# Floquet waves and classical plane waves in an anisotropic periodically multilayered medium: Application to the validity domain of homogenization

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The aim of this paper is to better understand the correspondence between classical plane waves propagating in each layer of an anisotropic periodically multilayered medium and Floquet waves. The last are linear combinations of the classical plane waves. Their wave number is obtained from the eigenvalues of the transfer matrix of one cell of the medium. A Floquet polarization which varies with its position in the periodically multilayered medium has been defined. This allows one to define a Floquet wave displacement by analogy with the displacement of classical plane waves, and to check the equality of the two displacements at any interface separating two layers. The periodically multilayered medium is then an equivalent material, considered as homogeneous, and one can draw dispersion curves and slowness surfaces which are dispersive. In the low-frequency range, when the relation between the Floquet wave numbers and the frequency is linear, the multilayered medium can be homogenized; the Floquet polarization at different interfaces tends to a limit which is the polarization of the classical plane wave in the homogenized medium.

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## INTRODUCTION

Acoustic propagation through anisotropic multilayered media has become a subject of intensive study, because of its application to nondestructive evaluation, geophysics, etc.... Generally speaking, multilayered media are made by stacking distinct anisotropic media. These multilayered media are now studied by the use of the propagator matrix formalism which was first developed by Thomson,[1] then furthered by Haskell[2] and afterwards by Gilbert and Backus.[3] By writing boundary conditions at each interface separating two successive layers, a transfer matrix of the whole medium can be obtained. This matrix relates the stresses and displacements at the last interface to those at the first one. A very interesting case is the one of anisotropic periodically multilayered media which are $P$ times an anisotropic multilayered medium cell, named "superlayer." As an extensive background has already been done in previous papers in Refs. 4 and 5, we will not do it again. The study of such media leads to Floquet waves which correspond to the propagation modes in the infinite periodically multilayered medium. They are linear combinations of the classical plane waves propagating in each layer of the multilayered medium. Many researchers such as Gilbert,[6] Schoenberg,[7,8] and Rousseau and Gatignol[9] have studied periodically multilayered media made up of fluid layers. Others, like Richard[10] and also Gatignol, Rousseau, and Moukemaha[11,12] have studied the case of isotropic layers. They have obtained solutions involving Floquet waves, as did Lhermitte with the propagation of an elastic shear wave, normal to the interfaces in a cross-ply fiber reinforced composite.[13] The dispersive behavior of the waves in such media has been known for a long time: Brillouin's works[14] and then Haskell's works in 1953[2] lead to an interpretation of the behavior of periodically multilayered media. These media behave like mechanical filters:[15] the system does not allow the propagation of Floquet waves for stopping bands in frequency or in angle though it allows it for others. In 1984, transversally isotropic media were studied by Helbig[16] who presented dispersive curves and slowness surfaces and who used the long-wavelength approximation. This approximation permits equivalent elastic constants for a multilayered medium to be obtained. The same year Shoenberg,[8] who studied alternating fluid/solid layers, specified that the homogenized medium which models the behavior of the periodically multilayered medium in the long-wavelength domain must have the same slowness surfaces as those of the multilayered medium.

Several works[17,18] have dealt with the dispersion in fiber-reinforced composites. More recently, Gatignol, Rousseau, and Moukemaha[9,11,12] have obtained solutions of the dispersion equation in fluid or isotropic periodically stratified media. For normal incidence, the number of troughs between two stopping bands of the reflection coefficient in water is directly related to the number of superlayers in the medium. Rousseau[11] has shown propagative and evanescent zones for Floquet waves in fluid or solid layers, for a given oblique incident wave. In 1988[15] and in 1992,[19] Braga and Hermann obtained the sixth-order dispersion equation as a function of the first three invariants of the transfer matrix of one superlayer. It was obtained for multilayered media in which the layers are, first, orthotropic and, second, have a plane of symmetry parallel to their interfaces. By using the Stroh matrix formalism for such media,[20] they have shown that the eigenvalues of the transfer matrix are either real and inverse each to each or complex conjugate. Moreover, they have related these properties to the dispersion curves: one can obtain either propagative waves (i.e., passing bands in fre-

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quency) or nonpropagative waves which correspond to the stopping bands of the Brillouin's dispersion spectrum. The displacements of these waves which are Floquet waves can also be divided into plane-strain and antiplane-strain movements. For orthotropic layers oriented in a plane of symmetry, four Floquet waves are enough to describe the behavior of the plane strains of the stratified medium, from a fourth-order dispersion equation for which the algebraic solution is given in Ref. 15. The curves obtained in the case of a 0°/90° material for which each layer is at 90° to the previous one have been checked by Lhermitte at normal incidence,[13,21] when an elastic shear wave propagates. At the same time, the works of Chimenti and Nayfeh[22,23] enabled us to give the exact solutions for the propagation of an horizontally polarized shear wave along an axis of symmetry for each orthotropic layer. Dispersion curves for several incidences were obtained. In 1991, Nayfeh gave[24] the properties of the transfer matrix of one superlayer, for monoclinic media where the second-order axis is perpendicular to the interfaces. He deduced from them dispersion curves and velocity surfaces, for the most general case of periodically multilayered media. In 1993, Hosten and Castaings[25,26] have studied absorbing multilayered media and used the propagator matrix formalism. The dispersion of surface waves has formed the subject of thorough studies such as the study of the Rayleigh waves[27,28] and Lamb waves.[29–34]

Here we treat the case of a multilayered medium in which all the layers are anisotropic and arbitrary oriented. The first aim of this paper is to be more specific about the correspondence between Floquet wave displacement and classical wave displacement, at any interface of the multilayered medium. A Floquet polarization vector can then be defined at any interface separating two successive layers. By analogy with classical plane-wave displacement, one can define Floquet wave displacement and check the identity between these two displacements at the interface separating two successive layers. The notion of an equivalent medium to the periodically multilayered medium is then stronger and dispersion curves and slowness surfaces can be drawn. An interesting case is the one of the behavior of the multilayered medium in the long-wavelength domain. By reconstructing the time echographic signal, it can be shown that the periodically multilayered medium behaves like an homogeneous medium.

## I. FORMALISM OF FLOQUET WAVES

Let us consider a periodically multilayered medium which is a reproduction of $P$ superlayers, each one made by the stacking of $Q$ distinct anisotropic media (see Fig. 1). Media 0 and $N+1$ above and below the periodically multilayered medium are semi-infinite. We study the acoustic propagation of waves which are generated by an oblique incident wave propagating in the media 0.

Let us define $x_3$ as the axis of stacked layers, $q$ as the number of the layer in a superlayer: $1 \leq q \leq Q$, $p$ as the number of a superlayer: $1 \leq p \leq P$, and $h$ as the thickness of a superlayer.

The propagation equations in each layer use the same form as the one developed by Rokhlin *et al.*[35,36] and com-

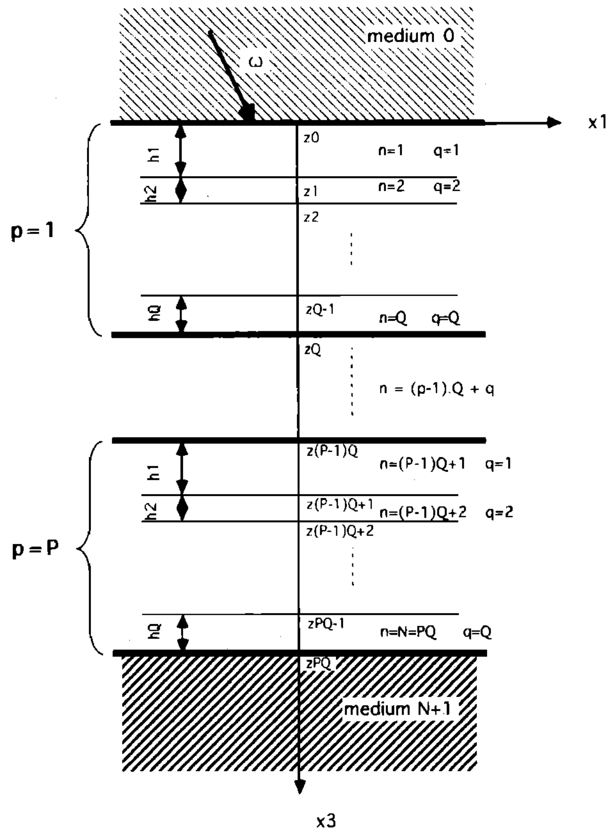


FIG. 1. Periodically multilayered medium.

pleted by Ribeiro *et al.*[37,38] with the help of the inhomogeneous wave form.[39,40] Generally speaking, an oblique incident wave propagating in the plane $x_1 0 x_3$ defined in Fig. 1 generates six plane waves with different velocities in an anisotropic layer surrounded by two media.

Let us note that $\eta$ is the label of the waves propagating in each layer $q$: $1 \leq \eta \leq 6$, $\omega$ is the natural frequency of the incident wave, ${}^{(\eta)}\mathbf{n}^q$ is the direction of the propagation vector of the wave $(\eta)$ in the layer $q$, ${}^{(\eta)}\mathbf{k}^q$ is the wave number vector of the wave $(\eta)$ in the layer $q$, ${}^{(\eta)}\mathbf{m}^q$ is the slowness vector of the wave $(\eta)$ in the layer $q$, ${}^{(\eta)}\mathbf{P}^q$ is the polarization vector of the wave $(\eta)$ in the layer $q$, ${}^{(\eta)}V^q$ is the propagation velocity of the wave $(\eta)$ in the layer $q$, ${}^{(\eta)}a^{p,q}$ is the complex amplitude of the particle displacement corresponding to the wave $(\eta)$ in the layer $n=(p-1)Q+q$, and ${}^{(\eta)}\mathbf{u}^{p,q}$ is the displacement vector of the wave $(\eta)$ in the layer $n$.

The slowness vector is related to the direction of the propagation vector and to the wave number vector by the following relation:

$$ {}^{(\eta)}\mathbf{m}^q = \frac{{}^{(\eta)}\mathbf{n}^q}{{}^{(\eta)}V^q} = \frac{{}^{(\eta)}\mathbf{k}^q}{\omega}. \tag{1} $$

The vectors ${}^{(\eta)}\mathbf{n}^q$ and ${}^{(\eta)}\mathbf{P}^q$ are normalized by the hermitian norm defined by

$$ \|\mathbf{R}\|^2 = R_k R_k^*, \tag{2} $$

where $R_k^*$ is the conjugate complex of $R_k$.

By introducing the slowness vector, the displacement vector of the wave $(\eta)$ in the layer $q$ of the superlayer $p$ can be written as

$$ {}^{(\eta)}\mathbf{u}^{p,q} = {}^{(\eta)}a^{p,q}\ {}^{(\eta)}\mathbf{p}^{q} e^{-i\omega({}^{(\eta)}\mathbf{m}^{q}\mathbf{x}-t)}. \tag{3} $$

The total displacement vector in the layer $n$ is expressed by

$$ \mathbf{u}^{p,q} = \sum_{\eta=1}^{6} {}^{(\eta)}\mathbf{u}^{p,q}. \tag{4} $$

Subsequently, we can note that $\{\ \}$ is a six-dimensional column vector, $\langle\ \rangle$ is a six-dimensional line vector, $[\ ]$ is a $(6\times 6)$ matrix, $T$ is a transpose operation, and $X^{q}_{\alpha\beta}$ is the coefficient of the matrix $[X^{q}]$ at the $\alpha$th row and $\beta$th column.

Let $\{\mathscr{A}^{p,q}\}$ be the $(6\times 1)$ column vector containing the amplitudes ${}^{(\eta)}a^{p,q}$ of the six waves propagating in the layer $q$ of a superlayer $p$:

$$ \{\mathscr{A}^{p,q}\} = \langle {}^{(1)}a^{p,q}, {}^{(2)}a^{p,q}, {}^{(3)}a^{p,q}, {}^{(4)}a^{p,q}, {}^{(5)}a^{p,q}, {}^{(6)}a^{p,q} \rangle^{T}. \tag{5} $$

Gradually, by equalizing the displacements and the stresses at the interface separating two successive layers, one can obtain the transfer matrix $[\Phi]$ of one superlayer,[4,41] which allows the amplitude displacements of the waves propagating or decreasing in the first layer of the superlayer $p+1$ to be expressed as a function of those in the first layer of the former superlayer $p$. That is to say,

$$ \{\mathscr{A}^{p+1,1}\} = [\Phi]\{\mathscr{A}^{p,1}\}, \tag{6} $$

with $[\Phi]$ given in Ref. 4.

The transfer matrix of the whole periodically multilayered medium is $[\Phi]^{P}$, where $P$ is the number of superlayers.[5,42–44]

As an extension of Floquet theory,[4,15] the eigenvalues ${}^{(\eta)}\lambda$ of $[\tau]$, which are of course the same as those of $[\Phi]$, are related to the Floquet slownesses through an exponential form

$$ {}^{(\eta)}\lambda = e^{-i\omega\, {}^{(\eta)}m_f h}. \tag{7} $$

The component following the $x_3$ axis of the Floquet slowness vectors is ${}^{(\eta)}m_f$, for a given incidence corresponding to $m_1$. The Floquet slowness vectors are of course related to the Floquet wave numbers ${}^{(\eta)}k_f$ by Eq. (1).

Let us define
$[\Xi]$ as the eigenvector matrix of $[\Phi]$ and $\{\mathscr{F}^{p,1}\}$ as the $(6\times 1)$ column vector containing the six complex amplitudes of the Floquet waves, at the first interface of the layer 1 in the superlayer $p$. $\{\mathscr{F}^{p,1}\}$ is related to $\{\mathscr{A}^{p,1}\}$ by the following relation, which is a change of basis:

$$ \{\mathscr{A}^{p,1}\} = [\Xi]\{\mathscr{F}^{p,1}\}. \tag{8} $$

From Eqs. (6) and (8), we finally obtain a matricial relation between the amplitudes of the Floquet waves at the first interface of the layer 1 in the superlayer $p+1$ as a function of those in the superlayer $p$:

$$ \{\mathscr{F}^{p+1,1}\} = [\mathscr{H}_f]\{\mathscr{F}^{p,1}\} $$

with

$$ [\mathscr{H}_f] = [\Xi]^{-1}[\Phi][\Xi] = \mathrm{diag}({}^{(\eta)}\lambda). \tag{9} $$

We then have six Floquet waves which correspond to the classical plane waves propagating in an infinite periodically multilayered medium, considered as an homogeneous material.[15,17,19]

Equation (10) allows the amplitudes of the Floquet waves in the first layer of the superlayer $p+1$ to be expressed as a function of those in the first layer of the first superlayer:

$$ \{\mathscr{F}^{p+1,1}\} = [\mathscr{H}_f]^{p}\{\mathscr{F}^{1,1}\}. \tag{10} $$

## II. RELATION BETWEEN CLASSICAL PLANE WAVES AND FLOQUET WAVES AT EACH INTERFACE SEPARATING TWO LAYERS

The displacement amplitudes of Floquet waves can also be defined in each layer $q$ of a superlayer $p$ from the displacement amplitudes of the classical plane waves in the same layer. Indeed, the matrix $[\Xi]$ is the eigenvector matrix of $[\Phi]$ which permits one to express the displacement amplitudes of the plane waves in the first layer of a superlayer as a function of those in the first layer of the next superlayer. But one can also define the transfer matrix $[\Phi^{q}]$, which allows the displacement amplitude of the waves in the layer $q$ of the superlayer $p$ to be expressed as a function of those in the layer $q$ of the next superlayer. The matrix $[\Phi]$ we have previously defined is then $[\Phi^{1}]$. These matrices $[\Phi^{q}]$ are expressed as

$$ [\Phi^{q}] = [A^{q}]^{-1}\left(\prod_{\substack{\alpha=q-1\\ q\neq 1}}^{1} [A^{\alpha}][\mathscr{H}^{\alpha}][A^{\alpha}]^{-1}\right) \times\left(\prod_{\alpha=Q}^{q} [A^{\alpha}][\mathscr{H}^{\alpha}][A^{\alpha}]^{-1}\right)[A^{q}], \tag{11} $$

where $[A^{\alpha}]$ and $[\mathscr{H}^{\alpha}]$ are given in Ref. 4. All the matrices $[\Phi^{q}]$ have the same eigenvalues defined by Eq. (7) but have different eigenvectors. The Floquet waves remain the same but are expressed by a linear combination of the classical plane waves which is different according to the layer $q$.

Let us define: $[\Xi^{q}]$ as the eigenvector matrix of $[\Phi^{q}]$ and $\{\mathscr{F}^{p,q}\}$ as the $(6\times 1)$ column vector containing the six complex amplitudes of the Floquet waves, at the first interface of the layer $q$ in the "superlayer" $p$. Similar to Eq. (8), let us define

$$ \{\mathscr{A}^{p,q}\} = [\Xi^{q}]\{\mathscr{F}^{p,q}\}. \tag{12} $$

## III. DISPLACEMENT OF ONE FLOQUET WAVE

Subsequently, we will consider a periodically multilayered medium in which only one Floquet wave propagates. Let $(\beta)$ be the datum index of this wave. Among the waves $(\eta)$, the column vector $\{\mathscr{F}^{p,q}\}$ thus has only one component $(\beta)$ which is not equal to zero. Indeed, by choosing judicious boundary conditions, one can manage to have only one Floquet wave in the multilayered medium. As the system is linear, the real propagation can again be found by the superposition of all the $(\beta)$ "states."

### A. Preliminary equations

*Remark*: Einstein's convention is not used because of the ambiguity on $\beta$.

The eigenvector associated with the eigenvalue ${}^{(\beta)}\lambda$ of the matrix $[\Phi^q]$ is the $\beta$th column $\{V^q_\beta\}$ of the eigenvector matrix $[\Xi^q]$. That is to say,

$$[\Phi^q]\{V^q_\beta\} = {}^{(\beta)}\lambda\{V^q_\beta\}. \tag{13}$$

The $\eta$th component of this equation is

$$\sum_{\alpha=1}^{6} \Phi^q_{\eta\alpha}\Xi^q_{\alpha\beta} = {}^{(\beta)}\lambda\,\Xi^q_{\eta\beta}. \tag{14}$$

From Eq. (6) and by recurrence, one can find

$$\{\mathscr{A}^{p+1,1}\} = [\Phi]^p\{\mathscr{A}^{1,1}\}, \tag{15}$$

which amounts to writing, for each layer $q$,

$${}^{(\eta)}a^{p+1,q} = \sum_{\alpha=1}^{6} ([\Phi^q]^p)_{\eta\alpha}\,{}^{(\alpha)}a^{1,q}. \tag{16}$$

For one Floquet wave ($\beta$), Eq. (12) can be written as

$${}^{(\eta)}a^{p,q} = \Xi^q_{\eta\beta}\,{}^{(\beta)}\mathscr{F}^{p,q}. \tag{17}$$

### B. Definition of a Floquet polarization vector

The displacement of the classical plane wave at the interface separating the layer $q$ from the layer $q+1$ of the first superlayer at $x_3 = z_q$ can be written as the following, omitting the factor $e^{-i\omega(m_1x_1-t)}$:

$$\mathbf{u}^{1,q+1}(z_q) = \sum_{\eta=1}^{6} {}^{(\eta)}a^{1,q+1}\,{}^{(\eta)}\mathbf{P}^{q+1} \quad \text{from Eqs. (3) and (4)}$$

$$= \sum_{\eta=1}^{6} \Xi^{q+1}_{\eta\beta}\,{}^{(\beta)}\mathscr{F}^{1,q+1}\,{}^{(\eta)}\mathbf{P}^{q+1} \quad \text{with Eq. (17)}$$

$$= {}^{(\beta)}\mathscr{F}^{1,q+1}\sum_{\eta=1}^{6} \Xi^{q+1}_{\eta\beta}\,{}^{(\eta)}\mathbf{P}^{q+1}.$$

Let us define the Floquet polarization vector ${}^{(\beta)}\mathbf{P}^{q+1}_f$ by

$${}^{(\beta)}\mathbf{P}^{q+1}_f = \sum_{\eta=1}^{6} \Xi^{q+1}_{\eta\beta}\,{}^{(\eta)}\mathbf{P}^{q+1}$$

$$= \frac{1}{{}^{(\beta)}\mathscr{F}^{1,q+1}}\sum_{\eta=1}^{6} {}^{(\eta)}a^{1,q}\,{}^{(\eta)}\mathbf{P}^{q+1}. \tag{18}$$

This Floquet polarization vector at the interface separating the layer $q$ from the layer $q+1$ is thus a linear combination of the polarization vectors of the waves propagating in the material which constitutes the layer $q+1$.[45] We obtain

$$\mathbf{u}^{1,q+1}(z_q) = {}^{(\beta)}\mathscr{F}^{1,q+1}\,{}^{(\beta)}\mathbf{P}^{q+1}_f. \tag{19}$$

When written like this, the displacement has the form of the displacement of the Floquet wave ($\beta$) at the interface separating the layer $q$ from the layer $q+1$; its amplitude is ${}^{(\beta)}\mathscr{F}^{1,q+1}$. Let us name $\mathscr{u}^{p,q+1}(z_n)$ the displacement of the Floquet wave ($\beta$) defined by Eq. (20) with the help of the Floquet polarization vector at $x_3 = z_n$ with $n = (p-1)Q + q$, i.e., at any interface separating two layers:

$$\mathscr{u}^{p,q+1}(z_n) = {}^{(\beta)}\mathscr{F}^{p,q+1}\,{}^{(\beta)}\mathbf{P}^{q+1}_f. \tag{20}$$

The interest of this formulation is that one can define a complex polarization vector of the Floquet waves at each interface separating two layers. Indeed, Eq. (18) defines the Floquet polarization vector in the first superlayer. Since the medium is periodical and as the Floquet waves are the modes of this infinite medium, the Floquet polarization vector must be unchanged by a translation of the period. This amounts to demonstrating that the Floquet wave displacement defined by Eq. (20) corresponds to the real wave displacement at any interface deduced from translation of a period.

*Remark*: The Floquet polarization vector ${}^{(\beta)}\mathbf{P}^{q+1}_f$ we have defined by Eq. (18) is not normalized, in order not to complicate the formula. ${}^{(\beta)}\mathscr{F}^{p,q+1}$ is thus the displacement amplitude of the Floquet wave ($\beta$), related to the non-normalized vector; the exact displacement amplitude is then ${}^{(\beta)}\mathscr{F}^{p,q+1}\|{}^{(\beta)}\mathbf{P}^{q+1}_f\|$.

### C. Verification of the validity of the writing at each interface separating two successive layers

We have just seen that $\mathbf{u}^{1,q+1}(z_q) = \mathscr{u}^{1,q+1}(z_q)$ in the first superlayer. Physically, as the Floquet waves are linear combinations of the classical plane waves, the displacements of these two kinds of waves are also equal at each interface of the multilayered medium. Thus we have to check the equality between the displacement of the classical plane waves and the displacement of the Floquet waves, in the superlayer $p+1$ at $x_3 = z_{n'} - 1$ with $n' = pQ + q$, omitting the factor $e^{-i\omega(m_1x_1-t)}$. In order to do that, we assume that the Floquet polarization vector, which is different at each interface, is known.

From Eqs. (10) and (20), the displacement of the Floquet wave ($\beta$) at $x_3 = z_{n'} - 1$ is

$$\mathscr{u}^{p+1,q}(z_{n'-1}) = {}^{(\beta)}\mathscr{F}^{1,q}\,{}^{(\beta)}\mathbf{P}^q_f({}^{(\beta)}\lambda)^p. \tag{21}$$

At the same interface, the displacement of a classical plane wave is

$$\mathbf{u}^{p+1,q}(z_{n'-1}) = \sum_{\eta=1}^{6} {}^{(\eta)}a^{p+1,q}\,{}^{(\eta)}\mathbf{P}^q$$

$$= \sum_{\eta=1}^{6}\left(\sum_{\alpha=1}^{6} ([\Phi^q]^p)_{\eta\alpha}\,{}^{(\alpha)}a^{1,q}\right){}^{(\eta)}\mathbf{P}^q$$

by applying (16)

$$= \sum_{\eta=1}^{6}\left(\sum_{\alpha=1}^{6} ([\Phi^q]^p)_{\eta\alpha}\Xi^q_{\alpha\beta}\,{}^{(\beta)}\mathscr{F}^{1,q}\right){}^{(\eta)}\mathbf{P}^q$$

with (17) and $\eta = \alpha$

$$\mathbf{u}^{p+1,q}(z_{n'-1}) = \sum_{\eta=1}^{6} \left( \sum_{\alpha=1}^{6} ([\Phi^q]^p)_{\eta\alpha} \; \Xi^q_{\alpha\beta} \right) {}^{(\beta)}\mathscr{F}^{1,q} \, {}^{(\eta)}\mathbf{p}^q$$

because ${}^{(\beta)}\mathscr{F}^{1,q}$ does not depend on $\alpha$

$$= \sum_{\eta=1}^{6} ({}^{(\beta)}\lambda)^p \; \Xi^q_{\eta\beta} \, {}^{(\beta)}\mathscr{F}^{1,q} \, {}^{(\eta)}\mathbf{p}^q$$

by applying (14) and because $({}^{(\beta)}\lambda)^p$ is an eigenvalue of $[\Phi^q]^p$

$$= \sum_{\eta=1}^{6} ({}^{(\beta)}\lambda)^p \, {}^{(\eta)}a^{1,q} \, {}^{(\eta)}\mathbf{p}^q \quad \text{with Eq. (17)}$$

$$= ({}^{(\beta)}\lambda)^p \sum_{\eta=1}^{6} {}^{(\eta)}a^{1,q} \, {}^{(\eta)}\mathbf{p}^q$$

because $({}^{(\beta)}\lambda)^p$ does not depend on $\eta$

$$= ({}^{(\beta)}\lambda)^p \, {}^{(\beta)}\mathscr{F}^{1,q} \, {}^{(\beta)}\mathbf{P}^q_f \quad \text{by applying (17)}$$

$$= \mathscr{u}^{p+1,q}(z_{n'-1}). \quad \text{from (21)}$$

Hence

$$\boxed{\mathbf{u}^{p+1,q}(z_{n'-1}) = \mathscr{u}^{p+1,q}(z_{n'-1})} \quad . \tag{23}$$

### D. Variation of the Floquet polarization vector

Equation (18) defines a Floquet polarization vector ${}^{(\beta)}\mathbf{P}^{q+1}_f$ at the interface separating the layer $q$ from the layer $q+1$, as a linear combination of the polarization vectors of the waves propagating in the material which constitutes the layer $q+1$. By defining a virtual interface which separates a layer into two parts and keeps the same properties, one can define a Floquet polarization vector at this interface in an analogous way to the one of Sec. III B and then have a Floquet polarization vector at any point of the periodically multilayered medium. We have checked on an example that the Floquet polarization vector varies according to its position in the multilayered medium. This variation is weak in the long-wavelength domain as we will see in Sec. IV C. The Floquet waves have an elliptic polarization and thus an elliptic movement of the particles for each mode, even if the wave is propagative.

We have just demonstrated that a Floquet polarization vector which is different according to its position in the multilayered medium enables us to write, by analogy with the classical plane waves, a displacement of the Floquet waves. As the Floquet waves are linear combinations of the classical plane waves in each layer, the displacement of the latter is equal to the one of the Floquet waves. Defining a Floquet polarization vector permits one to check it. This result can be found again in a physical way: indeed, if we choose boundary conditions at the first interface of the multilayered medium in order to have only one Floquet wave propagating, we will again find the same boundary conditions at the interface separating the first superlayer from the next, since the Floquet waves are propagation modes of the infinite periodically multilayered medium.

### E. Results

We can now assess our results on Floquet waves and classical plane waves. Just as classical plane waves, Floquet waves are the modes of the infinite medium in which they propagate. Moreover, when each layer of the multilayered medium is a monoclinic crystal system medium with a second-order axis perpendicular to the interfaces, the multilayered medium also has a second-order axis perpendicular to the interfaces. In such a medium, the waves propagate in the same way in two opposite directions,[4] and then the components following the $x_3$ axis of the slowness vectors are opposite to each other. The determinant of the propagation matrix of the medium is then equal to one and its eigenvalues are complex conjugate.[4,15,19,24] When the medium is homogeneous, the propagation matrix is $[\mathscr{H}^q]$ and when it is a multilayered medium, the propagation matrix is $[\mathscr{H}_f]$. Both matrices are defined in Sec. I. On the other hand, the Floquet slowness vectors and the Floquet wave number vector are defined by analogy with those in an homogeneous medium. The Floquet polarization vector is indeed the direction of the wave displacement and is a constant vector in planes parallel to the interfaces. But since it varies according to $x_3$, it is not a constant vector in planes perpendicular to the wave direction of propagation or to the slowness vector, which amounts to saying that the Floquet waves are not plane waves.

## IV. DISPERSION CURVES AND SLOWNESS SURFACES

As we have just seen the strong analogy between classical plane waves and Floquet waves, we can draw slowness surfaces, as for any waves. Similar surfaces have already been drawn by Nayfeh.[24] The method we propose in order to draw such surfaces is available even if the axis of symmetry is not perpendicular to the interfaces, which is the case Nayfeh[24] has treated.

From Eq. (10), the eigenvalues of $[\Phi]$ can be written in a trigonometric form

$${}^{(\eta)}\lambda = e^{-i\omega^{(\eta)}m_f h} = \rho e^{-i\theta}, \tag{24}$$

and thus

$$-i\omega^{(\eta)}m_f h = \ln \rho - i\theta[2\pi]. \tag{25}$$

It has been shown in Ref. 15 that a periodically multilayered media behaves as a mechanic filter: for some bands in frequency named stopping bands, Floquet waves are not propagative, though they are for others bands. Stopping bands occur when ${}^{(\eta)}m_f$ is complex. Floquet waves are thus propagative when ${}^{(\eta)}m_f$ is real.

${}^{(\eta)}m_f$ real implies

$$\begin{cases} \rho = 1 \\ {}^{(\eta)}m_f = \theta/\omega h, \end{cases} \quad \text{or else} \quad \begin{cases} \rho = 1 \\ {}^{(\eta)}k_f h = \theta, \end{cases} \tag{26}$$

TABLE I. Elastic constants in GPa for a carbon/epoxy medium.

| $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{33}$ | $c_{44}$ |
|---|---|---|---|---|
| 13.5 | 6.3 | 5.5 | 125.9 | 6.2 |

where ${}^{(\eta)}k_f$ is the component following the $x_3$ axis of the Floquet wave number vector for a given incidence corresponding to $m_1$,

$$^{(\eta)}k_f=\omega^{(\eta)}m_f, \tag{27}$$

and $\theta$ is the angle included between $-\pi$ and $\pi$.

The variations of the Floquet wave numbers or slownesses as a function of the frequency for a given incidence (dispersion curves), or as a function of the incident angle for a given frequency (slowness surfaces), give an indication of the possibility of considering a possible homogenization of the material. In this way, as Schoenberg specifies[8] in the case of alternating fluid/solid layers, the homogenized medium which models the behavior of the periodically multilayered medium in the long-wavelength domain must have the same slowness surface as the one of the multilayered medium and must be nondispersive.

We have seen in Sec. III E that when each layer of the multilayered medium is a monoclinic crystal system medium with a second-order axis perpendicular to the interfaces, the eigenvalues of the matrix $[\Phi]$ are complex conjugate.[4,15,19,24] In the case of ${}^{(\eta)}m_f$ being real, the projections on the $x_3$ axis of the Floquet slowness vectors are opposite to each other. The dispersion curves and the slowness surfaces are then symmetrical to the $x_1$ axis. This is the case of media made up of stacked identical hexagonal layers of carbon/epoxy, each being at 90° to the previous one (0°/90° medium) or at 45° to the previous one (0°/45°/90°/−45° medium).

The material used is an hexagonal crystal system medium with five independent elastic constants. If the sixth-order symmetry $A_6$ axis is parallel to the $\mathbf{O}x_3$ axis, these constants are given in Table I.[21] The volumetric mass of the carbon–epoxy is 1577 kg/m$^3$; each layer of the superlayer is 0.13 mm thick.

### A. Dispersion curves

For a given incident angle, one can draw the variations of the Floquet wave numbers with frequency, or else, which is equivalent from Eq. (26), the variations of the argument $\theta$ of ${}^{(\eta)}\lambda$ with frequency. Nayfeh[24] has drawn the phase velocity as a function of the wave number, which gives the same information. For an homogeneous material, the relation between $\theta$ and $\omega$ is linear. For carbon/epoxy 0°/90° or 0°/45°/90°/−45° media in normal incidence, we obtain the same curves as those presented by Lhermitte.[21]

For an incident angle equal to 10° in water, Fig. 2 presents the dispersion curves of an infinite periodically multilayered medium in carbon/epoxy 0°/45°/90°/−45°. Six Floquet waves propagate. Until 1 MHz, $\theta$=function ($\omega$) are straight lines. The medium can thus be considered as homogeneous. The more the frequency increases, the more the slownesses are great, i.e., the more the Floquet velocities decrease. Moreover, the first critical angle in a 0° layer of

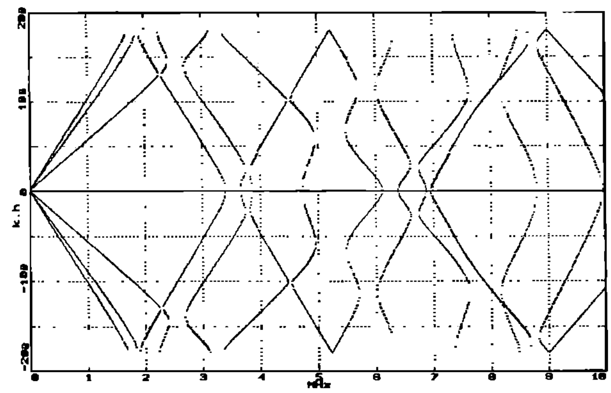

FIG. 2. Dispersion curves for a carbon/epoxy 0°/45°/90°/−45° multilayered medium at an incident angle in water equal to 10°.

carbon/epoxy is equal to 9.4°. Thus there are two inhomogeneous waves in the first layer of a superlayer. Yet there are six curves at the considered angle, 10°: the six Floquet waves in the multilayered medium are thus propagative. As the Floquet waves are linear combinations of the classical plane waves propagating in each layer, there is a reconstruction of the plane waves of which some are inhomogeneous in the first layer, to give Floquet waves which are all propagative.

It is possible to make an analogy with the propagation of a disturbance along a chain of atoms separated by springs which represent the joining power between atoms.[14,46–48] Until the first stopping band in frequency, the dispersion curves have a sinusoidal variation along the $\kappa$ axis, in the first Brillouin zone. The curves are named acoustical branches and each branch corresponds to a longitudinal mode or to a transversal mode. The Floquet wave propagation velocity can be deduced from the slopes of the curves. If the chain of atoms consists of two kinds of atoms, optical branches add to the acoustical branches after the first stopping band in frequency. The look of these acoustical and optical branches in the case of a diatomic chain, as can be seen in Ref. 14, for instance, enables us to consider the curves in Fig. 2 until 1.8 MHz as acoustical branches and from 1.9 MHz as optical branches.

We will go back over the homogenization in Sec. V.

### B. Slowness surfaces

For a given frequency, the variations of the Floquet slownesses can be drawn as a function of the angle of incidence in the multilayered medium. A development in the long-wavelength domain which has been done by Lhermitte[21] for 0°/90° and 0°/45°/90°/−45° materials gives equivalent elastic constants (see Table II) for these two materials. Both have the properties of tetragonal crystal system

TABLE II. Elastic constants in GPa for homogenized 0°/90° and 0°/45/90°/−45° materials.

| | $c_{11}$ | $c_{12}$ | $c_{13}$ | $c_{33}$ | $c_{44}$ | $c_{66}$ |
|---|---|---|---|---|---|---|
| 0°/90° | 69.7 | 5.6 | 5.9 | 13.5 | 4.6 | 6.2 |
| 0°/45°/90°/−45° | 56.8 | 18.5 | 5.9 | 13.5 | 4.6 | 19.1 |

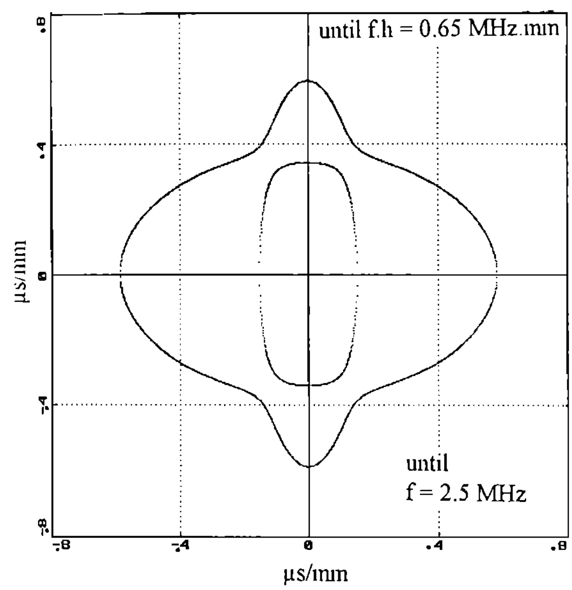


FIG. 3. Floquet slowness surfaces for a 0°/90° medium, valid until 2.5 MHz ($fh$=0.65 MHz mm).

media. Of course, this homogenization is only valid in a low-frequency range: the wavelength is much greater than the thickness of one superlayer.

(1) For a carbon/epoxy 0°/90° multilayered medium, Fig. 3 gives the Floquet slowness surfaces for an incident frequency inferior or equal to 2.5 MHz (i.e., $fh$=0.65 MHz mm). These surfaces are exactly the same as those obtained by drawing the slowness surfaces of the homogenized material in a classical way. The elastic constants of this material have been calculated by a development in the long-wavelength domain (see Table II). From 3 MHz (i.e, $fh$ =0.78 MHz mm), the slowness surfaces progressively alter in the $\mathbf{Ox_3}$ direction and the angular stopping bands neatly appear in Fig. 4. If the slowness surfaces of Fig. 3 (for a frequency inferior or equal to 2.5 MHz, i.e., $fh$=0.65 MHz mm) are superimposed to the surfaces obtained for a frequency equal to 4 MHz (i.e., $fh$=1.04 MHz mm), Fig. 4 is obtained. We notice a weak distortion of the surface corresponding to the quasilongitudinal waves around the $x_3$ axis, the axis of stacked layers. For weak incidence, the homogenization is thus valid for the quasilongitudinal waves. This result will be found again in Sec. V. On the other hand, the surfaces corresponding to the quasitransversal waves are not much distorted in the vicinity of the $x_1$ axis, parallel to the interfaces. The homogenization is thus valid for grazing shear waves.

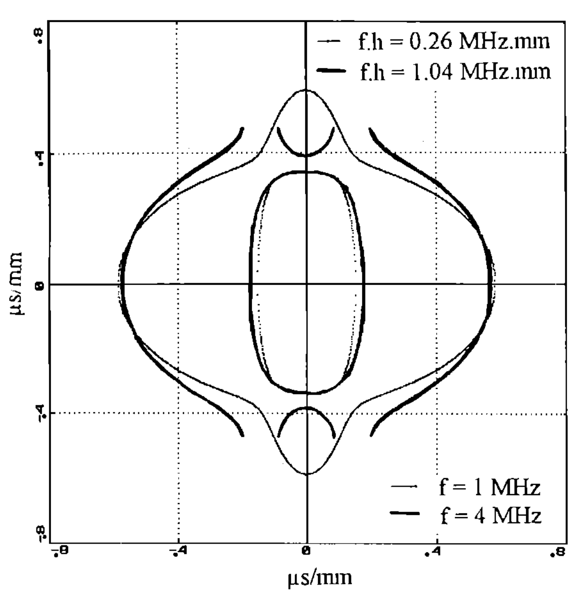


FIG. 4. Floquet slowness surfaces for a 0°/90° medium at 1 and 4 MHz.

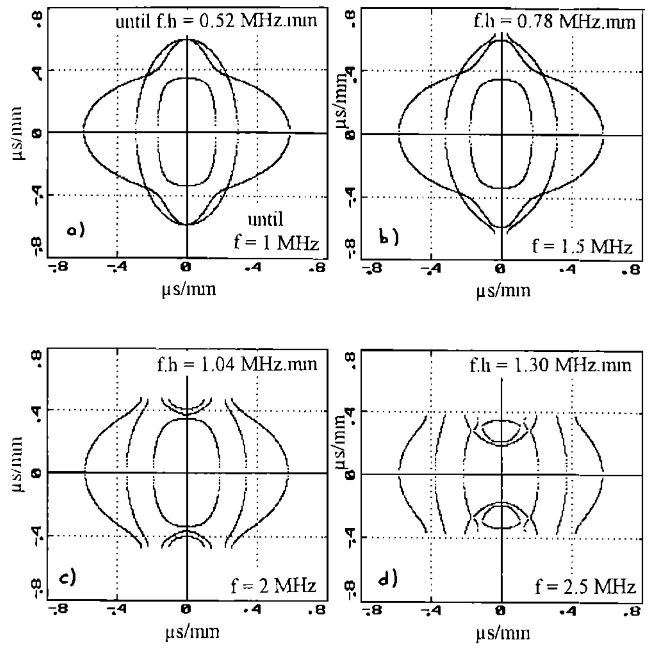


FIG. 5. Floquet slowness surfaces for a 0°/45°/90°/−45° material according to the frequency.

(2) For a carbon/epoxy 0°/45°/90°/−45° medium, Fig. 5(a) gives the Floquet slowness surfaces for an incident frequency inferior or equal to 1 MHz (i.e., $fh$=0.52 MHz mm). These surfaces are exactly the same as those obtained by drawing the slowness surfaces of the homogenized material in a classical way. The elastic constants of this material have been calculated by a development in the long-wavelength domain (see Table II). From 1.5 MHz (i.e., $fh$=0.78 MHz mm), the slowness surfaces progressively alter and the angular stopping bands neatly appear in Fig. 5.

### C. Floquet wave polarization

We have seen in Sec. III D that the Floquet polarization vector varies according to its position in the multilayered medium. As an example, for the 0°/90° medium studied in the former paragraph, Fig. 6 presents the Floquet polarization vector of a Floquet wave ($\beta$) at different interfaces. The index $q$=1 refers to ${}^{(\beta)}\mathbf{P}_f^1$ and $q$=2 refers to ${}^{(\beta)}\mathbf{P}_f^2$. We clearly observe at 4 MHz (i.e, $fh$=1.04 MHz mm) that the Floquet polarizations are not equal in the whole multilayered medium. In the long-wavelength domain, the multilayered medium can be homogenized as in the former paragraph. As this medium is homogeneous, the polarizations of the waves

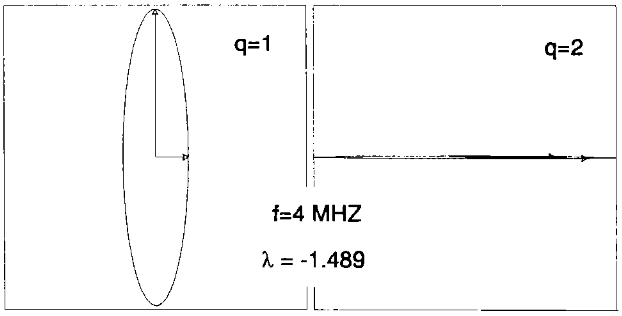


FIG. 6. Floquet polarizations for a carbon/expoy 0°/90° medium at an incident angle equal to 10° in water, for an incident frequency equal to 4 MHz (i.e., $fh=1.04$ MHz mm), and ${}^{(\beta)}\lambda=-1.489$.

propagating in it are the same whatever the position in the medium is. In this case, the polarization of the Floquet wave ($\beta$) tends to a limit which is the polarization of the classical plane wave in the homogenized medium. Then, for the 0°/90° medium studied in the former paragraph, we have seen that the slowness surfaces were the same as those of the homogenized material until 2.5 MHz (see Fig. 3). Yet, the Floquet polarizations at different interfaces are not at all equal (to within a complex constant with its modulus equal to 1) to the polarizations of the waves propagating in the homogenized medium. On the other hand, the polarizations are equal to within about 1% at 0.1 MHz. As an example, Fig. 7 presents Floquet polarization vectors at different frequencies and at different interfaces: the index $q=1$ refers to ${}^{(\beta)}\mathbf{P}_f^1$ and $q=2$ refers to ${}^{(\beta)}\mathbf{P}_f^2$ for a carbon/epoxy 0°/90° multilayered medium. We observe that in the low-frequency range, the Floquet polarizations tend to a rectilinear polarization which is the one of the corresponding waves in the homogenized medium. In the low-frequency range, the multilayered medium behaves as an homogeneous medium without being rigorously identical to it. This explains some phenomena, notably in the case of the reconstruction of the time echographic signal, as we are going to see in the next paragraph.

The same happens for a greater incident angle: Fig. 8 presents Floquet polarization vectors at different frequencies and at different interfaces, for an incident angle equal to 55°. We observe that from 2 to 5 MHz, these polarizations are quite constant whereas they tend to the rectilinear polarization which is the one of the corresponding waves in the homogenized medium from 1 MHz: the ellipses become flatter.

## V. RECONSTRUCTION OF THE TIME ECHOGRAPHIC SIGNAL IN THE LOW-FREQUENCY RANGE

In order to better understand the behavior of composite materials such as carbon/epoxy materials, we have reconstructed the time echographic signal in reflection and in transmission from the time signal set by the transducer and from the modelized reflection and transmission coefficients.[49] The calculation principle is the following: the frequency spectrum of the transducer is multiplied by the reflection or the transmission modelized coefficient. The inverse fast Fourier transform then gives the reflected or transmitted time signals.

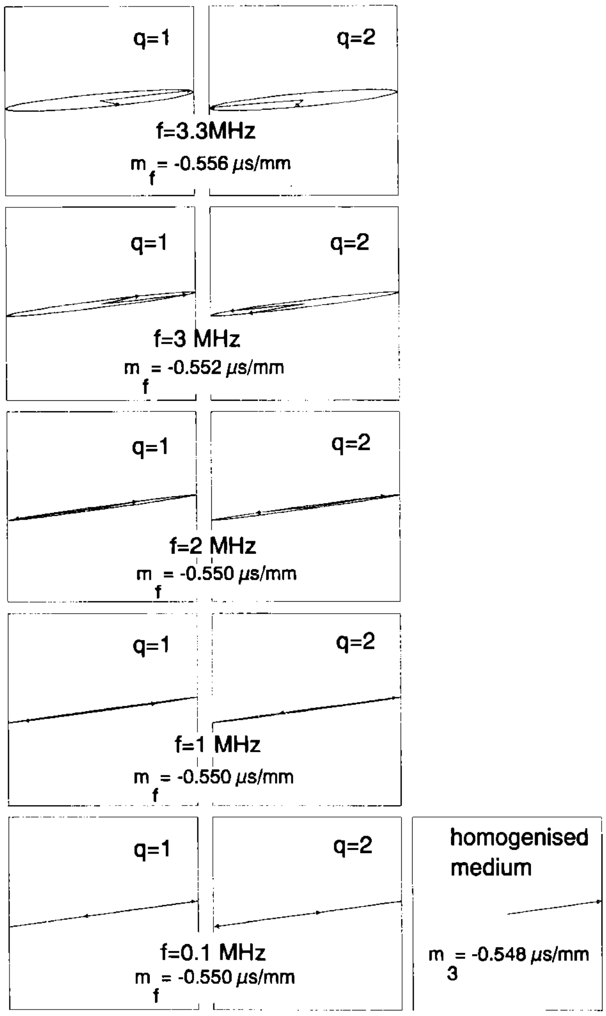


FIG. 7. Floquet polarizations for a carbon/expoy 0°/90° medium at an incident angle equal to 5° in water, for different frequencies.

Let us consider a carbon/epoxy 0°/90° medium immersed in water and made up of 42 superlayers; each layer is 0.13 mm thick.

Figure 9(a) presents the transmitted time signal for a central frequency of the transducer equal to 2.5 MHz and a bandwidth at $-6$ dB equal to 75%. The incident angle is equal to 8°. The transmission of Floquet quasilongitudinal or quasitransversal waves can be considered. The first echo corresponds to the course through the homogenized material by the quasilongitudinal wave. Let us name $t_1$ the flight time of this wave noted L. The second echo immediately follows the first one and corresponds to the crossing of the material by the quasitransversal wave (noted T) during a time $t_2$. Echo 3 occurs after a flight time equal to $3t_1$, echo 4 after $2t_1+t_2$, and echo 5 after $t_1+2t_2$. This is summarized in Fig. 10; the waves are numbered in order of appearance. For example, the fourth refracted wave corresponds to a course and return course of the quasilongitudinal wave and to a course of the quasitransversal wave. Echo 4 then occurs at $2t_1+t_2$ and we will name this wave LLT.

For incidences higher than 8°, the curves become less and less legible because of time overlaps and interferences, as is shown in Fig. 11(a) at 12°. Considering the bandwidth

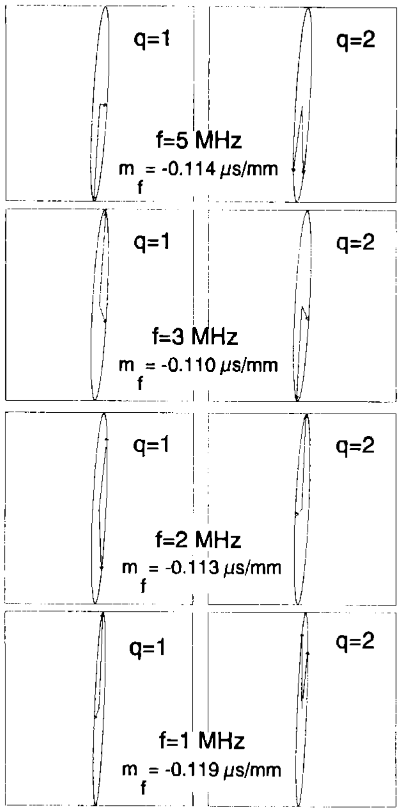


FIG. 8. Floquet polarizations for a carbon/expoy 0°/90° medium at an incident angle equal to 55° in water, for different frequencies.

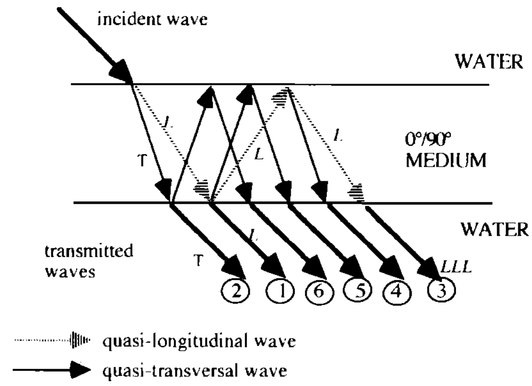


FIG. 10. Transmission through a 0°/90° medium.

of the transducer, an eventual homogenization of the multilayered medium is thus not possible at this incidence; this has already been found in Sec. IV C: the slowness surfaces of the quasilongitudinal waves at different frequencies only superimpose at a weak incidence (see Fig. 4). The central frequency of the transducer then must be diminished. In order to obtain results that may be compared, we have kept $H/\lambda$ constant, where $H$ is the thickness of the whole medium and $\lambda$ the wavelength in the material. That is what Hosten and Castaing also did in Refs. 26, 50, and 51. For a transducer with a central frequency equal to 1 MHz and for a 0°/90° material made up of 105 superlayers, we obtain the same curves as those in the former case with a 2.5-MHz transducer and with a 0°/90° medium made up of 42 superlayers. The resolution of the reflected and transmitted signals is then good until 12° as is shown in Fig. 11(b), which justifies the homogenization for this frequency and for this incident angle. From 14° onward, we only observe the quasi-transversal waves, as the quasilongitudinal wave becomes inhomog-eneous after the first critical angle at 12.31°.

All the results presented in this section are valid if the wavelengths are great in view of the superlayer thickness. When the carbon/epoxy multilayered medium is stimulated

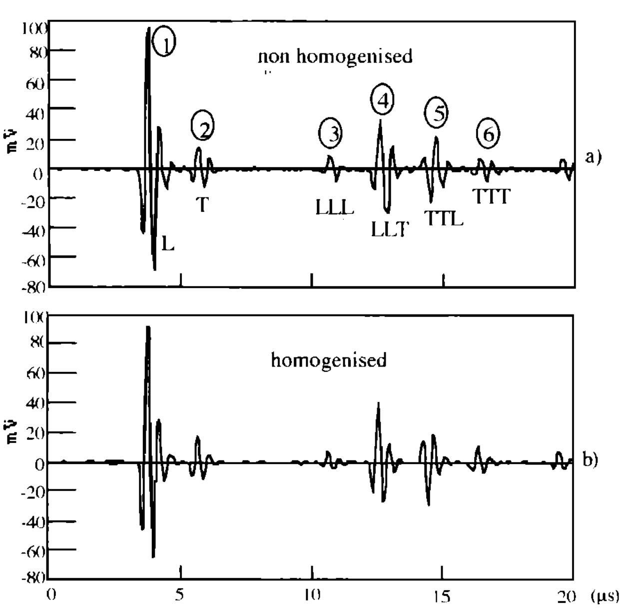


FIG. 9. Transmitted time signals for a 0°/90° material made up of (a) 42 superlayers and (b) for a single layer, 10.92 mm thick, of homogenized carbon/epoxy at an incidence equal to 8° and for a central frequency equal to 2.5 MHz.

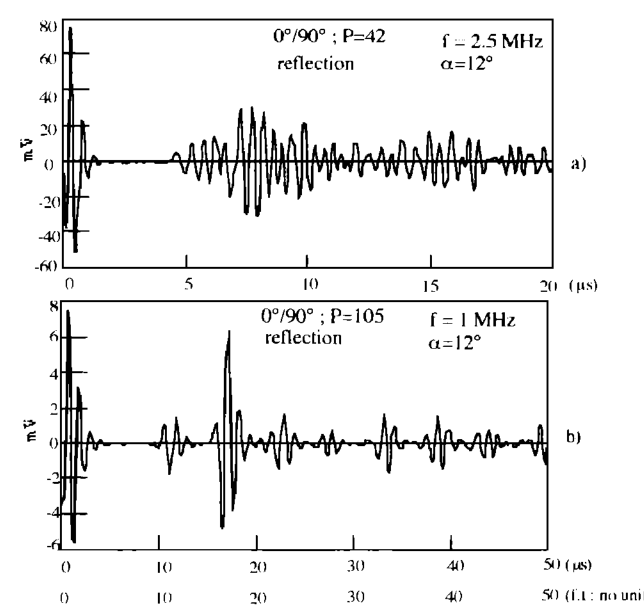


FIG. 11. Reflected time signals for a 0°/90° material made up of (a) 42 and (b) 105 superlayers at an incidence equal to 12°. A third scale has been added as a function of the nondimensional product $f\times t$: the frequency multiplied by the time.

in a low-frequency range, it behaves as if it was homogenized. In order make these results explicit, we have reconstructed the reflected and transmitted time signals in the same conditions as previously, but this time for an homogenized layer of carbon/epoxy, so as to eliminate the different interferences due to the layered structure of the composite material. The elastic constants used are those of Table II, calculated by Lhermitte.[21]

Figure 9(b) presents the transmitted time signals for an homogenized layer 10.92 mm thick, stimulated by an incident wave of which the central frequency is 2.5 MHz, for incident angle equal to 8°, which corresponds to a multilayered medium made up of 42 superlayers. At 8°, we obtain the same transmitted echoes as those in Fig. 9(a).

The fact that the little oscillations between the echoes No. 4 and No. 5 or No. 5 and No. 6 of the time signal for a 0°/90° medium [see Fig. 9(a)] are not found on the time signal of the homogenized medium does show, as explained in Sec. IV C, that even at very low frequencies, the composite medium tends to an homogenized medium, without being rigorously identical to it.

## VI. CONCLUSIONS

From the propagator matrix formalism, we have built a propagation model in an anisotropic periodically multilayered medium. The eigenvalues of the transfer matrix of one superlayer lead to the Floquet waves. These waves are linear combinations of the classical plane waves propagating in each layer of the multilayered medium. The aim of this paper was to understand the correspondence between classical plane waves and Floquet waves better. Defining a Floquet polarization vector which is different according to the layer and to its position permits expressing the Floquet wave displacement, which is of course a real displacement, by a formalism very similar to the one used for the classical plane waves. Then we checked that the Floquet polarization vector is unchanged by a translation of the period of the multilayered medium, by demonstrating that the Floquet wave displacement vector is equal, at each interface separating two successive layers, to the plane wave displacement in the layer at the same interface. The propagation of Floquet waves in a multilayered medium is then very similar to the propagation in an homogeneous medium and one can draw dispersion curves and slowness surfaces as a function of the frequency. For some stopping bands in frequency, the Floquet waves are no longer propagative, which occurs for a complex Floquet slowness vector. On the other hand, Floquet waves can be propagative though classical plane waves in one layer are not propagative: the reconstruction of inhomogeneous classical plane waves then gives propagative Floquet waves. When the multilayered medium gets a second-order axis perpendicular to its interfaces, the dispersion curves and the slowness surfaces are symmetrical to the axis parallel to the interfaces. When the relation between the Floquet wave number and the frequency is linear, the multilayered medium can be considered as homogeneous. Indeed, the reconstruction of the time echographic signal of a carbon/epoxy 0°/90° material shows that when this multilayered medium is stimulated in a low-frequency range, it behaves as an homogenized medium of the same thickness: there is propagation of quasilongitudinal and quasitransversal Floquet waves and disappearance after the first critical angle of the quasilongitudinal wave. The reconstruction of the time signal for one homogenized layer of carbon/epoxy gives the same results. The Floquet polarization vector at different interfaces tends to a limit which is the polarization vector of the classical plane wave in the homogenized medium.


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