# Surface waves in an anisotropic periodically multilayered medium: Influence of the absorption 

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#### Abstract

The influence of absorption upon behavior of the reflection coefficient in water for an anisotropic periodically multilayered medium is studied. From observing a trough of the experimental reflection coefficient of a carbon/epoxy composite immersed in water, the propagation modes of the Floquet waves in an infinite anisotropic periodically multilayered medium in vacuum have been studied. This was explained by a mode that we called the multilayered Rayleigh mode. This mode results from a combination of the Floquet waves which propagate in a periodically multilayered medium. This occurs when all the Floquet waves are inhomogeneous. Analyzing the results obtained in the isotropic case permits the more complicated cases to be explained. By analogy with the isotropic case, the multilayered Rayleigh wave is related to the poles and the zeros of the reflection coefficient. The existence of a critical attenuation becomes evident when the reflection coefficient in water for the multilayered medium reaches zero. © 1995 American Institute of Physics.


## I. INTRODUCTION

Ultrasonic surface waves are separated into different categories, corresponding to the type of interface separating the two different media: Rayleigh waves (vacuum/infinite isotropic medium), Love waves (vacuum/thin layerlinfinite substrate), Scholte waves (liquid/infinite substrate), Stoneley waves (solid/solid), and others. ${ }^{1-3}$ Subsequently, we are particularly interested in Rayleigh waves, especially in generalized Rayleigh waves or leaky Rayleigh waves. In a homogeneous isotropic medium, the Rayleigh wave is made up of the combination of two inhomogeneous waves, a longitudinal and a transversal wave. Their phase velocities in a direction contained in the interface plane are identical. The amplitude of the Rayleigh wave decreases with depth, measured out from the free surface. When the vacuum is replaced by a fluid, the Rayleigh wave becomes the generalized Rayleigh wave, also named "leaky Rayleigh wave" because its energy leaks into the fluid. ${ }^{4}$ Quentin, Derem, and Poiree ${ }^{5}$ have shown that the generalized Rayleigh wave can be modeled as a composite of three evanescent plane waves: one in the fluid and two in the isotropic solid. Moreover, the behavior of the leaky Rayleigh wave is related to the pole of the reflection coefficient. The wave numbers corresponding to the pole and to the zero of the reflection coefficient are conjugated complexes. ${ }^{6}$ In this case the modulus of the reflection coefficient is equal to 1 whereas its phase varies from $\pi$ to $-\pi$. In experiments a trough of the reflection coefficient is observed. This can be explained in a physical way: the transducer beams are bounded and the media are more or less absorbent. ${ }^{7}$ This can be modeled by taking into account the evanescent plane waves: ${ }^{8}$ The solution should take the form of six evanescent plane waves. Another method consists of modeling the absorption of the solid. ${ }^{3}$ The principal factor that controls the depth of the minimum of the reflected amplitude is the "shear wave attenuation per wavelength," 9,10 whereas the variation of the longitudinal wave attenuation or of the volumetric mass of the solid medium does not change
this depth significantly. There is a critical value of the shear wave attenuation for which zero reflection occurs at the Rayleigh critical angle. For liquid/solid/liquid interfaces, the amplitude of the minima of the reflection coefficient (Lamb waves), depends on the product $F d$ : the frequency of the incident wave multiplied by the thickness of the solid. ${ }^{11,12}$ When this product is small, the absorption has little effect on the shape of the curves, and when it is large the last minimum occurs at the critical Rayleigh angle and corresponds to the propagation of a Rayleigh wave. The ratio of the solid density to the fluid density also modifies the reflection quefficient behavior. ${ }^{13}$ For a liquid/solid/semi-infinite substrate structure, an infinite number of modes exists if the velocity of the shear waves in the layer is inferior to the velocity of the shear waves in the substrate. ${ }^{3}$ This has been detailed in Ref. 14. Chimenti and co-workers ${ }^{4}$ as well as Bogy and Gracewski ${ }^{15}$ have studied the reflection coefficient as a function of the product $F d$. In the case of a very thin viscoelastic layer between two semi-infinite substrates, the reflection coefficient presents a trough similar to that obtained at the Rayleigh critical angle for an interface liquid/solid. ${ }^{16}$ In the case of a vacuum/isotropic stratified media/semi-infinite substrate structure, Rousseau, Pouliquen, and Defebvre et al..$^{17.18}$ have considered a plane surface wave propagating along an axis parallel to the interfaces. They have established the corresponding system of propagation equations by writing boundary conditions: the vanishing of a determinant permits the velocity of this surface wave to be established. This velocity approaches asymptotically the Rayleigh velocity in the layer in contact with the vacuum for high frequencies, and that in the substrate for low frequencies. Bogy and Gracewski ${ }^{19}$ have studied the propagation in a lossy water/isotropic stratified media/semi-infinite substrate structure and have applied it to a lossy water/silver/nickel/copper structure. In addition, the results have been confirmed by Levesque and Piché, ${ }^{20}$ who were able to eliminate the numerical problems that previously existed. The propagative Rayleigh modes are all the more numerous when the product $F d$ increases. Nayfeh and


FIG. 1. Interface vacuum-semi-infinite isotropic solid.

Chimenti ${ }^{21}$ have studied composites made up of isotropic or transversally isotropic layers immersed in water.

The aim of this article is to study the propagation of what we call multilayered Rayleigh waves, and the influence of the absorption upon the reflection coefficient behavior in water for anisotropic periodically multilayered media. Such media are a $P$ times an anisotropic multilayered medium cell, called "superlayer." These media are now studied by the propagator matrix form. ${ }^{22-26}$ The study of such media leads to Floquet waves that are linear combinations of the classical plane waves propagating in each layer of the multilayered medium. ${ }^{27-32}$ They correspond to the propagation modes of the infinite periodically multilayered medium, considered as a homogeneous material, and have been studied in previous works. ${ }^{33-37}$

Experiments on composites made of carbon/epoxy layers have led us to explain the presence of some minima of the reffection coefficient by the propagation of a multilayered Rayleigh wave. This designation is justified in Sec. III C 1. The aim of this article is to confirm this theory by showing that the Floquet waves can be combined to give leaky multilayered Rayleigh waves when the former are all inhomogeneous. We then first study the propagation modes of the Floquet waves in a lossy and nonlossy infinite anisotropic periodically multilayered medium in a vacuum, then in the presence of fluid, and finally immersed in water. In order to make an analogy with the isotropic case, we present in Sec. II the background of the results obtained with lossy and nonlossy isotropic media.

## II. RAYLEIGH WAVES IN ISOTROPIC SOLIDS

The Rayleigh wave is related to the poles and zeros of the reflection coefficient. We saw in Sec. I that at the Rayleigh critical angle the wave numbers corresponding to the pole and to the zero of the reflection coefficient for an interface liquid/solid are conjugated complexes. ${ }^{6}$ As explained in the Appendix, this reflection coefficient has the form of a complex function. The position of poles and zeros of this complex function has an influence upon the shape of its modulus and its phase. ${ }^{38}$

## A. Interface vacuum/infinite isotropic solid

Let us consider now an isotropic medium unbounded in the $x_{1}$ and $x_{2}$ directions and occupying the half-space $x_{3}>0$ (see Fig. 1). We define the following parameters: $V_{L}$ is


FIG. 2. Modulus of the ( $3 \times 3$ ) determinant for an interface vacuumnonlossy and lossy isotropic steel.
the velocity of the longitudinal wave, $V_{T}$ the velocity of the shear wave, $V_{R}$ the velocity of the Rayleigh wave, $\eta=V_{R} / V_{T}, \xi=V_{T} / V_{L}, \mathbf{k}^{L, T, R}$ the wave-number vector of the longitudinal wave ( $L$ ), shear wave ( $T$ ), or Rayleigh wave ( $R$ ), and $\rho$ the volumetric mass. At the interface at $x_{3}=0$, equating to zero the normal and tangential stresses in order to have nonzero solutions leads to the vanishing of a $(3 \times 3)$ determinant which yields an equation of the sixth degree for $\eta_{1}^{1-3}$

$$
\begin{equation*}
\eta^{6}-8 \eta^{4}+8\left(3-2 \xi^{2}\right) \eta^{2}-16\left(1-\xi^{2}\right)=0 \tag{1}
\end{equation*}
$$

The inhomogeneous longitudinal and shear waves combine to yield the Rayleigh wave.

## 1. Nonlossy isotropic solid

If the solid is not viscoelastic, a good approximation of the velocity of the Rayleigh wave is given by

$$
\begin{equation*}
\eta_{R}=\frac{0.87+1.12 \nu}{1+\nu} \quad \text { with } \quad 0 \leqslant \nu \leqslant 0.5 \text {, } \tag{2}
\end{equation*}
$$

where $\nu$ is the Poisson ratio.
A critical Rayleigh angle can be defined in relation to a medium of reference. The critical Rayleigh angle is given by the Snell-Descartes law; if this medium is water, the equation gives

$$
\begin{equation*}
\theta_{R}=\operatorname{Arcsin}\left(V_{\text {water }} / V_{R}\right) . \tag{3}
\end{equation*}
$$

The projection on the $x_{1}$ axis of the wave-number vector of the Rayleigh wave is real. Thus, the graphical representation of the determinant as a function of a fictitious angle of incidence $\theta$ (which amounts to the projection on the $x_{1}$ axis of a wave-number vector) reaches a minimum equal to zero at $\theta=\theta_{R}$. The solid-line curve of Fig. 2 presents such a determinant for nonlossy steel. The constants are given in Ref. 20: $V_{L}=5760 \mathrm{~m} / \mathrm{s} ; V_{T}=3075 \mathrm{~m} / \mathrm{s} ; \rho=7930 \mathrm{~kg} / \mathrm{m}^{3}$. The Rayleigh critical angle is equal to $31.17^{\circ}$ by applying Eqs. (2) and (3).


FIG. 3. Interface fluid-isotropic solid.

## 2. Lossy isotropic solid

The Rayleigh wave attenuates during propagation. The projection on the $x_{1}$ axis of the wave-number vector of the Rayleigh wave is not real and, therefore, the graphical representation of the determinant as a function of a fictitious angle of incidence $\theta$ (which amounts to the projection on the $x_{1}$ axis of a wave-number vector) reaches a minimum not equal to zero at $\theta=\theta_{R}$. The dashed-line curve of Fig. 2 presents such a determinant for lossy steel. The constants are given in Ref. 20: $V_{L}=5760(1+0.008 j) \mathrm{m} / \mathrm{s}$; $V_{T}=3075(1+0.012 j) \mathrm{m} / \mathrm{s} ; \rho=7930 \mathrm{~kg} / \mathrm{m}^{3}$.

## B. Interface fluid/infinite isotropic solid

Let us consider two semi-infinite media separated by a plane interface (see Fig. 3). An oblique incident wave propagates in the fluid. The incident plane is defined by ( $0, x_{1}, x_{3}$ ) and the interface by $\left(0, x_{1}, x_{2}\right)$.

By writing boundary conditions at the interface, one can obtain the reflection coefficient $\mathfrak{F}$ of the longitudinal wave in the fluid of the following form: ${ }^{3,6,10}$

$$
\begin{equation*}
\mathfrak{R}=\frac{C-\tau}{C+\tau} \tag{4}
\end{equation*}
$$

A surface wave exists if the denominator of $\mathfrak{A}$ is equal to zero. The equation $C=-\tau$, which allows us to determine the pole of the reflection coefficient, corresponds to a generalized Rayleigh wave. The wave number $k_{p}^{R}$ corresponding to the solution of this equation is complex. In the cases we are interested in, the real part is quite equal to the Rayleigh wave number found in Sec. II A1 for an interface vacuum/nonlossy isotropic solid. The imaginary part is related to the leakage of the wave in the liquid.

## 1. Nonlossy isotropic solid

The wave number $k_{z}^{R}$ corresponding to the solution of the equation $C=\tau$, which gives the zero of the numerator of the reflection coefficient, is the conjugate complex of the pole. The reflection coefficient is then proportional to $\left(k_{1}-k_{z}^{R}\right) /\left(k_{1}-k_{p}^{R}\right)$, where $k_{1}$ is the projection on the $x_{1}$ axis of the wave-number vector of the homogeneous incident wave which is real. This function is of the form of the function $f(z)$ studied in the Appendix. As $k_{p}^{R}$ and $k_{z}^{R}$ are conju-


FIG. 4. (a) Modulus and (b) phase of the reflection coefficient in water for semi-infinite nonlossy steel (solid line) and semi-infinite lossy steel (dashed line).
gated complexes, the modulus of the reflection coefficient is equal to one, and its phase varies from $\pi$ to $-\pi$. If the solid is finite, a trough in the reflection coefficient appears. It vanishes when the thickness of the medium increases.

The equation $C=0$ leads to Eq. (1) which permits us to find the velocity of the Rayleigh wave.

## 2. Lossy isotropic solid

The pole and the zero of the reflection coefficient are no longer conjugated complexes. As can be seen in the Appendix, the modulus of the reflection coefficient presents a minimum at the critical Rayleigh angle. This is observed in experimental conditions. ${ }^{3}$

Figure 4 presents the modulus and the phase of the reflection coefficient in water for the nonlossy steel (solid line) and for the lossy steel (dashed line) we studied in the former paragraph. The velocity of the longitudinal wave in water is equal to $1480 \mathrm{~m} / \mathrm{s}$ and the volumetric mass is equal to 1000 $\mathrm{kg} / \mathrm{m}^{3}$.

As mentioned in Sec. $\mathrm{I}^{7,9,10}$ the principal factor that controls the depth of the minimum of the reflected amplitude is $l_{t}$. This is the shear wave attenuation per wavelength. The velocity of the shear wave is given by the following equation:

$$
\begin{equation*}
V_{T}=\frac{V_{t}}{1+j l_{t} / 2 \pi} \tag{5}
\end{equation*}
$$

where $V_{t}$ is the velocity of the shear wave in the nonlossy medium.


FIG. 5. Modulus and phase of the reflection coefficient in water for a semiinfinite lossy steel, according to the value of the shear wave attenuation per wavelength (figure from Ref. 10).

These phenomena are also linked to bounded beams. In 1973, Bertoni and Tamir ${ }^{7}$ provided the first comprehensive theoretical explanation of beam reflection from lossy and nonlossy half-spaces: "the reflected beam is shifted from the position predicted by geometrical acoustics" and is accompanied by a weaker ficld. "Within the reflected beam, the acoustic field exhibits an intensity minimum or null," whether the medium is lossy or nonlossy. They have calculated the reflected field near the Rayleigh angle by assuming that the incident beamwidth is large compared to the wavelength in the liquid, which allows the reflection coefficient to be approximated. They have also shown, among other things, that the depth of the minimum of the reflection coefficient depends on the absorption and on the angular deviation from the Rayleigh angle.

At the Rayleigh critical angle, a zero reflection occurs for a critical value of $l_{t}=l_{0}$. The depth of the minimum of the reflection coefficient is maximum for $l_{0}$ and $k_{\mathrm{Z}}^{R}$ is real. After this critical value of $l_{t}$, the depth of the minimum decreases and the sign of the imaginary part of $k_{z}^{R}$ changes. The variations of the depth of the minimum of the modulus and of the phase of the reflection coefficient is shown in Fig. 5 (extracted from Ref. 10): $V_{L}=5740 \mathrm{~m} / \mathrm{s} ; V_{t}=3142 \mathrm{~m} / \mathrm{s}$; $\rho=7930 \mathrm{~kg} / \mathrm{m}^{3}$; and $l_{t}$ varies from 0.0003 to $l .0 \mathrm{~Np} / \lambda$.

The modulus of the numerator and of the denominator of the reflection coefficient present a minimum of the Rayleigh critical angle, as can be observed in Fig. 6 for the same steel we used previously in Fig. 4. The value of the minimum of the denominator for the nonlossy steel is not equal to zero because of the presence of the fluid in the place of a vacuum. This value is equal to that of the numerator: the modulus of the reflection coefficient is thus equal to one. As far as distance between the pole and the zero is concerned, it can be seen in the Appendix that this distance is equal to $\left|B_{1}\right| \pm\left|B_{2}\right|$ depending on whether $z_{1}^{\prime \prime}$ is like or unlike $z_{2}^{\prime \prime}$ : this distance is then obtained from the minima of the numerator and of the denominator of the modulus of $f(z)$. If we call $\left|B_{1}\right|$ the value of the numerator and $\left|B_{2}\right|$ the value of the denominator of the reflection coefficient at the Rayleigh critical angle, we observe that $\left|B_{1}\right|+\left|B_{2}\right|$ for $l_{t}<l_{0}$ and $\left|B_{1}\right|-\left|B_{2}\right|$ for $l_{t}>l_{0}$ are quite constant (see Table I). For $l_{t}=l_{0}$, the zero of the reflection coefficient is on the real axis. As a conclusion, the distance between the pole and the zero of the reflection co-

(a)


FIG. 6. Modulus of the (a) numerator and (b) denominator of the reflection coefficient in water semi-infinite nonlossy steel (solid line) and semi-infinite lossy steel (dashed line).
efficient is obtained from the minimum of the numerator and the minimum of the denominator of the modulus of this coefficient. It practically does not vary.

## III. MULTILAYERED RAYLEIGH WAVES IN ANISOTROPIC PERIODICALLY MULTILAYERED MEDIA

## A. Formalism of Floquet waves for such a medium (background)

Let us now consider a periodically multilayered medium which is a reproduction of $P$ "superlayers," each one made by the stacking of $Q$ distinct anisotropic media (see Fig. 7). Media 0 and $N+1$ above and below the periodically multilayered medium are semi-infinite. The study of the acoustic propagation of waves that are generated by an oblique incident wave propagating in the media 0 has been carried out in

TABLE I. Values of the sum or of the substraction of the numerator and of the denominator of the reflection coefficient for different values of attenuation, for steel.

| $l_{t}(N p / \Lambda)$ | $\left\|B_{2}\right\| \pm\left\|B_{1}\right\|\left(\times 10^{-6}\right)$ |
| :---: | :---: |
| $l_{0} / 10$ | 5.039 |
| $l_{0} / 4$ | 5.039 |
| $l_{0} 2$ | 5.039 |
| $l_{0}=0.0727$ | 5.052 |
| $2 l_{0}$ | 5.006 |
| $4 l_{0}$ | 5.000 |



FIG. 7. Periodically multilayered medium.
previous works; ${ }^{33-37}$ we therefore first give a quick summary of these main results. We define the following: $\theta$ is the angle of incidence, $\left\{W^{p, q}\left(z_{q}\right)\right\}$ the $(6 \times 1)$ displacement and stresses column vector at $x_{3}=z_{q}$, and [ $\Phi$ ] the ( $6 \times 6$ ) transfer matrix of one superlayer defined in Refs. 33 and 34.

The transfer matrix of the whole periodically multilayered medium is $[\Phi]^{P}$ where $P$ is the number of superlayers. The matrix [ $\Phi$ ] allows the displacement amplitudes of the plane waves in the first layer of a superlayer to be expressed as a function of those in the first layer of the next superlayer. The waves that allow this by the means of a diagonal matrix are the Floquet waves. They are thus obtained by diagonalizing $[\Phi]$. These six Floquet waves are the propagation modes of an infinite periodically multilayered medium, considered as a homogeneous material. They are linear combinations of the classical waves propagating in each layer of
the multilayered medium. The linear combination is simply different according to the layer. ${ }^{36,37}$ As in every homogeneous medium, three of the six Floquet waves propagate (or decrease) in the $x_{3}$ direction and three of them propagate (or decrease) in the opposite direction. As the Floquet wave numbers are related to the eigenvalues of $[\Phi]$ the value of the modulus of these eigenvalues with respect to 1 will allow the direction of decreasing of the inhomogeneous Floquet waves to be known.

One of the interests in using the Floquet analysis is that the propagation equations are expressed in the Floquet wave basis. Indeed, these waves are the waves propagating in an homogeneous medium which satisfies the boundary conditions at the top and at the bottom of the multilayered medium. By writing them down, one can obtain the reflection and transmission coefficients in the media 0 and $N+1$.

The displacements and stresses vectors at $x_{3}=0$ are given by

$$
\begin{equation*}
\left\{W^{1,1}(0)\right\}=\left[B^{1}\right][\Xi][\varnothing]\left\{G^{1,1}\right\} e^{-i \omega\left(m_{1} x_{1}-t\right)} \tag{6}
\end{equation*}
$$

where the used matrices are defined in Ref. $34 .\left\{9^{4,1}\right\}$ is the $(6 \times 1)$ vector containing the complex amplitudes of the Floquet waves and then represents the new basis in which the propagation equations are expressed. Subsequently, Floquet waves propagating in a multilayered medium will be used as classical waves propagating in a homogeneous medium, which is physically more transparent. The major difference results from the dispersive character of the Floquet waves.

## B. Experimental multilayered Rayleigh mode

We have measured the reflection coefficient in water at $\theta=28.2^{\circ}$, for a medium made up of stacked identical hexagonal layers of carbon/epoxy, each being at $45^{\circ}$ to the previous one ( $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium). This composite plate consists of six superlayers, each layer being 0.12 mm thick. It was supplied by Aerospatiale (France). As the aim of the experiment was not to calculate the elastic constants of the plate, but to validate our model, we used the elastic constants determined by Hosten and Castaings ${ }^{39,40}$ (see constants A in Table II): These constants are complex, which amounts to saying that the medium is a lossy one. The volumetric mass of each layer is equal to $1577 \mathrm{~kg} / \mathrm{m}^{3}$. The experimental reflection coefficient presents a trough at 3.2 MHz (see Fig. 8). This trough is found on the modeled coefficient for a lossy medium at 3.17 MHz but is not found when the medium is considered as non lossy (i.e., the elastic constants are taken as real). The vanishing of this mode can be interpreted as a lack of radiation in the medium opposed to the insonification. ${ }^{4-43}$ As this mode appears for an incident angle greater than the critical angles in each layer of the

TABLE II. Elastic constants in GPa for a carbon/epoxy medium from Refs. 4 and 44 if the sixth-order symmetry $A_{6}$ axis is parallel to the $\left(O x_{3}\right)$ axis.

|  | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{33}$ | $c_{44}$ | $\rho\left(\mathrm{~kg} / \mathrm{m}^{3}\right)$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Constants $A$ <br> for $\Re=\operatorname{fct}(f)$ | $13.7+0.13 j$ | $7.1+0.04 j$ | $6.7+0.04 j$ | $126+0.73 j$ | $5.8+0.1 j$ | 1577 |
| Constants $B$ <br> for 9 | $13.5+0.13 j$ | $6.3+0.04 j$ | $5.5+0.04 j$ | $125.9+0.73 j$ | $6.2+0.1 j$ | 1580 |



FIG. 8. Modulus of the reflection coefficient for a $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium made of six superiayers at $\theta=28.2^{\circ}$. Calculus done with constants $A$ of Table II.
medium, the side opposed to the insonification of the plate has very small displacement amplitudes and so has a very weak radiation. The radiation is all the more weak since the thickness is sizeable. Moreover, the energetic balance for plane waves and for a medium which is nonlossy and which has no radiation on the side sheltered from the emission, shows that all the energy is reflected. In the case of water there is only one type of wave in the incident medium: The reflection coefficient is thus equal to one. This hypothesis is confirmed by the calculation of the reflection coefficient for a nonlossy $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium made of two superlayers and thus being three times less thicker: ${ }^{35}$ The displacement amplitudes of the Floquet waves on the sheltered side of the plate are much greater than those when the medium is made of six superlayers. The beginning of this mode can be observed in Fig. 9. As the thickness of the medium is not sizeable this mode is a Lamb mode. Above a certain thickness the medium responds as if it were infinite and so the Lamb mode becomes a multilayered Rayleigh mode. This phenomenon can also be observed on a reflection coefficient as a function of the incident angle instead of the incident frequency. This is presented as an example, in Fig. 10, where for a $0^{\circ} / 45^{\circ} 90^{\circ} / 135^{\circ}$ medium consisted of five superlayers,


FIG. 9. Modulus of the reflection coefficient for $0 \% / 45^{\circ} / 90^{\circ} / 135^{\circ}$ media at $\theta=28.2^{\circ}$. Calculus done with constants $A$ of Table II. Thick line: $\mathrm{P}=2$, nonlossy medium; thin line: $P=6$, nonlossy medium; dotted line: $P=6$, lossy medium.


FIG. 10. Modulus of the reflection coefficient for $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ media; $f=3 \mathrm{MHz}$. Calculus done with constants $B$ of Table I .
each layer being 0.13 mm thick. The elastic constants were determined by Lhermitte ${ }^{44}$ (see constants B, Table II). The volumetric mass of each layer is equal to $1580 \mathrm{~kg} / \mathrm{m}^{3}$.

A mode that we call the multilayered Rayleigh mode appears on Fig. 10 for a lossy medium at $\theta=27.75^{\circ}$ whereas it does not appear when the medium is a nonlossy one. We justify this designation below, in Sec. III C 1. Just as before, when the nonlossy stratified medium consists of two superlayers, this mode appears.

In order to confirm that this mode is a multilayered Rayleigh mode and that it corresponds to a combination of the Floquet waves so as to give a leaky multilayered Rayleigh wave, we first study the propagation modes of the Floquet waves in a lossy and nonlossy infinite anisotropic periodically multilayered medium in a vacuum, then in the presence of fluid. There is a strong analogy with the isotropic case, but there are three Floquet waves instead of two in the isotropic case.

## C. Interface vacuum/infinite multilayered medium

Let us now consider an anisotropic periodically multilayered medium unbounded in the $x_{1}$ and $x_{2}$ directions and occupying the half-space $x_{3}>0$. Equation (6) given in Sec. III A expresses the stresses and the displacements of the Floquet waves at the first interface at $x_{3}=0$. As the multilayered medium is semi-infinite, we must choose the three Floquet waves (among six) which propagate or decrease in the $x_{3}$ direction, just as if the medium were homogeneous. In the case we are interested in all the Floquet waves are inhomogeneous. We can thus apply the infinite radiation criterion, ${ }^{30.45,46}$ which amounts to saying that we keep the three Floquet waves linked to the eigenvalues of the transfer matrix [ $\Phi$ ] of which the modulus is inferior to $1 .{ }^{34}$ Equating to zero the normal and tangential stresses in order to have nonzero solutions leads to the vanishing of a $(3 \times 3)$ determinant.

## 1. Nonlossy stratified medium

By analogy with the isotropic case, we can say that if this ( $3 \times 3$ ) determinant, represented as a function of a fictitious angle of incidence $\theta$, reaches a minimum equal to zero,


FIG. 11. Modulus of the $(3 \times 3)$ determinant for an interface vacuuminfinite nonlossy $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium; $f=3 \mathrm{MHz}$. Calculus done with constants $B$ of Table II.
there is propagation of a leaky multilayered Rayleigh wave. Let us apply this reasoning to the $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium we studied before in Sec. III B for a natural frequency equal to 3 MHz . Indeed, a stratified medium is a dispersive medium ${ }^{37}$ and therefore the multilayered Rayleigh waves are also dispersive, which constitutes one difference to the isotropic case. A minimum equal to zero is reached in Fig. 11 at $\theta=27.75^{\circ}$. At this angle, a multilayered Rayleigh mode was observed in Fig. 10 for a lossy medium. A dispersive leaky multilayered Rayleigh wave therefore propagates. This multilayered Rayleigh wave is consequently not only characterized by an angle $\theta=\theta_{R}$, but also by a frequency $f=f_{R}$.

The same phenomenon is observed when the natural frequency varies instead of the incident angle. As an example, Fig. 12 represents the modulus of the $(3 \times 3)$ determinant at $\theta=28.2^{\circ}$ for the $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ previously studied in Sec. III B. The curve reaches a minimum equal to zero at $f=3.17$ MHz , which is the frequency for which a multilayered Rayleigh mode appeared in experiments.

We saw in Sec. I that a Rayleigh wave is a combination of two inhomogeneous waves, a longitudinal and a transversal wave, which propagate in a homogeneous isotropic me-


FIG. 12. Modulus of the ( $3 \times 3$ ) determinant for an interface vacuuminfinite $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium; $\theta=28.2^{\circ}$. Calculus done with constants $A$ of Table II.


FIG. 13. Modulus of the ( $3 \times 3$ ) determinant for an interface vacuum-lossy infinite $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium for different values of attenuation; $f=3$ MHz . Calculus done with constants $B$ of Table II .
dium. In the case we are now interested in there is a combination of three inhomogencous Floquet waves which are dispersive and which decrease as a function of $x_{3}$. This is why we call it a multilayered Rayleigh wave. In the isotropic case a Rayleigh wave may only appear after the two critical angles; in the multilayered case, critical angles do not exist, however, there are stopping bands in which all the Floquet waves are inhomogeneous: A multilayered Rayleigh wave may appear only in these bands. The use of Floquet waves allows a parallel to be drawn between the behavior of an isotropic homogeneous medium and the behavior of a multilayered medium in the vicinity of a Rayleigh angle.

## 2. Lossy stratified medium

As in the isotropic case, the graphical representation of the ( $3 \times 3$ ) determinant as a function of a fictitious angle of incidence $\theta$ (which amounts to the projection on the $x_{1}$ axis of a wave-number vector) reaches a minimum not equal to zero at $\theta=\theta_{R}$ and $f=f_{R}$. In order to study the influence of the absorption upon the determinant and later upon the reflection coefficient, we have varied the attenuation, which amounts to changing the imaginary part of the elastic constants. The imaginary part determined by Castaings ${ }^{40}$ and given in 'Table II is what we call "standard" attenuation. We have therefore used elastic constants of which the imaginary part had been divided by 10,5 , or 2 , or multiplied by 2,3 , or 5. Subsequently, the captions of Figs. 13-15 composed of $" / 10,15,12$, standard, ${ }^{*} 2, * 3, * 5$ " refer to these modifications.

The more the attenuation increases, the less the minimum of the modulus of the ( $3 \times 3$ ) determinant of Fig. 14 is marked; however, this minimum occurs at $\theta=\theta_{R}$ and $f=f_{R}$.

## D. Interface fluid/infinite multilayered medium

Let us consider two semi-infinite media separated by a plane interface. The upper medium is water and the lower medium is a multilayered medium. An oblique incident wave propagates in the fluid.


FIG. 14. (a) Modulus and (b) phase of the reflection coefficient in water for an infinite lossy $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium, for different values of attenuation; $f=3 \mathrm{MHz}$. Calculus done with constants $B$ of Table $\Pi$.

By writing boundary conditions at the interface, with the help of Eq. (6), one can obtain the reflection coefficient $\mathfrak{\Re}$ of the longitudinal wave in the fluid, from a system of four


FIG. 15. (a) Modulus and (b) phase of the reflection coefficient in water for an infinite $0 \% / 45^{\circ} / 90 / 135^{\circ}$ medium, for different values of attenuation; $\theta=28.29$. Caleulus done with constants $A$ of Table II

TABLE III. Values of the numerator and of the denominator of the reflection coefficient for different values of attenuation, for a $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ medium. $\theta_{R}=27.75^{\circ}$ and $f_{R}=3 \mathrm{MHz}$.

| Attenuation | $\left\|B_{1}\right\|$ (num) | $\left\|B_{2}\right\|$ (den) | $\left\|B_{2}\right\| \pm\left\|B_{1}\right\|$ | $R=\left\|B_{1} / B_{2}\right\|$ |
| :---: | :---: | :---: | :---: | :---: |
| Without | 5.936 | 5.936 | 11.872 | 1 |
| $/ 2$ | 4.515 | 7.308 | 11.823 | 0.6179 |
| 15 | 5.349 | 6.469 | 11.818 | 0.8268 |
| 110 | 5.628 | 6.190 | 11.818 | 0.9091 |
| Standard | 3.133 | 8.707 | 11.84 | 0.3599 |
| $* 2$ | 0.4254 | 11.48 | 11.905 | 0.0371 |
| $* 3$ | 2.174 | 14.19 | 12.016 | 0.1532 |
| $* 5$ | 6.925 | 19.29 | 12.365 | 0.359 |

equations. The three Floquet waves that correspond to the infinite radiation criterion are thus chosen, as mentioned above in Sec . III C .

## 1. Nonlossy stratified medium

When all the Floquet waves are inhomogeneous, the modulus of the reflection coefficient in water is equal to 1 . At $\theta=\theta_{R}$ and $f=f_{R}$ its phase varies from $\pi$ to $-\pi$. The zero and the pole of the reflection coefficient are conjugated complexes. As in the isotropic case, if the stratified medium is finite, a trough of the reflection coefficient appears. It vanishes when the thickness of the medium increases. That is what happens in Figs. 9 and 10: The Lamb mode appears when the medium consists of two superlayers, but vanishes when the number of superlayers and thus the thickness increases. As explained in Sec. III B, this Lamb mode is converted into a multilayered Rayleigh mode.

## 2. Lossy stratified medium

As in the isotropic case, the pole and the zero of the reflection coefficient are no longer conjugated complexes. The modulus of the reflection coefficient presents a minimum at the critical Rayleigh angle. When the attenuation of each layer of the medium varies we can observe in Fig. 14 the same variations of the modulus and of the phase of the reflection coefficient as in Fig. 5 in the isotropic case (see Sec. II B 2). A critical value of the attenuation exists in the multilayered medium, for which the depth of the minimum of the reflection coefficient is at a maximum. In our case, this critical value is nearly the double of what we call "standard" attenuation (see Sec. III C 2). After this critical value the depth of the minimum decreases and the sign of the imaginary part of the pole changes.

We saw in Sec. II B 2 that the distance between the pole and the zero of the reflection coefficient does not vary a great deal and is obtained from the minimum of the numerator and the minima of the denominator of the modulus of the reflection coefficient. If we call $\left|B_{1}\right|$ the value of the numerator and $\left|B_{2}\right|$ the value of the denominator of the reflection coefficient at $\theta=\theta_{R}$ and $f=f_{R}$, we observe that $\left|B_{1}\right|+\left|B_{2}\right|$, when the attenuation is multiplied by a number inferior to 2 , and $\left|B_{1}\right|-\left|B_{2}\right|$, when the attenuation is multiplied by 3 or 5 , are quite constant for a $0^{\circ} / 45^{\circ} \% 0^{\circ} / 135^{\circ}$ at $\theta_{R}=27.75^{\circ}$ and $f_{R}=3 \mathrm{MHz}$ (see Table III).

When the reflection coefficient is represented as a function of the natural frequency, the conclusions are the same (see Fig. 15).

## E. Stratified medium submerged in water

In the multilayered Rayleigh area, that is to say, around $\theta=\theta_{R}$ and $f=f_{R}$, the reflection coefficient when the multilayered medium is submerged in water is the same as the one when it is semi-infinite. This is provided that the whole thickness is great enough. Indeed, above a certain thickness the medium responds as if it were infinite; however, the fluid below the stratified medium causes some perturbation, and as a result the modulus of the denominator of the reflection coefficient does not pass through a minimum.

## IV. CONCLUSION

The observation of an experimental trough in a reflection coefficient in water for a periodically stratified medium has led us to study the propagation of modes. Indeed, the modeled coefficient without absorption did not show a trough, whereas there is one with absorption. Thesc modes have a strong analogy with the leaky Rayleigh waves found in the isotropic case; The depth of the trough of the reflection coefficient is linked to the attenuation in the solid; the Rayleigh wave is related to the poles and the zeros of the reflection coefficient in water. The displacement of them is related to the minimum of the numerator and the minimum of the denominator of the reflection coefficient. In the isotropic case the Rayleigh wave is a combination of two inhomogeneous waves. By analogy, in the periodically multilayered case, the wave we called the multilayered Rayleigh wave is a combination of three inhomogeneous Floquet waves. These waves, which are the propagation modes of the infinite periodically multilayered medium, are thus used as classical waves propagating in homogeneous medium. As the Floquet waves are dispersive, the multilayered Rayleigh wave is dispersive. It is related to the poles and the zeroes of the reflection coefficient. When the nonlossy stratified medium is submerged in water, the vanishing of the Rayleigh mode can be interpreted as a lack of radiation in the medium sheltered from the emission. If the medium is not thick enough, the

| X | $-\mathrm{z}_{i 0} 0$ | $+\infty$ |
| :---: | :---: | :---: |
| z' | $0 \quad z_{0}^{\prime}$ | $+\infty$ |
| $\|\mathrm{f}(\mathrm{z})\|$ | $\longrightarrow \frac{\left\|\mathrm{B}_{1}\right\|}{\left\|\mathrm{B}_{2}\right\|}$ |  |
| Arg f(z) | $+\left.\pi\right\|_{-\pi} ^{-}$ | 0 |
| $N(x)$ | $\longrightarrow\left\|\mathrm{B}_{1}\right\|$ | $+\infty$ |
| $D(x)$ | $\longrightarrow\left\|\mathrm{B}_{2}\right\|$ | $+\infty$ |

FIG. 16. Variations of the modulus and the phase of the function $f(z)$, as the numerator $N(x)$ and the denominator $D(x)$ of this modulus.

Lamb mode appears: The pole and the zero of the reflection coefficient are not yet conjugated complexes. When the medium is thick enough the Lamb mode becomes a multilayered Rayleigh mode. It can be seen on different curves related to $0^{\circ} / 45^{\circ} / 90^{\circ} / 135^{\circ}$ carbon/epoxy composites. When the stratified medium is a lossy one, a critical attenuation, related to the imaginary part of the elastic constants, exists: The depth of the minimum of the coefficient reaches a maximum.

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## APPENDIX

Let us examine the following function of a single complex variable $z$ :

$$
\begin{align*}
& f(z)=\frac{z-z_{1}}{z-z_{2}} \\
& \quad \text { with }\left\{\begin{array}{l}
z=z^{\prime}+i z^{\prime \prime} \\
z_{1}=z_{0}^{\prime}+i z_{1}^{\prime \prime} \\
z_{2}=z_{0}^{\prime}+i z_{2}^{\prime \prime}
\end{array} \text { and }\left|z_{2}^{\prime \prime}\right| \geqslant\left|z_{1}^{\prime \prime}\right| .\right. \tag{A1}
\end{align*}
$$

If $f(z)$ is the reflection coefficient, $z$ represents the projection of the wave number of the incident wave on an axis parallel to the interface and contained in the propagation plane. $f(z)$ is a function of the single real variable $x$ $=z^{\prime}-z_{0}^{\prime}$ if it is represented only as a function of the real part of $z$. This is the case when the reflection coefficient is represented as a function of the angle of incidence for a homogeneous incident wave. We thus have

$$
\begin{gather*}
f(z)=\frac{z^{\prime}-z_{0}^{\prime}+i\left(z^{\prime \prime}-z_{1}^{\prime \prime}\right)}{z^{\prime}-z_{0}^{\prime}+i\left(z^{\prime \prime}-z_{2}^{\prime \prime}\right)}=\frac{x+i B_{1}}{x+i B_{2}} \\
\text { with }\left\{\begin{array}{l}
x=z^{\prime}-z_{0}^{\prime} \\
B_{1}=z^{\prime \prime}-z_{1}^{\prime \prime} \\
B_{2}=z^{\prime \prime}-z_{2}^{\prime \prime}
\end{array}\right. \tag{A2}
\end{gather*}
$$

The modulus of $f(z)$ is written in the following form:

$$
\begin{equation*}
|f(z)|=g(x)=\left(\frac{x^{2}+B_{1}^{2}}{x^{2}+B_{2}^{2}}\right)^{1 / 2}=\frac{N(x)}{D(x)} \tag{A3}
\end{equation*}
$$

and its phase is of the following form:

$$
\begin{equation*}
\arg [f(z)]=\operatorname{Arctan}\left(\frac{B_{1}}{x}\right)-\operatorname{Arctan}\left(\frac{B_{2}}{x}\right) \tag{A4}
\end{equation*}
$$

The variations of the modulus and of the phase of the function $f(z)$, as well as the numerator $N(x)$ and the denominator $D(x)$ of this modulus, are represented in Fig. 16 as a function of the variables $x$ or $z^{\prime}$.

This set of equations leads to the following conclusions.
Forcing $z^{\prime \prime}$ to zero corresponds to having a homogeneous incident wave: The wave number of the incident wave is real. We thus have $B_{1}=-z_{1}^{\prime \prime}$ and $B_{2}=-z_{2}^{\prime \prime}$.

If $z_{1}=z_{2}^{*}$, i.e., if the pole and the zero of the function are conjugated complexes, then $B_{1}=B_{2}$ and $\left|f\left(z_{0}^{\prime}\right)\right|=1$.

The distance between the pole and the zero of the function is equal to $\left|B_{1}\right| \pm\left|B_{2}\right|$ according to whether $z_{1}^{\prime \prime}$ is like or unlike $z_{2}^{\prime \prime}$ : This distance is then obtained from the minima of the numerator and of the denominator of the modulus of $f(z)$.

$$
\text { For } z^{\prime}=z_{0}^{\prime} \text { the phase of } f(z) \text { varies from } \pi \text { to }-\pi
$$

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