# Propagation in an anisotropic periodically multilayered medium 

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#### Abstract

An anisotropic multilayered medium is studied using the method of transfer matrices, developed by Thomson [J. Appl. Phys. 21, 89 (1950)] and Haskell [Bull. Seismol. Soc. Am. 43, 17 (1953)]. The propagation equations in each layer of the multilayered medium use the form developed by Rokhlin et al. [J. Acoust. Soc. Am. 79, 906-918 (1986); J. Appl. Phys. 59 (11), 3672-3677 (1986)]. Physical explanations are given, notably when a layer is made up of a monoclinic crystal system medium. The displacement amplitudes of the waves in one layer may be expressed as a function of those in another layer using a propagation matrix form, which is equivalent to relating the displacement stresses of a layer to those in another layer. An anisotropic periodically multilayered medium is then studied by using a propagation matrix that has particular properties: a determinant equal to one and eigenvalues corresponding to the propagation of the Floquet waves. An example of such a medium with the axis of symmetry of each layer perpendicular to the interfaces is then presented together with the associated reflection coefficients as a function of the frequency or of the incident angle.


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## INTRODUCTION

The aeronautic industry uses more and more composite materials, because of their better mechanical properties for a given weight, such as carbon epoxy. They are often made from an assembling of layers in which the carbon fibers all have the same orientation, each layer being itself at a given angle from those which surround it. These multilayered media are thus anisotropic, and the ultrasonic propagation of such media has formed the subject of a number of works, often restricted to isotropic layers. The solutions are thus obtained simply, notably by the use of the propagation matrix form first developed by Thomson, ${ }^{1}$ then furthered by Haskell ${ }^{2}$ and afterwards by Gilbert and Backus. ${ }^{3}$ This form allows one to express the displacement and the stresses at an interface between two layers from those assumed known at the interface between the other two layers. Notably, Schoenberg ${ }^{4}$ has studied fluid periodically stratified media and further, Gatignol, Rousseau, and Moukemaha have studied the propagation in an isotropic periodically stratified media, ${ }^{5-7}$ for a given incidence, and have obtained solutions involving Floquet waves. Lhermitte has obtained results for the propagation of an elastic shear wave normal to the interfaces in a crossply fiber reinforced composite, for elastic wavelengths comparable to or different from the composite period. ${ }^{8}$ Schoenberg has studied stratified anisotropic media, ${ }^{9}$ with applications to geophysics, and Helbig has treated the case of transversely isotropic media. ${ }^{10}$ The important works of Chimenti and Nayfeh ${ }^{11,12}$ have related to anisotropic media in which the most general symmetry corresponds to a monoclinic crystal system in which the axis of symmetry is perpendicular to the interfaces. They have obtained particular properties about propagation matrices: determinant equal to 1 , reciprocal eigenvalues to each other, and have
shown up the relationships between generalized Floquet waves and solutions to the wave equation. We have built upon this foundation, a very general form on the propagation in stratified media where each layer has any given anisotropy and any given thickness. The amplitudes of displacements of the waves in each layer are expressed in terms of those in the next layer. The propagation equations use the form developed by Rokhlin et al. ${ }^{13,14}$ and completed by Ribeiro et al. ${ }^{15,16}$ We have then studied a periodically multilayered media made from the repetition of the above-studied stratified media, which leads us, by using propagation matrices, to study Floquet waves and to show the specific properties of such materials.

## I. FORMALISM USED FOR PROPAGATION EQUATIONS

In any stratified media (see Fig. 1), we can study the acoustic propagation of waves that are generated by an oblique incident wave propagating in the media 0 . Each layer of the stratified media has any given thickness. We define the following: $x_{3}$ is the axis of stacked layers; $q$ is the layer number: $1 \leqslant q \leqslant Q ; h_{q}$ is the thickness of the layer; $z_{q}$ is the position on the $x_{3}$ axis of the interface between two layers $q$ and $q+1$ such as $z_{q}=z_{q-1}+h_{q}$ and $z_{0}=0$.

Media 0 and $Q+1$ above and below the stratified media are semi-infinite. The propagation equations in each layer use the same form as the one developed by Rokhlin et al. ${ }^{13,14}$ and completed by Ribeiro et al. ${ }^{15,16}$ with the help of inhomogeneous waveform. ${ }^{17,18}$

## A. Slowness vector and slowness surface

Generally speaking, three plane waves with different velocities propagate in an anisotropic media $q$, for any given direction of propagation $\mathbf{n}^{q}$ : two are named quasi-


FIG. 1. Multilayered media.
transverse and the other quasilongitudinal. ${ }^{19,20}$ A normalized polarization vector $\mathbf{P}^{q}$ is associated with each wave. The three polarization vectors constitute an orthonormal basis. It is useful, for each layer $q$, to introduce the slowness vector $\mathrm{m}^{q}$, which can be defined by the following relation:

$$
\begin{equation*}
\mathbf{m}^{q}=\mathbf{n}^{q} / V^{q}, \tag{1}
\end{equation*}
$$

where $\mathbf{n}^{q}$ is the wave direction of propagation in the layer $q$ and $V^{q}$ is the wave propagation velocity in the layer $q$, often called phase velocity.

If the wave is monochromatic, it is related to the wavenumber vector by the following relation:

$$
\begin{equation*}
\mathbf{m}^{q}=\mathbf{k}^{q} / \omega \tag{2}
\end{equation*}
$$

where $\mathbf{k}^{q}$ is the wave-number vector of the wave in the layer $q$ and $\omega$ is the natural frequency of the incident wave.

The ends of the slowness vector $\mathrm{m}^{4}$, drawn from a fixed point 0 is named the slowness surface. One sheet of the slowness surface matches with each wave. For each material making up the layers, three sheets of the slowness surface are therefore obtained, of which the intersections with a plane are presented in Fig. 2.

Let us consider an oblique incident wave and choose the $0 x_{1}$ axis such that the direction of propagation of this wave is in the plane $x_{1} 0 x_{3}$ defined in Fig. 1. This choice does not change the generality of the problem. In each layer the wave generates several waves numbered by ( $\eta$ ). In the chosen reference system (see Fig. 1), the SnellDescartes law imposes that the projection of the wavenumber vector (or of the slowness vector) upon a plane parallel to interfaces would be maintained, i.e., ${ }^{(\eta)} m_{1}^{1}={ }^{(\eta)} m_{1}^{2}=\cdots={ }^{(\eta)} m_{1}^{q}$ and ${ }^{(\eta)} m_{1}^{q}=0, \forall \eta, \forall q$. There is an equality of the parallel components to the $x_{1}$ axis of the slowness vectors of all existing waves; we will call this component $m_{1}$.

The intersection of the line $x_{1}=m_{1}$ with the three sheets of the slowness surface gives us six slowness vectors. A maximum number of six waves therefore exist in each layer. In the case of two semi-infinite media separated by


FIG. 2. Three sheets of a slowness surface.
an interface, it is necessary to choose among the six waves which are consistent with free field boundary conditions. ${ }^{13,15}$ Among the six waves, three will be selected by these criterion and matched to the waves actually generated at the considered interface. In our case, each layer lies between two plane interfaces, the different reflections between them implying that, generally, six waves exist. The datum index $\eta$, of each one of these waves, thus varies from 1 to 6.

For a given wave and incidence, it may not have any intersection with one or several slowness surfaces traces. Nevertheless, the matching waves exist, even though they cannot be determined by this geometric construction and their slowness vectors are complex. ${ }^{15-18}$ The waves are attenuated following a direction perpendicular to the interfaces and are called inhomogeneous waves.

## B. Propagation equation

The plane wave propagation equation or Christoffel equation is written ${ }^{19,20}$ by the use of Einstein's convention consisting in summing indexes twice repeated:

$$
\left(c_{i j k l}^{q} \cdot{ }^{(\eta)} n_{j}^{q} \cdot(\eta)_{l}^{q}-\rho^{q} \cdot\left({ }^{(\eta)} V^{q}\right)^{2} \cdot \delta_{i k}\right) \cdot(\eta) P_{k}^{q}=0,
$$

where ${ }^{(\eta)} \mathrm{P}^{q}$ is the polarization vector of the wave $(\eta)$ in the layer $q,{ }^{(\eta)} V^{q}$ is the propagation velocity of the wave $(\eta)$ in the layer $q,{ }^{(\eta)} n^{q}$ is the direction of propagation vector of the wave ( $\eta$ ) in the layer $q, \rho^{q}$ is the volumetric mass of the material in the layer $q$, and $c_{i j k l}^{q}$ is the elastic constants of the layer $q$.

This equation can also be written as a function of the slowness vector ${ }^{13}$

$$
{ }^{(\eta)} G_{i k}^{q} \cdot{ }^{(\eta)} P{ }_{k}^{q}=0
$$

with

$$
\begin{equation*}
{ }^{(\eta)} G_{i k}^{q}=c_{i j k l}^{q} \cdot(\eta) m_{j}^{q} \cdot(\eta) m_{l}^{q}-\rho^{q} \cdot \delta_{i k} . \tag{3}
\end{equation*}
$$

The elastic constants $c_{i j k l}^{q}$ can be registered by two other indexes $\alpha$ and $\beta$ varying from 1 to 6 such as: $c_{a \beta}^{q}$ : $=c_{i j k l}^{q}$ (Ref. 20) with $(i j) \leftrightarrow \alpha(k l) \leftrightarrow \beta$, and (11) $\leftrightarrow 1$ $(22) \leftrightarrow 2(33) \leftrightarrow 3(23) \leftrightarrow 4(13) \leftrightarrow 5(12) \leftrightarrow 6$.

## C. Determination of slowness and polarization vectors

Following the above, for a given incident wave, $m_{1}$ is known $\forall \eta, \forall q$ (Snell-Descartes' law). Moreover, ${ }^{(\eta)} m_{2}^{q}=0 \forall \eta, \forall q$. The only unknown slowness vector component of the wave ( $\eta$ ) in the layer $q$ is thus ${ }^{(\eta)} m$. But Eq . (3) has nonzero solutions for ${ }^{(\eta)} P{ }^{q}$ if the determinant of ${ }^{(7)} G_{i k}^{q}$ is zero. Reducing the $6 \times 6$ determinant to zero yields an equation of the sixth degree for ${ }^{(\eta)} m$ with real factors if the attenuation is neglected so that the rigidities are real. The resolution of this equation, made clear in Ref. 14, gives six real or complex conjugate each to each root, which allow us to obtain the six slowness vectors for each wave ( $\eta$ ) in the layer $q$. Another method, developed by Mandal ${ }^{21}$ yields both eigenvalues and eigenvectors.

The slowness vector allows us as well, according to Eq. (1), to obtain the wave propagation velocity, ${ }^{(\eta)} V^{q}$, and the direction of propagation,

$$
\begin{equation*}
{ }^{(\eta)} \mathbf{n}^{q}={ }^{(\eta)} \mathbf{m}^{q} /\left\|^{(\eta)} \mathbf{m}^{q}\right\| . \tag{4}
\end{equation*}
$$

The Hermitian norm || || is defined by

$$
\begin{equation*}
\|\mathbf{m}\|^{2}=m_{k} \cdot m_{k}^{*}, \tag{5}
\end{equation*}
$$

where $m_{k}^{*}$ is the conjugate complex of $m_{k}$.
If the slowness vector is complex, ${ }^{15,16}$ i.e., if the wave is inhomogeneous, ${ }^{(\eta)} \mathbf{m}^{q}$ can be written in the form: ${ }^{17,18(\eta)} \mathbf{m}^{q}$ $=\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime}+i\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime \prime}$ with $\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime}$ and $\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime \prime}$ belonging to $\mathbf{R}^{3}$. If the wave is monochromatic, $\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime}$ is the propagative part and $\left({ }^{(\eta)} \mathrm{m}^{q}\right)^{\prime \prime}$ the attenuative part of ${ }^{(\eta)} \mathbf{m}^{q}:$ a harmonic plane wave propagates in the direction of $\left({ }^{(\eta)} \mathbf{m}^{q}\right)^{\prime}$ and attenuates exponentially in the direction of $\left({ }^{(\eta)} \mathrm{m}^{q}\right)^{\prime \prime}$.

For each wave ( $\eta$ ) for which the slowness vector has been determined, Eq. (3) enables us to calculate ${ }^{(\eta)} P_{k}^{q}=\left({ }^{(\eta)} P_{k}^{q}\right)^{\prime}+i\left({ }^{(\eta)} P{ }_{k}\right)^{\prime \prime}$ to the nearest factor. This vector is then normalized by means of a Hermitian norm. The wave $(\eta)$ is polarized elliptically: the particle motion follows an ellipse that has axes $\left({ }^{(\eta)} \mathbf{P}^{q}\right)^{\prime}$ and $\left({ }^{(\eta)} \mathbf{P}^{q}\right)^{\prime \prime}$ (Refs. 16 and 18).

## D. Monoclinic crystal system media

If each layer is a monoclinic crystal system media with a second-order axis perpendicular to the interfaces (or with a mirror plane parallel to these), the equation for ${ }^{(\eta)} m_{3}^{q}$ is an even polynomial: the roots are opposite to each other; It is the case treated by Ref. 11. Indeed, any rotation $\pi$ about this axis (or any symmetry with respect to this plane) leads the media to an indistinguishable configuration from the first one. Such a transformation is equivalent to changing the sign of $m_{1}$ or ${ }^{(7)} m_{3}^{q}$, which is equal in value, taking into account that the waves propagate in the

FIG. 3. Propagation in a layer with a binary axis perpendicular to its interface.
same way in two opposite directions. Then ${ }^{(\eta)} m_{3}^{q}$ and $-^{(\eta)} m_{3}^{q}$ are both roots of Eq. (3), which means that this equation is an even polynomial ( ${ }^{(\eta)} m_{3}^{q}$ can be either real or complex). These facts can be summarized in Fig. 3: if a wave with a ${ }^{(1)} \mathrm{m}$ vector propagates in the considered layer and if this layer has a binary axis perpendicular to their interfaces, a wave with a ${ }^{(2)} \mathrm{m}$ vector or a wave with an opposite vector ${ }^{(2)} \mathrm{m}^{\prime}$ will propagate in this layer; one can change from ${ }^{(1)} m$ to ${ }^{(2)} m^{\prime}$ by changing the sign of the third component, $m_{3}$ and $-m_{3}$ are both roots of the equation which is therefore an even polynomial.

With a mirror symmetry (reciprocal binary axis), the reasoning is even simpler than the preceding case. This expresses in a physical sense that for any incident wave on a plane interface separating both such media, a symmetrical reflected wave with respect to the normal exists. It is of course usual for an isotropic media.

All the materials similar to crystal materials, which have an even-order axis perpendicular to the interface, have this property (tetragonal, hexagonal, cubic crystal systems). Only a third-order axis (trigonal system) does not imply this property.

## II. TRANSMISSION EQUATIONS FROM ONE LAYER TO THE NEXT

The change from a layer $q$ to a layer $q+1$ can be done by expressing the relation between the displacements and the stresses from one interface to another, for a layer $q$, or by expressing the displacement amplitudes of the six waves in each layer $q$ as a function of those in layer $q+1$. First, we shall see how to express a six-dimensional vector that characterizes the displacements stresses in a layer $q$ (Sec. II A), then we shall relate this vector at the lowest interface of the layer $q$ to the one at the upper interface of the same layer, with a propagation matrix (Sec. II B). Lastly, we will relate this propagation matrix of displacement stresses to the one of total displacement amplitudes ( Sec . II C).

## A. Displacements and stresses

In a classical way, for a monochromatic plane wave with natural frequency $\omega$, the displacement of the wave $\eta$ in the layer $q$ can be expressed as a function of the wavenumber vector ${ }^{(\eta)} \mathbf{k}^{q}$ :

$$
\begin{equation*}
{ }^{(\eta)} \mathbf{u}^{q}={ }^{(\eta)} a^{q \cdot(\eta)} \cdot \mathbf{P}^{q} \cdot e^{\left.-i i^{(\eta)} \mathbf{k}^{q} \cdot \mathbf{x}{ }^{\omega t}\right)}, \tag{6}
\end{equation*}
$$

where ${ }^{(\eta)} a^{q}$ is the complex amplitude of the particle displacement tied to the wave $(\eta)$ in the layer $q$.

By introducing the slowness vector, the displacement vector of the wave ( $\eta$ ) in the layer $q$ can be written as

$$
\begin{equation*}
{ }^{(\eta)} \mathbf{u}^{q}={ }^{(\eta)} a^{q \cdot(\eta)} \mathbf{P}^{q} \cdot e^{\left.-i \omega)^{(\eta)} \mathbf{m}^{q} \cdot \mathbf{x}-t\right)} \tag{7}
\end{equation*}
$$

The total particle displacement is the sum of the displacements tied to each wave:

$$
\begin{equation*}
\mathbf{u}^{q}=\sum_{\eta=1}^{\mathbf{6}}{ }^{(\eta)} \mathbf{u}^{q} \tag{8}
\end{equation*}
$$

Stresses are expressed as a function of displacements by

$$
T_{i j}^{q}=c_{i j k l}^{q} \cdot \frac{\partial u_{k}^{q}}{\partial x_{1}}
$$

We will set $T_{\alpha}^{q}=T_{l j}^{q}$ with the same convention of change of index as in Sec. I B.

## B. Propagation matrix of displacements and stresses

\{ \} a six-dimensional column vector
( ) a six-dimensional line vector
[] a $(6 \times 6)$ matrix
$T$ transpose operation
$X_{\alpha \beta}^{q} \quad$ the coefficient of the matrix $[X]$ at $\alpha$ th row and $\beta$ th column.
Subsequently, we will use the following notation:
Let $\left\{\mathbf{W}^{q}\left(x_{3}\right)\right\}$ be the ( $6 \times 1$ ) column vector made up of the three components of the vector $\mathbf{u}^{q}$ and the three components ( $T_{3}^{q}, T_{4}^{q}, T_{5}^{q}$ ) of the stress vector applied to a surface parallel to the interfaces, and in the layer $q$ for $z_{q-1}<x_{3}<z_{q}$ and let $\left\{\mathscr{A}^{q}\right\}$ be the ( $6 \times 1$ ) column vector containing the amplitudes ${ }^{(\eta)} a^{q}$ of the six waves propagating in the layer $q$ :

$$
\left\{\mathbf{W}^{q}\left(x_{3}\right)\right\}=\left\langle u_{1}^{q}, u_{2}^{q}, u_{3}^{q}, T_{3}^{q}, T_{4}^{q}, T_{5}^{q}\right\rangle^{T}
$$

and

$$
\left\{\mathscr{A}^{q}\right\}=\left\langle{ }^{(1)} a^{q},{ }^{(2)} a^{q},{ }^{(3)} a^{q},{ }^{(4)} a^{q},{ }^{(5)} a^{q},{ }^{(6)} a^{q}\right\rangle^{T}
$$

The column vector $\left\{\mathbf{W}^{q}\left(x_{3}\right)\right\}$ can be expressed as a function of $\left\{\mathscr{A}^{q}\right\}$ using the following equation:

$$
\begin{align*}
\left\{\mathbf{W}^{q}\left(x_{3}\right)\right\}= & {\left[\mathbf{B}^{q}\right] \cdot\left[\mathscr{H}^{q}\left(x_{3}-z_{q-1}\right)\right] } \\
& \cdot\left\{\mathscr{A}^{q}\right\} \cdot e^{-i \omega \cdot\left(m_{1} \cdot x_{1}-t\right)}, \tag{9}
\end{align*}
$$

with

$$
\left[\mathscr{H}^{q}(s)\right]=\operatorname{diag}\left(e^{-i \omega \cdot(\eta)} m_{3}^{q} \cdot s\right),
$$

( $6 \times 6$ ) diagonal matrix $\eta=1, \ldots, 6 \mathbf{B}_{\alpha \eta}^{q}=\mathbf{A}_{\alpha \eta}^{q}$, for $\alpha=1,2,3$ and $\mathbf{B}_{\alpha \eta}^{q}=-i \omega \mathbf{A}_{\alpha \eta}^{q}$, for $\alpha=4,5,6$, where $\left[\mathbf{A}^{q}\right]$ is given in the Appendix.

The matrix relation (9) enables us to express the displacement stresses at $x_{3}=z_{q}$, as a function of those at $x_{3}$ $=z_{q-1}$, by eliminating the common amplitudes $\left\{\mathscr{A}^{q}\right\}$ :

$$
\begin{aligned}
& \left\{\mathbf{W}^{q}\left(z_{q-1}\right)\right\}=\left[\mathbf{B}^{q}\right] \cdot\left\{\mathscr{A}^{q}\right\} \cdot e^{-i \omega \cdot\left(m_{1} \cdot x_{1}-t\right)} \\
& \left\{\mathbf{W}^{q}\left(z_{q}\right)\right\}=\left[\mathbf{B}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left\{\mathscr{A}^{q}\right\} \cdot e^{-i \omega \cdot\left(m_{1} \cdot x_{1}-t\right)}
\end{aligned}
$$

where

$$
\left[\mathscr{H}^{q}\right]=\left[\mathscr{H}^{q}\left(h_{q}\right)\right],
$$

hence

$$
\begin{equation*}
\left\{\mathbf{W}^{q}\left(z_{q}\right)\right\}=\left[\mathbf{B}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left[\mathbf{B}^{q}\right]^{-1}\left\{\mathbf{W}^{q}\left(z_{q-1}\right)\right\} \tag{10}
\end{equation*}
$$

The matrix $\left[X^{q}\right]=\left[B^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left[B^{q}\right]^{-1}$ is the propagation matrix in the layer $q$.

## C. Propagation matrix of displacement amplitudes

One can as well express the displacement amplitudes in the layer $q$ as a function of those in the layer $q+1$, instead of expressing the displacement stresses of a layer as a function of those in the next layer. In order to do that, boundary conditions are written at the interface $x_{3}=z_{q}$, i.e., equal displacement and stress vectors at the interface.

The principle of the calculation is the following: displacements and stresses are expressed for the two successive layers as $q\left(z_{q-1}<x_{3}<z_{q}\right)$ and $q+1\left(z_{q}<x_{3}<z_{q+1}\right)$.

According to Eq. (8), the displacement of each wave in two successive layers $q$ and $q+1$ can be expressed. The displacements are equal at the interface $x_{3}=z_{q}$, which gives us three equations:

$$
\begin{gather*}
\sum_{\eta=1}^{6}{ }^{(\eta)} a^{q \cdot(\eta)} \mathbf{P}^{q} \cdot e^{-i \omega \cdot(\eta)} m_{3}^{q} \cdot h_{q} \\
=\sum_{\eta=1}^{6}{ }^{(\eta)} a^{q+1} \cdot(\eta) \mathbf{P}^{q+1} \tag{11}
\end{gather*}
$$

By expressing the stress vector applied to a surface parallel to the interface, which has components ( $T_{3}^{q}, T_{4}^{q}, T_{5}^{q}$ ) in both successive layers $q$ and $q+1$, and by setting them equal at the interface $x_{3}=z_{n}$, three other equations are obtained, with $\alpha=3,4,5$ :

$$
\begin{align*}
& \sum_{\eta=1}^{6}\left(\left(c_{\alpha 1}^{q} \cdot m_{1} \cdot{ }^{(\eta)} P_{1}^{q}+c_{\alpha 3}^{q} \cdot(\eta) m_{3}^{q} \cdot{ }^{(\eta)} P_{3}^{q}+c_{\alpha 4}^{q} \cdot(\eta) m_{3}^{q} \cdot(\eta) P_{2}^{q}+c_{\alpha 5}^{q} \cdot\left({ }^{(\eta)} m_{3}^{q} \cdot{ }^{(\eta)} P_{1}^{q}+m_{1} \cdot{ }^{(\eta)} P_{3}^{q}\right)\right.\right. \\
& \left.\left.+c_{\alpha 6}^{q} \cdot m_{1} \cdot{ }^{(\eta)} P_{2}^{q}\right)\right) \cdot{ }^{(\eta)} a^{q} \cdot e^{-i \omega \cdot(\eta)} m_{3}^{q} \cdot h_{q}=\sum_{\eta=1}^{6}\left(\left(c_{\alpha 1}^{q+1} \cdot m_{1} \cdot(\eta) P_{1}^{q+1}+c_{\alpha 3}^{q+1} \cdot(\eta) m_{3}^{q+1} \cdot(\eta) \quad P_{3}^{q+1}\right.\right. \\
& +c_{a 4}^{q+1} \cdot(\eta) m_{3}^{q+1} \cdot{ }^{(\eta)} P_{2}^{q+1}+c_{\alpha 5}^{q+1} \cdot\left({ }^{(\eta)} m_{3}^{q+1} \cdot{ }^{(\eta)} P_{1}^{q+1}+m_{1} \cdot{ }^{(\eta)} P_{3}^{q+1}\right) \\
& \left.\left.+c_{\alpha 6}^{q+1} \cdot m_{1} \cdot(\eta) P_{2}^{q+1}\right)\right) \cdot{ }^{(\eta)} a^{q+1} . \tag{12}
\end{align*}
$$

In these equations, the only unknowns are the displacement amplitudes of the waves ${ }^{(\eta)} a^{q}$.

Equations (11) and (12) permit us to express the amplitudes of the waves in the layer $q$ as functions of those in the layer $q+1$, by matrices $\left[\mathbf{A}^{q}\right]$ and $\left[\mathscr{H}^{q}\right]$ which have been defined previously, such as:

$$
\begin{equation*}
\left\{\mathscr{A}^{q+1}\right\}=\left[\mathbf{A}^{q+1}\right]^{-1} \cdot\left[\mathbf{A}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left\{\mathscr{A}^{q}\right\} \tag{13}
\end{equation*}
$$

The matrix $[\mathscr{H} q]$ is just the propagation matrix of the layer $q$ at $s=h_{q},{ }^{5,22}$ it has the following properties:

$$
\left[\mathscr{H}^{q}(0)\right]=\mathscr{I},
$$

where $\mathscr{I}$ is the $(6 \times 6)$ identity matrix,

$$
\begin{aligned}
& {\left[\mathscr{H}^{q}(s)\right]^{-1}=\left[\mathscr{H}^{q}(-s)\right],} \\
& {\left[\mathscr{H}^{q}\left(s_{1}+s_{2}\right)\right]=\left[\mathscr{H}^{q}\left(s_{1}\right)\right] \cdot\left[\mathscr{H}^{q}\left(s_{2}\right)\right] .}
\end{aligned}
$$

The matrix $\left[\mathscr{H}^{q}\right]$ caracterizes the amplitude propagation of the waves in the layer $q$. It is expressed in another basis by the matrix $\left[X^{q}\right]$. These matrices only do not depend on the other layers. Both methods made clear in Secs. II B and II C are equivalent: indeed, in the first case, the matrix $\left[\mathrm{B}^{q}\right.$ ] is expressed as a function of the matrix [ $\mathbf{A}^{q}$ ] used in the second case for the amplitude displacements.

The column vector $\left\{\mathscr{A}^{q}\right\}$ of the displacement amplitudes of waves enables a more direct approach to the propagation in the layer $q$ than the column vector $\left\{\mathbf{W}^{q}\left(x_{3}\right)\right\}$ of the displacement stresses. ${ }^{5,7}$


FIG. 4. Periodically multilayered media.

## III. APPLICATION TO A PERIODICALLY MULTILAYERED MEDIA

Now let us consider a periodically multilayered media that is a reproduction of $P$ "superlayers," each one made up by the stacking of $Q$ distinct anisotropic media, studied previously in Secs. I and II (see Fig. 4).

Here, $q$ is the number of the layer: $1 \leqslant q \leqslant Q ; p$ is the number of a "superlayer": $1 \leqslant p \leqslant P ; n$ is the number of the layer, $n=(p-1) Q+q ; h_{q}$ is the thickness of the layer; and $z_{n}$ is the position on the $x_{3}$ axis of the interface between two layers $q$ and $q+1$ such as: $z_{n}=z_{n-1}+h_{q}$ and $z_{0}=0$.

In order to precisely characterize the waves propagating in a layer $q$ in a "superlayer" $p$, we are led to complicate a little the notation introduced previously for the displacement vector and the displacement of the wave $(\eta)$ in the layer $n$. All the other parameters depend only upon the nature of the material $q$ and not on its position in the multilayered media.

We will use the following notation:
${ }^{(\eta)} a^{p, q} \quad$ complex amplitude of the particle displacement tied to the wave ( $\eta$ )
in the layer $n$,
$\left\{\mathscr{A}^{p, q}\right\} \quad(6 \times 1)$ column vector of ${ }^{(\eta)} a^{p, q}$,
${ }^{(\eta)} \mathbf{u}^{p, q} \quad$ displacement vector of the wave $(\eta)$ in the layer $n$.
Equation (13) is written, with the new notations:

$$
\begin{equation*}
\left[\mathbf{A}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left\{\mathscr{A}^{p, q}\right\}=\left[\mathbf{A}^{q+1}\right] \cdot\left\{\mathscr{A}^{p, q+1}\right\} \tag{14}
\end{equation*}
$$

Equation (13) also allows us to write the equation corresponding to an interface separating two superlayers, by substituting in the left-hand member $q$ for $Q$, and in the right-hand member $q$ for 1 and $p$ for $p+1$.

Looking for the propagation matrix [ $\Phi$ ], which permits us to express the amplitudes of the first layer of the superlayer $p$ as a function of those in the first layer of the "superlayer" $p+1$, i.e.,

$$
\begin{equation*}
\left\{\mathscr{A}^{p+1,1}\right\}=[\Phi] \cdot\left\{\mathscr{A}^{p, 1}\right\}, \tag{15}
\end{equation*}
$$

we obtain

$$
\begin{align*}
{[\Phi]=} & {\left[\mathbf{A}^{1}\right]^{-1} \cdot\left(\prod_{q=Q}^{2}\left[\mathbf{A}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left[\mathbf{A}^{q}\right]^{-1}\right) } \\
& \cdot\left[\mathbf{A}^{1}\right] \cdot\left[\mathscr{H}^{1}\right] \tag{16}
\end{align*}
$$

or else

$$
\begin{align*}
{[\tau] } & =\left[\mathbf{A}^{1}\right] \cdot[\Phi] \cdot\left[\mathbf{A}^{1}\right]^{-1} \\
& =\prod_{q=Q}^{1}\left[\mathbf{A}^{q}\right] \cdot\left[\mathscr{H}^{q}\right] \cdot\left[\mathbf{A}^{q}\right]^{-1} . \tag{17}
\end{align*}
$$

From Ref. 23, [ $\tau$ ] can be written in the following form:

$$
\begin{equation*}
[\tau]=e^{i[\mathrm{~A} \mid h}, \quad h=h_{1}+\cdots+h_{Q} \tag{18}
\end{equation*}
$$

where [ $\mathbf{A}$ ] is a constant matrix in which the eigenvalues are the Floquet wave numbers.

It can be seen that $[\tau]$ is the product of propagation matrices that depend, for a given frequency $\omega$ and incidence (thus fixed $m_{1}$ ), only on material properties for each layer $q$ of the "superlayer" $p$.

The eigenvalues ${ }^{(\eta)} \lambda$ of $[\tau]$, which are of course the same as those of [Ф], are related to the Floquet wave numbers through the medium of the Floquet slownesses associated to the Floquet wave numbers ${ }^{(\eta)} m_{f}$ :

$$
\begin{equation*}
{ }^{(\eta)} \lambda=e^{i \omega \cdot(\eta)} m_{f} \cdot h . \tag{19}
\end{equation*}
$$

The Floquet slowness vectors have thus, for a given incidence corresponding to $m_{1},{ }^{(\eta)} m_{f}$ as the component following the $x_{3}$ axis. That is to say that the dispersion equation for a periodically multilayered media corresponds to the characteristic equation of the matrix $[\tau]{ }^{23-25}$ The Floquet waves are linear combinations of the waves propagating in each homogeneous layer. These waves are propagation modes of the infinite periodically multilayered media. It is thus possible to consider a finite periodically multilayered media as a homogeneous material in which the Floquet waves propagate. The resolution of such a problem therefore, after having determined the characteristics of propagation of Floquet waves, amounts to looking for a solution, a linear combination of these waves, which satisfies the boundary conditions of the media.

By recurrence, one can find

$$
\begin{equation*}
\left\{\mathscr{A}^{P, Q}\right\}=\left[\mathscr{H}^{Q}\right]^{-1} \cdot\left[\mathbf{A}^{Q}\right]^{-1} \cdot\left[\mathbf{A}^{1}\right] \cdot[\Phi]^{P}\left\{\mathscr{A}^{1,1}\right\} . \tag{20}
\end{equation*}
$$

We have developed a very general program in order to obtain, among others, the reflection and transmission coefficients in the medium surrounding the periodically multilayered medium.

## IV. INFLUENCE OF THE MEDIA SURROUNDING THE MULTILAYERED MEDIA

## A. Reflection and transmission coefficients in the water

The amplitude of the incident wave is supposed equal to 1 , which avoids a change in notations for the coefficients of the waves. The periodically multilayered media is surrounded by water (see Fig. 5).

By applying to water the boundary condition equations (11) and (12), we obtain 8 boundary condition equations: 4 for the water/(layer $n=1, p=1, q=1$ ) interface and 4 for the (layer $n=N, p=P, q=Q$ )/water interface.

Equation (20) provides 6 supplementary equations, giving us in all 14 equations, allowing us to obtain the 14 unknowns:

| $\left\{{ }^{(\eta)} a^{1,1}\right\}$ | $: 6$ unknowns |
| ---: | :--- |
| $\left\{{ }^{(\eta)} a^{P, Q}\right\}$ | $: 6$ unknowns |
| $\mathscr{R}$ and $\mathscr{T}$ | $: 2$ unknowns |

14 unknowns.

As we have, by the matrices [ $\mathscr{H}^{q}$ ] and/or [ $\Phi$ ], the displacement amplitudes ${ }^{(\eta)} a^{p, q}$ of the wave ( $\eta$ ) in each


FIG. 5. Periodically multilayered mediasurrounded by water.
layer $n$ of the multilayered media, it is possible to calculate the amplitudes for each layer $n$ and therefore to know the profile of the displacement of the waves.

## B. Reflection and transmission coefficients in anisotropic media

If the media surrounding the periodically multilayered media are anisotropic, the method of resolution is the same as above, but the number of equations is different; in effect, by applying boundary condition equations (11) and (12) to an anisotropic media, 12 not 8 boundary condition equations are obtained: 6 for the anisotropic media/(layer $n=1, p=1, q=1$ ) interface and 6 for the (layer $n=N$, $p=P, q=Q) /$ anisotropic media interface.

Equation (20) provides the 6 supplementary equations, giving us in all 18 equations, allowing us to obtain the 18 unknowns. We shall call $\mathscr{R}$ the three reflection coefficients and $\mathscr{T}$ the three transmission coefficients:

| $\left\{\begin{array}{c}(\eta) \\ \left.a^{1,1}\right\} \\ \left\{\begin{array}{c}(\eta) \\ \left.a^{P, Q}\right\}\end{array}\right. \\ : 6 \text { unknowns } \\ \mathscr{R}\end{array}\right.$ | $: 3$ unknowns |
| :---: | :---: |
| $\mathscr{T}$ | $: 3$ unknowns |

## 18 unknowns.

## C. Case of Lamb waves

If the medium surrounding the periodically multilayered media is a vacuum, boundary condition equations lead to a homogeneous system, as the stresses at $x_{3}=0$ and $x_{3}=P \cdot h$ are zero values. Reducing to zero the determinant $(12 \times 12)$ gives us then Lamb modes.

## V. APPLICATION TO A PERIODICALLY MULTILAYERED MEDIA $0^{\circ} / 90^{\circ}$

Now, as an illustration of the possibilities of our general program, let us consider a media made up by stacked identical orthotropic layers, each layer being at $90^{\circ}$ from


FIG. 6. Reflection coefficient in water for a periodically multilayered medium $0^{\circ} / 90^{\circ}$ made from layers of carbon epoxy as a function of the frequency.
the previous, immersed in water. The symmetry of the media leads to slownesses of which the third components are opposed or conjugate to each other, as explained in Sec. I D.

If we consider that the motion is effected in the plane ( $x_{1} 0 x_{3}$ ), by only keeping equations corresponding to $u_{1}, u_{3}$, $T_{3}$, and $T_{5}, 4 \times 4$ matrices are obtained.

In this case, the propagation matrix has particular properties: $\operatorname{det}[\tau]=1, \eta$ and $1 / \eta$ are eigenvalues of $[\tau]$, which is the result of the physical explanation in Sec. I D: ${ }^{(\eta)} m^{q}$ and $-{ }^{(\eta)} m^{q}$ are both roots of Eq. (3). ${ }^{11}$

An example of a reflection coefficient in water for a periodically multilayered medium made from layers of carbon epoxy is given in Fig. 6, as a function of the frequency. The material used is an hexagonal crystal system medium with five independent elastic constants. If the sixth-order symmetry $A_{6}$ axis is parallel to the $0 x_{3}$ axis, these constants are: ${ }^{26}$

$$
\begin{aligned}
& c_{11}=13.5 \mathrm{GN} / \mathrm{m}^{2} ; \quad c_{12}=6.3 \mathrm{GN} / \mathrm{m}^{2} \\
& c_{13}=5.5 \mathrm{GN} / \mathrm{m}^{2} ; \quad c_{33}=125.9 \mathrm{GN} / \mathrm{m}^{2} ; \\
& c_{44}=6.2 \mathrm{GN} / \mathrm{m}^{2} .
\end{aligned}
$$

The volumetric mass of the carbon-epoxy is 1577 $\mathrm{kg} / \mathrm{m}^{3}$ and the thickness of each layer of a superlayer is equal to 0.13 mm . This representation permits us to observe easilly stopping bands in frequency $(\mathscr{R}=1)$. The minima of the reflection coefficient correspond to Lamb modes. Taking many reflection coefficients as the one shown in Fig. 6 and recording the minima as a function of the incident angle, permits us to obtain a dispersionlike plot of the ultrasonic reflection behavior. Nayfeh and Chimenti have presented results of the same kind. ${ }^{10}$ In the isotropic case and at normal incidence, Moukemaha ${ }^{7}$ has shown that the number of oscillations between two stopping bands is equal to the number of superlayers minus one. As an extension of this result we just can say that the number of oscillations is related to the number of superlayers.

Two examples of reflection coefficients in water for the same periodically multilayered medium are given in Fig. 7,


FIG. 7. Reflection coefficients in water for a periodically multilayered medium $0^{\circ} / 90^{\circ}$ made from layers of carbon epoxy as a function of the incident angle.
as a function of the angle of incidence. The minimum of the reflection coefficient ( $P=6, f=2 \mathrm{MHz}$ ) and the maximum of $\mathscr{R}(P=11, f=3 \mathrm{MHz})$ at $31^{\circ}$ correspond to the second critical angle of carbon epoxy. It can be verified for other frequencies and other superlayers.

## VI. CONCLUSIONS

From a form developed by Rokhlin et al., we have built a propagation model in an anisotropic multilayered media. This model uses the notion of slowness rather than the one of velocity, which permits us to write all the propagation equations as a function of the slowness vector, notably if the waves are inhomogeneous. In the particular case of a monoclinic crystal system media with a symmetry axis perpendicular to the interfaces, reducing the determinant to zero coming from the propagation equation gives us an even order equation, which is justified by the physical explanation given in Secs. I-IV.

The form of the propagation matrices for one layer then allows us to express the displacement stresses of one layer as a function of those in the next layer. Rather than express the displacement stresses at one interface as a function of those at the next, we have chosen to relate the displacement amplitudes of the waves in one layer to those in the next layer. Notably, this permits us to obtain more directly the profile of displacement in each layer of the
multilayered media. The writing of the boundary conditions at each interface permits us to obtain a propagation matrix (product of propagation matrices in each layer) of the multilayered media. These matrices depend only upon the nature of the material, the frequency, and the incident angle of the incident wave.

Next we have applied this model to a periodically multilayered media made from the reproduction $P$ times of the multilayered media studied previously. The propagation matrix of this periodically multilayered media is expressed as a function of the one of the initial multilayered media. Therefore the dispersion equation of the media corresponds to the characteristic equation of the propagation matrix of the periodically multilayered media, itself tied to Floquet waves.

Reflection and transmission coefficients and possibly Lamb modes of the media are obtained from boundary conditions, depending upon the media surrounding the periodically multilayered media. The use of propagation matrices thus allows us to limit the size of the system to be solved, which does not exceed $(18 \times 18)$ in the most complex case.

An application to the aeronautic industry is the one of a media made up by stacked identical orthotropic layers, each layer being at $90^{\circ}$ from the previous, for which examples of reflection coefficients are given. The use of periodically multilayered media will permit us to study thick media.

## APPENDIX

We give here the $\eta$ th column of the matrix $\left[A^{q}\right]$ :
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