Acoustic propagation in anisotropic periodically multilayered media: A method to solve numerical instabilities

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Acoustic propagation through thick composites has become a subject of intensive study due to their application to nondestructive evaluation. The anisotropic multilayered media are now usually studied by the propagator matrix formalism. Though this formalism is very convenient, it leads to numerical instabilities for thick composites at high frequencies. These numerical instabilities come from the combination of very high exponential terms which reduces the dynamics of the calculation. A very interesting case is the one of anisotropic periodically multilayered media. The method developed in this paper uses Floquet waves which correspond to the modes of an infinite periodically multilayered medium. They are linear combinations of the real waves propagating in each layer of the medium. The Floquet wave numbers are the eigenvalues of the propagation matrix of one period of the medium. The anisotropic periodically multilayered medium can then be considered as a dummy medium in which the Floquet waves propagate. High exponential terms can be avoided through a judicious choice of reference of the Floquet waves' amplitudes. This method enabled us to calculate reflection coefficients up until 40 MHz, of thick composites of carbone/epoxy placed in water. Furthermore, it has permitted us to not have a limitation for a single layer of any given material, at any given frequency.

I. INTRODUCTION

Acoustic propagation through anisotropic multilayered media has become a subject of intensive study, because of their application to nondestructive evaluation, geophysics, etc. Generally speaking, multilayered media are made by the stacking of distinct anisotropic media. A very interesting case is the one of anisotropic periodically multilayered media which are a *P*-times reproduction of an anisotropic multilayered medium cell.

The propagator matrix formalism which was first developed by Thomson,¹ then furthered by Haskell,² and afterwards by Gilbert and Backus,³ is a convenient general method for treating such media. Periodically layered fluid media were studied by Gilbert⁴ and by Schoenberg⁵ who has also examined alternating fluid/solid layers.⁶ Helbig has analyzed transversely isotropic periodically media with the long-wavelength approximation.⁷ Further, Gatignol, Rousseau, and Moukemaha have studied the propagation in an isotropic periodically stratified medium⁸⁻¹⁰ for a given incidence. They have obtained solutions involving Floquet waves, as did Lhermitte with the propagation of an elastic shear wave, normal to the interfaces in a cross-ply fiber reinforced composite.¹¹

Though the propagator matrix formalism is very useful, it yields numerical instabilities when the whole thickness of the medium and/or the frequency of the incident wave become large. These difficulties are caused by the components of certain matrix products¹² which are infinite or very small and bring about losses of accurency during the calculation.¹³ Chimenti and Nayfeh¹⁴ have noticed that a direct approach to the problem, i.e., the resolution of a 6nequations system with 6n unknowns where *n* is the number of layers, leads to numerically stable results but is limited by the computer memory. In the isotropic case, Hosten¹⁵ has limited exponential terms when they become high, because they will balance each one with the other. Dunkin¹² expanded upon by Levesque and Piche¹⁶ both separate the propagation matrix into submatrices. In the isotropic case, the latest have solved the problem.

In the anisotropic case, we propose a physical approach using Floquet waves and using a judicious choice of reference for the amplitudes of these waves, according to their direction of propagation in the whole medium. In the case of an anisotropic homogeneous medium of any thickness, surrounded by two different media, our method permits us to eliminate infinite terms and thus to have no limitation in frequency or in angle of incidence. In the case of anisotropic periodically multilayered media, the method permits us to significantly improve our results, and to push back the limit of calculation.

II. FORMALISM USED FOR PROPAGATION EQUATIONS

Let us consider a periodically multilayered medium which is a reproduction of P "superlayers," each one made by the stacking of Q distinct anisotropic media (see Fig. 1). Media 0 and N+1 above and below the periodically multilayered medium are semi-infinite. We study the acoustic propagation of waves which are generated by an oblique incident wave propagating in the media 0. Let us define:

 x_3 , the axis of stacked layers;

- q, the number of the layer in a superlayer: $1 \le q \le Q$;
- p, the number of a "superlayer": $1 \le p \le P$;
- *n*, the number of the layer (p,q): n = (p-1)Q+q;

 h_q , the thickness of the layer;



FIG. 1. Periodically multilayered medium.

 z_n , the position on the x_3 axis of the interface between two layers *n* and *n*+1 such as $z_n = z_{n-1} + h_q$ and $z_0 = 0$.

The propagation equations in each layer use the same form as the one developed by Rokhlin *et al.*^{17,18} and completed by Ribeiro *et al.*^{19,20} with the help of the inhomogeneous waveform.^{21,22}

Generally speaking, an oblique incident wave propagating in the plane x_10x_3 defined in Fig. 1 generates six plane waves with different velocities in an anisotropic layer surrounded by two media.²³

Let us note:

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 η , the datum index of each wave: $1 \leq \eta \leq 6$;

 (η) **n**^q, the direction of propagation vector of the wave (η) in the layer q;

 ${}^{(\eta)}\mathbf{k}^{q}$, the wave number vector of the wave (η) in the layer q;

 $^{(\eta)}\mathbf{m}^{q}$, the slowness vector of the wave (η) in the layer q;

q; $^{(\eta)}\mathbf{P}^{q}$, the polarization vector of the wave (η) in the layer q;

 $^{(\eta)}\mathbf{V}^{q}$, the propagation velocity of the wave (η) in the layer q.

The slowness vector is related to the direction of the propagation vector and to the wave number vector by the following relation:

$${}^{(\eta)}\mathbf{m}^{q} = {}^{(\eta)}\mathbf{n}^{q}/{}^{(\eta)}\mathbf{V}^{q} = {}^{(\eta)}\mathbf{k}^{q}/\omega, \tag{1}$$

where ω is the natural frequency of the incident wave.

The plane wave propagation equation or Christoffel equation is written^{24,25} by the use of Einstein's convention

consisting in summing inferior indexes twice repeated (no summation on q):

$$(c_{ijkl}^{q}{}^{(\eta)}n_{j}^{q(\eta)}n_{l}^{q}-\rho^{q}({}^{(\eta)}V^{q})^{2}\delta_{ik}){}^{(\eta)}P_{k}^{q}=0, \qquad (2)$$

where ρ^q is the volumetric mass of the material in the layer q and c_{ijkl}^q are elastic constants of the layer q.

This equation can also be written as a function of the slowness vector:¹⁷

$${}^{(\eta)}G^{q}_{ik}{}^{(\eta)}P^{q}_{k}=0$$

with

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$${}^{\eta)}G^{q}_{ik} = c^{q}_{ijkl} {}^{(\eta)}m^{q(\eta)}_{j}m^{q}_{l} - \rho^{q}\delta_{ik}.$$
(3)

In the chosen reference system (see Fig. 1), the Snell-Descartes law imposes that the projection of the wave number vector (or of the slowness vector) upon a plane parallel to interfaces would be maintained, i.e., ${}^{(\eta)}m_1^1 = {}^{(\eta)}m_1^2 = \cdots = {}^{(\eta)}m_1^q$ and ${}^{(\eta)}m_2^q = 0$, $\forall \eta, \forall q$. There is equality of the parallel components to the x_1 axis of the slowness vectors of all existing waves; we will call this component m_1 . Reducing the 6×6 determinant of ${}^{(\eta)}m_3^q$ with real factors if the attenuation is neglected. If ${}^{(\eta)}m_3^q$ is real, the wave is homogeneous (or propagative); if ${}^{(\eta)}m_3^q$ is complex, the wave is inhomogeneous.^{20,22} For each wave (η) of which the slowness vector has been determined, Eq. (3) enables us to calculate ${}^{(\eta)}P_k^q$ normalized by the hermitian norm || || defined by

$$\|\mathbf{P}\|^2 = P_k P_k^*,\tag{4}$$

where P_k^* is the conjugate complex of P_k .

By introducing the slowness vector, the displacement vector of the wave (η) in the layer q can be written as

$${}^{(\eta)}\mathbf{u}^{p,q} = {}^{(\eta)}a^{p,q} \cdot {}^{(\eta)}\mathbf{P}^{q}e^{-i\omega({}^{(\eta)}\mathbf{m}^{q}\cdot\mathbf{x}-t)}.$$
(5)

Stresses are expressed as a function of displacements by

$$T_{ij}^{p,q} = c_{ijkl}^{q} \frac{\partial u_{k}^{p,q}}{\partial x_{1}}, \qquad (6)$$

where ${}^{(\eta)}a^{p,q}$ is the complex amplitude of the particle displacement tied to the wave (η) in the layer *n* and ${}^{(\eta)}\mathbf{u}^{p,q}$ is the displacement vector of the wave (η) in the layer *n*. The total displacement vector in the layer *n* is expressed by

$$\mathbf{u}^{p,q} = \sum_{\eta=1}^{6} {}^{(\eta)} \mathbf{u}^{p,q}.$$
 (7)

III. FORMALISM OF TRANSFER MATRICES

Subsequently, we can note the following:

- { } a six-dimensional column vector;
- $\langle \rangle$ a six-dimensional line vector;

[] a (6×6) matrix;

T a transpose operation;

 $X^{q}_{\alpha\beta}$ the coefficient of the matrix [X] at α th row and β th column.

Let $\{W^{p,q}(x_3)\}$ be the (6×1) column vector made up of the three components of the vector $\mathbf{u}^{p,q}$ and the three

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components $(T_{33}^{p,q}, T_{32}^{p,q}, T_{31}^{p,q})$ of the stress vector applied to a surface parallel to the interfaces, and in the layer q of a "superlayer" p for $z_{q-1} < x_3 < z_q$. Furthermore, let $\{\mathscr{A}^{p,q}\}$ be the (6×1) column vector containing the amplitudes $(\eta)a^{p,q}$ of the six waves propagating in the layer q of a "superlayer" p.

$$\{W^{p,q}(x_3)\} = \langle u_1^{p,q}, u_2^{p,q}, u_3^{p,q}, T_{33}^{p,q}, T_{32}^{p,q}, T_{31}^{p,q} \rangle^T$$

and

$$\{\mathscr{A}^{p,q}\} = \langle {}^{(1)}a^{p,q}, {}^{(2)}a^{p,q}, {}^{(3)}a^{p,q}, {}^{(4)}a^{p,q}, {}^{(5)}a^{p,q}, {}^{(6)}a^{p,q} \rangle^{T}$$

 $\{W^{p,q}(x_3)\}$ can be expressed as a function of $\{\mathscr{A}^{p,q}\}$ using the following equation:

$$\{W^{p,q}(z_q)\} = [B^q][\mathscr{H}^q]\{\mathscr{A}^{p,q}\}e^{-i\omega(m_1x_1-t)},$$
(8)

with

$$[\mathscr{H}^{q}] = \operatorname{diag}(e^{-i\omega^{(\eta)}m_{3}^{q}h_{q}}),$$
(6×6) diagonal matrix $n=1,...,6$

 $B_{\alpha\eta}^q = A_{\alpha\eta}^q$ for $\alpha = 1, 2, 3$ and $B_{\alpha\eta}^q = -i\omega A_{\alpha\eta}^q$ for $\alpha = 4, 5, 6$ where $[A^q]$ is given in the Appendix.

The displacements and stresses at $x_3=z_q$ can be expressed as a function of those at $x_3=z_{q-1}$ by the following relation:

$$\{W^{p,q}(z_q)\} = [B^q][\mathscr{H}^q][B^q]^{-1}\{W^{p,q}(z_{q-1})\}.$$
 (9)

The matrix $[X^q] = [B^q][\mathcal{H}^q][B^q]^{-1}$ is the propagation matrix in the layer $q^{.^{8,23,26}}$

By expressing displacements and stresses for two successive layers q ($z_{q-1} < x_3 < z_q$) and q+1 ($z_q < x_3 < z_{q+1}$), followed by equaling them at the interface $x_3 = z_q$, we obtain the amplitudes of the waves in the layer q+1 as functions of those in the layer q, in a superlayer p.

$$\{\mathscr{A}^{p,q+1}\} = [A^{q+1}]^{-1}[A^q][\mathscr{H}^q]\{\mathscr{A}^{p,q}\}.$$
 (10)

It is possible to express the amplitudes of the first layer of the "superlayer" p as a function of those in the first layer of the "superlayer" p+1, by means of a propagation matrix $[\Phi]$ defined by

$$\{\mathscr{A}^{p+1,1}\} = [\Phi]\{\mathscr{A}^{p,1}\},\tag{11}$$

where

$$[\Phi] = [A^1]^{-1} \left(\prod_{q=Q}^2 [A^q] [\mathscr{H}^q] [A^q]^{-1} \right) [A^1] [\mathscr{H}^1], \quad (12)$$

or else

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$$[\mathcal{T}] = [A^1][\Phi][A^1]^{-1} = \prod_{q=Q}^1 [A^q][\mathcal{H}^q][A^q]^{-1}.$$
(13)

From Refs. 23 and 27, the eigenvalues ${}^{(\eta)}\lambda$ of $[\mathcal{T}]$, which are of course the same as those of $[\Phi]$, are related to the Floquet wave numbers through the medium of the Floquet slownesses associated to the Floquet wave numbers ${}^{(\eta)}m_f$:

$$^{(\eta)}\lambda = e^{i\omega^{(\eta)}m_fh},\tag{14}$$

with

 $h = h_1 + h_2 + \cdots + h_0.$

The component following the x_3 axis of the Floquet slowness vectors is ${}^{(\eta)}m_f$ for a given incidence corresponding to m_1 .

By recurrence, one can find

$$\{\mathscr{A}^{P+1,1}\} = [\Phi]^{P}\{\mathscr{A}^{1,1}\}.$$
(15)

In fact, the amplitudes of the waves in the first layer of the "superlayer" P+1 are the amplitudes of fictitious waves, because the "superlayer" P+1 does not exist.

IV. NUMERICAL INSTABILITIES

The transfer matrix form is very useful, notably when P or Q are great: the size of the final system, coming from boundary conditions, will never exceed (18×18). But it yields important numerical instabilities¹²⁻¹⁴ when the waves vanish inside a layer.

It is quite easy to understand why numerical instabilities appear for a great kh: we have seen Sec. III that the eigenvalues of the propagation matrix $[\Phi]$, tied to the Floquet wave numbers, are exponential and function of the product $(i\omega^{(\eta)}m_fh)$. When ωh and P are great and when the waves (η) become inhomogeneous, the real part of $\exp[(i\omega^{(\eta)}m_fh)P]$ becomes very high in absolute value. These factors, combined with the first order values, bring out losses in precision.

As mentioned in the Introduction, Hosten¹⁵ suggests that it is possible, in the isotropic case, to limit the exponential terms when they become high, because they will balance each one with the other. But in the anisotropic case, it is quite impossible to foresee which term is going to balance the other. Dunkin¹² expanded upon by Levesque¹⁶ both separate the propagation matrix $[\Phi]$ into submatrices.

We propose to solve the problem by a physical approach using Floquet waves.

A. The use of the Floquet waves

Let us define:

 $[\Xi]$, the eigenvector matrix of $[\Phi]$;

 $\{\mathcal{F}^{p,1}\}\$, the (6×1) column vector containing the six complex amplitudes of the Floquet waves, at the first interface of the layer 1 in the "superlayer" p.

 $\{\mathcal{F}^{p,1}\}$ is related to $\{\mathcal{A}^{p,1}\}$ by the following relation, which is a change of basis:

$$\{\mathscr{A}^{p,1}\} = [\Xi]\{\mathscr{F}^{p,1}\},\tag{16}$$

that is to say,

$$\{\mathcal{F}^{p,1}\} = [\Xi]^{-1} \{\mathcal{A}^{p,1}\}.$$
(17)

From Eq. (11) we obtain at the end a matricial relation between the amplitudes of the Floquet waves at the first interface of the layer 1 in the "superlayer" p+1 as a function of those in the "superlayer" p:

$$\{\mathscr{F}^{p+1,1}\} = [\Xi]^{-1}[\Phi][\Xi]\{\mathscr{F}^{p,1}\}.$$
(18)

These waves are Floquet waves because the matrix $[\mathcal{H}^{f}] = [\Xi]^{-1}[\Phi][\Xi]$ is diagonal:

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$$[\mathscr{H}^{f}] = \operatorname{diag}({}^{(\eta)}\lambda). \tag{19}$$

We have then six Floquet waves which correspond to the classical plane waves propagating in an infinite periodically multilayered medium, considered as a homogeneous material.²⁷⁻²⁹ Let us consider this dummy medium in which the Floquet waves propagate. At any given plane parallel to the interface, it is possible to express a displacements and stresses vector which does not have any real meaning: indeed, these waves are linear combinations of the real waves propagating in each layer of the medium.

But this vector corresponds to the real displacements and stresses vector at the reference interface by means of a change of basis, and thus at any interface which can be deduced from the periodicity of the medium.

B. Boundary conditions

From Eqs. (8) and (16) the displacements and stresses vector at the first interface of the first layer in the first "superlayer" $(x_3=z_0)$ as a function of the amplitudes of the Floquet waves at the same interface:

$$\{W^{1,1}(z_0)\} = [B^1][\Xi]\{\mathscr{F}^{1,1}\}e^{-i\omega(m_1x_1-t)}.$$
 (20)

Though the "superlayer" P+1 does not exist, we can also express the displacements and stresses vector at the interface $x_3=z_{PQ}$: the boundary conditions just need to be satisfied at the interface.

$$\{W^{P+1,1}(z_{PQ})\} = [B^1][\Xi]\{\mathscr{F}^{P+1,1}\}e^{-i\omega(m_1x_1-t)}.$$
 (21)

Moreover, by using Eqs. (18) and (19) and by recurrence, one can find

$$\{\mathcal{F}^{p+1,1}\} = [\mathcal{H}^f]^P \{\mathcal{F}^{1,1}\}.$$
 (22)

Finally, using Eqs. (21) and (22) we obtain the displacements and stresses at the last interface as a function of the amplitude of the Floquet waves at the first interface:

$$\{W^{P+1,1}(z_{PQ})\} = [B^1][\Xi][\mathscr{H}^f]^P\{\mathscr{F}^{1,1}\}e^{-i\omega(m_1x_1-t)}.$$
(23)

As $[\mathcal{H}^{f}]$ is a diagonal matrix, $[\mathcal{H}^{f}]^{P}$ =diag[exp($i\omega^{(\eta)}m_{f}hP$)] where *Ph* is the thickness of the entire periodically multilayered medium. $[\mathcal{H}^{f}]^{P}$ is thus the propagation matrix of a dummy homogeneous medium.

C. Change of reference

Now let us consider a homogeneous medium surrounded by two different media (see Fig. 2). An incident wave propagating in the medium 0 generates six waves in the medium 1 (see Sec. II): three of them, numbered by $\eta = 1, 2, 3$ for instance, propagate (or decrease if the wave η is inhomogeneous) in the x_3 direction and the three others, numbered by $\eta = 4, 5, 6$ for instance, propagate (or decrease) in the opposite direction.

Suppose that the waves (1) and (4) are inhomogeneous and that the reference is taken at $x_3=z_0$. The amplitude of the wave (1) is negligible at the interface $x_3=z_1$ and so, it is easy to equal it to zero. The amplitude of the



FIG. 2. Homogeneous medium surrounded by two other media.

wave (4) is finite at the interface $x_3=z_1$ because of the mode conversion, but is infinite at the interface $x_3=z_0$ because of the reference taken at $x_3=z_0$. This amplitude is then multiplied by very small factors, which leads to losses of precision and then to numerical instabilities.

In order to avoid this, the reference for the waves propagating (or decreasing) in the x_3 direction is taken at $x_3=z_0$. The reference for the waves propagating (or decreasing) in the opposite direction is taken at $x_3=z_1$.

The total displacement is then expressed by the following equation:

$$\mathbf{u} = \left(\sum_{\eta=1}^{3} {}^{(\eta)} a^{(\eta)} \mathbf{P} e^{-i\omega^{(\eta)} m_3(x_3 - z_0)} + \sum_{\eta=4}^{6} {}^{(\eta)} a^{(\eta)} \mathbf{P} e^{-i\omega^{(\eta)} m_3(x_3 - z_1)} \right) e^{-i\omega(m_1 x_1 - t)}.$$
 (24)

The numerical instabilities that we have in this case are solved by the change of reference for the amplitudes of the waves in a single layer.

Let us apply the above method to the dummy homogeneous medium defined in Sec. IV B: according to the direction of propagation of the Floquet waves, the reference will be taken at $x_3=z_0$ or at $x_3=z_{PQ}$, which amounts to expressing the amplitudes of the Floquet waves in another basis, by means of a diagonal matrix $[\mathscr{C}]$ defined by

If modulus of $(\mathcal{H}_{\eta\eta}^f)^P$ is superior to 1

then
$$\mathscr{C}_{\eta\eta} = 1/(\mathscr{H}_{\eta\eta}^f)^P$$
 else $\mathscr{C}_{\eta\eta} = 1.$ (25)

The change of basis gives

$$[\mathcal{F}^{1,1}] = [\mathscr{C}]\{\mathscr{G}^{1,1}\},\tag{26}$$

where $\{\mathscr{G}^{1,1}\}\$ is the (6×1) vector containing the complex amplitudes of the Floquet waves of which references are taken at different interfaces.

Finally, we obtain the displacements and stresses vectors at $x_3 = z_0$ and at $x_3 = z_{PO}$:

$$\{W^{1,1}(z_0)\} = [B^1][\Xi][\mathscr{C}]\{\mathscr{G}^{1,1}\}e^{-i\omega(m_1x_1-t)}$$

$$\{W^{P+1,1}(z_{PQ})\} = [B^1][\Xi][\mathscr{H}^f]^P[\mathscr{C}]\{\mathscr{G}^{1,1}\}e^{-i\omega(m_1x_1-t)}.$$

(27)

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FIG. 3. Modulus of the reflection coefficient in water for one layer of carbon cpoxy which the sixth folded axis is oriented parallel to the Ox_1 axis, as a function of the angle of incidence. P=1; f=10 MHz.

If the wave η is inhomogeneous in the direction $x_3 < 0$, the product of the two matrices $[\mathscr{H}^f]^P$ and $[\mathscr{E}]$ gives a diagonal matrix which the $\eta\eta$ term is equal to 1 instead of being infinite.

By writing boundary conditions at $x_3=z_0$ and at $x_3=z_{PQ}$, one can obtain at most 18 equations with 18 unknowns including reflection and transmission coefficients. The connection between the Floquet waves and the reflection coefficient(s) is given in Sec. V B through an example.

V. APPLICATION TO CARBON/EPOXY COMPOSITES

A. Case of a single layer of carbon/epoxy

In the case of a medium made by a single layer of any given material, there is no limitation of thickness of the layer and/or of the frequency of the incident wave. An example of a reflection coefficient in amplitude in water for a medium made up by a layer of carbon epoxy, immersed in water, is given in Fig. 3 as a function of the angle of incidence.

The material used is a hexagonal crystal system medium with five independent elastic constants. If the sixthorder symmetry A_6 axis is parallel to the Ox_3 axis, these constants are³⁰

$$c_{11} = 13.5 \text{ GN/m}^2; \quad c_{12} = 6.3 \text{ GN/m}^2;$$

 $c_{13} = 5.5 \text{ GN/m}^2; \quad c_{33} = 125.9 \text{ GN/m}^2;$
 $c_{44} = 6.2 \text{ GN/m}^2.$

The volumetric mass of the carbon-epoxy is 1577 kg/m³, the thickness of the layer is equal to 20 mm, and the frequency of the incident wave is equal to 10 MHz.

B. Case of a periodically multilayered medium 0°/90°

Now let us consider a medium made by stacked identical hexagonal layers, each layer being at 90° to the previous, and immersed in water.

An example of the reflection and transmission coefficients in amplitude in water for a periodically multilayered medium made from layers of carbon epoxy is given in Fig. 4, as a function of the frequency. Each layer of a superlayer



FIG. 4. Modulus of the reflection and transmission coefficients in water for a periodically multilayered medium 0°/90° made from layers of carbon epoxy, as a function of the frequency. P=50; $\alpha=10^{\circ}$.

is 0.13 mm thick and the periodically multilayered medium is 13 mm thick.

This representation allows us to observe easily stopping bands in frequency when the reflection coefficient is equal to 1. The minima of the reflection coefficient correspond to Lamb modes. If the method described above is not applied, the computation is valid only until about 6 MHz. Though Levesque and Piché¹⁶ have observed that it is more difficult to obtain a correct transmission coefficient than a reflection coefficient, the transmission coefficient presented in Fig. 4 corresponds exactly to the reflection coefficient for the energy conservation. These coefficients permit us to give directions to nondestructive evaluation: indeed, when the reflection coefficient is equal to 1, the wave does not penetrate the end of the medium. So, in order to detect a defect, it would be preferable to take a natural frequency of the incident wave equal to 20 MHz rather than to 11 MHz.

Another example of the reflection and transmission coefficients as a function of the incident angle, for the same periodically multilayered medium surrounded above by water and below by the titane, is given in Figs. 5(a), 5(b), and 5(c).

The titane is an hexagonal crystal system medium with five independent elastic constants. If the sixth-order symmetry A_6 axis is parallel to the Ox_3 axis, these constants are²⁴

$$c_{11} = 162.4 \text{ GN/m}^2; c_{12} = 92.0 \text{ GN/m}^2;$$

 $c_{13} = 69.0 \text{ GN/m}^2; c_{33} = 180.7 \text{ GN/m}^2;$
 $c_{44} = 46.7 \text{ GN/m}^2.$

The volumetric mass of the titane is 4506 kg/m^3 , the periodically multilayered medium is 2.6 mm thick, and the frequency of the incident wave is equal to 10 MHz.

Figure 5(d) represents the reflection coefficient in water as a function of the incident angle for the same periodically multilayered medium immersed in water.





FIG. 6. Modulus of the reflection coefficient in water for a periodically multilayered medium $0^{\circ}/45^{\circ}/90^{\circ}/-45^{\circ}$ made from layers of carbon epoxy as a function of the angle of incidence. P=10; f=4 MHz.

FIG. 5. Modulus (a) of the reflection coefficient in water, (b) of the transmission coefficient of the longitudinal wave in the titane, (c) of the transmission coefficient of the shear wave in the titane, for a periodically multilayered medium $0^{\circ}/90^{\circ}$ made from layers of carbon epoxy as a function of the angle of incidence. The medium is surrounded by water above and by titane below. (d) Modulus of the reflection coefficient in water for the same periodically multilayered medium immersed in water. (e) Number of the Floquet waves which are inhomogeneous in the periodically multilayered medium. P=10; f=10 MHz.

In order to give a physical interpretation of these results we have put Fig. 5(e) below (a)-(d). This figure represents the numbers of the Floquet waves which are inhomogeneous in the periodically multilayered medium. As this medium is made up by identical hexagonal layers, each layer being at 90° to the previous, only four waves exist in the medium. That is the reason why the inhomogeneous waves in Fig. 5(e) are numbered from 0 to 4. On this figure, one can see several angular bands which correspond to four inhomogenous Floquet waves in the periodically multilayered medium: in these bands, for any given incident wave in water with any given natural frequency, there is total reflection in water, however the medium below the periodically multilayered medium may be. When the incident angle is contained between 24° and 26°, two Floquet waves are propagative again: there is transmission in the titane because the incident angle is inferior to the second critical angle of the titane. When the incident angle is contained between 41° and 50°, two Floquet waves become propagative again: the reflection in water is not total for the periodically multilayered medium immersed in water, although it is total when the medium below the periodically multilayered medium is the titane because the waves are inhomogeneous in it.

C. Case of periodically multilayered media $0^{\circ}/45^{\circ}/90^{\circ}/-45^{\circ}$ and $0^{\circ}/45^{\circ}/-45^{\circ}/-90^{\circ}$

Let us consider a medium made by stacked identical hexagonal layers, each layer being at 45° to the previous, immersed in water: it is a $0^{\circ}/45^{\circ}/90^{\circ}/-45^{\circ}$ medium. An example of the reflection coefficient in amplitude, in water, for a periodically multilayered medium made from layers of carbon epoxy is given in Fig. 6, as a function of the angle of incidence. Each layer of a superlayer is 0.13 mm thick, the periodically multilayered medium is 5.72 mm thick, and the frequency of the incident wave is equal to 4 MHz.

Now let us consider a medium made by four stacked identical orthotropic layers: the sixth-order symmetry A_6 axis of the first layer is parallel to the Ox_1 axis, the second layer is at 45° to the first layer, the third layer is at 90° to the previous, and the last layer is at 45° to the previous: it is a $0^{\circ}/45^{\circ}/-45^{\circ}/-90^{\circ}$ medium.

An example of the reflection coefficient in amplitude in water for a periodically multilayered medium made from layers of carbon epoxy and immersed in water is given in Fig. 7, as a function of the angle of incidence. As for the $0^{\circ}/45^{\circ}/90^{\circ}/-45^{\circ}$ medium, each layer of a superlayer is 0.13 mm thick, the periodically multilayered medium is 5.72 mm thick, and the frequency of the incident wave is equal to 4 MHz. As the modulus of the amplitude reflection coefficient of the Fig. 6 is different from the one of Fig. 7, one can say that the stacking order of the layers in one superlayer has an effect on the reflection coefficient.

Another example of the reflection coefficient in amplitude in water for the same periodically multilayered medium $0^{\circ}/45^{\circ}/-45^{\circ}/-90^{\circ}$ immersed in water is given in Fig. 8. Each layer of a superlayer is 0.13 mm thick, the

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FIG. 7. Modulus of the reflection coefficient in water for a periodically multilayered medium $0^{\circ}/45^{\circ}/-45^{\circ}/-90^{\circ}$ made from layers of carbon epoxy as a function of the angle of incidence. P=10; f=4 MHz.

FIG. 8. Modulus of the reflection coefficient in water for a periodically multilayered medium $0^{\circ}/45^{\circ}/-45^{\circ}/-90^{\circ}$ made from layers of carbon epoxy as a function of the angle of incidence. P=20; f=5 MHz.

periodically multilayered medium is 10.92 mm thick, and the frequency of the incident wave is equal to 5 MHz.

VI. CONCLUSIONS

From the propagator matrix formalism, we have built a propagation model in an anisotropic periodically multilayered medium. This form permits us to express the amplitudes of the waves propagating in one layer of a superlayer as a function of those in the same layer of the previous superlayer; it is thus possible to study the propagation in the whole medium by means of the propagation matrix of one superlayer. Therefore, the dispersion equation of the media corresponds to the characteristic equation of the propagation matrix of the periodically multilayered medium, itself linked to Floquet waves.

Though the propagator matrix formalism is very convenient, it leads to numerical instabilities when the frequency and the thickness become great. In order to avoid this, we have proposed a physical approach using Floquet waves. As these waves correspond to the classical plane waves propagating in an infinite periodically medium, we have expressed the propagation equations in the Floquet waves basis. The Floquet waves can be considered as waves propagating in an equivalent medium which satisfies the boundary conditions at the two extreme interfaces. So, the reference of the amplitudes of the waves propagating downward through the medium, was chosen at the top of it and the reference of the amplitudes of the waves propagating upward through the medium was chosen at the bottom of it. This amounts to expressing the amplitudes of the Floquet waves in a different vector basis. As a summary, infinite terms in propagation matrices can be balanced by means of two basis changes. It is thus easy to return to the real amplitudes after the calculation. This method which is based on a physical understanding of phenomena permits us to present reflection and transmission coefficients at high frequencies for thick anisotropic periodically multilayered media.

APPENDIX

We give here the η th column of the matrix $[A^q]$:



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