Two Families of Modal Waves for Periodic Structures with Two Field Functions: A Cayleigh-Hamilton Approach

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Summary
This paper aims at providing an analytical study of periodic structures depending on two field functions, using the Cayleigh-Hamilton theorem. This approach allows to emphasize the role played in the constitution of modal waves (i.e. guided and surface modal waves), on one hand by the period cell, and on the other hand by the number of periods of the structure. Introducing the formalism of transfer matrix and the theory of Floquet waves leads, for guided modal waves, to two families of modal waves: the structure modal waves, which take place in the passing bands and which are strongly dependent on the number of periods in the structure, and the period modal waves, which take place in the stopping bands and which depend only on the period cell and on the nature of the external condition. In the case of semi-infinite periodic structures, surface modal waves are found, despite of the fact that all the layers are fluid media. Some results are also given when the two external boundary conditions are different one from the other, in particular, in the case of structures for which the period cell of the last period is modified (one layer added or suppressed). At low frequencies, the "effective" continuous medium (homogenization technique) appears to be anisotropic. In order to make it easier to understand, the presentation is focused on the case of fluid-type periodically layered structures, but the same conclusions can be drawn for any periodic structure described with two field functions (2nd-order transfer matrix): one-dimensional periodic lattice such as diatomic chain of coupled spheres for example, SH waves in isotropic layers, fluid layers, etc. In the case of one-dimensional structures, these results concern eigen vibrations.

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Introduction

Anisotropic periodically layered media have been studied extensively since they are a good representation of composite materials. In particular, the propagation of waves along such structures has retained much attention, as the basis for numerical simulations of the ultrasonic techniques with a view to non destructive evaluation of these materials. Thus, dispersion properties have been established for Lamb-type waves in finite thickness structure, or for Rayleigh-type waves in semi-infinite periodic media. One will find general considerations in Nayfeh [1], and numerical results for Rayleigh and Lamb waves in Podlipenets *et al.* [2], and for Lamb waves in Safaeinili *et al.* [3] for the isotropic case.

Many properties of Lamb-type waves or Rayleigh-type waves can be found in the literature (see [1, 4, 5] and references contained therein), and are now very well known.

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But the unavoidable use of numerical studies when the media are too complicated, does not highlight the role played, on one hand by the period cell, and on the other hand by the number of periods of the structure. In order to emphasize these basic phenomena, it is necessary to get analytical developments which can be carried on, all along the study. As a consequence, we focus in the present paper on fluid-type periodically layered media which are good illustration of periodic structures depending on two field functions (one-dimensional periodic lattice such as diatomic chain of coupled spheres for example, SH waves in isotropic layers...). For such structures which yields second-order transfer matrices, the use of the Tchebychev polynomials appears to be relevant, as it may be found in [6, 7, 8]. To our knowledge, this approach is new in the field of Acoustics and the associated results obtained for modal waves do not have been yet reported.

These waves are often called "free waves" since they propagate without any acoustic source. Following Hayes [9], we prefer the terminology of "*modal waves*" to refer simultaneously to surface waves, interface waves and guided waves. Indeed, whereas free waves may propagate also in infinite homogeneous media (the plane waves), modal waves need one or more boundaries to construct themselves. Then, for modal waves, the acoustic energy propagates in the directions of some subspace of the physical space, whereas these waves present a stationary character in the supplementary subspace (in the sense of the linear space theory).

When the relevant structure includes semi-infinite media which extend to infinity in the directions of the stationary subspace, getting a modal wave needs that the acoustic fields be evanescent in these infinite regions, with exponentially decreasing amplitudes. Of this nature are Rayleigh waves, and Stoneley waves at a fluid/solid or a solid/solid interface. Also, Osborne and Hart [10] have studied the propagation of modal waves in an elastic layer immersed in an infinite fluid.

As it has been mentioned at the beginning of this introduction, this paper aims at providing an analytical study of periodically fluid-type layered media, using Cayleigh-Hamilton theorem and Tchebychev polynomials, in order to emphasize the role played in the constitution of modal waves, on one hand by the period cell, and on the other hand by the number of periods of the structure.

We will introduce the formalism of the transfer matrix (see [1] for example) and the theory of Floquet waves for monochromatic waves. It will be shown that a major role is played by the eigenvalues of the transfer matrix of the period cell. According to whether these eigenvalues are imaginary or real, the associated Floquet waves (the relevant eigenvectors) are propagative or evanescent, and one speaks of passing bands or stopping bands respectively. We will show below that, for a finite thickness structure, composed of a finite number of periods, with the two external boundary conditions considered as identical, the guided modal waves split into two families of modal waves. One family, the structure modal waves, takes place in the passing bands. These waves are strongly dependent on the number of periods in the structure but they fit with a large variety of external conditions. The second family, the period modal waves, takes place in the stopping bands; it depends only on the period cell, but on the other hand, it depends strongly on the nature of the external condition. The previous numerical studies of Lamb waves do not seem to have described this second type of modal waves.

The present paper deals also with the case of semi-infinite periodic structures, for which surface modal waves are found, despite of the fact that all the layers are fluid media.

At low frequencies, when the wavelength is large compared to the thickness of the layers, a homogenization technique is presented in order to obtain the “effective” continuous medium which appears to be anisotropic.

This paper gives also some results when the two external boundary conditions are different one from the other. In particular, the case of structures, for which the period cell of the last period is modified (one layer added or suppressed), is studied. One shows that, in such situations, the two modal wave families are coupled, and that the number of modal waves in the passing bands and in the stopping bands may be modified [11].

After a brief sum-up of the classical results obtained by the transfer matrix formalism when studying periodically stratified media (section 1) the paper focuses on the existence of modal waves in structures with a finite number of periods, when it is bounded by vacuum, by the use of Tchebychev polynomials (section 2) and in infinite structures (section 3). In particular, the existence of surface modal waves is highlighted, though these waves do not exist in an homogeneous infinite fluid medium. Various external boundary conditions are then studied (section 4). Finally, an example is given in the particular case of a periodically two-layered fluid structure (section 5) with a comparison, in normal incidence, with the results obtained by Hladky-Hennion *et al.* [11] in one-dimensional diatomic chain of coupled spheres.

1. Sum-up of classical results in periodically multilayered media - anisotropic character of a homogenized fluid-type periodic medium

The aim of this section is to recall the transfer matrix formalism classically used for studying multilayered structures (section 1.2, 1.3 and 1.4 contain well known results), and to introduce the different notations. Section 1.5 highlights the anisotropic character of the stacking of fluid layers, in the long wavelength domain.

1.1. Geometry of the problem and notations

The structure considered here is a periodically fluid multilayered medium. This layered structure is the replication of a number of “periods”, each one made of the stacking of Q distinct media numbered q ($1 \leq q \leq Q$), the period cell. Figure 1 shows, in the particular case of a semi-infinite structure, how these periods are organized in order to constitute the periodically multilayered medium. With a view to applications for which there is an external medium, the wave numbers will be normalised with respect to physical quantities of a medium of reference, numbered ⓪.

1.2. The displacement-stress vector: a state vector

In media with fluid behavior, propagation equations can be written in terms of the pressure variation and of the normal (to the interfaces) particle velocity. As the extension to elastic structures is foreseen, the two field quantities chosen here for propagation equations are the particle normal displacement noted w (z-component of the displacement vector) and the normal stress noted T (the opposite of the pressure variation).

A modal wave of angular frequency ω is associated to an “axial” wave number k_x, in the direction of the layers. Thus, k_x is the projection of all the wave number vectors on the x-axis. In each homogeneous layer q of a period cell p, the displacement potential can be written as:

$$\psi(x, z, t) = \left(A e^{\mathrm{i}k_z z} + B e^{-\mathrm{i}k_z z} \right) e^{\mathrm{i}(k_x x - \omega t)}, \qquad (1)$$

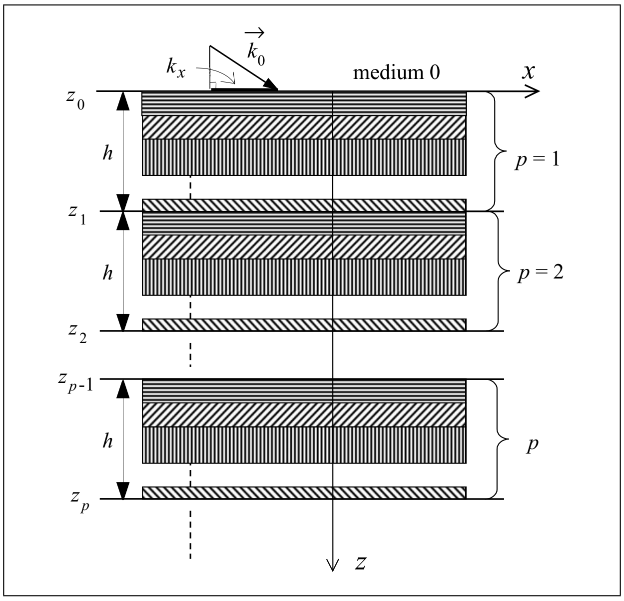


Figure 1. Geometry of the problem: semi-infinite structure.

where k_z is the solution of the dispersion relation in the corresponding medium, with speed of sound V:

$$k_x^2 + k_z^2 = \omega^2/V^2. \tag{2}$$

The particle displacement $w(z)$ and the stress $T(z)$ can be obtained from equation (1) by the following relations:

$$w(z) = \frac{\partial\psi}{\partial z} = \mathrm{i}k_z\Big(A\mathrm{e}^{\mathrm{i}k_z z} - B\mathrm{e}^{-\mathrm{i}k_z z}\Big)\mathrm{e}^{\mathrm{i}(k_x x-\omega t)}, \tag{3}$$

$$T(z) = \rho\frac{\partial^2\psi}{\partial t^2} = -\rho\omega^2\Big(A\mathrm{e}^{\mathrm{i}k_z z} + B\mathrm{e}^{-\mathrm{i}k_z z}\Big)\mathrm{e}^{\mathrm{i}(k_x x-\omega t)}. \tag{4}$$

Let $\mathcal{W}(z)$ be the (2×1) vector of the particle displacement $w(z)$ and the normal stress $T(z)$:

$$\mathcal{W}(z) = \left\{ \begin{array}{c} w(z) \\ T(z) \end{array} \right\}. \tag{5}$$

This vector will be termed subsequently the displacement-stress vector or the state vector. In order to simplify the notations, the displacement-stress vector $\mathcal{W}(z_p)$ will be denoted $\mathcal{W}_p$.

1.3. Transfer matrix of the period

The boundary conditions at each interface separating two successive layers, i.e. continuity of normal displacement and normal stress at the interface, lead to the following matrix relation between $\mathcal{W}_{p+1}$ and $\mathcal{W}_p$:

$$\mathcal{W}_{p+1} = \boldsymbol{\tau}\mathcal{W}_p. \tag{6}$$

Equation (6) allows the stress-displacement vector at a period interface to be expressed as a function of that at the former period interface. The matrix $\boldsymbol{\tau}$ is thus the transfer matrix of one period. It may be shown [1, 12, 13], through the construction of $\boldsymbol{\tau}$, that its determinant is equal to unity (fulfilling of the principle of reciprocity). It is convenient to write down this transfer matrix in the following form:

$$\boldsymbol{\tau} = \begin{bmatrix} a & b \\ -c & d \end{bmatrix} \quad \text{with} \quad ad + bc = 1. \tag{7}$$

The coefficients of $\boldsymbol{\tau}$ are real valued functions of the angular frequency ω (associated to K_0) and of the "axial" wave number k_x (associated to K_x).

1.4. Eigenvectors - Floquet waves - Stopping and passing bands of the structure

We have just seen that whatever the state vector $\mathcal{W}_p$ at the first interface of the period $(p+1)$, the state vector $\mathcal{W}_{p+1}$ at the next period interface can be deduced from the matrix relation (6). Among all the possible state vectors $\mathcal{W}_p$, the case of the two eigenvectors $\mathcal{V}$ of the matrix $\boldsymbol{\tau}$ is of interest:

$$\boldsymbol{\tau}\mathcal{V} = \lambda\mathcal{V}. \tag{8}$$

The eigenvalues of the transfer matrix $\boldsymbol{\tau}$ are solutions of the characteristic equation

$$\lambda^2 - 2s\lambda + 1 = 0, \tag{9}$$

and $$s = (a+d)/2. \tag{10}$$

The roots of equation (9) are either conjugated complexes with their modulus equal to unity, or real with like signs and reciprocal to each other, and can take the following form:

$$\lambda = \mathrm{e}^{\pm\mathrm{i}\kappa_z h}, \tag{11}$$

where the (real or imaginary) wave number κ_z can be interpreted as the projection on the z-axis of the wave number vector of the so-called *Floquet waves* which propagate (or decrease) in the infinite periodically medium. This wave number κ_z depends on the parameters "a" and "d", i.e. on the parameters K_0 and K_x.

As the function $s(K_0, K_x)$ is the sum of the roots of equation (9) divided by two, it may be written:

$$2s = \mathrm{e}^{\mathrm{i}\kappa_z h} + \mathrm{e}^{-\mathrm{i}\kappa_z h}$$

i.e. $$s(K_0, K_x) = \cos(\kappa_z h), \tag{12}$$

which may be written:

$$s(K_0, K_x) = \cos(\tilde{K}_z K_0), \tag{13}$$

with $$\tilde{K}_z = \kappa_z/k_0. \tag{14}$$

The Floquet waves, associated to eigenvectors, are either propagative or evanescent [12], depending on respectively the conjugated complex or real roots of equation (9).

As a consequence, since these two Floquet eigenvectors are independent solutions of the wave propagation equations, the general solution may be expressed as a linear combination of them [14], and this solution has their propagative or evanescent character. When the Floquet waves are evanescent, the corresponding domain in the frequency/angle plane (K_0, K_x) is called *stopping band* (or stop band). When the Floquet waves are propagative, the corresponding domain is called *passing band* (or pass band). These names correspond to the fact that, for a semi-infinite periodic structure, the acoustic energy is totally reflected in the external medium for stopping bands, whereas, for passing bands, a part of the energy goes into the semi-infinite periodic structure and radiates to the infinity. The stopping and passing bands of the structure correspond respectively to the case $|s| > 1$ and $|s| \leq 1$ [12, 15]. These zones are separated in the plane (K_0, K_x) by curves $s(K_0, K_x) = \pm 1$ and therefore, the curves defined by $s(K_0, K_x) = \pm 1$ are of interest for the study. At low frequency, i.e. when K_0 is small enough, the relevant domain in the (K_0, K_x) plane is a passing band. This domain includes the limit case of homogenization, when K_0 tends to zero, for which the periodic structure may be replaced by an effective anisotropic medium. This topic is introduced in the following section 1.5 and extensively developed further in section 5.1.

1.5. Long wavelength domain: anisotropic character of the effective fluid medium

In the particular case of the long wavelength domain, i.e. when K_0 tends to zero, the wave number κ_z of equation (11) can be interpreted as the projection on the z-axis of a wave number vector for the effective medium (i.e. the homogenized medium), and the expression (13) of the function $s(K_0, K_x)$ may be approximated by:

$$s(K_0, K_x) \approx 1 - \frac{1}{2}(\tilde{K}_z, K_0)^2 = 1 - \frac{1}{2}(\kappa_z h)^2. \quad (15)$$

Developing the right hand side $(a + d)$ of equation (10) in a series in the neighboring of $K_0 = 0$, and comparing to equation (15) permits to obtain the following form of the dispersion equation for the effective homogenized medium:

$$\tilde{K}_z^2 + \mathcal{A}^2 K_x^2 = \tilde{K}^2, \quad (16)$$

where $\mathcal{A}$ and $\tilde{K}$ are functions of the properties of each fluid constituting the period. In particular, the parameter $\mathcal{A}$ expresses the *anisotropic character* of the homogenized fluid layered structure (see section 5.1 and Appendix A4 in the case of two layers in a period cell).

When the wave propagates in the direction x of the layers, i.e. when $\tilde{K}_z$ vanishes, the limiting form of equation (16) leads to

$$K_x = \tilde{K}/\mathcal{A}, \quad (17)$$

from which the phase velocity of the wave may be calculated (see details in section 5.1).

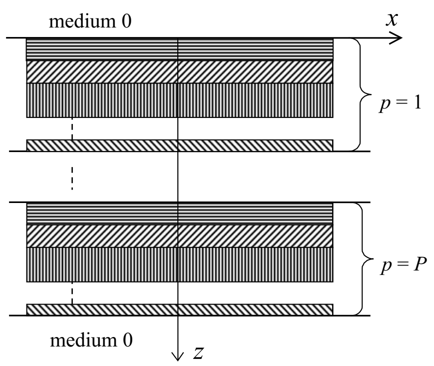


Figure 2. Finite thickness structure.

2. Guided modes in a finite structure in vacuum

The structure considered here is a finite periodic structure in vacuum (which implies null stresses at the first and the last interfaces); the case of other boundary conditions will be studied in section 4. The aim of section 2 is to show that the dispersion equation can be factorized with the help of Tchebychev polynomials, and that this factorization leads to two families of guided modes: the first one depends on the whole structure via the zeroes of Tchebychev polynomials (they will be called *structure modes*) and the second one depends only on the period cell (we will call them *period modes*).

2.1. Transfer matrix of the whole layered medium

Consider a finite periodically multilayered medium made up of P periods (see Figure 2). Using equation (6), the displacement-stress vector at the last interface of the layered medium can be deduced from the one at the first interface by the following relation:

$$\mathcal{W}_P = \tau^P \mathcal{W}_0, \quad (19)$$

where the transfer matrix of the whole structure, being thus the P-th power of the matrix $\boldsymbol{\tau}$, is denoted

$$\boldsymbol{\tau}^P = \begin{bmatrix} a_P & b_P \\ -c_P & d_P \end{bmatrix}. \quad (20)$$

The transfer matrix $\boldsymbol{\tau}^P$ can be calculated in terms of $\boldsymbol{\tau}$, by means of Tchebychev polynomials of the second kind [6, 7, 8, 15, 16] (see Appendix A1):

$$\boldsymbol{\tau}^P = U_{P-1}(s)\boldsymbol{\tau} - U_{P-2}(s)\mathcal{I}, \quad (21)$$

where $\mathcal{I}$ is the identity matrix and "s" is the function given by equation (10).

Substituting expression (7) of $\boldsymbol{\tau}$ and expression (20) of $\boldsymbol{\tau}^P$ into equation (21) leads to:

$$a_P(K_0, K_x) = a(K_0.K_x)U_{P-1}(s) - U_{P-2}(s), \quad (22a)$$
$$b_P(K_0, K_x) = b(K_0.K_x)U_{P-1}(s), \quad (22b)$$
$$c_P(K_0, K_x) = c(K_0.K_x)U_{P-1}(s), \quad (22c)$$
$$d_P(K_0, K_x) = d(K_0.K_x)U_{P-1}(s) - U_{P-2}(s). \quad (22d)$$

2.2. Structure and period modal waves

In order to simplify the notations, the particle displacement $w(z_P)$ and the stress $T(z_P)$ will be respectively denoted w_P and T_P. Substituting $\boldsymbol{\tau}^P$ from equation (20) into equation (19) leads to

$$w_P = a_P w_0 + b_P T_0, \tag{23a}$$
$$T_P = -c_P w_0 + d_P T_0. \tag{23b}$$

If the periodic finite structure is surrounded by vacuum, the boundary conditions at the external interfaces imply $T_P = 0$ and $T_0 = 0$, which leads, by means of equation (23) and (22), to the eigenmode equation:

$$c_P(K_0, K_x) = c(K_0, K_x)U_{P-1}(s) = 0. \tag{24}$$

One sees that this equation is factorized, so that two families of modal waves are obtained.

2.2.1. Structure modal waves

The first family of modal waves corresponds to what may be termed *structure modal waves*. It is given by the condition:

$$U_{P-1}(s) = 0. \tag{25}$$

From the properties of Tchebychev polynomials (see Appendix A1), the $(P-1)$ roots of equation (25) are such that $|s| < 1$, which amounts to saying that these modes do occur in *passing bands*. Accross a given passing band in the plane (K_0, K_x), the function $s(K_0, K_x)$ varies between the boundary values $+1$ and -1; therefore, each structure modal wave (corresponding to each zero of the Tchebychev polynomial) is present in each passing band. If, for a given K_x, the function $s(K_0, K_x)$ is monotonic as a function of K_0 from these boundary values $+1$ and -1, each of the $(P-1)$ structure modal wave is present only one time. This monotonic character may be demonstrated in the case $Q = 2$ (see section 5, equation (68) for the expression of "s").

Using equations (22) and (25), one obtains

$$a_P = d_P = -U_{P-2}(s) \quad \text{and} \quad b_P = c_P = 0, \tag{26}$$

which amounts to saying that the transfer matrix $\boldsymbol{\tau}$ of the whole structure is a scalar matrix. Moreover, its determinant is always equal to 1, which leads to

$$a_P^2 = 1 \quad \text{i.e.} \quad a_P = -U_{P-2}(s) = \pm 1. \tag{27}$$

The transfer matrix $\boldsymbol{\tau}^P$ of the whole structure (given by equation 20), is therefore equal to the identity matrix or to its opposite. If λ' and λ'' denote the eigenvalues of the transfer matrix $\boldsymbol{\tau}$ of one period (given by equation 7), with $\lambda'\lambda'' = 1$, these eigenvalues must verify

$$\lambda'^P = \lambda''^P = a_P = \pm 1. \tag{28}$$

The solutions $\lambda' = \lambda'' = 1$ and $\lambda' = \lambda'' = -1$ have to be withdrawn because they lead respectively to $s = 1$ and to $s = -1$ which cannot cancel the Tchebychev polynomial $U_{P-2}(s)$. The possible solutions are located on the unit circle of the complex plane and have their argument included between 0 and π. Specifically, they take the following values

$$\lambda' = \exp\left(\mathrm{i}\frac{m}{P}\pi\right), \quad m = 1, \dots, P-1, \tag{29a}$$
$$\lambda'' = \exp\left(-\mathrm{i}\frac{m}{P}\pi\right), \quad m = 1, \dots, P-1. \tag{29b}$$

The corresponding values for the function $s(K_0, K_x)$ are thus

$$s(K_0, K_x) = \cos\left(\frac{m}{P}\pi\right), \quad m = 1, \dots, P-1, \tag{30}$$

which correspond indeed to the $(P-1)$ roots of the Tchebychev polynomial U_{P-1}. As a consequence, the structure modes essentially depend on the number P of periods. They are such that the phase shift of the Floquet waves, at the crossing of each period, be a multiple of π/P, strictly included between 0 and π in absolute value.

Under these conditions, the structure eigenvalues λ'^P and λ''^P are thus equal, the common value being $+1$ or -1. Any state vector is an eigenvector and we have

$$w_P = a_P w_0, \tag{31a}$$
$$T_P = a_P T_0, \tag{31b}$$

so that, for such frequency K_0 and wave number K_x, the input and output acoustic fields are equal except for a multiplicative factor.

We see, from equation (31), that these structure modal waves are compatible with more general boundary conditions at the first and at the last interfaces. These conditions could be written such as

$$T_0 = \alpha w_0, \tag{32a}$$
$$T_P = \alpha w_P, \tag{32b}$$

with the same proportional coefficient α. In particular, $\alpha = 0$ when the structure is in vacuum, and when the structure is bounded by rigid walls.

These results are enlarged in section 4, with impedance conditions at the extreme interfaces.

2.2.2. Period modal waves

The second factor of equation (24) simply leads to the modal equation

$$c(K_0, K_x) = 0, \tag{33}$$

which is independent of the number P of periods constituting the whole structure: the function $c(K_0, K_x)$ which is involved in this equation is one of the four elements of the period transfer matrix $\boldsymbol{\tau}$. Then, the modal equation (33) corresponds to modal waves which are directly linked to the period cell: these modal waves will thus be termed *period modal waves*. Under the condition (33), the relation (7) is reduced to

$$ad = 1. \tag{34}$$

As the functions $a(K_0, K_x)$ and $d(K_0, K_x)$ are real, it can be deduced from equation (10) that the function $s(K_0, K_x)$ is real and satisfies

$$|s| \geq 1. \tag{35}$$

As a consequence, the period modes are located in the stopping bands or on their boundaries.

Conversely, for the state vectors $\mathcal{W}_0$ and $\mathcal{W}_1$ at the entrance and at the end of a period (the first one for instance), equation (33) implies the following relations:

$$w_1 = aw_0 + bT_0, \tag{36a}$$
$$T_1 = dT_0, \tag{36b}$$

hence

$$w_1T_1 = adw_0T_0 + bdT_0^2. \tag{37}$$

Relation (34) and the conservation of energy which implies

$$w_1T_1 = w_0T_0, \tag{38}$$

permit to reduce equation (37), with $d \neq 0$ from (34), to the simple following form:

$$bT_0^2 = 0. \tag{39}$$

If $b \neq 0$, equation (39) implies

$$T_0 = 0, \tag{40a}$$
$$\text{and thus} \quad T_1 = 0. \tag{40b}$$

As a consequence, in this case, the period modal waves defined by equation (33) are compatible only with stress free conditions at the boundary of the structure (structure in vacuum). From equation (36), at each interface, the state vector $\mathcal{W} = [w, 0]^T$ thus appears as an eigenvector of the period transfer matrix $\boldsymbol{\tau}$, associated to the eigenvalue "a" (the superscript T standing for the transpose operator).

The particular case where, simultaneously,

$$c = 0 \quad \text{and} \quad b = 0 \tag{41}$$

deserves to be highlighted. The relations (36) then reduce to

$$w_1 = aw_0, \tag{42a}$$
$$T_1 = dT_0. \tag{42b}$$

Moreover, if

$$a = d, \tag{43}$$

relations (42) are identical to relations (32) which correspond to the same boundary conditions at the extreme interfaces.

From relation (34), one can obtain

$$a = d = \pm 1, \tag{44}$$
$$\text{and thus} \quad s = \pm 1. \tag{45}$$

These particular period modes, compatible with boundary conditions (32) at the extreme interfaces, are similar to the structure modes studied in section 2-2-1. From (45), they are located on the boundary lines of stopping bands.

In facts, it can easily be shown that, under the conditions (41) and (43), the function $s(K_0, K_x)$ reaches a stationary value (equal to ± 1). In practice, these particular period modes are located either at the intersection between two lines $s = 1$ or $s = -1$, which separate the stopping bands from the passing bands, or along a double line $s = 1$ or $s = -1$ which bounds, inside a passing band, a null-width stopping band. In particular, this last circumstance occurs if the period cell is itself a periodic structure.

2.2.3. Inverse stacking order of the layers

As an obvious remark, we see that, when the layers of the multilayered structure are stacked in inverse order, i.e. when a period is made up of *layer Q / layer Q − 1 / . . . / layer 2 / layer 1*, and when the first and last interfaces obey to the same boundary conditions, the structure and period modal waves are unchanged: in fact, the modal wave problem is invariant when z is changed into $-z$.

This result no longer holds in the case of a semi-infinite structure, studied subsequently in section 3.3.

3. Surface modal waves in a semi-infinite structure in contact with vacuum

The aim of this section is to show that a surface modal wave may exist (the Floquet wave associated to the eigenvalue $\lambda' = a$) in a semi-infinite fluid periodic structure in contact with vacuum, whereas it does not exist in a single fluid medium. The case of the inverse stacking order of the layers is also studied.

3.1. Surface modal waves

Consider now a semi-infinite periodically multilayered medium, the external interface of which is in contact with vacuum (see Figure 1). If we come back to the study of period modal waves (see section 2.2.2), it can be seen that equation (33) leads to the relations (36) and hence that the boundary condition $T_0 = 0$ on the external interface implies

$$w_1 = aw_0, \tag{46a}$$
$$T_1 = 0. \tag{46b}$$

Thus, under equation (33), when the first interface of the structure is in contact with vacuum, the condition $T_0 = 0$ implies the condition $T_1 = 0$ and so on for other period interfaces. As it has already been mentioned in section 2.2.2, the state vector at each period interface is an eigenvector of the period transfer matrix $\boldsymbol{\tau}$, associated to the eigenvalue "a". In particular, after p periods,

$$w_p = a^p w_0, \tag{47a}$$
$$T_p = 0. \tag{47b}$$

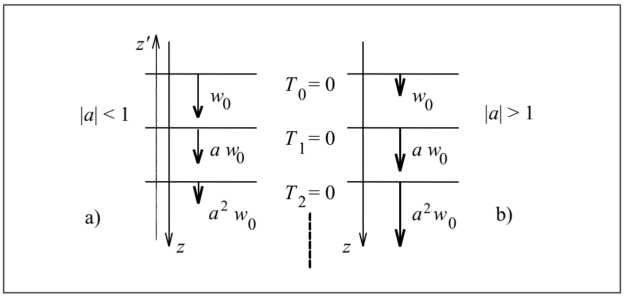


Figure 3. Period mode: surface modal wave a) and "*anti modal*" wave b).

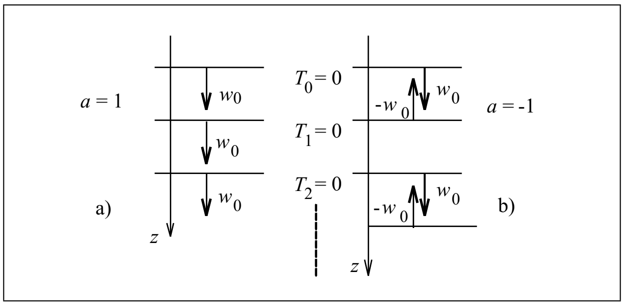


Figure 4. Period mode when $b = 0$ and $a = \pm 1$: $a = 1$ "*strobe plane wave*" a) and $a = -1$ b).

Since the structure is semi-infinite, the integer p grows indefinitely as z tends to infinity. Thus, if $|a| < 1$ then the amplitude of the wave, observed at each period interface, will decrease as a function of z, whereas if $|a| > 1$ it will increase (see Figure 3). The first case corresponds to a surface modal waves (which has been called a "*multilayered Rayleigh mode*" in [17, 18]).

The second one corresponds to what may be called an "*anti modal wave*", in the terminology suggested by the present authors: such a modal solution has poor physical meaning since its amplitude is not bounded at infinity. As mentioned in the Introduction, it should be noted the crucial role that plays here the periodic character of the structure: actually, in the case of semi-infinite homogeneous fluid media, there is no surface modal wave, similar to that which has just been described for a periodically structure. When the layers are stacked in inverse order, the surface modal wave and the "anti modal wave" are exchanged (see section 3.3).

3.2. Plane-type wave (degenerated case: $a = d = \pm 1$)

When equation (33) (i.e $c = 0$) is associated to the condition (44) of equality of the coefficients "a" and "d" of the period transfer matrix $\boldsymbol{\tau}$, i.e.

$$a = d = \pm 1, \tag{48}$$

the state vectors $\mathcal{W}_p$ have a constant amplitude as the integer p increases; these vectors are all equal if $a = 1$, or they have alternating signs according to the parity of p if $a = -1$ (see Figure 4).

In that sense, the surface modal wave studied in section 3.1 degenerates into a "plane" wave (within the meaning of its values of amplitude on the period interfaces) which propagates in the direction x of the layers.

Moreover, if the condition

$$b(K_0, K_x) = 0 \tag{49}$$

is added to equation (33) and to condition (48), all the state vectors $\mathcal{W}_P$ are eigenvector of the period transfer matrix $\boldsymbol{\tau}$, associated to the double eigenvalue $a = d = \pm 1$; the previous "plane" wave is compatible with any boundary condition on the external interface of the form

$$T_0 = \alpha w_0.$$

In this particular case, at each period of the semi-infinite structure, the particular period modal waves which are similar to structure modal waves as described in section 2.2.2, and which are located on the boundary lines $s = \pm 1$, are found again. In the case $Q = 2$, it can be shown from the expressions written in Appendix A3, that equation (33) and condition (48) imply the condition (49).

3.3. Inverse stacking order of the layers

We now examine the modal surface waves for the semi-infinite structure in vacuum when the period cell is made of the same homogeneous layers but in inverse stacking order: *layer Q / layer Q−1 / ... / layer 2 / layer 1*. Unlike the case of a finite structure in vacuum for which the structure modal waves and the period modal waves are unchanged, as noticed in section 2.2.3, the modal surface waves in the semi-infinite structure change in their nature.

The new period transfer matrix $\boldsymbol{\tau}'$ may be easily expressed in terms of the original transfer matrix for the initial stacking *layer 1 / layer 2 / ... / layer Q − 1 / layer Q*. Indeed, stacking the layers in inverse order amounts to observing the original structure in the direction $z < 0$, i.e. in the direction $z' > 0$ (see Figure 3a). As a consequence, the particle normal displacement w_p has to be changed in $-w'_p$ while the normal stress remains the same: $T'_p = T_p$. For one period, using equation (6), it is easy to write on one hand:

$$\mathcal{W}_0 = \boldsymbol{\tau}^{-1}\mathcal{W}_1, \tag{50}$$

and on the other hand:

$$\mathcal{W}'_0 = \boldsymbol{\tau}'\mathcal{W}'_1, \tag{51}$$

$$\text{where} \quad \mathcal{W}'_p = \left\{ \begin{array}{c} w'_p \\ T'_p \end{array} \right\} = \left\{ \begin{array}{c} -w_p \\ T_p \end{array} \right\}. \tag{52}$$

Noticing that the inverse matrix $\boldsymbol{\tau}^{-1}$ of $\boldsymbol{\tau}$ can be expressed as

$$\boldsymbol{\tau}^{-1} = \begin{bmatrix} d & -b \\ c & a \end{bmatrix}. \tag{53}$$

and using equations (50) and (51), the transfer matrix has the following form:

$$\boldsymbol{\tau}' = \begin{bmatrix} a' & b' \\ -c' & d' \end{bmatrix} = \begin{bmatrix} d & b \\ -c & a \end{bmatrix}. \tag{54}$$

Comparing the new transfer matrix $\boldsymbol{\tau}'$ given by (54) to the original transfer matrix $\boldsymbol{\tau}$ given by (7), we see that

$$c' = c, \tag{55a}$$

$$s' = s, \tag{55b}$$

so that it is found again, through equation (24) that the structure and the period modal waves are identical for the two finite structures.

On the contrary, for semi-infinite structures, as far as surface modal waves are concerned, the change from one structure to the inverse stacked order one, amounts to exchanging the coefficient $a(K_0, K_x)$ with the coefficient $d(K_0, K_x)$. As these coefficients are reciprocal to each other (when $c = 0$, equation 33), it can be seen that if $|a| < 1$ then $|d| > 1$; as a consequence, when the layers are stacked in inverse order, a surface modal wave is converted into an "*anti modal wave*", whereas an anti modal wave, with $|a| > 1$, is converted into a surface modal wave.

4. Impedance-like boundary conditions

It has been seen in section 2.2 that when the structure is bounded by vacuum, the boundary conditions lead to a factorized equation, and that one family of modal waves is compatible with more general boundary conditions. The aim of this Section is to take an inventory of other boundary conditions which lead or do not lead to a factorized equation, including the case of total transmission in the presence of an external medium.

The different cases corresponding to this section are summarized in Table I.

4.1. Eigenmode equations

The boundary conditions at $z = z_0$ and by (surface) impedance-type relation

$$T_0 = -\zeta_0 w_0, \tag{56a}$$

$$T_P = +\zeta_P w_P, \tag{56b}$$

where ζ_0 and ζ_P can be complex. Since the parameters ζ_0 and ζ_P relate a stress to a displacement, they are not classical acoustic impedance, but can be deduced from the latest by a multiplication by the factor $\mathrm{i}\omega$.

From equation (22) and (23), the eigenmode equation (dispersion equation) can be straightforwardly deduced:

$$\begin{aligned}\big(\zeta_P a + \zeta_0 d - \zeta_0\zeta_P b + c\big)U_{P-1}(s)& \\ -\big(\zeta_0 + \zeta_P\big)U_{P-2}(s) &= 0,\end{aligned} \tag{57}$$

which is not *a priori* a *factorized equation* because of the additional term $(\zeta_0 + \zeta_P)U_{P-2}(s)$.

The case $\zeta_0 = \zeta_P = 0$ corresponds to a structure in vacuum and has been studied previously in section 2.2.1. In this particular case, equation (57) leads to *period and structure modal waves* given by equation (24):

$$cU_{P-1}(s) = 0. \tag{58}$$

The case $\zeta_0 + \zeta_P = 0$ leads to the factorization of equation (57). The family of structure modal waves, given by the dispersion equation $U_{P-1}(s) = 0$, is found again in this more general context. On the contrary, the dispersion equation for the period modal waves will be different; it is given by

$$\zeta_P a + \zeta_0 d - \zeta_0\zeta_P b + c = 0. \tag{59}$$

This particular case, where ζ_0 and ζ_P are necessarily real in the case of passive boundary conditions, is studied in section 4.3 and corresponds to purely reactive impedance walls. Also, a different situation is encountered in the case of total transmission with the presence of an external medium, as considered in section 4.4.1.

It can be shown (see Appendix A2) that this situation is also encountered when the layers are circularly permuted: the structure modal waves remain the same whereas the period modal waves, given by equation (59), are not given necessarily by equation (33).

The case $\zeta_0 = 0$ corresponds to a first layer in contact with vacuum. Equation (57) leads to a non factorized equation:

$$\big(\zeta_P a + c\big)U_{P-1}(s) - \zeta_P U_{P-2}(s) = 0. \tag{60}$$

This particular result will be used in section 5.3.

Let us see now which practical boundary conditions of the form (56) lead to a factorized or not factorized dispersion equation.

4.2. Rigid wall

When the extreme media ⓪ are rigid walls, the surface impedance ζ_0 and ζ_P of equations (56) are infinite, in order to satisfy the boundary conditions at the upper and lower interfaces (namely $w_P = 0$ and $w_0 = 0$). By means of equation (22) and (23), the following eigenmode equation is obtained:

$$b_P(K_0, K_x) = 0, \tag{61a}$$

$$\text{i.e.}\quad b(K_0, K_x)U_{P-1}(s) = 0. \tag{61b}$$

Because this equation is factorized, two families of modal waves can occur for this type of boundary condition: the *structure modal waves* and *specific period modal waves*.

4.3. Purely reactive impedance wall

When the medium ⓪ is reduced to purely reactive walls with the same impedance $Z_W = \mathrm{i}Z''_W$ on each side of the structure, the writing of boundary conditions implies:

$$T_0 = -\mathrm{i}\omega Z_W w_0, \quad \text{i.e.}\quad T_0 = \omega Z''_W w_0, \tag{62a}$$

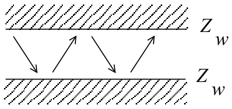
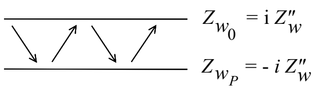
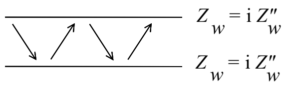
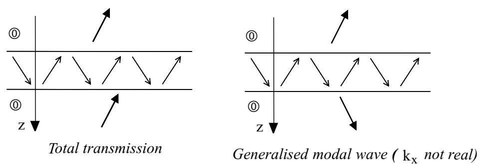
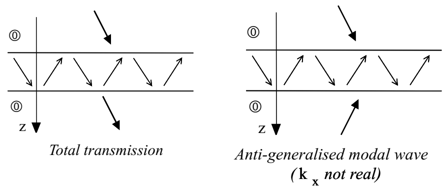
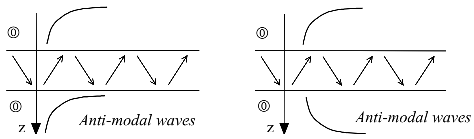
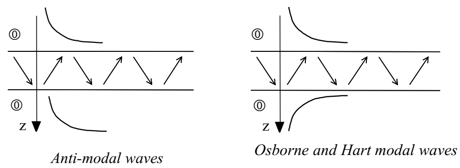
$$T_P = \mathrm{i}\omega Z_W w_P, \quad \text{i.e.}\quad T_P = -\omega Z''_W w_P, \tag{62b}$$

which is equivalent to equations (56) with

$$\zeta_P = \zeta_0 = -\omega Z''_W. \tag{63}$$

In this case, there is *no factorization property* for the dispersion equation which is of the form (57).

Table I. Different boundary conditions for the periodically multilayered structure which lead to modal waves, the eigenmode equation of which is factorized or not factorized.

Boundary conditions $\begin{cases} T_0 = -\zeta_0 w_0 \\ T_P = +\zeta_P w_P \end{cases}$	factorized eigen mode (57) $\zeta_0 + \zeta_P = 0$	non-factorized eigen mode (57) $\zeta_P = \zeta_0$
Rigid wall: $Z_W \to \infty$ or vacuum: $Z_W = 0$		
Pure reactive impedance		
Propagative waves in the same external fluid medium		
Evanescent waves in the same external fluid medium		

On the contrary, when the medium ⓪ is reduced to a purely reactive wall of impedance $Z_{W_0} = \mathrm{i}Z''_W$ at the upper interface $z = z_0$ and Z_{W_P} at the lower interface $z = z_P$, the writing of boundary conditions implies

$$T_0 = -\mathrm{i}\omega Z_{W_0} w_0, \quad \text{i.e.} \quad T_0 = \omega Z''_W w_0, \tag{64a}$$

$$T_P = \mathrm{i}\omega Z_{W_P} w_P, \quad \text{i.e.} \quad T_P = -\omega Z''_W w_P, \tag{64b}$$

which is equivalent to equation (56) with $\zeta_P = -\zeta_0 = \omega Z''_W$.

In this case, equation (57) is a *factorized dispersion equation* which leads in particular to the structure dispersion condition (25).

4.4. Presence of external media

When the medium ⓪ in the semi-infinite upper and lower external regions is a fluid and when only one wave (propagative or evanescent) "propagates" in each of these regions, two cases have to be considered, according to whether the directions of "propagation" are the same or opposite.

4.4.1. Same directions of propagation along the z-axis

The particle normal displacement and the normal stress in each external medium can be obtained from equations (3) and (4) with $B = 0$.

As a consequence,

$$T_0 = \mathrm{i}\frac{\rho_0\omega^2}{k_{z_0}} w_0, \tag{65a}$$

$$T_P = \mathrm{i}\frac{\rho_0\omega^2}{k_{z_0}} w_P, \tag{65b}$$

where k_{z_0} is the projection on the z-axis of the wave number vector of the wave propagating in the medium ⓪.

Equation (65) is equivalent to equation (56) and the modal equation (57) is factorized, since

$$\zeta_P = -\zeta_0 = \mathrm{i}\frac{\rho_0\omega^2}{k_{z_0}}.$$

When the waves are propagative, i.e. when k_{Z_0} is real, one obtains the *total transmission* modes (no reflection, see Table I) for which it may be deduced from equation (57) that the nature of acoustic fields in the layers corresponds to structure modal waves.

4.4.2. Opposite directions of propagation along the z-axis

In this case, equation (65) must be rewritten in the form

$$T_0 = -\mathrm{i}\frac{\rho_0\omega^2}{k_{z_0}} w_0, \tag{66a}$$

$$T_P = \mathrm{i}\frac{\rho_0\omega^2}{k_{z_0}} w_P. \tag{66b}$$

Table II. Properties of the structure with $h/V_0 = 1\ \mu$s.

	V_q (m/s)	ρ_q (kg/m^3)	η_q
medium 1, $q = 1$	4717	8900	2/3
medium 2, $q = 2$	6340	2786	1/3

Since now $\zeta_P = \zeta_0$, the modal equation (57) is not factorized.

The case when the waves are evanescent is of interest: it corresponds to a generalization of the modal waves studied by Osborne and Hart [10].

5. Application to a periodically two-layered structure

The aim of this section is to provide some results in the particular case of periodically two-layered fluid medium ($Q = 2$), in order to have analytical expressions for a, b, c, d and s, and to plot the dispersion curves corresponding to the general above-presented results.

The expressions of the coefficients a, b, c, and d of the transfer matrix $\boldsymbol{\tau}$ are given in Appendix A3, as functions of the dimensionless parameters defined in section 1.1. It should be noted the dispersion equation (2) can be written in each layer as

$$K_x^2 + K_{z_q}^2 = K_q^2. \tag{67}$$

The function s defined by equation (10) has the following expression:

$$\begin{aligned} s(K_0, K_x) = {} & \cos\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\cos\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right) \\ & -\frac{1}{2}\left[\tilde{\rho}\frac{\sqrt{K_2^2-K_x^2}}{\sqrt{K_1^2-K_x^2}} + \frac{1}{\tilde{\rho}}\frac{\sqrt{K_1^2-K_x^2}}{\sqrt{K_2^2-K_x^2}}\right] \\ & \cdot \sin\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\sin\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right). \end{aligned} \tag{68}$$

From this expression, an analytical reasoning makes it possible to show that, for any given value of K_x, the function $s(K_0, K_x)$ is a monotonous function of K_0 in each passing band. When the period is made up of a number of layers $Q > 2$, a similar study of the function $s(K_0, K_x)$ can be made, at least numerically. However, it can be seen *a priori* that the monotonic character of the function $s(K_0, K_x)$ which is established here in the case of a two-layers period is not universal, as shown by the particular case of a four-layers period built by the juxtaposition of two structures with two identical layers (the function "s" passes through a stationary value inside a passing band, see comment at the end of section 2.2.2).

As an example, the stopping bands corresponding to a periodically two-layered structure, the properties of which are listed in Table II, are shown as hatched regions in Figure 5. It should be noted that, as the extension to elastic structures is foreseen, the numerical applications are performed with physical values relevant for longitudinal wave propagation in solids. A cross section of these curves is given in Figure 6, in the particular case $K_x = 0.1$. The stopping bands are also shown as hatched regions.

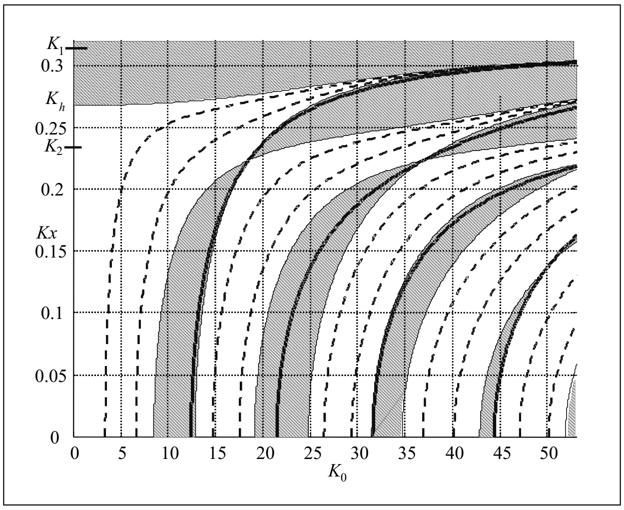


Figure 5. Structure layer ① / layer ② / layer ① / layer ② / ... Superposition of Floquet waves spectrum of an infinite multilayer and modal waves spectrum of a finite multilayer having an even number of layers (case of 2-layer period). Thin solid lines: curves $s = \pm 1$, thick solid lines: period modes such as $c = 0$, dashed lines: structure modes such as $U_2(s) = 0$.

5.1. Long wavelength domain: effective medium

Following the method presented in section 1.5 (the detailed calculus are given in Appendix A4), the dispersion equation for the homogenized effective medium (in the long wavelength domain) corresponding to a periodically two-layered medium takes the following form:

$$\begin{aligned} & \tilde{K}_z^2 + \left[\eta_1^2 + \eta_2^2 + \left(\tilde{\rho} + \frac{1}{\tilde{\rho}}\right)\eta_1\eta_2\right]K_x^2 = \\ & K_1^2\eta_1^2 + K_2^2\eta_2^2 + \left(\tilde{\rho}K_2^2 + \frac{1}{\tilde{\rho}}K_1^2\right)\eta_1\eta_2 \end{aligned} \tag{69}$$

Some particular cases of this equation can be pointed out as follows:

- The particular value K_h of the wave number K_x, as defined in equation (17), corresponding to a wave propagating in the direction of the layers (see Figure 5) has the following expression:

$$K_h = \sqrt{\frac{K_1^2\eta_1^2 + K_2^2\eta_2^2 + \left(\tilde{\rho}K_2^2 + \frac{1}{\tilde{\rho}}K_1^2\right)\eta_1\eta_2}{\eta_1^2 + \eta_2^2 + \left(\tilde{\rho} + \frac{1}{\tilde{\rho}}\right)\eta_1\eta_2}}. \tag{70}$$

  From equation (70), the value of the phase velocity $V_h = V_0/K_h$ may be deduced, which corresponds to the sound speed for an homogenized medium, in the direction of the layers.

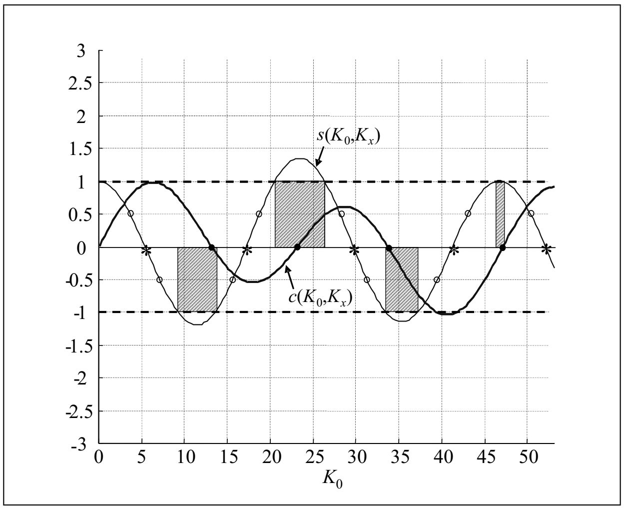


Figure 6. Cross section of Figure 5 for $K_x = 0.1$. Thin solid line: curves $s(K_0, K_x = 0.1)$, thick solid line: curve $c(K_0, K_x = 0.1)$, filled circles: period modal waves $c = 0$, stars: structure modes for 2 periods ($U_1(s) = 0$ i.e. $s = 0$), empty circles: structure modes for 3 periods ($U_2(s) = 0$ i.e. $s = \pm 1/2$).

- As mentioned in section 1.5, the term $\mathcal{A}^2$ on the left hand side of equations (16) and (69) expresses the *anisotropic character* of the homogenized fluid layered structure. However, when the density of each medium is the same, i.e. $\tilde{\rho} = 1$, equation (69) can be written as:

$$\tilde{K}_z^2 + K_x^2 = K_1^2 \eta_1 K_2^2 \eta_2, \tag{71}$$

which is the dispersion equation of an isotropic medium with speed of sound

$$V_h = \frac{V_1 V_2}{\sqrt{\eta_1 V_2^2 + \eta_2 V_1^2}}. \tag{72}$$

5.2. Guided waves

The dispersion curves for a periodically two-layered structure made up of three periods are presented in Figure 5. Using equations (21) and (A2), the two structure modal waves obtained in this case correspond to $s = \pm 1/2$ (empty circles on Figure 6), the values of zeroes of the relevant Tchebychev polynomial (see section 2.2). In the same way, for the case of a two period structure, there is only one structure modal wave in each passing band, corresponding to $s = 0$ (stars on Figure 6).

As mentioned in section 2.2.2, the period modal waves only occur in *stopping bands* (see hatched regions in Figure 5). For example, when $K_x = 0.1$, the curve "c" crosses the K_0-axis only in stopping bands (filled circles in Figure 6).

In normal incidence ($K_x = 0$), all these results are in agreement with the conclusions of Hladky-Henion *et al.* [11] concerning the localized modes in a unidimensional diatomic chain of coupled spheres made up of an even number of spheres: only one period mode occurs in each stopping band, and $(P - 1)$ structure modes occur in the passing bands.

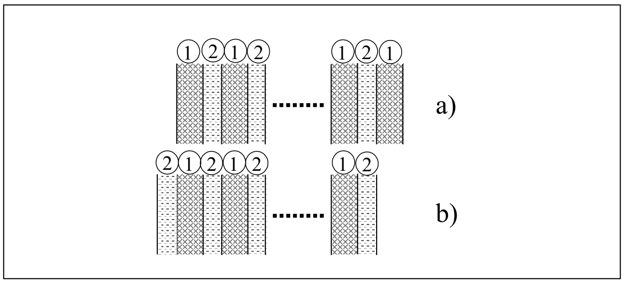


Figure 7. Odd-layer structure. a) Configuration 1: extra layer ① added at the end of the periodic structure. b) Configuration 2: extra layer ② added at the beginning of the periodic structure.

5.3. Guided waves in an odd-layer structure

Consider the case of an odd-layer structure in vacuum, made up, for the first and the last layer, of the same fluid medium. For example, ***P*** *periods and an extra layer* ① *at the end* (see Figure 7a) or *an extra layer* ② *at the beginning and* ***P*** *periods* (see Figure 7b), called respectively "configuration 1" and "configuration 2" below. The impedances $Z_1 = \rho_1 V_1$ and $Z_2 = \rho_2 V_2$ of the constitutive layers are different.

Denoting $\boldsymbol{\tau}_q$ the transfer matrix of the layer q, with

$$\boldsymbol{\tau}_q = \begin{bmatrix} \alpha_q & \beta_q \\ -\gamma_q & \alpha_q \end{bmatrix}, \tag{73}$$

and using the same reasoning as in section 2.2, it is easy to find out that the eigenmode equations, for configurations 1 and 2 respectively, are

$$\big(\gamma_1 a + \alpha_1 c\big) U_{P-1}(s) - \gamma_1 U_{P-2}(s) = 0, \tag{74}$$

and

$$\big(\alpha_2 c + \gamma_2 d\big) U_{P-1}(s) - \gamma_2 U_{P-2}(s) = 0, \tag{75}$$

the expressions of the coefficients of the matrix $\boldsymbol{\tau}_q$ being given in Appendix A3. It can be noticed that, following the discussion of section 3.3, equation (75) can be deduced from equation (74) by exchanging a with d and by replacing subscript 1 by subscript 2.

In the general case, these equations cannot be factorized, and thus, no conclusion can be drawn for both the presence of a mode and the number of modes in or out passing bands (see Figure 8 when $P = 2$ and Figure 9 when $P = 1$). The structure and period modal waves are coupled by the extra layer.

Another faster way for obtaining equation (74), using the results of section 4.1, is to notice that a supplementary layer can be modeled by a purely reactive impedance: let us consider the configuration 1, for which a layer ① is added at $z = z_P$. Dividing the corresponding eigenmode equation (74) by α_1 leads to

$$\left(\frac{\gamma_1}{\alpha_1} a + c\right) U_{P-1}(s) - \frac{\gamma_1}{\alpha_1} U_{P-2}(s) = 0. \tag{76}$$

Comparing equation (76) with equation (60) leads to identify ζ_P to

$$\zeta_P = \frac{\gamma_1}{\alpha_1} = \omega \frac{\rho_1 V_0}{\sqrt{K_1^2 - K_x^2}} \tan\left(\sqrt{K_1^2 - K_x^2} K_0 \eta_1\right), \tag{77a}$$

i.e.

$$\zeta_P = \omega Z_1 \tan\left(\omega h_1/V_1\right) \quad \text{for normal incidence.} \tag{77b}$$

Therefore, adding a supplementary layer at $z = z_P$ amounts to replacing this layer by a pure reactive surface impedance (ζ_P is real).

The case of normal incidence ($K_x = 0$) However, we see from Figure 9 that, in normal incidence ($K_x = 0$), for 3 layers ($P = 1$), all the modes are located outside the stopping bands. For $P = 2$ (and more), it can be observed in this particular case that, when a mode is located in a stopping band (see Figure 8), it always corresponds to a mode associated to configuration 2; in this configuration, the periodic structure is surrounded by layer ② which has the weakest impedance (see Table II). This result has to be compared with that of [11] in the case of one dimensional diatomic chain of coupled spheres: when there is an odd number of spheres, with a small sphere at each extremity, there are modes in passing bands and in stopping bands, but when there is a big sphere at each extremity, there are modes only in passing bands.

The case of normal incidence with $Z_1 \ll Z_2$ As the impedance ratio Z_1/Z_2 between the impedances of each layer seems to be an important parameter, it is interesting to study, in normal incidence, how this ratio influences the eigenmodes of the odd-layer structure, with respect to those of the periodic structure.

As an example, the eigenmode equation (74) corresponding to configuration 1 can be written as

$$\begin{aligned}
&\Bigg\{\omega\frac{Z_1}{Z_2}\sin\left(\omega h_1/V_1\right)\Bigg[\cos\left(\omega h_1/V_1\right)\cos\left(\omega h_2/V_2\right) \\
&\qquad - \frac{Z_1}{Z_2}\sin\left(\omega h_1/V_1\right)\sin\left(\omega h_2/V_2\right)\Bigg] \\
&\quad + \omega\cos\left(\omega h_1/V_1\right)\Bigg[\frac{Z_1}{Z_2}\sin\left(\omega h_1/V_1\right)\cos\left(\omega h_2/V_2\right) \\
&\qquad + \cos\left(\omega h_1/V_1\right)\sin\left(\omega h_2/V_2\right)\Bigg]\Bigg\}U_{P-1}(s) \\
&\qquad - \omega\frac{Z_1}{Z_2}\sin\left(\omega h_1/V_1\right)U_{P-2}(s) = 0.
\end{aligned} \tag{78}$$

When $Z_1 \ll Z_2$, equation (78) reduces to

$$\begin{aligned}
&\omega\cos\left(\omega h_1/V_1\right)\left[\cos\left(\omega h_1/V_1\right)\right. \\
&\qquad \left.\sin\left(\omega h_2/V_2\right)\right]U_{P-1}(s) = 0,
\end{aligned} \tag{79}$$

which is a factorized equation.

The zeroes of this equation correspond

i) to the structure modes ($U_{P-1}(s) = 0$); in that particular case, where Z_1/Z_2 is very small, the approximate value of "s", given by equation (68), and the fact that the zeroes of Tchebychev polynomials do not depend on Z_1/Z_2, show that these structure modes reduce to the eigen modes of each basic layer ($q = 1, 2$) separately: closed/closed for layer ①, open/open for layer ② (following the classical terminology in the framework of the acoustics of ducts and waveguides).

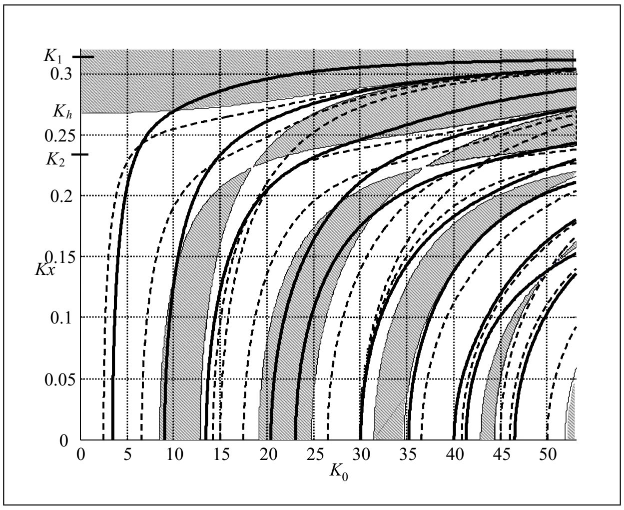


Figure 8. Superposition of Floquet waves spectrum of an infinite multilayer and modal waves spectrum of a finite multilayer having an odd number of layers (case of 2-layer period). Thin solid lines: curves $s = \pm 1$, dashed lines: configuration 1 for $P = 2$ (layer ① / layer ② / layer ① / layer ② / layer ①), thick solid line: configuration 2 for $P = 2$ (layer ② / layer ① / layer ② / layer ① / layer ②).

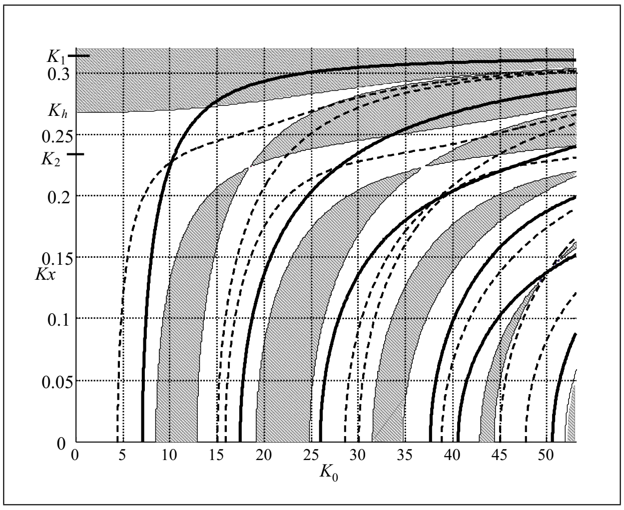


Figure 9. Superposition of Floquet waves spectrum of an infinite multilayer and modal waves spectrum of a finite multilayer having an odd number of layers (case of 2-layer period). Thin solid lines: curves $s = \pm 1$, dashed lines: configuration 1 for $P = 1$ (layer ① / layer ② / layer ①), thick solid line: configuration 2 for $P = 1$ (layer ② / layer ① / layer ②).

ii) to the open/closed and closed/open modes of layer ① ($\cos(\omega h_1/V_1) = 0$) which correspond respectively to a first layer ① surrounded by vacuum at the left and by layer ② with high impedance Z_2 at the right (see Figure 7a), and to a last layer ① surrounded by layer ② at the left and by vacuum at the right;

iii) again to the open/open modes of layer ② ($\sin(\omega h_2/V_2)$) which correspond to the situation when each layer ② is surrounded by layers ① with weak impedance Z_1 (see Figure 7a). The nullity of the factor $\sin(\omega h_2/V_2) = 0$ leads again to the open/open modes for layer ②. These

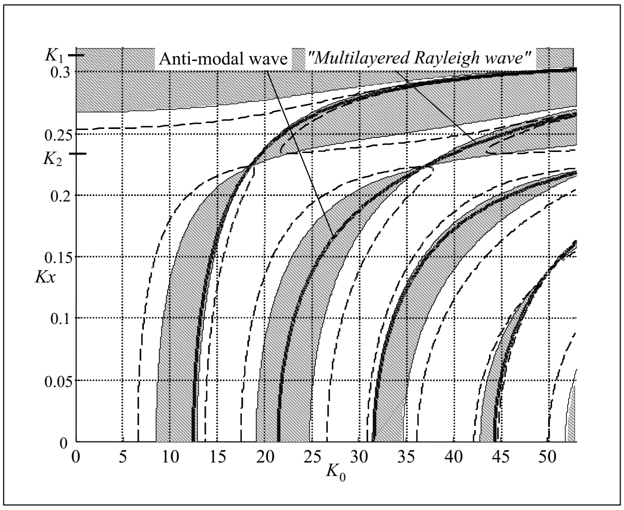


Figure 10. Structure ① / layer ② / layer ① / layer ②) / ... Superposition of Floquet waves spectrum of an infinite multilayer and modal waves spectrum of a finite multilayer having an even number of layers (case of 2-layer period). Thin solid lines: curves $s = \pm 1$, thick solid lines: period modes such as $c = 0$, dashed lines: curves such as $a = \pm 1$.

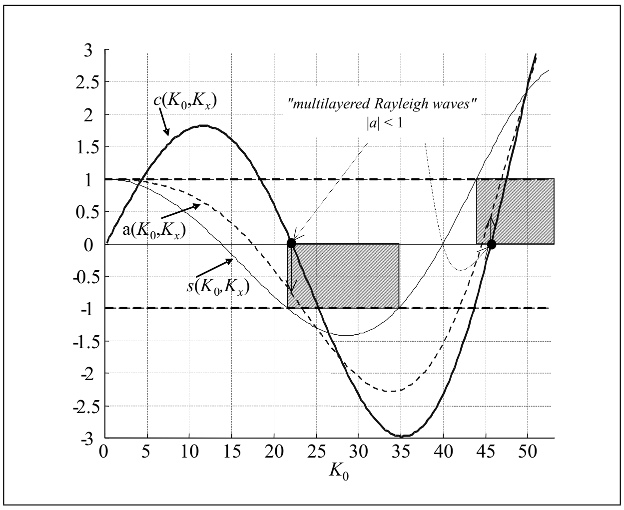


Figure 11. Cross section of Figure 10 for $K_x = 0.25$. Thin solid line: curves $s(K_0, K_x = 0.25)$, thick solid line: curve $c(K_0, K_x = 0.25)$, dashed line: curve $a(K_0, K_x = 0.25)$, filled circles: surface modal waves $c = 0$.

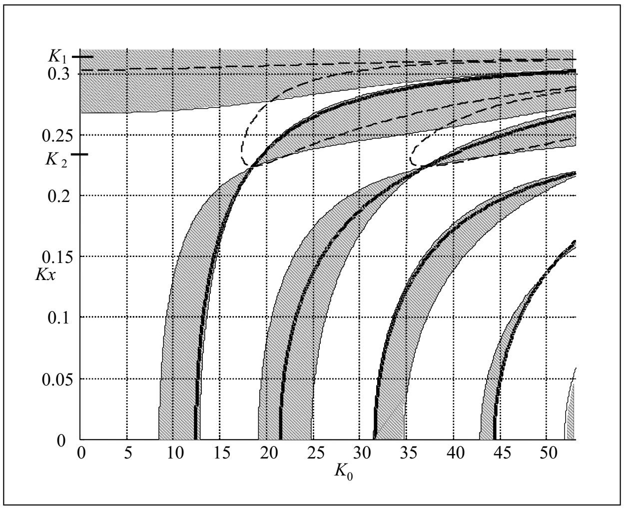


Figure 12. Structure ② / layer ① / layer ② / layer ①) / ... Superposition of Floquet waves spectrum of an infinite multilayer and modal waves spectrum of a finite multilayer having an even number of layers (case of 2-layer period). Thin solid lines: curves $s = \pm 1$, thick solid lines: period modes such as $c = 0$, dashed lines: curves such as $a = \pm 1$.

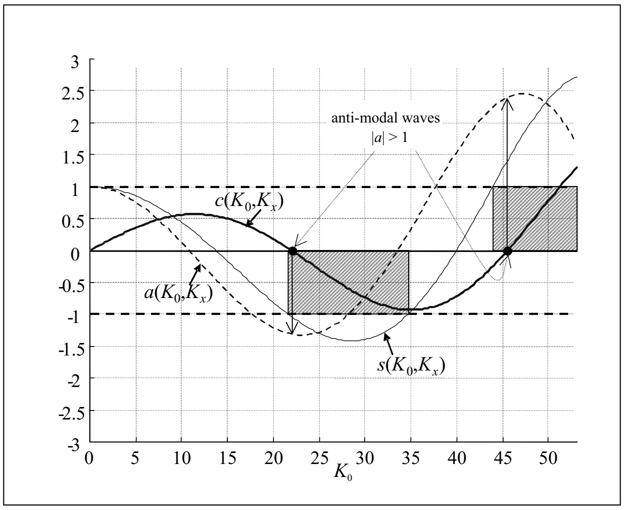


Figure 13. Cross section of Figure 12 for $K_x = 0.25$. Thin solid line: curves $s(K_0, K_x = 0.25)$, thick solid line: curve $c(K_0, K_x = 0.25)$, dashed line: curve $a(K_0, K_x = 0.25)$, filled circles: pseudo-modal waves $c = 0$.

modes are among those obtained in i); they correspond to the situation when each layer ② is surrounded by layers ① with weak impedance Z_1 (see Figure 7a).

5.4. Surface waves

The dashed curves corresponding to $a(K_0, K_x) = \pm 1$ are drawn in Figure 10. These curves describe some semi-closed regions in which we have $|a| > 1$. Only the parts of the curves $c(K_0, K_x) = 0$ outside these regions correspond to a surface modal wave which is an evanescent Floquet wave (see section 3.1). This can be seen more easily on Figure 11 which presents a cross section of Figure 10, in the particular case $K_x = 0.25$: the two filled circles corresponding to "$c = 0$" are located in hatched regions (stopping bands) where $|a| < 1$. The corresponding surface modal waves are thus surface modal waves.

Inverse stacking order (semi-infinite structure) Figure 12 presents the dispersion curves of a structure *vacuum / layer ② / layer ① / layer ② / layer ① / ...* It can be easily seen that the stopping and passing bands are the same as in Figure 8 when layers ① and ② are exchanged.

The curves $c = 0$ which were "outside" the semi-closed regions delimited by the curves $a = \pm 1$ and which gave thus surface modal waves in Figure 10, are now "inside" the semi-closed regions delimited by curves $a = \pm 1$ in Figure 12 when the layers are exchanged. A cross section of these curves is given in Figure 13, in the particular case $K_x = 0.25$. Here, the two filled circles corresponding to the surface modal waves are located in regions where $|a| > 1$, contrary to the case of Figure 11. As a consequence, an inverse stacking order changes surface modal waves into "*anti-modal waves*", and vice-versa (see section 3.3). This result is linked to the fact that functions "a" and "d" exchange each other; in other words, the impedance ratio is replaced by its inverse.

6. Conclusions

Propagation modes (the so-called "modal waves"), the acoustical energy of which propagates along the layers of a periodically layered fluid structure, whereas it remains bounded in a direction perpendicular to the layers, have been studied. The propagation has been studied by using the transfer matrix method which permits to find passing and stopping bands of the structure. When the structure is bounded (P periods), the transfer matrix of the whole medium is the period transfer matrix raised to the power P. Using the Cayleigh-Hamilton theorem, this matrix has been calculated by means of the Tchebychev polynomials of the second kind. When the structure is in vacuum, the dispersion equation for modal waves can be split into two terms, which lead to two families of modal waves. These modes are either *structure modal waves* or *period modal waves*. The first family is given by the cancellation of a Tchebychev polynomial and only exists in passing bands. The second family does not depend on the number of periods and only exists in stopping bands. The corresponding period modal waves are, as a matter of fact, a Floquet wave. When this Floquet wave decreases in the direction $z > 0$, the displacement amplitude of the period modal waves decreases as a function of the depth, and if the structure is semi-infinite, this modal wave may be interpreted as a surface modal wave. When the Floquet wave increases in the opposite direction, the solution has poor real physical meaning since it does not remain bounded at infinity and corresponds to what has been termed an "anti-modal wave". For inverse stacking order of the layers in the period, opposite results are obtained.

When the structure is bounded, other boundary conditions have been introduced in order to derive the eigenmode equation. This equation can be either a factorized or a non factorized equation. In the first case, the family corresponding to *structure modal waves* is involved in the equation. This is for example the case of total transmission when the structure is bounded by the same fluid. The case of an extra layer (added at the beginning or the end of the structure) has also been studied for a periodically two-layered structure; this extra layer can be modeled by pure reactive impedance, and the dispersion equation cannot be factorized in the general case.

It should be noted that most of the results so obtained do apply to more general periodic structures, provided the depend on two field functions (2nd-order transfer matrix): this is the case of one-dimensional periodic lattice such as diatomic chain of coupled spheres for example [11], of SH waves in isotropic layers, of fluid layers, etc.

Appendix

A1. Tchebychev polynomials

A1.1. p-th power of a matrix using Tchebychev polynomials

From Cayleigh-Hamilton theorem, a $(n \times n)$ matrix satisfies its own characteristic equation. In the case we are interested in, this equation is given by equation (9). As a consequence, $\boldsymbol{\tau}$ satisfies

$$\boldsymbol{\tau}^2 - 2s\boldsymbol{\tau} + \mathcal{I} = 0, \quad \text{i.e.} \quad \boldsymbol{\tau}^2 = 2s\boldsymbol{\tau} - \mathcal{I}, \tag{A1}$$

where $\mathcal{I}$ is the (2×2) identity matrix.

The calculus of powers superior to 2 can thus be done by recurrence as follows:

$$\begin{aligned} \boldsymbol{\tau}^2 &= 2s\boldsymbol{\tau} - \mathcal{I}, \\ \boldsymbol{\tau}^3 &= (4s^2 - 1)\boldsymbol{\tau} - 2s\mathcal{I}, \\ \boldsymbol{\tau}^4 &= (8s^3 - 4s)\boldsymbol{\tau} - (4s^2 - 1)\mathcal{I}, \\ &\vdots \end{aligned} \tag{A2}$$

Then, assuming that $\boldsymbol{\tau}^p$ can be expressed as

$$\boldsymbol{\tau}^p = \alpha_{p-1}(s)\boldsymbol{\tau}^2 - \alpha_{p-2}(s)\mathcal{I}, \tag{A3}$$

where $\alpha_p(s)$ are polynomials of the p-th order in s, leads to

$$\boldsymbol{\tau}^{p+1} = \alpha_{p-1}(s)\boldsymbol{\tau}^2 - \alpha_{p-2}(s)\boldsymbol{\tau}, \tag{A4}$$

or, invoking the first equation (A2),

$$\boldsymbol{\tau}^{p+1} = \left[2s\alpha_{p-1}(s) - \alpha_{p-2}(s)\right]\boldsymbol{\tau} - \alpha_{p-1}(s)\mathcal{I}, \tag{A5}$$

yielding

$$\boldsymbol{\tau}^{p+1} = \alpha_p(s)\boldsymbol{\tau} - \alpha_{p-1}(s)\mathcal{I}, \tag{A6}$$

with

$$\alpha_p(s) = 2s\alpha_{p-1}(s) - \alpha_{p-2}(s). \tag{A7}$$

Equation (A7) appears to be the recurrence relation for *Tchebychev polynomials*.

These Tchebychev polynomials are:

- either the Tchebychev polynomials of the first kind obtained through this recurrence relation with the initial values $\alpha_0(s) = 1$ and $\alpha_1(s) = s$, denoted $\alpha_p(s) = T_p(s)$,
- or the Tchebychev polynomials of the second kind obtained for the initial values $\alpha_0(s) = 1$ and $\alpha_1(s) = 2s$, denoted $\alpha_p(s) = U_p(s)$.

From equation (A1), it is obvious that the initial values are $\alpha_0(s) = 1$ and $\alpha_1(s) = 2s$ and thus the p-th power of $\boldsymbol{\tau}$ has the following form:

$$\boldsymbol{\tau}^p = U_{p-1}(s)\boldsymbol{\tau} - U_{p-2}(s)\mathcal{I}. \tag{A8}$$

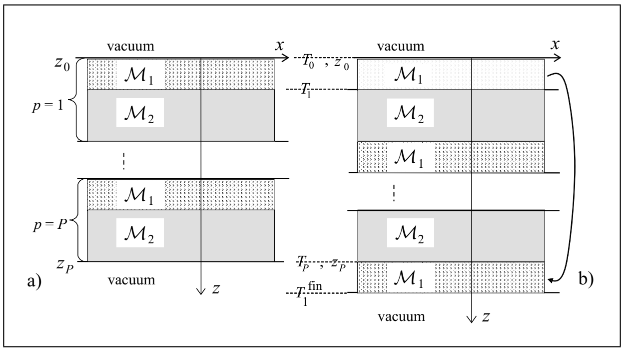


Figure A1. Finite periodic multilayered structure in vacuum, a period being made up by the stacking of two multilayered media respectively denoted a) $\mathcal{M}_1$ and $\mathcal{M}_2$, and -b) $\mathcal{M}_2$ and $\mathcal{M}_1$ [translation of the multilayered medium $\mathcal{M}_1$ from the beginning to the end of the multilayered medium of Figure 14a)].

A1.2. Properties of Tchebychev polynomials

The two functions $T_p(x)$ and $U_p(x)$ are involved in the general solution of the linear differential equation

$$(1-x^2)y'' - xy' + p^2 y = 0, \tag{A9}$$

as follows:

$$y(x) = AT_p(x) + B\sqrt{1-x^2}U_{p-1}(x). \tag{A10}$$

For $|x| < 1$, $U_p(x)$ admits the following trigonometric representation

$$U_p(x) = \frac{\sin[(p+1)\arccos x]}{\sin[\arccos x]}, \tag{A11}$$

which permits to deduce that the polynomial of the pth-order $U_p(x)$ has its p roots in the range $x \in]-1, 1[$, and that its p zeroes are given by

$$\cos\left(\frac{m}{p+1}\pi\right), \quad m = 1, \ldots, p.$$

This trigonometric representation is widely used in physics [19].

The polynomial $U_p(x)$ involved in the paper can also be expressed as

$$U_p(x) = \sum_{k=0}^{[p/2]} (-1)^k C_{p-k}^k (2x)^{p-2k}, \tag{A12}$$

where $[p/2]$ stands for the integer part of $p/2$ and where C_{p-k}^k are the binomial coefficients.

A2. Circular permutation of the layers

The period of the periodic multilayered structure (Figure 2) can be seen as the stacking of two multilayered media respectively denoted $\mathcal{M}_1$ and $\mathcal{M}_2$ (see Figure A1a). The transfer matrix of each multilayered medium $\mathcal{M}_i$ ($i = 1, 2$) is denoted $\overline{\tau}_i$ and has the form (7):

$$\overline{\tau}_i = \begin{bmatrix} \overline{a}_i & \overline{b}_i \\ -\overline{c}_i & \overline{d}_i \end{bmatrix}, \quad i = 1, 2 \tag{A13}$$

with $\overline{a}_i\overline{d}_i + \overline{b}_i\overline{c}_i = 1$.

If $\{w_i^{\text{ini}}, T_1^{\text{ini}}\}$ and $\{w_i^{\text{fin}}, T_1^{\text{fin}}\}$ denote respectively the displacement-stress vector at the first (initial) and last (final) interfaces of multilayered medium $\mathcal{M}_1$, we have

$$\begin{aligned} w_i^{\text{fin}} &= \overline{a}_1 w_1^{\text{ini}} + \overline{b}_1 T_1^{\text{ini}}, \\ T_i^{\text{fin}} &= -\overline{c}_1 w_1^{\text{ini}} + \overline{d}_1 T_1^{\text{ini}}. \end{aligned} \tag{A14}$$

The translation of the multilayered medium from the beginning to the end of the multilayered medium of Figure A1a leads to a new multilayered medium as shown on Figure A1b. As the structures are in vacuum, the condition $T_1 = 0$ at the interface $z = z_1$ leads, using equation (A14) and substituting T_1^{fin} for T_1, T_1^{ini} for T_0 and w_1^{ini} for w_0, to a fictitious condition for T_0:

$$T_0 = \frac{\overline{c}_1}{\overline{d}_1} w_0, \tag{A15a}$$

which has the form of equation (56-a) with a fictitious parameter (surface impedance)

$$\zeta_0 = -\overline{c}_1/\overline{d}_1. \tag{A15b}$$

An analogous reasoning for the condition $T_1^{\text{fin}} = 0$ at the last interface of the structure of Figure A1b leads, using equation (A14) and substituting T_1^{ini} for T_P and w_1^{ini} for w_P, to a condition for T_p:

$$T_P = \frac{\overline{c}_1}{\overline{d}_1} w_P, \tag{A16a}$$

which has the form of equation (56b) with a parameter (surface impedance)

$$\zeta_P = -\overline{c}_1/\overline{d}_1. \tag{A16b}$$

Hence, using equations (A15b) and (A16b), the following relation can be written

$$\zeta_0 + \zeta_p = 0. \tag{A17}$$

As a consequence, the dispersion equation for the structure of Figure A1b can be straightforwardly obtained from equation (57):

$$\big(\zeta_P a + \zeta_0 d - \zeta_0\zeta_P b + c\big)U_{P-1}(s) = 0, \tag{A18}$$

which is a factorized equation.

Therefore, the dispersion equation for structural modes is again given by

$$U_{P-1}(s) = 0, \tag{A19}$$

for any circular permutation of the layers, so that the structure modal waves are unchanged by circular permutation of the layers.

On the contrary, the dispersion equation for period modes, given by

$$\zeta_P a + \zeta_0 d - \zeta_0\zeta_P b + c = 0, \tag{A20a}$$

i.e., using equations (A15b) and (A16b)

$$\overline{c}_1\overline{d}_1(a-d)+\overline{c}_1^2 b+\overline{d}_1^2=0, \tag{A20b}$$

for the structure of Figure A1b is not the same as that for the structure of Figure A1a.

However, in the particular case of a bilayered period, the exchange between the layers $q = 1$ and $q = 2$ leads to an equation (A20) which reduces to the initial equation "$c = 0$", since in this case, a circular permutation of the layers is identical to reverse the z-direction (see section 2.2.3).

A3. Coefficients of the transfer matrix τ in the case of two layers in a period

When $Q = 2$, the coefficients of the period transfer matrix $\boldsymbol{\tau}$ are:

$$\begin{aligned}a(K_0,K_x)=&\\ &\cos\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\cos\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\\ &-\tilde{\rho}\frac{\sqrt{K_2^2-K_x^2}}{\sqrt{K_1^2-K_x^2}}\\ &\quad\cdot\sin\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\sin\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right),\end{aligned} \tag{A21}$$

$$\begin{aligned}b(K_0,K_x)=&\frac{h}{\rho_1K_0V_0^2}\Bigg[\sqrt{K_1^2-K_x^2}\\ &\cdot\sin\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\cos\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\\ &+\tilde{\rho}\sqrt{K_2^2-K_x^2}\\ &\quad\cdot\sin\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\cos\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\Bigg],\end{aligned} \tag{A22}$$

$$\begin{aligned}c(K_0,K_x)=&\frac{\rho_2K_0V_0^2}{h}\Bigg[\frac{\tilde{\rho}}{\sqrt{K_1^2-K_x^2}}\\ &\quad\cdot\sin\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\cos\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\\ &+\frac{1}{\sqrt{K_2^2-K_x^2}}\\ &\quad\cdot\sin\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\cos\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\Bigg],\end{aligned} \tag{A23}$$

$$\begin{aligned}d(K_0,K_x)=&\\ &\cos\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\cos\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right)\\ &-\frac{1}{\tilde{\rho}}\frac{\sqrt{K_1^2-K_x^2}}{\sqrt{K_2^2-K_x^2}}\\ &\quad\cdot\sin\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\sin\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right).\end{aligned} \tag{A24}$$

The coefficients of the transfer matrix $\boldsymbol{\tau} : p$ of each layer q, defined by equation (73), are

$$\alpha_q(K_0,K_x)=\cos\left(\sqrt{K_q^2-K_x^2}\,K_0\eta_q\right), \tag{A25}$$

$$\begin{aligned}\beta_q(K_0,K_x)=&\frac{1}{\rho_q}\frac{h}{K_0V_0^2}\sqrt{K_q^2-K_x^2}\\ &\cdot\sin\left(\sqrt{K_q^2-K_x^2}\,K_0\eta_q\right),\end{aligned} \tag{A26}$$

$$\gamma_q(K_0,K_x)=\frac{\rho_qK_0V_0^2}{h\sqrt{K_q^2-K_x^2}}\sin\left(\sqrt{K_q^2-K_x^2}K_0\eta_q\right). \tag{A27}$$

It should be noted that, even if $\sqrt{K_q^2-K_x^2}$ is imaginary, all these coefficients are real.

A4. Detailed calculus for the homogenized structure, in the case of two layers in a period

This appendix is related to section 5.1.

In the case of a periodically two-layered medium, when K_0 tends to zero, equation (68) gives the following approximated expression for "s":

$$\begin{aligned}s(K_0,K_x)\approx&\ 1-\frac{1}{2}\left(K_1^2-K_x^2\right)K_0^2\eta_1^2-\frac{1}{2}\left(K_2^2-K_x^2\right)K_0^2\eta_2^2\\ &+\frac{1}{2}\left[\tilde{\rho}\frac{\sqrt{K_2^2-K_x^2}}{\sqrt{K_1^2-K_x^2}}+\frac{1}{\tilde{\rho}}\frac{\sqrt{K_1^2-K_x^2}}{\sqrt{K_2^2-K_x^2}}\right]\\ &\cdot\left(\sqrt{K_1^2-K_x^2}K_0\eta_1\right)\left(\sqrt{K_2^2-K_x^2}K_0\eta_2\right),\end{aligned} \tag{A28}$$

i.e.

$$\begin{aligned}s(K_0,K_x)\approx&\ 1-\frac{1}{2}\left(K_1^2-K_x^2\right)K_0^2\eta_1^2\\ &-\frac{1}{2}\left(K_2^2-K_x^2\right)K_0^2\eta_2^2\\ &-\frac{1}{2}\left[\tilde{\rho}\left(K_2^2-K_x^2\right)+\frac{1}{\tilde{\rho}}\left(K_1^2-K_x^2\right)\right].\end{aligned} \tag{A29}$$

With the help of equation (15), the dispersion equation for the homogenized medium is thus obtained:

$$\begin{aligned}&\tilde{K}_z^2+\left[\eta_1^2+\eta_2^2+\left(\tilde{\rho}+\frac{1}{\tilde{\rho}}\right)\eta_1\eta_2\right]K_x^2=\\ &K_1^2\eta_1^2+K_2^2\eta_2^2+\left(\tilde{\rho}K_2^2+\frac{1}{\tilde{\rho}}K_1^2\right)\eta_1\eta_2.\end{aligned} \tag{A30}$$

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