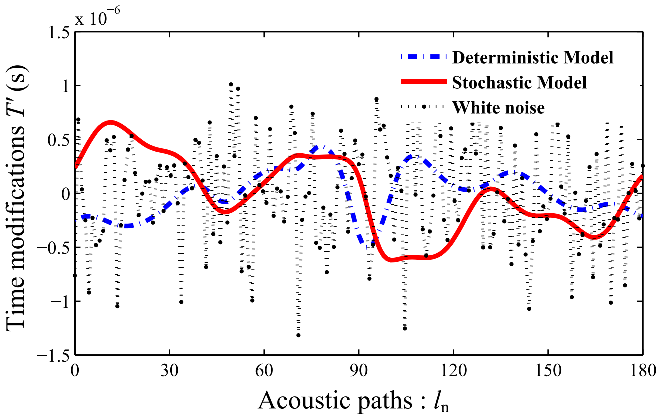

FIG. 16. Time fluctuations for each contribution on the transducer surface (Fig. 15) obtained by three kinds of methods: the deterministic model (geometrical optics), the stochastic model, and the white noise generator.

hemispherical sources over the whole radiating surface (see Fig. 15).

Fig. 15 shows the example of the acoustic field calculated in a homogeneous medium using the Rayleigh integral for a $\varnothing$30 mm circular transducer. The source signal is a Gaussian modulated sinusoidal pulse whose center frequency is 2 MHz and bandwidth is 60%.

To calculate the acoustic beam in an inhomogeneous medium, the acoustic paths are calculated using the geometrical optics model (use of Eqs. (10a) and (10b)) for a given inhomogeneous velocity map as shown in Fig. 1. Compared to a homogeneous medium, these acoustic paths are no longer straight lines as the rays illustrated in Figs. 2 and 3 and the travel time of each path will also be modified. The impulse response (IR) at each field point is the synthesis of all contributions from the radiating surface. Since the interferences of these contributions depend on their respective amplitude and time of flight delay (time difference between each contribution), the modification of the travel time from a homogeneous to an inhomogeneous medium can cause destructive interferences when synthesizing the final IR at each point. Therefore, the amplitude at one point can be randomly changed from one realization of the medium velocity field to another. For a large number of calculations at one point carried out with different independent realizations, the probability density of field amplitude is obtained at this point. This probability density is strongly related to the travel time of each contribution, which can be explained by the example below. This travel time delay can be modeled using three different methods: the deterministic model (geometrical optics), the stochastic model and a white noise randomly generated without spatial correlation. Travel time fluctuations for each method and for one single realization are shown in Fig. 16. We can see that the time fluctuations generated by the deterministic and the stochastic models are both smooth due to the time spatial correlations, contrary to those provided by the white noise generator. The spatial continuity of these time fluctuations can influence the final field amplitude on the observation point because of the phase interferences.

After 500 realizations of field calculation at one point, we get the probability density of maximum amplitude related to each model shown in Fig. 17 where the red lines represent the results in a homogeneous medium and the blue bars denote the amplitude distributions in the inhomogeneous media. We note that the distributions of maximum amplitudes at the observation point predicted by the stochastic model is consistent with the results of the deterministic model and their distributions are centered on the single value obtained in the mean homogeneous medium. But with the white noise, we find that most of amplitudes are smaller than that in a homogeneous medium; it is due to the fact that the time delays modeled without spatial correlation can give rise to destructive interferences.

We now compare the deterministic model and the stochastic model for the calculation of wave beam field in a two

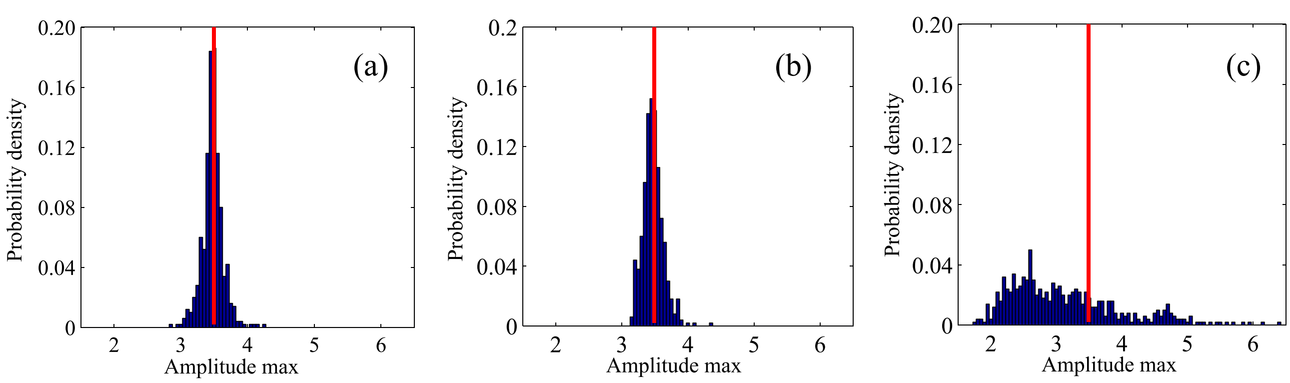

FIG. 17. Probability density of maximum amplitude at one point calculated by the (a) deterministic model, (b) stochastic model, and (c) white noise time delay.

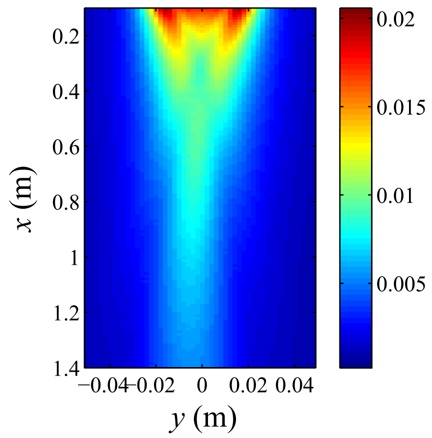

FIG. 18. Acoustic beam propagation in a inhomogeneous medium calculated by the deterministic ray tracing model.

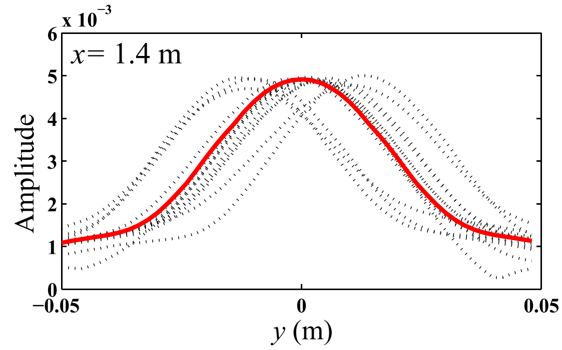

FIG. 20. Amplitude distributions along the line $x = 1.4$ m calculated by the deterministic model; the solid (red) curve denotes the result in a mean homogeneous medium and the dotted curves denotes the results in the random medium, mean deviation: ~10 mm, standard deviation of gaps: 7.3 mm.

dimensional computation zone. Fig. 18 shows an example of the acoustic beam prediction in an inhomogeneous medium calculated by geometrical optics for a single realization of random velocity field. This calculation used the same transducer as that used in Fig. 15. The random velocity field is defined by the standard deviation $\sigma_\epsilon = 0.0018$ and the characteristic length $l_\epsilon = 0.1$m. The beam has been calculated in a two-dimensional region $0.1\text{m} < x < 1.4\,\text{m}$; $-0.05\,\text{m} < y < 0.05\,\text{m}$. We notice that the main axis of the acoustic beam is no longer along the axis ($y = 0$). The inhomogeneities of the random velocity medium give rise to a slight deviation of the acoustic beam.

However, the computation time of the geometrical optics model is proportional to the number of paths, to the number of field point to calculate and to the discretization level on each path. Because of the iterative computation, the calculation of the acoustic beam by ray tracing in a random medium is much more expensive than those in a homogeneous medium. In this example, the calculating zone ($0.1\text{m} < x < 1.4\,\text{m}, -0.05\,\text{m} < y < 0.05\,\text{m}$) has 2000 points; the computation time is about 1.5 h (PC, CPU: Intel Core 2Duo CPU E6750 2.66 GHz, RAM: 3.25Go). However, the stochastic model developed can simulate faster the propagation of an acoustic beam in a random medium. This model considers that the wave propagation in the random media is the propagation field in a mean homogeneous medium corrected by some modifications of travel time. So in the stochastic modeling, the ray paths in such homogeneous medium are conserved but a correction provided by the stochastic model is operated on the travel times of these rays in order to take into account the influence of these inhomogeneities. The result shown in Fig. 19 is the stochastic simulation of the acoustic beam propagating in an inhomogeneous medium with the same statistical properties of the velocity field and the same calculation area as with the deterministic calculation of Fig. 18. The computation time for the stochastic model on the same computer is only 10 s.

We can see that the acoustic beam predicted by the stochastic model is also slightly distorted. As the stochastic model does not work on an individual velocity field but on the statistical functions of velocity, we cannot compare its results with those of the deterministic model for only one realization of velocity field. Thus, a large number of realizations should be made to compare the statistics of the results obtained with these two models.

In Figs. 20 and 21, the amplitude distributions on one horizontal line of the previous 2D computation zone are presented. The red curve represents the amplitude distribution in a mean homogeneous medium and dotted curves are the results from a large number of realizations in the random

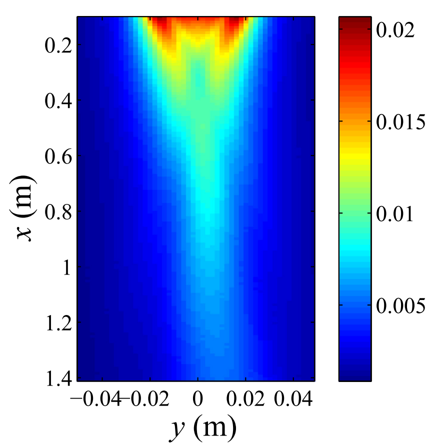

FIG. 19. Acoustic beam propagation in a inhomogeneous medium calculated by the stochastic simulation model.

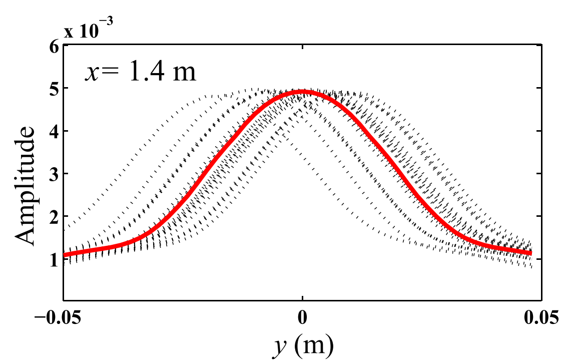

FIG. 21. Amplitude distributions along the line $x = 1.4$ m calculated by the stochastic model; the solid (red) curve denotes the result in a mean homogeneous medium and the dotted curves denotes the results in the random medium, mean deviation: ~10 mm, standard deviation of gaps: 7.6 mm.

medium. These amplitude distributions show the wave beam deviations randomly vary around the central homogeneous distribution. The stochastic model predicts wave beam deviations of the same order as the deterministic model, which means that their mean deviations are both about 10 mm for $x = 1.4$ m and the standard deviation of the gaps are respectively 7.3 mm for the deterministic model and 7.6 mm for the stochastic model.

V. CONCLUSION

A deterministic model based on geometrical optics has been devised to simulate the wave propagation in a Gaussian random celerity field. Using ray tracing, a distortion of an acoustic incident plane wave has been observed and corresponding fluctuations of travel time have also been obtained. But this model has the inconvenient to be time consuming. A new concept of high frequency wave propagation modeling in a random medium has been developed using a stochastic modeling. This model can generate the fluctuation part of the propagation field, due to the random velocity variations in the propagation medium. The inputs of the random field generator are the statistical moments of the wave field in a random medium which are determined analytically. The deterministic model and the stochastic model are both applied to calculate the propagation of a realistic acoustic beam in a Gaussian random velocity field. First, a large number of field calculations has shown that both the distribution of field amplitude and the obtained beam deviation are similar for the two models. These comparisons between these two models allow to validate the new stochastic model. This model developed here for acoustics can be extended to deal with other wave propagation phenomena. Furthermore, the stochastic model can be simply coupled with a deterministic simulation in order to model a more realistic propagation medium. For instance, a realistic medium can be decomposed into two parts: one deterministic part due to stationary inhomogeneities invariant in time (as a velocity gradient) and another part due to random fluctuations.

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