acta acustica · ACUSTICA
Vol. 87 (2001) 340 – 351

Energetic Criterion for the Radiation of Floquet Waves in Infinite Anisotropic Periodically Multilayered Media

Catherine Potel*, Philippe Gatignol, Jean-François de Belleval
Laboratoire Roberval, UMR UTC-CNRS 6066, Université de Technologie de Compiègne, BP 20 529, 60205 Compiègne Cedex, France

Summary
The aim of this paper is to express in a correct manner the radiation conditions in terms of Floquet waves in an infinite anisotropic periodically multilayered medium. These waves are linear combinations of the classical plane waves propagating in each layer of the medium and are defined by the eigen vectors of the transfer matrix of the unit cell of the medium, termed "period". As the propagation reference of each Floquet wave is only known at each interface separating two successive periods (stroboscopic effect), the corresponding pseudo Floquet wave number which thus can be defined just gives an apparent propagation direction and not the effective propagation direction of a propagative Floquet wave. The propagation direction for each propagative Floquet wave is given by the sign of its normal power flux, involving its corresponding eigen vector. The problem is first analysed in isotropic periodically multilayered media with normal incidence and then extended to anisotropic periodically multilayered media.

PACS no. 43.20.Fn, 43.35.Cg

1. Introduction

Ultrasonic propagation through multilayered media has become a subject of intensive study in the last few years. Generally speaking, multilayered media are made up by the stacking of distinct anisotropic media. Here, we are interested in anisotropic periodically multilayered media, bounded by a first layer which is contiguous to some fluid continuous half space. Such media are composed of an anisotropic multilayered medium cell, called "period" or "superlayer", which is P times repeated. These media are now studied by use of propagator matrices [1, 2, 3, 4, 5]: the transfer matrix of one period can be found from the boundary conditions at each interface separating two successive layers. It allows the displacement and stress vector at the interface separating two successive periods (i.e. a period interface) to be expressed as a function of that at the previous period interface. The transfer matrix eigen vectors lead to the Floquet waves. These waves are linear combinations of the classical plane waves propagating in each layer of the multilayered medium [6, 7, 8, 9, 10, 11, 12]. Though they are not plane waves in general, these particular solutions play the role of plane waves for infinite periodically multilayered media: any solution may be expressed as a linear combination of these waves. This permits to go from one period interface to another one by means of a diagonal matrix. As an extensive background has already been done in previous papers [13, 14, 15, 16, 17], we will not do it again. Nevertheless, it is worth mentioning the main contributions to the study of the propagation direction of the Floquet waves (see section 2.3).

The aim of the paper is to express in a correct manner the radiation conditions for Floquet waves in a infinite periodically multilayered medium, in order to solve the reflection-transmission problem of a periodically semi-infinite stratified structure. Many studies deal with a wave propagation in a direction parallel to the layers (see [12] and references contained therein). In this case, by using continuum mixture equations, the composite medium is replaced by a homogeneous, yet dispersive medium. In the case of this paper, the discontinuous character of the structure is conserved at any scale (except for the low frequency domain of homogenisation), and the waves we consider here partly propagate in the direction perpendicular to the layers. In this case, the physical feature of the radiation condition is not obvious, since the bulk waves are indefinitely reflected on the interfaces as they propagate far and far in the medium.

Section 2 recalls the basic problem of radiation conditions for homogeneous media and sets the problem for multilayered media. The purpose of section 3 is to show in the simplest case of one-dimensional medium how to write the correct radiation condition. Finally, the calculus is extended to the case of general anisotropic periodically multilayered media in section 4.

2. The basic problem of radiation conditions for multilayered media

Since Sommerfeld's work, it is well known that the solution of monochromatic propagation problems in (partly) unbounded media involves extra conditions at infinity. These conditions express the fact that the elementary solutions (monopoles or plane waves) which are involved in the integral representation of the field do propagate towards infinity, so excluding any reflection in the boundlessness directions. We focus here on the radiation conditions for plane waves since these waves are the basic concept in the case of multilayered media. Subsequently, the harmonic time dependence convention will be $e^{+i\omega t}$.

Received 7 July 2000,
accepted 21 March 2000.

* Also with: Université de Picardie Jules Verne, IUT de l'Aisne, Département OGP de Soissons, France. Catherine.Potel@utc.fr

2.1. The homogeneous case

When the unbounded space region is a homogeneous medium, there is no real difficulty in writing the expected radiation condition since the plane waves propagate in the medium at infinity. The problem then reduces to sort out the progressive plane waves which may be considered as propagating towards infinity, among all the possible waves.

In the case of an absorbing material, the amplitude of the generalised progressive plane waves decreases in the direction they propagate and a simple argument of boundedness gives the actual condition. If there is no absorption, the argument may be a little more subtle since the wave amplitude phase has to be examined. For an isotropic medium, the radiation condition is classically based on the direction of the wave vector. In fact, in this simple case, phase velocity direction and energy velocity direction are the same.

However, in the anisotropic case, these two directions are different in general and the choice must be done between either looking at the phase velocity direction or studying the propagation direction of the plane wave energy (Poynting vector or group velocity) [18, 19, 20]. It is clear that this last point of view is the right one. For example, let us consider the case of the reflection- transmission at the interface separating two distinct semi-infinite media, one of which is anisotropic. Then it may be shown that for some configurations, a radiation condition based on the wave vector would lead to a reflection coefficient with a magnitude greater than one. In particular, this situation occurs when the slowness surfaces of the anisotropic medium present inflexion points [21, 22].

2.2. The case of a stratified medium

When the medium which extends to infinity is no longer homogeneous, writing a consistent radiation condition becomes a real problem which may have no solution. Indeed, the scattering of waves by the inhomogeneities prevents progressive plane waves from existing in the medium, even at infinity.

When the wave length is much longer than the characteristic length of the inhomogeneity, the difficulty may be bypassed: for low frequency wave propagation, the inhomogeneous medium may be replaced by an equivalent homogeneous medium for which progressive plane waves do exist again and may be used to express radiation conditions.

Apart from the validity domain of homogenisation, there is no general method to express te radiation condition in the infinite medium. The difficulty is clearly enlightened in the case of an infinite plane multilayered medium. Indeed, at each interface separating two successive layers, any progressive plane wave undergoes a reflection effect, so that forwards and backwards progressive waves are simultaneously present in the layers at more and more far-off distances. If the layered medium does not follow a particular law for the stacking of the layers, then writing a radiation condition becomes an unsolvable problem. Indeed, the monochromatic hypothesis assumes that a transient propagation phenomenon has evolved for an infinite time. In the present case, no final situation may be reached by a transient wave since as long as it propagates towards infinity, the wave front encounters new and new structural configurations.

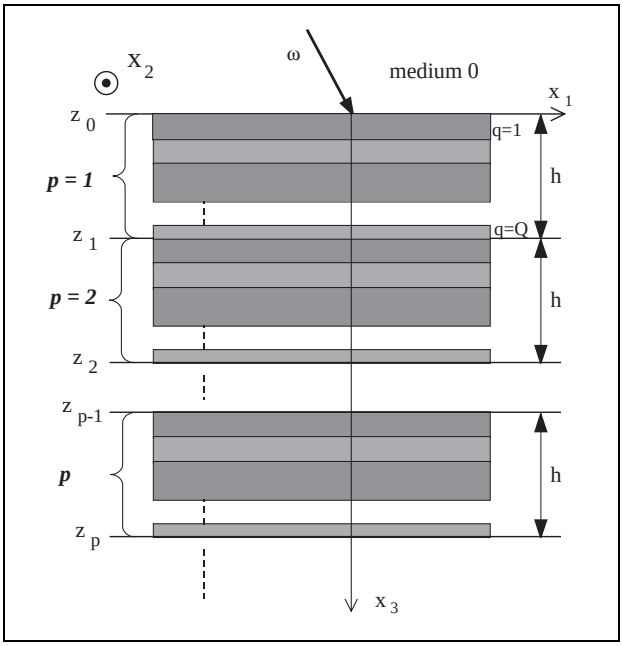

Figure 1. Geometry of the infinite periodically anisotropic multilayered medium.

2.3. The case of an infinite periodically multilayered medium

Fortunately, when an infinite multilayered medium presents a periodic law for the order of its layers, it will be seen subsequently that it is possible to express radiation conditions (see Figure 1 for the geometry of the medium). Same way as in the general case of an infinite multilayered medium (see section 2.2), no progressive wave is available at infinity and other basic solutions have to be found in order to express radiation condition. These basic solutions are the so-called Floquet waves which are particular arrangements of forward and backward plane waves in each layer.

Floquet waves are the eigen solutions of the transfer matrix τ of the unit cell of the periodic medium. For such solutions, the physical field variables are simultaneously multiplied by the same number λ (the relevant eigen value) when they are compared upon a distance equal to the length h of the period in the direction normal to the layers. In general, the eigen values of the matrix τ are different so that the Floquet waves are independent solutions. As a consequence, for a given incidence, the general solution in the multilayered medium may be expressed as a linear combination of these eigen solutions. A propagative Floquet wave is obtained when the eigen value is complex with its magnitude equal to one. In this case, it may be interesting to write this eigen value in the form:

$$\lambda = \mathrm{e}^{-\mathrm{i}\varphi}, \tag{1}$$

where φ appears as a common phase change in the amplitudes of the physical field variables. When the eigen value

does not have its magnitude equal to one, the associated Floquet wave is said to be inhomogeneous, as a generalisation of the so-called inhomogeneous plane waves. The Floquet wave form leads to interpret the behavior of periodically multilayered media as mechanical filters [2, 23, 24, 25]: the system does not allow Floquet waves propagation for frequencies in stopping bands (for a fixed incident angle) or for angles in stopping bands (for a fixed frequency), though it allows it for others. These bands correspond to the case where all the Floquet waves are inhomogeneous.

In order to write the radiation conditions in an infinite periodically multilayered medium, it is essential to be able to determine the propagation direction of the Floquet waves. It is very easy to know the decreasing direction of inhomogeneous Floquet waves, since it is found that half of the Floquet waves have their amplitude which increases exponentially. Then, a simple argument of boundedness of the solution leads to the choice of Floquet waves the amplitude of which tends to zero: this is the case when the magnitude of the eigen value is less than one as it will be seen in section 3.3.1. On the other hand, in the case of propagative Floquet waves (magnitude of the eigen value equal to one), the definition of the propagation direction is not a so simple task and the radiation condition can not be deduced from a single boundedness argument (see section 3.3.2). In this case, for the corresponding eigen solution, the physical variables undergo simultaneously a same change in the phase of their amplitude upon a distance h normal to the layers. The periodically multilayered medium could be thus considered as a homogeneous medium in which the Floquet waves would propagate. By analogy with homogeneous media, it is tempting to define a pseudo Floquet wave number in the direction normal to the layers, by using the phase variation across each period, and to determine the propagation direction from the sign of this pseudo Floquet wave number. The pseudo Floquet wave numberκ can be defined such that:

$$\varphi = \kappa h, \tag{2}$$

where h is the thickness of one period. Hence, using equation (1):

$$\lambda = \mathrm{e}^{-\mathrm{i}\kappa h}. \tag{3}$$

However, note that the phase reference comes from a discrete information, namely the field values at the interfaces separating two successive periods: so it is not a continuous concept. Moreover, the definition itself of the phase shift φ is subjected to some indeterminacy, since its value is known only within some additive constant equal to an integer number times 2π. As a consequence, phase data are too poor to allow the definition of an effective propagation direction. Therefore, it is not possible to determine the propagation direction of the Floquet wave by examining the pseudo Floquet wave vector. It happens that several authors [11, 24, 26] did not see the restrictive side of this stroboscopic effect. The situation is identical if one wants to determine if a wheel turns clockwise or anticlockwise, just by flashing it periodically. More inspection is necessary to determine the actual propagation direction of propagative Floquet waves. For fluid or isotropic layers, some authors [4, 5, 27, 28] have bypassed the difficulty by introducing a small absorption in the medium. In this case, the eigen values of the transfer matrix are complex but their magnitude is no longer equal to one. The corresponding Floquet waves are thus always inhomogeneous. As mentioned above, the decreasing direction is easily found and permits the radiation conditions to be written. For propagative Floquet waves, the key was briefly given by Rousseau and Gatignol [6, 8] for two fluids, and, afterwards, for isotropic layers in a superlayer: the sign of the normal power flux of each propagative Floquet wave gives its propagation direction.

It should be noted that, in the present case of propagative Floquet waves, this discrepancy between energy velocity and phase velocity has nothing to do with any anisotropic effect (see section 2.1): the observation may be displayed on the simple case of one-dimensional periodic media. For these simple periodic structures, it may be shown that once out of two, the energetic propagation direction is opposed to the propagation direction that would have been deduced from the study of the pseudo Floquet wave number's sign.

It should be noted that the same problem would be encountered in various types of periodically structures. As an example, in the case of periodically layered piezoelectric media, the only difference would be the order of the transfer matrix which would be changed from (6×6) to (8×8) due to the presence of extra two variables form the electric field [29]. In the case of periodic wave guides, Bradley [30] refers to Bloch wave functions and points out that the Bloch wave number (equivalent to the pseudo Floquet wave number) is multivalued and he does not propose a solution to the problem, except in the case of the uniform wave guide limit.

3. Floquet waves strobe effect in infinite isotropic periodically multilayered media

In order to understand why the sign of the real part of the pseudo Floquet wave vector does not give the Floquet wave propagation direction, let us now consider the case of normal incidence propagation in a periodic medium composed with the reproduction of two isotropic layers. Most of the results presented in this section have been given by Rousseau and Gatignol [6, 8] and afterwards by Moukemaha [9]. However, it seems interesting to us to emphasise some aspects of Floquet wave physical properties and to develop some results which were just briefly mentioned by these authors.

3.1. Period transfer matrix and Floquet waves

Let us consider a semi infinite fluid 0 in contact with a semi-infinite stratified medium, at $x_3 = 0$ (see Figure 2). The stratified medium is a periodically multilayered medium, reproduction of an infinite number of unit cells, termed "periods", each one made by the stacking of two distinct isotropic media numbered 1 and 2. As the wave in the fluid 0 propagates with normal incidence, the transmitted waves in the stratified medium are longitudinal waves.

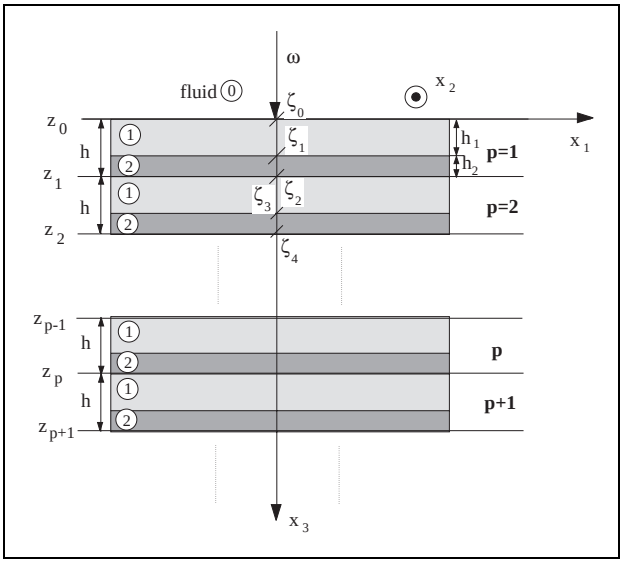

Figure 2. Geometry of the periodic medium composed of two isotropic layers. Normal incidence.

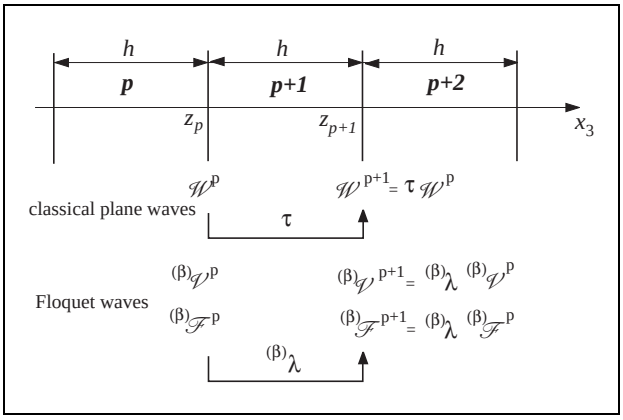

Figure 3. Passage from one superlayer to the next one for the general wave solutions and for the Floquet waves.

Define: V_0, V_1, V_2: Velocity of the longitudinal wave propagating in medium 0, 1 or 2. Z_0, Z_1, Z_2: Acoustic impedance of medium 0, 1 or 2. h_1, h_2: Thickness of the layers 1 and 2 with $h = h_1 + h_2$. τ_1, τ_2: Flight time in layers 1 and 2 with $\tau_1 = h_1/V_1$ and $\tau_2 = h_2/V_2$. ω: Natural angular frequency of the incident wave (harmonic time dependence convention $\mathrm{e}^{+\mathrm{i}\omega t}$). z_p: co-ordinate with respect to the x_3-axis of the interface separating period p from the next period $p+1$, with $z_{p+1} = z_p + h$ and $z_0 = 0$. The corresponding interface will be termed "*period interface*".

Let us define the state vector $\mathcal{W}^p$ at $x_3 = z_p$ by:

$$\mathcal{W}^p = \left\{ \begin{array}{c} w_p \\ T_p \end{array} \right\}, \tag{4}$$

where w_p and T_p are the normal displacement and stress at $x_3 = z_p$.

The use of the boundary conditions, at the interface separating two successive layers, i.e. equalling stresses and displacements, leads to express $\mathcal{W}^{p+1}$ at z_{p+1} as a function of $\mathcal{W}^p$ at z_p, by means of a matricial relation (see corresponding calculus in Appendix A1):

$$\mathcal{W}^{p+1} = \boldsymbol{\tau}\mathcal{W}^p, \tag{5}$$

where $\boldsymbol{\tau}$ is the period transfer matrix. In the simple case considered in this section, $\boldsymbol{\tau}$ is a (2×2) matrix.

As a consequence, whatever the state vector $\mathcal{W}^p$ at z_p, the state vector $\mathcal{W}^{p+1}$ at the next period interface z_{p+1} can be deduced from the matricial relation (5). Among all the possible state vectors $\mathcal{W}^p$, some are particular: these vectors are the eigen vectors ${}^{(\beta)}\mathcal{V}$ of the matrix $\boldsymbol{\tau}$ and define the Floquet waves. They are such that:

$$\boldsymbol{\tau}\mathcal{V} = \lambda\mathcal{V}. \tag{6}$$

Here, as the matrix $\boldsymbol{\tau}$ is of the 2nd-order, there are two eigen values ${}^{(1)}\lambda$ and ${}^{(2)}\lambda$ such that:

$${}^{(1)}\mathcal{V} = {}^{(1)}\lambda\ {}^{(1)}\mathcal{V} \quad \text{and} \quad {}^{(2)}\mathcal{V} = {}^{(2)}\lambda\ {}^{(2)}\mathcal{V}. \tag{7}$$

As a consequence, if the state vector is proportional to the particular state vector ${}^{(\beta)}\mathcal{V}$ $(\beta = 1, 2)$ at z_p, the state vector at the next period interface z_{p+1} will be obtained from equation (5), with $\mathcal{V}$ instead of $\mathcal{W}$:

$${}^{(\beta)}\mathcal{V}^{p+1} = \boldsymbol{\tau}\ {}^{(\beta)}\mathcal{V}^p, \tag{8}$$

which reads, using equation (6):

$${}^{(\beta)}\mathcal{V}^{p+1} = {}^{(\beta)}\lambda\ {}^{(\beta)}\mathcal{V}^p. \tag{9}$$

This is summarised in Figure 3.

Since the eigen values of the transfer matrix are distinct, the eigen vectors of the matrix are independent. As a consequence, the Floquet waves are independent solutions so that, for any solution, the state vector $\mathcal{W}^p$ at z_p may be expressed as a linear combination of the eigen vectors ${}^{(1)}\mathcal{V}$ and ${}^{(2)}\mathcal{V}$:

$$\mathcal{W}^p = {}^{(1)}\mathcal{F}^p\ {}^{(1)}\mathcal{V} + {}^{(2)}\mathcal{F}^p\ {}^{(2)}\mathcal{V}, \tag{10}$$

with $\left({}^{(1)}\mathcal{F}^p,\ {}^{(2)}\mathcal{F}^p\right) \in \mathbb{C}^2$,

and by omitting the $\mathrm{e}^{+\mathrm{i}\omega t}$ factor.

According to equations (5), (7) and (9), it is possible to obtain:

$$\mathcal{W}^{p+1} = \boldsymbol{\tau}\mathcal{W}^p = {}^{(1)}\mathcal{F}^p\ {}^{(1)}\lambda\ {}^{(1)}\mathcal{V} + {}^{(2)}\mathcal{F}^p\ {}^{(2)}\lambda\ {}^{(2)}\mathcal{V}. \tag{11}$$

By identification with equation (9) with $(p+1)$ in place of p, the Floquet amplitudes at the period interface z_{p+1} are deduced from those at the previous period interface z_p by the following relation:

$${}^{(\beta)}\mathcal{F}^{p+1} = {}^{(\beta)}\lambda\ {}^{(\beta)}\mathcal{F}^p, \quad \beta = 1, 2. \tag{12}$$

Step by step, it is possible to obtain:

$${}^{(\beta)}\mathcal{F}^p = \left({}^{(\beta)}\lambda\right)^p\ {}^{(\beta)}\mathcal{F}^0, \quad \beta = 1, 2. \tag{13}$$

Hence, using equations (10) and (13):

$$\mathcal{W}^p = {}^{(1)}\mathcal{F}^0\left({}^{(1)}\lambda\right)^p\ {}^{(1)}\mathcal{V} + {}^{(2)}\mathcal{F}^0\left({}^{(2)}\lambda\right)^p\ {}^{(2)}\mathcal{V}. \tag{14}$$

When ${}^{(\beta)}\lambda$ is written in the following form:

$$ {}^{(\beta)}\lambda = \mathrm{e}^{-\mathrm{i}\,{}^{(\beta)}\varphi}, \tag{15} $$

with ${}^{(\beta)}\varphi$ a linear function of the thickness of a period, $\mathcal{W}^p$ can be seen as the solution of a differential equation with constant coefficients, which is a very simple application of Floquet's theorem (see Appendix A2).

3.2. Period transfer matrix eigen values

The eigen values of $\boldsymbol{\tau}$ are the same as those of any similar matrix of $\boldsymbol{\tau}$. Let us define the period transfer matrix $\boldsymbol{\Phi}$ of the amplitudes such that:

$$ \mathcal{A}^{p+1} = \boldsymbol{\Phi}\mathcal{A}^p, \tag{16} $$

where $\mathcal{A}^p$ is the displacement amplitude vector such that:

$$ \mathcal{A}^p = \left\{ \begin{array}{c} {}^{(1)}a^p \\ {}^{(2)}a^p \end{array} \right\}, \tag{17} $$

with ${}^{(1)}a^p$ and ${}^{(2)}a^p$ the displacement amplitudes of the classical longitudinal plane waves propagating up and down in the first layer of the period p. Their phase origin reference is taken at $x_3 = z_{p-1}$.

$\boldsymbol{\tau}$ and $\boldsymbol{\Phi}$ are similar matrices, both period transfer matrices, related by the following relation:

$$ \boldsymbol{\Phi} = \left(B^1\right)^{-1}\boldsymbol{\tau}B^1, \tag{18} $$

In the particular case we are interested in, $\boldsymbol{\Phi}$ has the following form [6, 8, 9]:

$$ \boldsymbol{\Phi} = \left[\begin{array}{cc} \alpha & \gamma \\ \gamma^* & \alpha^* \end{array} \right], \tag{19} $$

where $*$ denotes the complex conjugate. The expressions of α, γ and B^1 are given in Appendix A1.

It can be shown [6, 8, 9] that the eigen values of the transfer matrix $\boldsymbol{\Phi}$, which are identical to those of the $\boldsymbol{\tau}$ matrix, are solutions of the following equation:

$$ \lambda^2 \Leftrightarrow 2\Re e(\alpha)\lambda + 1 = 0, \tag{20} $$

which gives

$$ {}^{(\beta)}\lambda = \Re e(\alpha) \pm \sqrt{\Re e(\alpha)^2 \Leftrightarrow 1}, \quad \beta = 1, 2. \tag{21} $$

The corresponding eigen vectors of $\boldsymbol{\Phi}$ are:

$$ {}^{(\beta)}\xi = \left\{ \begin{array}{c} \gamma \\ {}^{(\beta)}\lambda \Leftrightarrow \alpha \end{array} \right\}, \quad \beta = 1, 2, \tag{22} $$

and are related to the eigen vectors of $\boldsymbol{\tau}$ by the relation:

$$ {}^{(\beta)}\mathcal{V} = B^1\,{}^{(\beta)}\xi. \quad \beta = 1, 2, \tag{23} $$

From equation (A12) given in the Appendix, according to the variations of α as a function of the natural frequency ω, four zones in the complex plane [6, 9] can be defined (see Figure 4, with $Z_1 = 4.8$ MRayl, $h_1 = 0.13$ mm, $V_1 = 3000$ m/s, $Z_2 = 3$ MRayl, $h_2 = 0.013$ mm and $V_2 = 2500$ m/s, representative values for carbon/epoxy). In zones I and II, the eigen values ${}^{(1)}\lambda$ and ${}^{(2)}\lambda$ have unit magnitude and are complexe conjugates: the Floquet waves are thus propagative. In zones III and IV, the eigen values ${}^{(1)}\lambda$ and ${}^{(2)}\lambda$ are real and reciprocal: the Floquet waves are thus evanescent.

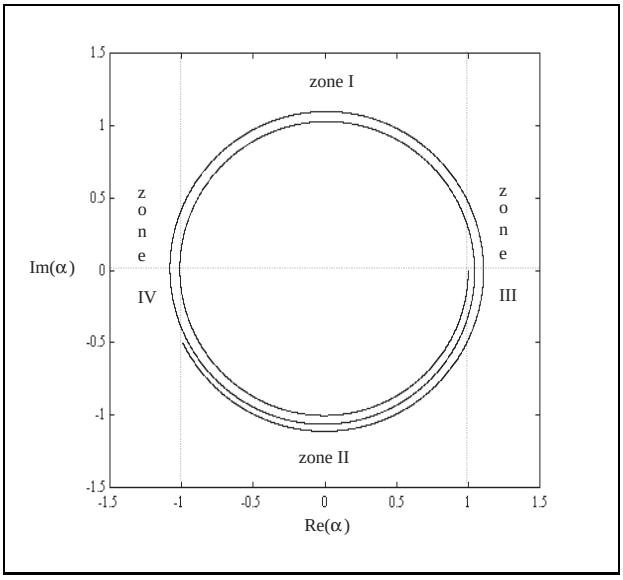

Figure 4. Representation of α in the complex plane, when the frequency varies from 0 to 50 MHz.

3.3. Strobe effect of Floquet waves

In each layer, there are two classical plane waves: one propagates in the $x_3 > 0$ direction, and the other propagates in the opposite direction. As the Floquet waves are a linear combination of these two plane waves, the linear combination being different according to the layer, it is not possible, a priori, to assign a propagation direction to the Floquet waves. Indeed, from the wave propagation point of view, the multiple reflections on each interface do not give any information on the propagation direction. In the case of evanescent Floquet waves, the radiation condition is very easy to write (see section 3.3.1). But, in the case of propagative Floquet waves, the normal power flux of each Floquet wave has to be calculated (see section 3.3.2)

3.3.1. The case of evanescent Floquet waves

Take ${}^{(1)}\lambda$ and ${}^{(2)}\lambda$ such that $|{}^{(1)}\lambda| < 1$ and $|{}^{(2)}\lambda| > 1$.

From equation (12) it can be seen that at each interface separating two successive superlayers, the amplitude ${}^{(1)}\mathcal{F}^0$ is multiplied by ${}^{(1)}\lambda$ of magnitude less than one: thus, the amplitude of this Floquet wave decreases in an exponential way in the direction $x_3 > 0$. On the contrary, the amplitude ${}^{(2)}\mathcal{F}^0$ is multiplied by ${}^{(2)}\lambda$ of magnitude greater than one: thus, the amplitude of this Floquet wave increases in an exponential way in the direction $x_3 > 0$. Therefore, the radiation condition consists in keeping the Floquet wave for which the amplitude decreases at infinity, that is to say such that $|{}^{(1)}\lambda| < 1$.

3.3.2. The case of propagative Floquet waves

We have seen in section 3.1 that for a Floquet wave, the state vector at the period interface z_{p+1} is obtained by a simple multiplication of the state vector at the previous period

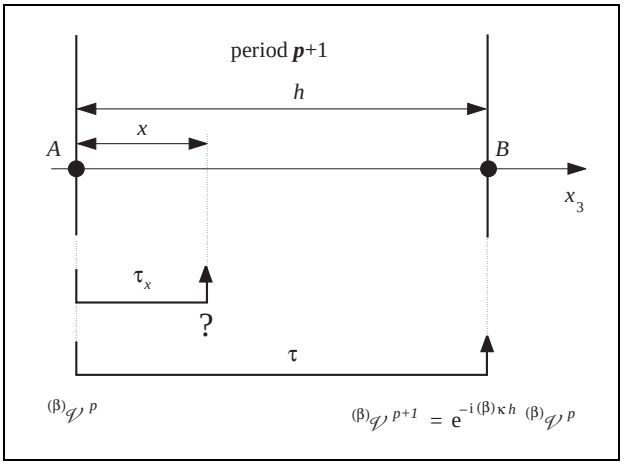

Figure 5. Strobe effect for Floquet waves. Inadequacy of a pseudo wave number.

interface z_p:

$$^{(\beta)}\mathcal{V}^{p+1} = {}^{(\beta)}\lambda\ {}^{(\beta)}\mathcal{V}^{p}, \quad \beta = 1, 2, \tag{24}$$

where here, $^{(\beta)}\lambda$ is complex with magnitude equal to one.

As the two interfaces are separated by distance h, by analogy with the propagation in homogeneous media, a pseudo Floquet wave number vector $^{(\beta)}\kappa$ can thus be defined by the following relation:

$$^{(\beta)}\lambda = \mathrm{e}^{-\mathrm{i}^{(\beta)}\kappa h}, \quad \beta = 1, 2. \tag{25}$$

$^{(\beta)}\kappa$ is not determined from equation (25) in only one way. Indeed,

$$^{(\beta)}\kappa = \Leftrightarrow \frac{\arg\left(^{(\beta)}\lambda\right) \text{ modulo} 2\pi}{h}. \tag{26}$$

The authors quoted below have fixed $^{(\beta)}\kappa$ by taking

$$\Leftrightarrow \pi \leq \arg(^{(\beta)}\lambda) \leq +\pi.$$

Through this convention, they consider that a Floquet wave (β) propagates in the direction $x_3 > 0$ when $^{(\beta)}\kappa > 0$, whereas it propagates in the opposite direction when $^{(\beta)}\kappa < 0$. However, one must be conscious of the conventional aspect of this definition since $^{(\beta)}\kappa$ would have different signs for different choices. Thus, due to the indeterminacy on the value of $^{(\beta)}\kappa$, it is illusory to try to define a phase velocity. Indeed, if $^{(\beta)}\kappa > 0$, the corresponding Floquet wave would propagate in the $x_3 > 0$ direction, but only in reference to points A and B, at each interface separating two successive superlayers (see Figure 5). In fact, it should be noticed that the multiplying factor between the state vector at a distance x from point A and the state vector at the point A can not be said to be equal to $\mathrm{e}^{-\mathrm{i}^{(\beta)}\kappa x}$. In other words, in general, if an eigen value of the transfer matrix $\boldsymbol{\tau}$ of a superlayer is $\mathrm{e}^{-\mathrm{i}^{(\beta)}\kappa h}$, the eigen value of the transfer matrix $\boldsymbol{\tau}_x$ is not $\mathrm{e}^{-\mathrm{i}^{(\beta)}\kappa x}$. This constitutes the strobe effect of Floquet waves propagation. Here, the phase information is a *discrete and not continuous information*. As mentioned in the Introduction, the situation is the same as determining if a wheel turns clockwise or anticlockwise, just by flashing it periodically. As a consequence, the sign of the pseudo Floquet wave number vector $^{(\beta)}\kappa$ just gives an apparent propagation direction and not the effective propagation direction. This is the reason why condition (27), first given in [24] and later in [26], is not correct to express the radiation condition in a semi infinite medium:

$$\Re e\left(^{(\beta)}\kappa\right) > 0 \Leftrightarrow \arg\left(^{(\beta)}\lambda\right) < 0 \Leftrightarrow \frac{\Im m\left(^{(\beta)}\lambda\right)}{\Re e\left(^{(\beta)}\lambda\right)} < 0. \tag{27}$$

Note here that due to the harmonic time dependence convention $\mathrm{e}^{+\mathrm{i}\omega t}$, equation (27) is slightly different in reference [24] and [26].

Condition (28), given afterwards in [11], by noticing that if $\Leftrightarrow \pi \leq \arg(^{(\beta)}\lambda) \leq +\pi$ then the imaginary part of $^{(\beta)}\lambda$ has the same sign as its argument, is not correct either:

$$\Re e\left(^{(\beta)}\kappa\right) > 0 \Leftrightarrow \arg\left(^{(\beta)}\lambda\right) < 0 \Leftrightarrow \Im m\left(^{(\beta)}\lambda\right) < 0. \tag{28}$$

We will see in section 3.4 that conditions (27) and (28) lead to a reflection coefficient of magnitude greater than one.

As mentioned in [6, 8, 9], the right radiation condition is given by the positive sign of the normal power flux of the relevant propagative Floquet wave. This power flux can be calculated by the following relation:

$$F_3 = \Leftrightarrow \frac{1}{4}\mathrm{i}\omega\left(\Leftrightarrow T w^* + T^* w\right), \tag{29}$$

where w and T are the normal displacement and stress.

As far as the Floquet wave (β) is concerned, these displacement and stress are given by the particular state vector $^{(\beta)}\mathcal{V}$ which is the eigen vector, defined by equations (22) and (23).

Finally, the normal power flux of each propagative Floquet wave (β) can be expressed [6, 8] as:

$$F_3 = Z_1\omega^2\left(\Re e\left(\alpha\ {}^{(\beta)}\lambda^*\right) \Leftrightarrow 1\right). \tag{30}$$

In zone I (see Figure 4), the Floquet wave which corresponds to $F_3 > 0$ is associated to the eigen value $^{(1)}\lambda = \Re e(\alpha) + \mathrm{i}\sqrt{1 \Leftrightarrow [\Re e(\alpha)]^2}$. In zone II, this is $^{(2)}\lambda = \Re e(\alpha) \Leftrightarrow \mathrm{i}\sqrt{1 \Leftrightarrow [\Re e(\alpha)]^2}$. The corresponding calculus are detailed in Appendixes A3 and A4. It should be noted that the use of condition (28) would lead to choose $^{(2)}\lambda$ in both zones I and II.

3.4. Reflection coefficient

Let us note λ^+ the eigen value associated to the Floquet wave which propagates $(F_3 > 0)$, or decreases in the direction $x_3 > 0$. The reflection coefficient $\mathcal{R}$ in fluid 0 on the semi-infinite two-layered periodic structure can be expressed by the following relation [6, 8, 9]:

$$\mathcal{R} = \frac{r_{01}\gamma + \lambda^+ \Leftrightarrow \alpha}{\gamma + r_{01}(\lambda^+ \Leftrightarrow \alpha)}, \tag{31}$$

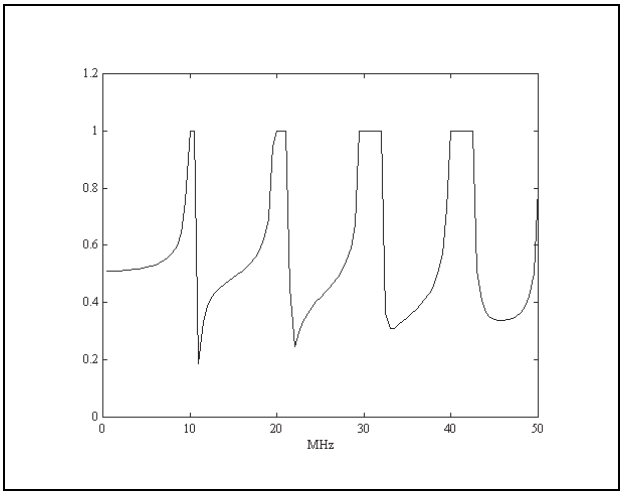

Figure 6. Magnitude of the reflection coefficient using the radiation condition given by the normal power flux.

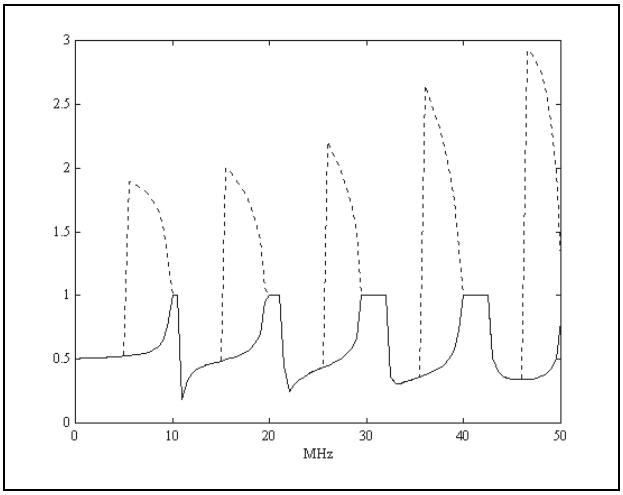

Figure 7. Comparison of the moduli of the reflection coefficient. Full line: calculus using the radiation condition given by the normal power flux ; dotted line: calculus using radiation condition (27).

where r_{01} is the reflection coefficient of the fluid 0 / medium 1 interface:

$$r_{01} = \frac{Z_1 \Leftrightarrow Z_0}{Z_1 + Z_0}. \tag{32}$$

The calculus have been performed with the same medium properties as in Figure 4: $Z_1 = 4.8$ MRayl, $h_1 = 0.13$ mm, $V_1 = 3000$ m/s, $Z_2 = 3$ MRayl, $h_2 = 0.013$ mm and $V_2 = 2500$ m/s. The fluid 0 is water with $Z_0 = 1.48$ MRayl and $V_0 = 1480$ m/s.

The reflection coefficient (see its magnitude in Figure 6) has been calculated by using, for propagative Floquet waves, the radiation condition given by the normal power flux. The magnitude is always less than or equal to one, and stopping bands can be observed. The comparison of this coefficient with the coefficient calculated using radiation condition (27) is shown in Figure 7. It appears clearly that when the two curves do not coincide, the reflection coefficient is much greater than one and may tend to infinity. The use of radiation condition (28) leads to the same conclusions (see Figure 8). In this case, it can be observed that the calculus is correct once out of two.

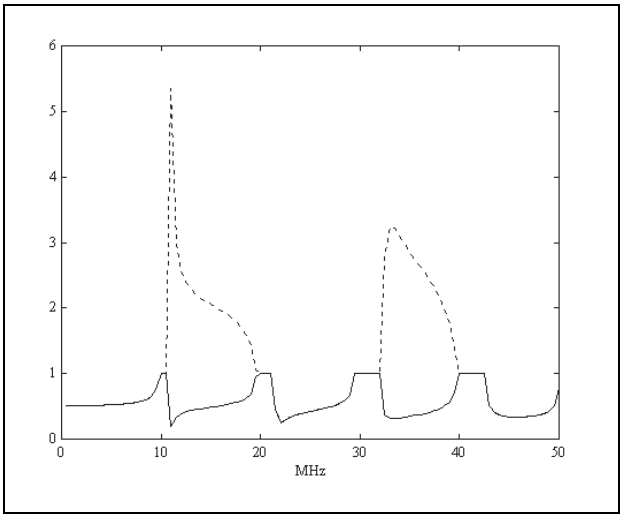

Figure 8. Comparison of the moduli of the reflection coefficient. Full line: calculus using the radiation condition given by the normal power flux ; dotted line: calculus using radiation condition (28).

4. Radiation conditions in an infinite anisotropic periodically multilayered medium

We have just seen for a very simple configuration how the radiation condition has to be handled in the case of propagative Floquet waves. The aim of this section is to extend this method to infinite anisotropic periodically multilayered media.

4.1. Background on transfer matrices

Let us now consider a semi-infinite periodically multilayered medium which is a reproduction of an infinite number of "periods", each one made up by the stacking of Q distinct elastic anisotropic media (see Figure 1). Each layer of the period may have any thickness. Medium 0 above the periodically multilayered medium is semi-infinite. The study of the acoustic propagation of waves which are generated by an oblique incident wave propagating in medium 0 with a propagation vector contained in the $(x_1 x_3)$ plane, as defined in Figure 1, has been carried out in previous works [13, 14, 15]: therefore, we will not explain it again with much details but we will restrict to give the keys to understand what follows.

Let $\mathcal{W}^p$ be the state vector at the period interface $x_3 = z_p$, made up of the three components of the displacement vector $\vec{u}^p$ and the three components $(T^p_{33}, T^p_{23}, T^p_{13})$ of the stress vector applied to a surface parallel to the interfaces. Furthermore, let $\mathcal{A}^p$ be the (6×1) column vector containing the displacement amplitude ${}^{(\eta)}a^p$ of the six classical plane waves propagating in the first layer of the period $p(1 \leq \eta \leq 6)$.

Namely, the state vector is written such that:

$$\mathcal{W}^p = \left\langle u_1^p, u_2^p, u_3^p, T_{33}^p, T_{23}^p, T_{13}^p \right\rangle^T, \tag{33}$$

and the amplitude vector such that:

$$\mathcal{A}^p = \left\langle {}^{(1)}a^p, {}^{(2)}a^p, {}^{(3)}a^p, {}^{(4)}a^p, {}^{(5)}a^p, {}^{(6)}a^p \right\rangle^T, \tag{34}$$

where T denotes the transpose operation.

The (6×6) period transfer matrix $\mathbf{\Phi}$ for amplitudes vectors as defined in [13] and [14], allows the displacement amplitudes of the classical plane waves in the first layer of the period $(p+1)$ to be expressed as a function of those in the first layer of the previous period p, by means of the following matricial relation, already written in section 3.2:

$$\mathcal{A}^{p+1} = \mathbf{\Phi}\mathcal{A}^p. \tag{35}$$

In the same way, equation (5) which expresses the matricial relation between the state vector $\mathcal{W}^{p+1}$ at the period interface z_{p+1} and the state vector $\mathcal{W}^p$ at the previous period interface z_p remains:

$$\mathcal{W}^{p+1} = \boldsymbol{\tau}\mathcal{W}^p, \tag{36}$$

where $\boldsymbol{\tau}$ is here of the 6-th order.

It has been seen in section 3.2 that $\boldsymbol{\tau}$ is related to $\mathbf{\Phi}$ by the following relation:

$$\mathbf{\Phi} = \left(B^1\right)^{-1}\boldsymbol{\tau}B^1, \tag{37}$$

where B^1 is here a (6×6) matrix which depends only on the incident wave and on the characteristics of the medium constituting the $q = 1$ layer. B^1 is defined in reference [14].

4.2. Calculus of the normal power flux of each Floquet wave

In the same way as in equation (10), section 3.1, the state vector $\mathcal{W}^p$ at z_p can be expressed on the Floquet wave basis as a linear combination of the six eigen vectors ${}^{(\beta)}\mathcal{V}$, $1 \leq \beta \leq 6$:

$$\mathcal{W}^p = \sum_{\beta=1}^{6} {}^{(\beta)}\mathcal{F}^p \; {}^{(\beta)}\mathcal{V}. \tag{38}$$

It should be noted that the $\mathrm{e}^{-\mathrm{i}(k_1 x_1 - \omega t)}$ factor has been omitted in equation (38), k_1 being the projection of the wave vector of the incident wave on the x_1-axis.

As explained in section 3.1, each eigen vector ${}^{(\beta)}\mathcal{V}$ is a particular state vector $\mathcal{W}^p$ (see equation (33) for the position of each element in the state vector matrix). As a consequence, in the following expression of the normal power flux [31] using Einstein's convention,

$$F_3 = \Leftrightarrow\frac{1}{4}\mathrm{i}\omega\left(\Leftrightarrow T_{3j}u_j^* + T_{3j}^* u_j\right), \tag{39}$$

u_j, $j = 1, 2, 3$, correspond to the first three components of $\mathcal{W}^p$ or ${}^{(\beta)}\mathcal{V}$ and T_{3j}, $j = 1, 2, 3$ to the last three components of $\mathcal{W}^p$ or ${}^{(\beta)}\mathcal{V}$. Therefore, for each Floquet wave (β), equation (39) can be written as:

$$\begin{aligned} F_3 = \Leftrightarrow\frac{1}{4}\mathrm{i}\omega\Big(& \Leftrightarrow {}^{(\beta)}\mathcal{V}_6 \; {}^{(\beta)}\mathcal{V}_1^* + {}^{(\beta)}\mathcal{V}_6^* \; {}^{(\beta)}\mathcal{V}_1 \\ & \Leftrightarrow {}^{(\beta)}\mathcal{V}_5 \; {}^{(\beta)}\mathcal{V}_2^* + {}^{(\beta)}\mathcal{V}_5^* \; {}^{(\beta)}\mathcal{V}_2 \\ & \Leftrightarrow {}^{(\beta)}\mathcal{V}_4 \; {}^{(\beta)}\mathcal{V}_3^* + {}^{(\beta)}\mathcal{V}_4^* \; {}^{(\beta)}\mathcal{V}_3\Big). \end{aligned} \tag{40}$$

Note that if Ξ and ϑ are respectively the eigenvector matrices of the period transfer matrix of amplitudes Φ and of the period matrix of stress-displacement $\boldsymbol{\tau}$, they are related by the following relation:

$$\Xi = B^1\vartheta, \tag{41}$$

$\mathbf{\Phi}$ and $\boldsymbol{\tau}$ being similar matrices as a consequence of equation (37).

In short, the period transfer matrix of amplitudes Φ is first calculated. Then, its eigen values and its eigen vector matrix are numerically calculated. Next, the eigen vector matrix of the period transfer matrix of stress-displacement $\boldsymbol{\tau}$ is calculated by means of equation (41). Each column ${}^{(\beta)}\mathcal{V}$ of this eigen vector matrix corresponds to the state vector $\mathcal{W}^p$ of the Floquet wave associated to the eigen value ${}^{(\beta)}\lambda$. Finally, the normal power flux of the Floquet wave (β) is numerically evaluated by means of equation (40).

4.3. Radiation conditions and reflection coefficients for semi-infinite anisotropic periodically multilayered media

In the case of the reflection and transmission problem at the interface separating the fluid and the semi-infinite stratified medium, radiation conditions have to be written. In the same way as in section 3.3, in the case of propagative Floquet waves, the pseudo Floquet wave number associated to each propagative Floquet wave gives only an apparent propagation direction. The correct radiation condition is given by the energy propagation direction of each propagative Floquet wave.

- In the case of propagative Floquet waves, $|{}^{(\beta)}\lambda| = 1$, and the radiation condition is given by the choice of the Floquet wave with:

$$ {}^{(\beta)}F_3 > 0. \tag{42}$$

- In the case of inhomogeneous Floquet waves, $|{}^{(\beta)}\lambda| \neq 1$, and the radiation condition is simply given by the choice:

$$|{}^{(\beta)}\lambda| < 1. \tag{43}$$

The validity of these radiation conditions is brought to the fore by calculating reflection coefficients.

By applying conditions (42) or (43), it is possible to choose the three Floquet waves which propagate or decrease in the $x_3 > 0$ direction and then to obtain the displacement and

Table I. Elastic constants in GPa for a carbon/epoxy medium from reference [33] when the sixth-order symmetry A_6 axis is parallel to the (Ox_3) axis.

c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	ρ (kg/m^3)
13.7	7.1	6.7	126	5.8	1577

stress vector $\mathcal{W}^0$ at the first interface at $x_3 = z_0$, which amounts to expressing the radiation condition:

$$\mathcal{W}^0 = \sum_{\beta=1}^{6} {}^{(\beta)}\mathcal{F}^{0+}\ {}^{(\beta)}\mathcal{V}^{+}, \tag{44}$$

where ${}^{(\beta)}\mathcal{F}^{0+}$ and ${}^{(\beta)}\mathcal{V}^{+}$, $(\beta = 1, 2, 3)$, are respectively the amplitudes and the eigen vectors associated to each of these three Floquet waves. Note that the Floquet amplitudes are referenced at $x_3 = z_0$. It is then possible, by use of the boundary conditions at the first interface, to calculate the reflection coefficients. As a result of the good choice of radiation condition, the calculated reflection coefficient appears with a magnitude less or equal to one.

The calculus is performed on a medium made up with stacked identical hexagonal layers of carbon/epoxy, each one 45° rotated with respect to the previous one (0°/45°/90°/135° medium). Each layer of this composite plate is 0.12 mm thick. The elastic constants used in the model are those determined by Hosten and Castaings [32, 33] (see Table I). The volumetric mass of each layer is equal to 1577 kg/m^3. The semi-infinite medium above the multilayered medium is water: the volumetric mass is equal to 1000 kg/m^3 and the velocity of the longitudinal wave is equal to 1480 m/s. The reflection coefficient which magnitude is presented in Figure 9a) has been calculated for a frequency equal to 3 MHz. It is possible to see on Figure 9b) that the stopping bands correspond to Floquet waves which are all inhomogeneous. Note that around 35°, one Floquet wave is propagative. Though the corresponding magnitude of the reflection coefficient on Figure 9a) seems to be equal to one, it is less than one.

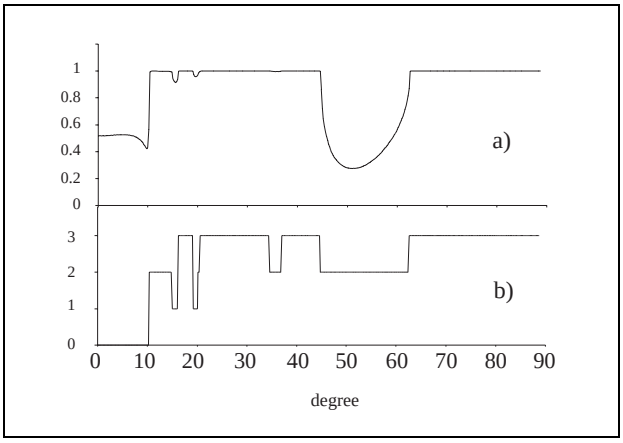

Figure 9. Infinite 0°/45°/90°/135° medium; $f = 3$ MHz. a) Magnitude of the reflection coefficient in water. b) Number of inhomogeneous Floquet waves as a funtion of the incidence angle.

5. Conclusion

The aim of this paper was to define the propagation direction of Floquet waves and to express in a correct way the radiation conditions. These Floquet waves propagate or decrease in the periodically multilayered medium and are particular propagation modes of the infinite periodically medium. Floquet waves are linear combinations of the classical plane waves which propagate or decrease in each layer of the multilayered medium. Any stress and displacement state vector at an interface separating two successive periods can be deduced from the one at the previous period interface by means of a matricial relation, involving the period transfer matrix. Floquet waves are associated to the eigen vectors of this matrix which are particular state vectors. For these particular solutions, the passage from one period interface to another one is then possible by a simple multiplication of the eigen vector by the corresponding eigen value of the matrix.

By analogy with the propagation of classical plane waves in homogeneous media, the propagation of the Floquet waves can be interpreted as propagation in a medium "equivalent" to the stratified medium. However, the notion of "equivalent" medium can only be understood in the literal sense for the low frequency domain of homogenisation. Apart from this case, this notion is only an image, related to the interpretation of the Floquet eigen values, from period to period. Moreover, this interpretation must not let forget that, even if the Floquet waves are defined in the whole stratified medium, they are only characterised from interface to interface, i.e. in a discrete way. As a consequence, there occurs a spatial strobe effect.

An illustration of this problem is given when one tries to define a pseudo Floquet wave number, using the phase variation across each period. As the propagation of the corresponding Floquet wave is only referenced at each interface separating two successive periods, the sign of the real part of the pseudo Floquet wave number gives only an apparent propagation direction and not the effective propagation direction of the propagative Floquet wave. As a consequence, the only possibility to define a propagation direction with a physical meaning is to consider the sign of the normal power flux of each Floquet wave.

To sum up, whereas the radiation condition for inhomogeneous Floquet waves is very easy to write (by keeping only the waves for which the magnitude of the corresponding eigen value is less than one), in the case of propagative Floquet waves, the radiation condition needs to depart from the consideration of the pseudo Floquet wave number and to turn towards an energy criterion. However, a few authors did propose to deduce the propagation direction from the study of the pseudo Floquet wave number sign. It is the main purpose of the present work to show that this is the wrong choice, when compared to the correct propagation direction deduced from energetic considerations. In order to clarify this point,

we first considered the simple case of the propagation at normal incidence in isotropic periodically multilayered media, each period being made up of two layers. The calculus of a reflection coefficient using the correct radiation condition has been compared with the one using the apparent propagation direction. With an energetic criterion, the magnitude of the reflection coefficient keeps equal or less than one, whereas it may tend to infinity with a pseudo wave number criterion. Finally the study has been extended to anisotropic periodically multilayered media.

Appendix

A1. Obtention of the period transfer matrix in the case of a two layers periodic medium [6, 8, 9]

Generally speaking, in the normal incidence case, the normal displacement and stress in the nth layer composed of the medium q, $q = 1, 2$, are given by:

$$w_n = \Big({}^{(1)}a_n \mathrm{e}^{-\mathrm{i}k_q(x_3-\zeta_{n-1})} - {}^{(2)}a_n \mathrm{e}^{\mathrm{i}k_q(x_3-\zeta_{n-1})}\Big)\mathrm{e}^{\mathrm{i}\omega t}, \tag{A1}$$

$$T_n = -\mathrm{i}\omega Z_q\Big({}^{(1)}a_n \mathrm{e}^{-\mathrm{i}k_q(x_3-\zeta_{n-1})} + {}^{(2)}a_n \mathrm{e}^{\mathrm{i}k_q(x_3-\zeta_{n-1})}\Big)\mathrm{e}^{\mathrm{i}\omega t}, \tag{A2}$$

where $k_q = \omega/V_q$ and Z_q are respectively the wave number and the acoustic impedance of medium q. ζ_{n-1} is the co-ordinate with respect to the x_3-axis of the upper interface of the nth layer (see Figure 2).

At $x_3 = \zeta_n$, the displacement-stress vector can be written such that:

$$\begin{Bmatrix} w_n \\ T_n \end{Bmatrix}_{\zeta_n} = B^q \mathcal{H}^q \begin{Bmatrix} {}^{(1)}a_n \\ {}^{(1)}a_n \end{Bmatrix} \mathrm{e}^{\mathrm{i}\omega t}, \tag{A3}$$

where B^q and $\mathcal{H}^q$ have the following form:

$$B^q = \begin{bmatrix} 1 & -1 \\ -\mathrm{i}\omega Z_q & -\mathrm{i}\omega Z_q \end{bmatrix}, \tag{A4}$$

and

$$\mathcal{H}^q = \begin{bmatrix} \mathrm{e}^{-\mathrm{i}k_q h_q} & 0 \\ 0 & \mathrm{e}^{+\mathrm{i}k_q h_q} \end{bmatrix}. \tag{A5}$$

If n is an odd number, then $q = 1$. The use of the continuity of normal displacement and stress at $x_3 = \zeta_n$ and at $x_3 = \zeta_{n+1}$ provides a relationship between the displacement amplitudes of the plane waves in the n-th layer and those in the $(n+2)$-th layer:

$$\begin{Bmatrix} {}^{(1)}a_{n+2} \\ {}^{(2)}a_{n+2} \end{Bmatrix} = \Phi \begin{Bmatrix} {}^{(1)}a_n \\ {}^{(2)}a_n \end{Bmatrix}, \tag{A6}$$

with

$$\Phi = \left(B^1\right)^{-1} B^2 \mathcal{H}^2 \left(B^2\right)^{-1} B^1 \mathcal{H}^1. \tag{A7}$$

Same way as for the amplitudes, a relationship may be found between the displacement-stress vector at $x_3 = \zeta_{n-1}$ and that at $x_3 = \zeta_{n+1}$:

$$\begin{Bmatrix} w_{n+2} \\ T_{n+2} \end{Bmatrix}_{\zeta_{n+2}} = \boldsymbol{\tau} \begin{Bmatrix} w_n \\ T_n \end{Bmatrix}_{\zeta_{n-1}}, \tag{A8}$$

with

$$\boldsymbol{\tau} = B^2 \mathcal{H}^2 \left(B^2\right)^{-1} B^1 \mathcal{H}^1 \left(B^1\right)^{-1}. \tag{A9}$$

$\boldsymbol{\tau}$ and $\boldsymbol{\Phi}$ are similar matrices, both period transfer matrices, related by the following relation:

$$\boldsymbol{\Phi} = \left(B^1\right)^{-1} \boldsymbol{\tau} B^1. \tag{A10}$$

With no difficulty, it can be found that $\boldsymbol{\Phi}$ has the following form [6, 8, 9]:

$$\boldsymbol{\Phi} = \begin{bmatrix} \alpha & \gamma \\ \alpha^* & \gamma^* \end{bmatrix}, \tag{A11}$$

where $*$ denotes the complex conjugate, with

$$\alpha = \Big[\cos(\omega\tau_2) - \mathrm{i}S_{12}\sin(\omega\tau_2)\Big]\mathrm{e}^{-\mathrm{i}\omega\tau_1}, \tag{A12}$$

$$\gamma = -\mathrm{i}D_{12}\sin(\omega\tau_2)\mathrm{e}^{+\mathrm{i}\omega\tau_1}, \tag{A13}$$

$$\alpha\alpha^* - \gamma\gamma^* = 1, \tag{A14}$$

$$S_{12} = \frac{1}{2}\left(\frac{Z_1}{Z_2} + \frac{Z_2}{Z_1}\right), \tag{A15}$$

$$D_{12} = \frac{1}{2}\left(\frac{Z_1}{Z_2} - \frac{Z_2}{Z_1}\right). \tag{A16}$$

Note that expressions (A12) and (A13) are slightly different from those of reference [6] because of the harmonic time dependence which is here $\mathrm{e}^{+\mathrm{i}\omega t}$.

If z_p denotes the co-ordinate with respect to x_3-axis of the interface separating period p from the next period $p+1$, with $z_{p+1} = z_p + h$ and $z_0 = 0$ (see Figure 1), equations (A6) and (A8) can be written such that:

$$\mathcal{A}^{p+1} = \boldsymbol{\Phi}\mathcal{A}^p, \tag{A17}$$

and

$$\mathcal{W}^{p+1} = \boldsymbol{\tau}\mathcal{W}^p, \tag{A18}$$

where

$$\mathcal{A}^p = \begin{Bmatrix} {}^{(1)}a^p \\ {}^{(2)}a^p \end{Bmatrix}, \tag{A19}$$

with ${}^{(1)}a^p$ and ${}^{(2)}a^p$ the displacement amplitudes of the classical longitudinal plane waves propagating up and down in the first layer of the period p. Their phase origin reference is taken at $x_3 = z_{p-1}$.

The state vector $\mathcal{W}^p$ at $x_3 = z_p$ is defined by:

$$\mathcal{W}^p = \begin{Bmatrix} w_p \\ T_p \end{Bmatrix}, \tag{A20}$$

where w_p and T_p are the normal displacement and stress at $x_3 = z_p$.

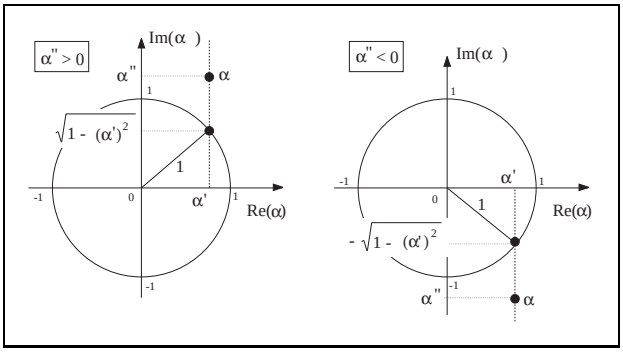

Figure A1. Position of α in the complex plane, in comparison with the radius one circle, in the case of propagative Floquet waves.

A2. Floquet's theorem

Floquet waves can be introduced by an extension of Floquet's theorem to the field equations governing the dynamic behavior of a *periodic continuous* medium, when they are written in a matrix differential form. Originally, when the periodic function contains a cosine term in the one-dimensional problem, the field equation can be written such as [23] :

$$\frac{\partial^2 u}{\partial x^2} + \big[a + b\cos(2x)\big]\, u = 0. \tag{A21}$$

Floquet discovered that the general solution of the equation could be written:

$$u = A_1 F(x)\mathrm{e}^{\alpha x} + A_2 G(x)\mathrm{e}^{-\alpha x}. \tag{A22}$$

In a three-dimensional problem, when the field equations are written in a matrix differential form, an extension of Floquet's theorem (A21) leads to write the solutions in terms of an exponential matrix, the eigen values of which are the Floquet wave numbers. In the case we are interested in (see Figure 1), the multilayered medium is not a periodically continuous medium: the periodic function is *constant piece by piece*, which amounts to equalling b to zero in equation (A20), with a depending on the layer medium. This equation becomes a simple differential equation with constant coefficients, so that, strictly speaking, Floquet's theorem reduces in this case to a very classical result.

A3. Calculus of the normal power flux of a propagative Floquet wave

The normal power flux can be calculated by use of the following relation:

$$F_3 = \Leftrightarrow\!\frac{1}{4}\mathrm{i}\omega\big(\Leftrightarrow\!T w^* + T^* w\big), \tag{A23}$$

where w and T are the normal displacement and stress.

As far as the Floquet wave (β) is concerned, these displacement and stress are given by the particular state vector which is the eigen vector ${}^{(\beta)}\mathcal{V}$, defined by equations (22) and (23). Hence,

$$w = \gamma \Leftrightarrow\!\big({}^{(\beta)}\lambda \Leftrightarrow\!\alpha\big), \tag{A24}$$

$$T = \Leftrightarrow\!\mathrm{i}\omega Z_1\big(\gamma + {}^{(\beta)}\lambda \Leftrightarrow\!\alpha\big). \tag{A25}$$

With w and T from equation (A24) and (A25) and using equation (A23), the normal power flux is thus given by:

$$F_3 = \frac{Z_1}{2}\omega^2\Big(\gamma\gamma^* \Leftrightarrow\!\alpha\alpha^* \Leftrightarrow\!{}^{(\beta)}\lambda{}^{(\beta)}\lambda^* + \alpha\,{}^{(\beta)}\lambda^* + \alpha^*\,{}^{(\beta)}\lambda\Big). \tag{A26}$$

Taking equation (A14) into account and noticing that $|{}^{(\beta)}\lambda| = 1$, one can finally obtain:

$$F_3 = Z_1\omega^2\Big(\Re e\big(\alpha\,{}^{(\beta)}\lambda^*\big) \Leftrightarrow\!1\Big). \tag{A27}$$

A4. Sign of the normal power flux of a propagative Floquet wave

Let us note in this section $\alpha = \alpha' + \mathrm{i}\alpha''$.

Propagative Floquet waves are associated to the eigen values ${}^{(\beta)}\lambda = \alpha' \pm \mathrm{i}\sqrt{1 \Leftrightarrow\!(\alpha')^2}$ with $|\alpha'| < 1$ It can also be noted that, due to equation (A14), $|\alpha| \geq 1$. As a consequence, considering the radius one circle in the complex plane, $|\alpha''| > \sqrt{1 \Leftrightarrow\!(\alpha')^2}$ (see Figure A1).

The sign of the normal power flux defined by equation (A27) is given by the sign of $\Re e(\alpha\,{}^{(\beta)}\lambda^*) \Leftrightarrow\!1$

- Let us consider the case of the Floquet wave associated to ${}^{(1)}\lambda = \alpha' + \mathrm{i}\sqrt{1 \Leftrightarrow\!(\alpha')^2}$.

$$\begin{aligned}\Re e\big(\alpha\,{}^{(\beta)}\lambda^*\big) \Leftrightarrow\!1 &= (\alpha')^2 + \alpha''\sqrt{1 \Leftrightarrow\!(\alpha')^2} \Leftrightarrow\!1\\ &= \sqrt{1 \Leftrightarrow\!(\alpha')^2}\Big[\alpha'' \Leftrightarrow\!\sqrt{1 \Leftrightarrow\!(\alpha')^2}\Big].\end{aligned}$$

  The sign of $\Re e(\alpha\,{}^{(\beta)}\lambda^*) \Leftrightarrow 1$ is thus the sign of $\alpha'' \Leftrightarrow\!\sqrt{1 \Leftrightarrow\!(\alpha')^2}$.
  - If $\alpha'' < 0$, which corresponds to zone II (see Figure 4), this expression is always negative.
  - If $\alpha'' > 0$, which corresponds to zone I (see Figure 4), since $|\alpha''| > \sqrt{1 \Leftrightarrow\!(\alpha')^2}$, this expression is always positive.
- Let us now consider the case of the Floquet wave associated to ${}^{(2)}\lambda = \alpha' \Leftrightarrow\!\mathrm{i}\sqrt{1 \Leftrightarrow\!(\alpha')^2}$.

$$\begin{aligned}\Re e\big(\alpha\,{}^{(\beta)}\lambda^*\big) \Leftrightarrow\!1 &= (\alpha')^2 \Leftrightarrow\!\alpha''\sqrt{1 \Leftrightarrow\!(\alpha')^2} \Leftrightarrow\!1\\ &= \sqrt{1 \Leftrightarrow\!(\alpha')^2}\Big[\alpha'' + \sqrt{1 \Leftrightarrow\!(\alpha')^2}\Big].\end{aligned}$$

  The sign of $\Re e(\alpha\,{}^{(\beta)}\lambda^*) \Leftrightarrow\!1$ is thus the opposite sign of $\alpha'' + \sqrt{1 \Leftrightarrow\!(\alpha')^2}$.
  - If $\alpha'' < 0$, which corresponds to zone II (see Figure 4), since $|\alpha''| > \sqrt{1 \Leftrightarrow\!(\alpha')^2}$, this expression is always negative and thus $\Re e\big(\alpha\,{}^{(\beta)}\lambda^*\big) \Leftrightarrow\!1$ is always positive.
  - If $\alpha'' > 0$, which corresponds to zone I (see Figure 4), this expression is always positive and thus $\Re e\big(\alpha\,{}^{(\beta)}\lambda^*\big) \Leftrightarrow\!1$ is always negative.

Finally, in zone I (see Figure 4), the Floquet wave which corresponds to $F_3 > 0$ is associated to the eigen value ${}^{(1)}\lambda = \Re e(\alpha) + \mathrm{i}\sqrt{1 \Leftrightarrow [\Re e(\alpha)]^2}$. In zone II, this is ${}^{(2)}\lambda = \Re e(\alpha) \Leftrightarrow \mathrm{i}\sqrt{1 \Leftrightarrow [\Re e(\alpha)]^2}$.

References

[1] W. T. Thomson: Transmission of elastic waves through a stratified solid medium. J. Appl. Phys. **21** (1950) 89–93.
[2] N. A. Haskell: The dispersion of surface waves in multilayered media. Bull. Seismol. Soc. Am. **43** (1953) 17–34.
[3] F. Gilbert, G. E. Backus: Propagator matrices in elastic wave and vibration problem. Geophysics **31** (1966) 326–332.
[4] K. E. Gilbert: A propagator matrix method for periodically stratified media. J. Acoust. Soc. Am. **73** (1983) 137–142.
[5] M. Schoenberg: Properties of a periodically stratified acoustic half-space and its relation to a Biot fluid. J. Acoust. Soc. Am. **73** (1983) 61–67.
[6] M. Rousseau, P. Gatignol: Propagation acoustique dans un milieu périodiquement stratifié. Acustica **64** (1987) 188–194.
[7] P. G. Richards: Elastic wave solutions in stratified media. Geophysics **36** (1971) 798–809.
[8] M. Rousseau: Floquet wave properties in a periodically layered medium. J. Acoust. Soc. Am. **86** (1989) 2369–2376.
[9] J. S. Moukémaha: Modélisation des signaux acoustiques réfléchis par un multicouche périodique - application à l'identification de défauts par méthodes ultrasonores. PhD Thesis, Univ. Techn. Compiègne, France, 1991.
[10] T. Lhermitte, B. Perrin, M. Fink: Ultrasonic spectroscopy in cross-ply reinforced composites applied to dispersion effects characterisation of elastic shear wave. First French Conf. on Acoust., Suppl. J. Phys., tome 51,, 1990. C2 1265–C2 1268.
[11] A. M. B. Braga, G. Hermann: Floquet waves in anisotropic periodically layered composites. J. Acoust. Soc. Am. **91** (1992) 1211–1227.
[12] A. H. Nayfeh: Wave propagation in layered anisotropic media with applications of composites. North-Holland, 1995.
[13] C. Potel, J. F. de Belleval: Propagation in an anisotropic periodically multilayered medium. J. Acoust. Soc. Am. **93** (1993) 2669–2677.
[14] C. Potel, J. F. de Belleval: Acoustic propagation in anisotropic periodically multilayered media. A method to solve numerical instabilities. J. App. Phys. **74** (1993) 2208–2215.
[15] C. Potel, J. F. de Belleval, Y. Gargouri: Floquet waves and classical plane waves in an anisotropic periodically multilayered medium: application to the validity domain of homogenization. J. Acoust. Soc. Am. **97** (1995) 2815–2825.
[16] C. Potel, J. F. de Belleval: Surface waves in an anisotropic periodically multilayered medium: influence of the absorption. J. Appl. Phys. **77** (1995) 6152–6161.
[17] C. Potel, J. F. de Belleval, E. Genay, P. Gatignol: Behavior of Lamb waves and multilayered Rayleigh waves in an anisotropic periodically multilayered medium: application to the long-wave length domain. Acustica-Acta acustica **82** (1996) 738–748.
[18] E. G. Henneke II: Reflection-refraction of stress wave at a plane boundary between anisotropic media. J. Acoust. Soc. Am. **51** (1972) 210–217.
[19] L. G. Merkulov: Ultrasonic waves in crystals. Appl. Mat. Res. (1963) 321–240.
[20] M. J. P. Musgrave: Crystal acoustics. Holden-Day, San Francisco, 1970.
[21] D. Royer, E. Dieulesaint: Ondes élastiques dans les solides, Vol 1: Propagation libre et guidée. Masson, 1996.
[22] S. I. Rokhlin, T. K. Bolland, L. Adler: Reflection and refraction of elastic waves on a plane interface between two generally anisotropic media. J. Acoust. Soc. Am. **79** (1986) 906–918.
[23] Brillouin: Wave propagation in periodic structures. 2nd ed. Dover, New-York, 1953.
[24] A. M. B. Braga, G. Herman: Plane waves in anisotropic layered composites. – In: Wave Propagation In Structural Composites. A. K. Mal, T. C. T. Ting (eds.). ASME, New-York, 1988, AMD Vol.90, 81–98.
[25] P. J. Shull, D. E. Chimenti, S. K. Datta: Elastic guided waves and the floquet concept in periodically layered plates. J. Acoust. Soc. Am. **95** (1994) 99–108.
[26] F. Sun, P. Banks-Lee, H. Peng: Wave propagation theory in anisotropic periodically layered fluid-saturated porous media. J. Acoust. Soc. Am. **93** (1993) 1277–1285.
[27] M. Schoenberg: Reflection of elastic waves from periodically stratified media with interfacial slip. Geophysical prospecting **3** (1983) 265–292.
[28] M. Schoenberg: Plane wave propagation in stratified anisotropic media. J. Acoust. Soc. Am. **55** (1974) 5.
[29] E. L. Adler: Matrix methods applied to acoustic waves in multilayers. IEEE Trans. Ultrason. Ferroelec. Freq. Contr. **37** (1990) 485–490.
[30] C. E. Bradley: Time harmonic acoustic bloch wave propagation in periodic wave guides. Part I. Theory. J. Acoust. Soc. Am. **96** (1994) 1844–1853.
[31] J. L. Synge: Flux of energy for elastic waves in anisotropic media. Proc. Royal Irish Academy. Sect A, Elastic waves in anisotropic media **58** (1956) 13–21.
[32] B. Hosten, M. Castaings: Transfer matrix of multilayered absorbing and anisotropic media. Measurements and simulations of ultrasonic wave propagation through composite materials. J. Acoust. Soc. Am. **94** (1993) 1488–1495.
[33] M. Castaings: Propagation ultrasonore dans les milieux stratifiés plans constitués de matériaux absorbants et orthotropes. Thèse de doctorat, Univ. Bordeaux I, 1993.