C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

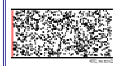
Université du Maine

Chapter 3

Analytical formulation of fundamental problems in acoustics in fluid media: the fundamental laws

Well-posed acoustic problem

- **The three fundamental equations**
  - Acoustical motion



http://www.kettering.edu/~drussell/Demos/demos.html

1) inertia of the system → *PFD: Euler equation*

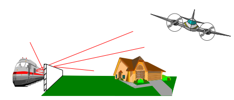
2) elasticity of the system (compressibility) → **div $\vec{v} \neq 0$** *: mass conservation law*

  - Nature of the coefficient of compressibility

3) "behavior" law → *adiabatic transformations, relation between* **p** *and* **ρ** (and **s**)

Propagation equation (1, 2, 3)

- **Boundaries of the domain**
  - Spatial domain
    - ✓ boundary conditions
    - ✓ Sommerfeld condition
  - Time domain
    - ✓ initial conditions



\+ • Energy conservation law
  • Sources

Thermomechanical parameters and variables (1/2)

- Thermodynamic parameters of a fluid medium
  - Properties of the fluid medium
    - ✓ density ρ_E
    - ✓ "static" pressure P_E
    - ✓ temperature T_E
  - Nature of the fluid medium
    - ✓ shear viscosity coefficient of the fluid μ
    - ✓ bulk viscosity coefficient η
    - ✓ coefficient of thermal conductivity λ
    - ✓ heat capacities per unit of mass C_p, C_v
    - ✓ ratio of specific heats $\gamma = C_p/C_v$
    - ✓ thermal expansion coefficient (at constant pressure) α
    - ✓ coefficient of thermal pressure variation (isochoric) β
    - ✓ coefficients of adiabatic and isothermal compressibility $\chi_S = \chi_T/\gamma$

All these parameters depend on the point $\vec{r}$ and on the time t

If the parameters of the fluid do not depend of the point and of the time, subscript "E" → subscript "0"

Thermomechanical parameters and variables (2/2)

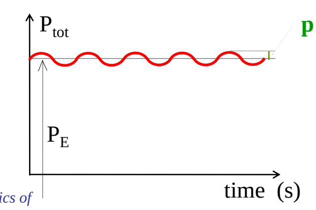
- Fundamental variables (instantaneous difference)
  - ✓ acoustic pressure $p(\vec{r};t)$
  - ✓ density $\rho(\vec{r};t)$
  - ✓ particle velocity $\vec{v}(\vec{r};t)$ → particle displacement $\vec{\xi}(\vec{r};t)$
  - ✓ entropy $s(\vec{r};t)$
  - ✓ temperature $\tau(\vec{r};t)$

Variations around a reference state "E" :

$$p(\vec{r};t) = P_{tot}(\vec{r};t) - P_E(\vec{r};t)$$
$$\rho(\vec{r};t) = \rho_{tot}(\vec{r};t) - \rho_E(\vec{r};t)$$
$$\vec{v}(\vec{r};t) = \vec{v}_{tot}(\vec{r};t) - \vec{v}_E(\vec{r};t)$$
$$s(\vec{r};t) = S_{tot}(\vec{r};t) - S_E(\vec{r};t)$$
$$\tau(\vec{r};t) = T_{tot}(\vec{r};t) - T_E(\vec{r};t)$$

P_0, ρ_0, $\vec{v}_0$, S_0, T_0

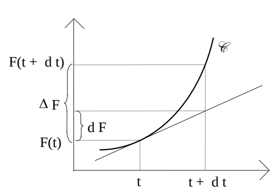
homogeneous *fluid, the characteristics of which do not depend on time*



Elementary variation - Instantaneous difference

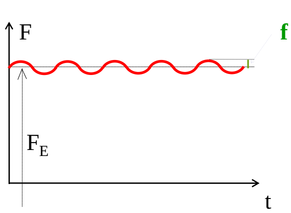
- Elementary variation

$$dF = \lim_{dt \to 0} \left[F(t+dt) - F(t)\right]$$



- Instantaneous difference, at any given time, from a given origin F_E

$$f(t) = \int_{F_E}^{F} dF = F(t) - F_E(t)$$



Application:

$$p(\vec{r};t) = \int_{P_E}^{P_{tot}} dP = P_{tot}(\vec{r};t) - P_E(\vec{r};t)$$

Subsequent hypotheses in the course

- Homogeneous fluid, the characteristics of which do not depend of time

$$p(\vec{r};t) = P_{tot}(\vec{r};t) - P_0$$
$$\rho(\vec{r};t) = \rho_{tot}(\vec{r};t) - \rho_0$$
$$\vec{v}(\vec{r};t) = \vec{v}_{tot}(\vec{r};t) - \vec{v}_0$$
$$s(\vec{r};t) = S_{tot}(\vec{r};t) - S_0$$
$$\tau(\vec{r};t) = T_{tot}(\vec{r};t) - T_0$$

fluid at rest ($\vec{v}_0$)

- Viscosity of the fluid medium and thermal conductivity are neglected
  - → ✓ shear viscosity coefficient μ
  - ✓ bulk viscosity coefficient η
  - ✓ coefficient of thermal conductivity λ

  neglected

because acoustical transformations are (quasi) ***adiabatic***

- Linear acoustics
  - ✓ small variations around an origin state
  - → ✓ equations limited to the *1st order of the acoustic quantities*

Thermodynamic state of a fluid (1/2)

- Equations of state

$$f\left(P_{tot}, V_{tot}, T_{tot}\right) = 0$$

pressure, *volume per unit of mass:* $V_{tot} = 1/\rho_{tot}$, *temperature*

Example: Eq. of perfect gases (PG)

$$P_{tot} V_{tot} - \frac{R}{M} T_{tot} = 0$$

constant of PG (R), *molar mass* (M)

- Bivariance of the fluid medium: 2 independent thermodynamic variables

$$\Rightarrow dS_{tot} = \frac{C_v}{T_{tot} P_{tot} \beta}\left[dP_{tot} - \frac{1}{\rho_{tot} \chi_s} d\rho_{tot}\right] \quad \textit{or} \quad dS_{tot} = \frac{C_p}{T_{tot}} dT_{tot} - \frac{C_p - C_v}{T_{tot} P_{tot} \beta} dP_{tot}$$

entropy, *pressure*, *density*, *temperature*, *pressure*

$$\chi_S = \chi_T/\gamma = -(\partial V/\partial P)_S / V$$

- Adiabatic transformations: no heat exchanges $\Rightarrow$ $\delta Q_{tot} = 0$

$$\delta Q_{tot} = T_{tot}\, dS_{tot} \Rightarrow dS_{tot} = 0$$

$$\Rightarrow dP_{tot} = \frac{1}{\rho_{tot} \chi_s} d\rho_{tot} \quad \textit{i.e.} \quad dP_{tot} = c^2\, d\rho_{tot} \quad \textit{with} \quad c^2 = \frac{\gamma}{\rho_{tot} \chi_T} = \frac{1}{\rho_{tot} \chi_s}$$

adiabatic velocity

Thermodynamic state of a fluid (2/2)

- Adiabatic transformations $\quad dP_{tot} = c^2\, d\rho_{tot} \quad$ *with* $\quad c^2 = \frac{\gamma}{\rho_{tot} \chi_T} = \frac{1}{\rho_{tot} \chi_s}$

adiabatic velocity

✓ **case of linear acoustics**

$$\frac{\gamma}{\rho_{tot} \chi_T} \approx \text{constant} = \frac{\gamma}{\rho_E \chi_T} \Rightarrow \int_{P_E}^{P_{tot}} dP_{tot} = \frac{\gamma}{\rho_E \chi_T} \int_{\rho_E}^{\rho_{tot}} d\rho_{tot}$$

$$\textit{i.e.} \quad \underbrace{P_{tot} - P_E}_{p} = \frac{\gamma}{\rho_E \chi_T} \underbrace{\left(\rho_{tot} - \rho_E\right)}_{\rho}$$

✓ **case of linear acoustics + homogeneous fluid medium which does not depend on time**

$$\rho_E = \rho_0 \Rightarrow p = \frac{\gamma}{\rho_0 \chi_T} \rho \Rightarrow p = c_0^2 \rho, \quad c_0 = \sqrt{\frac{\gamma}{\rho_0 \chi_T}}$$

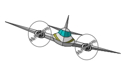
adiabatic velocity of sound ≈ 344,8 m/s in the air

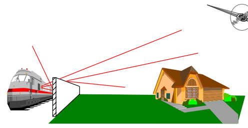
Types of sources (1/3): bulk sources

- force sources

$F(\vec{r};t)$

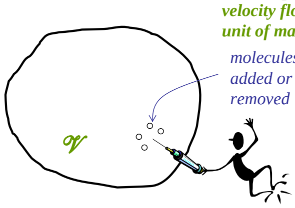
fluctuating force per unit of mass





- volume velocity sources $\quad q(\vec{r};t)$

velocity flow per unit of mass

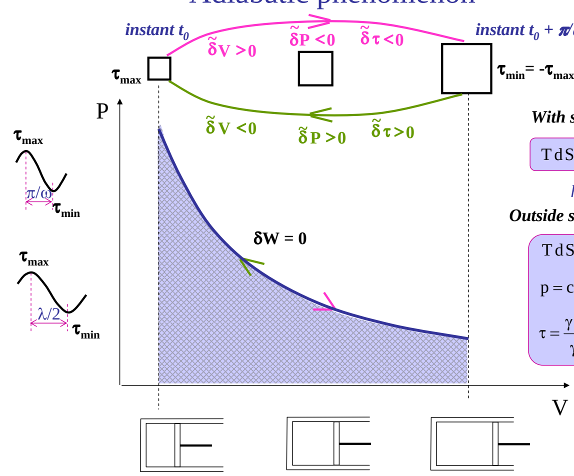


- heat source

$h(\vec{r};t)$

quantity of heat per unit of mass and per unit of time

Adiabatic phenomenon



With sources:

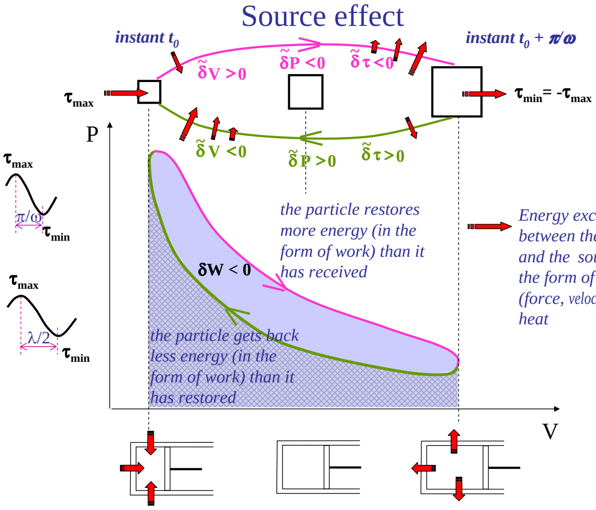
$$T\, dS = h\, dt$$

heat source

Outside sources:

$$T\, dS = 0, \quad p = c_0^2 \rho, \quad \tau = \frac{\gamma - 1}{\gamma \beta} p$$

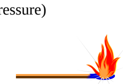
Source effect



Energy exchanged between the particle and the source, in the form of work (force, velocity flow) or heat

Principle diagram of the effect of a heat source

- Particle in a state of minimal volume, maximal acoustic pressure, maximal temperature
- Outside positive introduction of heat (consecutive increase of its internal energy and of its pressure)



- Absorbs less acoustic energy during the compression than it restores during the expansion
- Increase of the acoustical energy by conversion of heat energy into acoustic energy

- Particle in a state of maximal volume, minimal acoustic pressure, minimal temperature
- Gives heat outside $Q_c > 0$ (consecutive decrease of its internal energy and of its pressure)

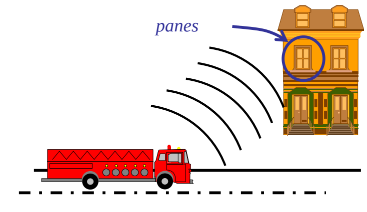
A mean for realizing this process is to maintain a static temperature gradient, which is high enough, and to predict the phase relation between the acoustic pressure and the displacement such that the preceding schematic situation be realized (in its principle).

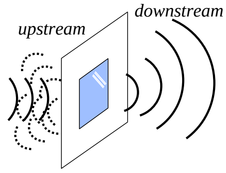
Types of sources (2/3): surface sources

- Loudspeakers on the walls

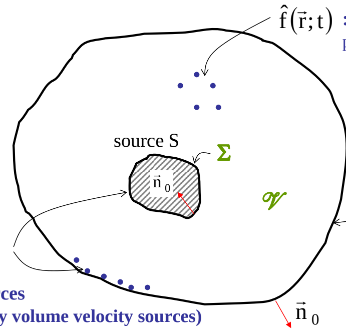
$u(\vec{r};t)$ { - force sources
- volume velocity sources
- heat sources }

- Vibrating surface





Types of sources (3/3)



$\hat{f}(\vec{r};t)$ **: bulk sources**
proportional to div $\vec{F}$, $\partial_t q$ or $\partial_t h$

- ✓ *music instruments on a stage*
- ✓ *aerodynamic sources*
- ✓ *acoustic gyrometer...*

$\hat{u}(\vec{r};t)$:
surface sources
(subsequently volume velocity sources)

- ✓ *loudspeakers on a wall*
- ✓ *vibrating surface*
- ✓ *machine in a shop*
- ✓ ...

$\hat{f}(\vec{r};t)$ and $\hat{u}(\vec{r};t)$ are non local sources (non "punctual")

The three fundamental equations

- Acoustical motion

http://www.kettering.edu/~drussell/Demos/demos.html

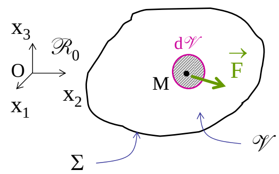
1) inertia of the system → *PFD : Euler equation*
2) elasticity of the system (compressibility) → **div $\vec{v} \neq 0$** *: mass conservation law*

- Nature of the compressibility coefficient

3) "behavior" law → *adiabatic transformations, relation between* **p** *and* **ρ** (and **s**)

A combination of these three fundamental equations permits to obtain the ***propagation equation***

Translation of inertia: Euler equation (1/4)



- **Fundamental Relation of Dynamics for resultants**

$$\vec{\mathcal{R}}(\text{ext} \to \mathcal{V}) = \vec{d}(\mathcal{V}/\mathcal{R}_0),\ \forall\ \mathcal{V}$$

- **Dynamic resultant**

$$\vec{d}(\mathcal{V}/\mathcal{R}_0) = \iiint_{\mathcal{V}} \rho_{tot} \frac{d\vec{v}_{tot}}{dt}\, d\mathcal{V}$$

- **Resultant of external forces**

$$\vec{\mathcal{R}}(\text{ext} \to \mathcal{V}) = \vec{F}_{vol} + \vec{F}_s$$

bulk force — *surface force*

summation of the external forces applied on the volume $d\mathcal{V}$: $d\vec{F} = \rho_{tot}\, \vec{F}\, d\mathcal{V}$ — *force per unit of mass*

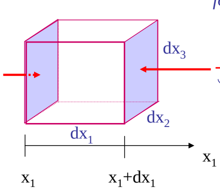
$$\vec{F}_{vol} = \iiint_{\mathcal{V}} \rho_{tot}\, \vec{F}\, d\mathcal{V}$$

$$\vec{F}_s = \iint_{\Sigma} -P_{tot}(\vec{r};t)\, \vec{n}\, d\sigma$$

Translation of inertia: Euler equation (2/4)

- **Surface forces**

Summation of all the external forces $d\vec{f}_s = \vec{f}_s\, ds$ applied on the elementary surfaces of the little piece, of small volume $d\mathcal{V}$. — *force per unit of area*



$P_{tot}(x_1,x_2,x_3;t)\, dx_2 dx_3\, \vec{e}_{x1}$: $d\vec{f}_s(x_1,x_2,x_3;t)\cdot\vec{e}_{x_1}$

$-P_{tot}(x_1+dx_1,x_2,x_3;t)\, dx_2 dx_3\, \vec{e}_{x1}$: $d\vec{f}_s(x_1+dx_1,x_2,x_3;t)\cdot\vec{e}_{x_1}$

$$d\vec{\mathcal{R}}_s(x_1,x_2,x_3;t)\cdot\vec{e}_{x_1} = [P_{tot}(x_1,x_2,x_3;t) - P_{tot}(x_1+dx_1,x_2,x_3;t)]\, dx_2\, dx_3 = -\frac{\partial P_{tot}}{\partial x_1} dx_1\, dx_2\, dx_3\,.$$

and so on for the other faces of the elementary cube (fluid particle)

Translation of inertia: Euler equation (3/4)

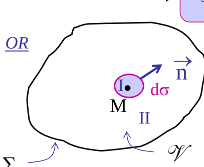
- **Surface forces, follow-up**

$$\begin{cases} d\vec{\mathcal{R}}_s(x_1,x_2,x_3;t)\cdot\vec{e}_{x_1} = -\dfrac{\partial P_{tot}}{\partial x_1} dx_1\, dx_2\, dx_3\,. \\ d\vec{\mathcal{R}}_s(x_1,x_2,x_3;t)\cdot\vec{e}_{x_2} = -\dfrac{\partial P_{tot}}{\partial x_2} dx_1\, dx_2\, dx_3 \\ d\vec{\mathcal{R}}_s(x_1,x_2,x_3;t)\cdot\vec{e}_{x_3} = -\dfrac{\partial P_{tot}}{\partial x_3} dx_1\, dx_2\, dx_3 \end{cases}$$

$$\Rightarrow\ d\vec{\mathcal{R}}_s(x_1,x_2,x_3;t) = -\overrightarrow{\text{grad}}\, P_{tot}\, dx_1\, dx_2\, dx_3 = -\overrightarrow{\text{grad}}\, P_{tot}\, d\mathcal{V}$$

$$\Rightarrow\ \vec{F}_s = \iiint_{\mathcal{V}} -\overrightarrow{\text{grad}}\, P_{tot}\, d\mathcal{V}$$

OR



Ostrogradsky (or Gauss) theorem

$$\vec{F}_s = \iint_{\Sigma} -P_{tot}(\vec{r};t)\, \vec{n}\, d\sigma$$

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Translation of inertia: Euler equation (4/4)

- **Fundamental Relation of Dynamics for resultants** $\vec{\mathcal{R}}(\text{ext}\to\mathcal{V}) = \vec{d}(\mathcal{V}/\mathcal{R}_0),\ \forall\ \mathcal{V}$

$$\iiint_{\mathcal V} \left(\rho_{tot}\vec F - \overrightarrow{grad}\, P_{tot}\right) d\mathcal V = \iiint_{\mathcal V} \rho_{tot} \frac{d\vec v_{tot}}{dt}\, d\mathcal V$$

i.e. $$\iiint_{\mathcal V}\left(\rho_{tot}\vec F - \overrightarrow{grad}\, P_{tot} - \rho_{tot}\frac{d\vec v_{tot}}{dt}\right)d\mathcal V = \vec 0,\ \forall\ \mathcal V$$

➡ $$\rho_{tot}\frac{d\vec v_{tot}}{dt} = \rho_{tot}\vec F - \overrightarrow{grad}\, P_{tot}$$ *i.e.* $$\rho_{tot}\frac{\partial\vec v_{tot}}{\partial t} + \rho_{tot}\left(\overline{\overline{grad}}\,\vec v_{tot}\right)\cdot\vec v_{tot} = \rho_{tot}\vec F - \overrightarrow{grad}\, P_{tot}$$

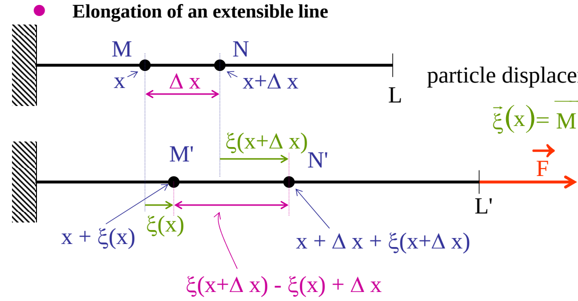
Euler equation with sources

➡ $$\rho_{tot}\frac{d\vec v_{tot}}{dt} = -\overrightarrow{grad}\, P_{tot}$$ *i.e.* $$\rho_{tot}\frac{\partial\vec v_{tot}}{\partial t} + \rho_{tot}\left(\overline{\overline{grad}}\,\vec v_{tot}\right)\cdot\vec v_{tot} = -\overrightarrow{grad}\, P_{tot}$$

Euler equation outside sources

Translation of compressibility: mass conservation law (1/6)

- **Elongation of an extensible line**



particle displacement: $\vec\xi(x) = \overrightarrow{MM'}$

relative variation length of the little piece MN:

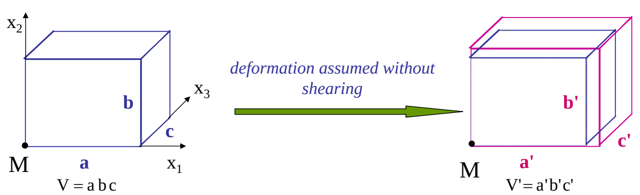
$$\frac{[\xi(x+\Delta x)-\xi(x)+\Delta x]-\Delta x}{\Delta x} = \frac{\Delta\xi}{\Delta x}$$

strain (elongation per unit of length):

$$\lim_{\Delta x\to 0}\frac{\xi(x+\Delta x)-\xi(x)}{\Delta x} = \frac{d\xi}{dx}$$

Translation of compressibility: mass conservation law (2/6)

- **Interpretation of the divergence of the particle velocity vector**



$$V' \approx V\left(1+\partial\xi_1/\partial x_1+\partial\xi_2/\partial x_2+\partial\xi_3/\partial x_3\right)$$ ➡ $$\frac{dV}{V} = \frac{V'-V}{V} \approx div\,\vec\xi$$

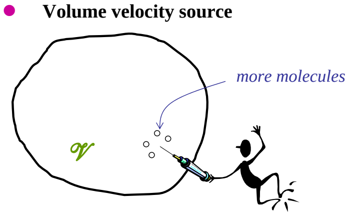
relative variation of volume

$\vec\xi = \vec v\, dt$ ➡ $$\frac{dV}{V} \approx div\,\vec v\, dt$$

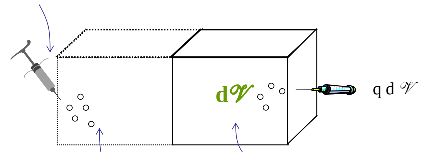
elementary displacement of point M during time dt

Translation of compressibility: mass conservation law (3/6)

- **Volume velocity source**



other outside source



neighboring particle *little elementary volume* $d\mathcal V$

$$q(\vec r;t) = \frac{q\,d\mathcal V}{d\mathcal V}$$ (s^{-1})

flow rate per unit of mass

volume of fluid introduced in the elementary volume $d\mathcal V$ *per unit of time*

Translation of compressibility: mass conservation law (4/6)

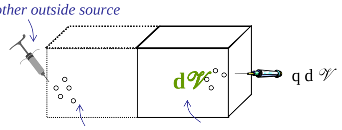
- **Mass conservation law**

Δ of mass contained in the volume $\mathcal V$ / t = masse introduced in this volume / t

$$\frac{dm_{contained}}{dt} = \frac{dm_{introduced}}{dt}$$

✓ Mass introduced by an exterior source in the volume $\mathcal V$ per unit of time

other outside source



volume of fluid introduced in the elementary volume $d\mathcal V$ *per unit of time, by an exterior source*

neighboring particle *little elementary volume* $d\mathcal V$

$\rho_{tot}q$: mass of fluid introduced per unit of volume and per unit of time ($kg.m^{-3}.s^{-1}$)

$\rho_{tot}q\,d\mathcal V$: mass introduced in the volume $d\mathcal V$ per unit of time ($kg.s^{-1}$)

➡ Mass introduced by the source in the $\mathcal V$ per unit of time

$$\frac{dm_i}{dt} = \iiint_{\mathcal V}\rho_{tot}\,q\,d\mathcal V$$

Translation of compressibility: mass conservation law (5/6)

✓ Masse of fluid introduced in the volume $\mathcal V$ through the fixed surface Σ, per unit of time

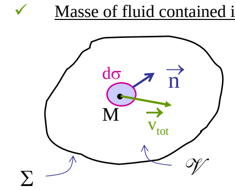
$$\frac{dm_{introduced}}{dt} = -\iint_\Sigma\left(\rho_{tot}\vec v_{tot}\right)\cdot\vec n\, d\sigma + \iiint_{\mathcal V}\rho_{tot}\,q\,d\mathcal V$$

input flux = opposite of the outward flux of $\rho_{tot}\vec v_{tot}$ through Σ

mass of fluid introduced by the volume velocity source $= \frac{dm_i}{dt}$

➡ $$\frac{dm_{introduced}}{dt} = -\iiint_{\mathcal V} div\left(\rho_{tot}\vec v_{tot}\right)d\mathcal V + \iiint_{\mathcal V}\rho_{tot}\,q\,d\mathcal V$$

✓ Masse of fluid contained in the volume $\mathcal V$, at the instant t



Fixed volume $\mathcal V$ *crossed by the fluid*

$$m_{contained} = \iiint_{\mathcal V}\rho_{tot}(\vec r;t)\,d\mathcal V$$

➡ variation per unit of time

$$\frac{dm_{contained}}{dt} = \iiint_{\mathcal V}\frac{\partial\rho_{tot}}{\partial t}\,d\mathcal V$$

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Translation of compressibility: mass conservation law (6/6)

- **Mass conservation law**

Δ of mass contained in the volume $\mathscr{V}$ / t = masse introduced in this volume / t

$$\frac{d\,m_{contained}}{dt} = \frac{d\,m_{introduced}}{dt}$$

i.e. $$\iiint_{\mathscr{V}} \frac{\partial \rho_{tot}}{\partial t}\,d\mathscr{V} = \iiint_{\mathscr{V}} \left[-\operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) + \rho_{tot}\,q\right] d\mathscr{V}\ ,\ \forall \mathscr{V}$$

i.e. $$\iiint_{\mathscr{V}} \left\{\frac{\partial \rho_{tot}}{\partial t} - \left[-\operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) + \rho_{tot}\,q\right]\right\} d\mathscr{V} = 0\ ,\ \forall\ \mathscr{V}$$

⇒ $$\frac{\partial \rho_{tot}}{\partial t} + \operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) = \rho_{tot}\,q \quad \textit{i.e.} \quad \frac{d\rho_{tot}}{dt} + \rho_{tot}\operatorname{div}\vec{v}_{tot} = \rho_{tot}\,q$$

Mass conservation law with sources

⇒ $$\frac{\partial \rho_{tot}}{\partial t} + \operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) = 0 \quad \textit{i.e.} \quad \frac{d\rho_{tot}}{dt} + \rho_{tot}\operatorname{div}\vec{v}_{tot} = 0$$

Mass conservation law outside sources

with $$\operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) = \rho_{tot}\operatorname{div}\vec{v}_{tot} + \vec{v}_{tot}\cdot\overrightarrow{\operatorname{grad}}\,\rho_{tot}$$

Behavior law of a fluid: compressibility coefficient

- Outside heat sources: adiabatic transformations $\quad d\,S_{tot} = 0$

$$d\,P_{tot} = c^2\,d\rho_{tot} \quad \textit{with} \quad c^2 = \frac{\gamma}{\rho_{tot}\,\chi_T} = \frac{1}{\rho_{tot}\,\chi_s}$$

adiabatic velocity

✓ **case of linear acoustics + homogeneous fluid medium which does not depend on time**

$$p = c_0^2\,\rho \quad \textit{with} \quad c_0 = \sqrt{\frac{\gamma}{\rho_0\,\chi_T}}$$

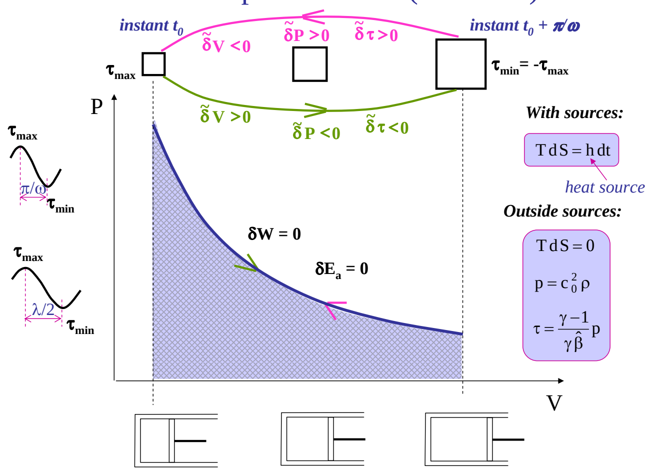
- With a heat source $\quad T_{tot}\,d\,S_{tot} = h\,dt$

(algebraic) quantity of heat introduced per unit of fluid mass, per unit of time

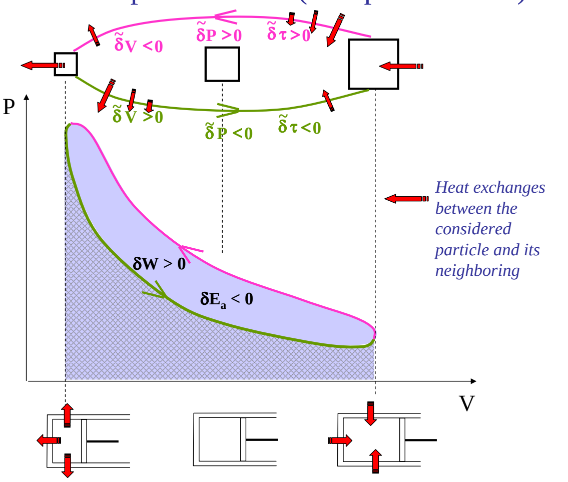
or $$\begin{cases} d\,S_{tot} = \dfrac{C_v}{T_{tot}\,P_{tot}\,\beta}\left[d\,P_{tot} - \dfrac{\gamma}{\rho_{tot}\,\chi_T}\,d\rho_{tot}\right] \\ \alpha = \beta\,\chi_T\,P_{tot} \\ C_p = \gamma\,C_v \end{cases}$$

⇒ $$\frac{1}{\rho_{tot}}\frac{d\rho_{tot}}{dt} = \frac{\chi_T}{\gamma}\frac{d\,P_{tot}}{dt} - \frac{\alpha}{C_p}h$$

Adiabatic phenomenon (reminder)



Non adiabatic phenomenon (dissipative effect)



Synthesis of the three fundamental laws of acoustics

- With sources

homogeneous fluid medium, independent of time, at rest, linear acoustics

✓ $\rho_{tot}\dfrac{d\vec{v}_{tot}}{dt} + \overrightarrow{\operatorname{grad}}\,P_{tot} = \rho_{tot}\,\vec{F}$	*Euler equation*	✓ $\rho_0\dfrac{\partial\vec{v}}{\partial t} + \overrightarrow{\operatorname{grad}}\,p = \rho_0\,\vec{F}$
✓ $\dfrac{\partial\rho_{tot}}{\partial t} + \operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) = \rho_{tot}\,q$	*mass conservation law equation*	✓ $\dfrac{\partial\rho}{\partial t} + \rho_0\operatorname{div}\vec{v} = \rho_0\,q$
✓ $\dfrac{1}{\rho_{tot}}\dfrac{d\rho_{tot}}{dt} = \dfrac{\chi_T}{\gamma}\dfrac{dP_{tot}}{dt} - \dfrac{\alpha}{C_p}h$	*"behavior" law*	✓ $\dfrac{\partial\rho}{\partial t} = \dfrac{1}{c_0^2}\dfrac{\partial p}{\partial t} - \dfrac{\alpha\rho_0}{C_p}h$ *with* $c_0^2 = \dfrac{\gamma}{\rho_0\,\chi_T}$

- Outside sources

✓ $\rho_{tot}\dfrac{d\vec{v}_{tot}}{dt} + \overrightarrow{\operatorname{grad}}\,P_{tot} = \vec{0}$	*Euler equation*	✓ $\rho_0\dfrac{\partial\vec{v}}{\partial t} + \overrightarrow{\operatorname{grad}}\,p = \vec{0}$
✓ $\dfrac{\partial\rho_{tot}}{\partial t} + \operatorname{div}(\rho_{tot}\,\vec{v}_{tot}) = 0$	*mass conservation law equation*	✓ $\dfrac{\partial\rho}{\partial t} + \rho_0\operatorname{div}\vec{v} = 0$
✓ $dP_{tot} = \underbrace{\dfrac{\gamma}{\rho_{tot}\,\chi_T}}_{c^2}\,d\rho_{tot}$	*"behavior" law*	✓ $\dfrac{\partial\rho}{\partial t} = \dfrac{1}{c_0^2}\dfrac{\partial p}{\partial t}$ *i.e.* $p = c_0^2\,\rho$

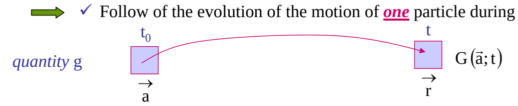
Particle derivative (1/2)

- **Particle located by the point M**

  - ✓ volume large enough for the hypothesis of continuous media to be valid (great number of molecules in the particle)
  - ✓ volume small enough with respect to λ so that the physical quantities can be considered as (quasi) constant

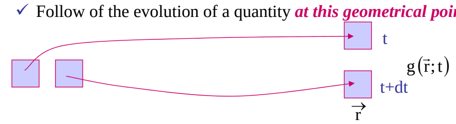
- **Lagrange description**

  - ✓ Variables linked to the considered particle: initial position $\vec{a}$, and time t
  - ⇒ ✓ Follow of the evolution of the motion of ***one*** particle during time



- **Euler description (used in classical acoustics)**

  - ✓ The variables are linked to the geometrical point $\vec{r}$ (or at least with infinitesimal displacement $d\,\vec{r}$)
  - ⇒ ✓ Follow of the evolution of a quantity ***at this geometrical point***, during time



Particle derivative (2/2)

particle derivative: derivative, with respect to time, of the quantity g linked to a particle the motion of which is followed during the time dt

- scalar quantity $g(\vec r;t)$

$$dg=\frac{\partial g}{\partial x_1}dx_1+\frac{\partial g}{\partial x_2}dx_2+\frac{\partial g}{\partial x_3}dx_3+\frac{\partial g}{\partial t}dt$$

i.e.
$$\frac{dg}{dt}=\frac{\partial g}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial g}{\partial x_2}\frac{dx_2}{dt}+\frac{\partial g}{\partial x_3}\frac{dx_3}{dt}+\frac{\partial g}{\partial t} \;\Rightarrow\; \frac{dg}{dt}=\left(\overrightarrow{\text{grad}}\,g\right)\cdot\vec v_{tot}+\frac{\partial g}{\partial t}$$

or
$$\frac{d}{dt}=\vec v_{tot}\cdot\overrightarrow{\text{grad}}+\frac{\partial}{\partial t}$$

convection term → $\left(\overrightarrow{\text{grad}}\,g\right)\cdot\vec v_{tot}$

local derivative, at a given fixed point $\vec r$

- vector quantity $\vec A(\vec r;t)$

$$\frac{d\vec A}{dt}=\frac{\partial\vec A}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial\vec A}{\partial x_2}\frac{dx_2}{dt}+\frac{\partial\vec A}{\partial x_3}\frac{dx_3}{dt}+\frac{\partial\vec A}{\partial t}$$

$$\Rightarrow\; \frac{d\vec A}{dt}=\left(\overline{\overline{\text{grad}}}\,\vec A\right)\cdot\vec v_{tot}+\frac{\partial\vec A}{\partial t}$$

or
$$\frac{d}{dt}=\vec v_{tot}\cdot\overline{\overline{\text{grad}}}+\frac{\partial}{\partial t}$$

with
$$\left(\overline{\overline{\text{grad}}}\,\vec A\right)=\begin{bmatrix}\frac{\partial A_1}{\partial x_1}&\frac{\partial A_1}{\partial x_2}&\frac{\partial A_1}{\partial x_3}\\ \frac{\partial A_2}{\partial x_1}&\frac{\partial A_2}{\partial x_2}&\frac{\partial A_2}{\partial x_3}\\ \frac{\partial A_3}{\partial x_1}&\frac{\partial A_3}{\partial x_2}&\frac{\partial A_3}{\partial x_3}\end{bmatrix}$$

Propagation equation

- With sources

$$\textbf{div}\left(\rho_0\frac{\partial\vec v}{\partial t}+\overrightarrow{\text{grad}}\,p=\rho_0\vec F\right)$$
$$\ominus$$
$$\frac{\partial}{\partial t}\left(\frac{\partial\rho}{\partial t}+\rho_0\,\text{div}\,\vec v=\rho_0 q\right)$$

$$\text{div}\left(\overrightarrow{\text{grad}}\,p\right)-\frac{\partial^2\rho}{\partial t^2}=\rho_0\left(\text{div}\,\vec F-\frac{\partial q}{\partial t}\right)$$

Δp

"Behavior" law
$$\frac{\partial}{\partial t}\left(\frac{\partial\rho}{\partial t}=\frac{1}{c_0^2}\frac{\partial p}{\partial t}-\frac{\alpha\rho_0}{C_p}h\right)\;\Rightarrow\;\frac{\partial^2\rho}{\partial t^2}=\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\frac{\alpha\rho_0}{C_p}\frac{\partial h}{\partial t}$$

$$\Rightarrow\;\Delta p-\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}=\rho_0\left(\text{div}\,\vec F-\frac{\partial q}{\partial t}-\frac{\alpha}{C_p}\frac{\partial h}{\partial t}\right)$$

$$\Rightarrow\;\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p=-f \quad \textit{with} \quad -f=\rho_0\left(\text{div}\,\vec F-\frac{\partial q}{\partial t}-\frac{\alpha}{C_p}\frac{\partial h}{\partial t}\right)$$

- Outside sources
$$\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p=0$$

d'Alembertian operator denoted □

Helmholtz equation

- Complex representation $p(\vec r;t)=\mathscr{Re}\left[\hat p(\vec r;t)\right]$

real part — *complex form*

- Monochromatic field, angular frequency ω

$$\hat p(\vec r;t)=\hat P(\vec r;\omega)e^{i\omega t}$$
$$\hat f(\vec r;t)=\hat F(\vec r;\omega)e^{i\omega t}$$
$$\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat p=-\hat f$$

Helmholtz equation

$$\Rightarrow\;\left(\Delta+\frac{\omega^2}{c_0^2}\right)\hat P(\vec r;\omega)=-\hat F(\vec r;\omega) \qquad \textit{with sources}$$
$$\Rightarrow\;\left(\Delta+\frac{\omega^2}{c_0^2}\right)\hat P(\vec r;\omega)=0 \qquad \textit{outside sources}$$

- Decomposition of a signal into a summation of monochromatic signals

$$\hat p(\vec r;t)=\int_{-\infty}^{+\infty}\hat P(\vec r;\omega)e^{i\omega t}d\omega \quad \textit{and} \quad \hat P(\vec r;\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat p(\vec r;t)e^{-i\omega t}dt$$

Fourier transform / ω — *Fourier transform / t*

$\hat P(\vec r;\omega)=TF_{/t}\left[\hat p(\vec r;t)\right]$ *and* $\hat F(\vec r;\omega)=TF_{/t}\left[\hat f(\vec r;t)\right]$ *satisfy Helmholtz equation*

Velocity potential

- Definition $\vec v(\vec r;t)=\overrightarrow{\text{grad}}\,\varphi(\vec r;t)$ *for any constant* $K_1(t)$
- Propagation equation

$$\rho_0\frac{\partial\vec v}{\partial t}+\overrightarrow{\text{grad}}\,p=\vec 0 \;\Rightarrow\; \rho_0\frac{\partial}{\partial t}\left(\overrightarrow{\text{grad}}\,\varphi\right)+\overrightarrow{\text{grad}}\,p=\vec 0 \;\Rightarrow\; \overrightarrow{\text{grad}}\left(\rho_0\frac{\partial\varphi}{\partial t}+p\right)=\vec 0,\ \forall\ (\vec r,t)$$

$$\Rightarrow\;\rho_0\frac{\partial\varphi}{\partial t}+p=K_2(t) \quad \textit{with} \quad \rho_0\frac{dK_1}{dt}=K_2(t)$$

$$\Rightarrow\;\rho_0\frac{\partial\varphi}{\partial t}+p=0 \;\Rightarrow\; p=-\rho_0\frac{\partial\varphi}{\partial t}$$

$$\frac{\partial\rho}{\partial t}=\frac{1}{c_0^2}\frac{\partial p}{\partial t} \;\Rightarrow\; \rho=-\frac{\rho_0}{c_0^2}\frac{\partial\varphi}{\partial t}$$

$$\frac{\partial\rho}{\partial t}+\rho_0\,\text{div}\,\vec v=0 \;\Rightarrow\; \Delta\varphi-\frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2}=0$$

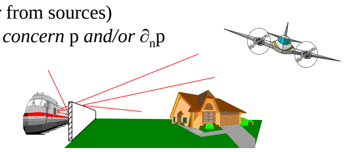
- Helmholtz equation

Monochromatic field: $\hat\varphi(\vec r;t)=\hat\Phi(\vec r;\omega)e^{i\omega t} \;\Rightarrow\; \left(\Delta+k_0^2\right)\hat\Phi(\vec r;\omega)=0$

⚠ *Notation* $k_0=\dfrac{\omega}{c_0}$

Boundary problems for acoustics

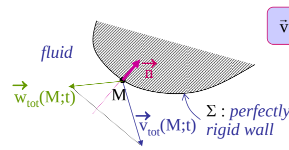
- Bounded or infinite spatial domain
  - ✓ boundary conditions:
    - boundary materialized by a separation surface between two media,
    - boundary described by the vibratory properties at the interface between the considered medium and the wall which constitutes the boundary.
  - ✓ Sommerfeld condition: condition of decreasing, which cancel the field at infinity (far from sources)
  - ✓ *NB: the conditions concern* p *and/or* ∂_np



- Time domain
  - ✓ initial conditions
  - ✓ *NB: the initial conditions concern* p *and* ∂_tp

Non linearised boundary conditions

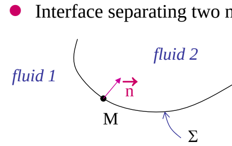
- Perfectly rigid interface



moving wall Σ:
$$\vec v_{tot}(M\in\Sigma;t)\cdot\vec n=\vec w_{tot}(M\in\Sigma;t)\cdot\vec n,\ \forall\ M\in\Sigma,\ \forall\ t$$

fixed wall Σ:
$$\vec v_{tot}(\vec x\in\Sigma;t)\cdot\vec n=0,\ \forall\ M\in\Sigma,\ \forall\ t$$

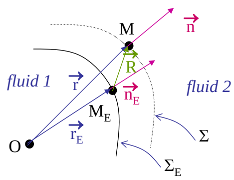
- Interface separating two non viscous fluid media



equality of the normal components of the velocity:
$$\vec v_{tot_1}(\vec x\in\Sigma;t)\cdot\vec n=\vec v_{tot_2}(\vec x\in\Sigma;t)\cdot\vec n,\ \forall\ M\in\Sigma,\ \forall\ t$$

equality of acoustic pressures:
$$P_{tot_1}(\vec r\in\Sigma;t)=P_{tot_2}(\vec r\in\Sigma;t),\ \forall\ M\in\Sigma,\ \forall\ t$$

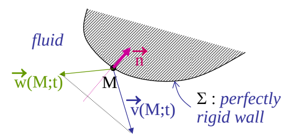
Linearisation of boundary conditions



Keep only the 1st-order acoustic quantities leads to ***write the boundary conditions for the fluctuating quantities on the surface, at its reference position***.

Usual boundary conditions (linear acoustics), homogeneous fluid medium, independent of time, at rest (1/4)

- Perfectly rigid interface



moving wall Σ:

$$\vec{v}(M\in\Sigma;t)\cdot\vec{n}=\vec{w}(M\in\Sigma;t)\cdot\vec{n},\ \forall\ M\in\Sigma,\ \forall\ t$$

fixed wall Σ:

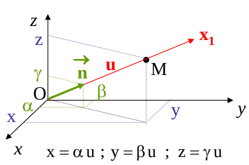
$$\vec{v}(\vec{r}\in\Sigma;t)\cdot\vec{n}=0,\ \forall\ M\in\Sigma,\ \forall\ t$$

$$\vec{n}\cdot\left(\rho_0\frac{\partial\vec{v}}{\partial t}+\overrightarrow{\text{grad}}\,p=\vec{0}\right)\ \Rightarrow\ \rho_0\frac{\partial(\vec{v}\cdot\vec{n})}{\partial t}+\frac{\partial p}{\partial n}=0$$

$$\Rightarrow\ \frac{\partial p(\vec{r}\in\Sigma;t)}{\partial n}=0,\ \forall\ M\in\Sigma,\ \forall\ t$$

Neumann condition

Mathematical reminder



$x=\alpha u\ ;\ y=\beta u\ ;\ z=\gamma u$

$$\overrightarrow{OM}\begin{vmatrix}x\\y\\z\end{vmatrix}_{\mathscr{B}} = \begin{vmatrix}u\\0\\0\end{vmatrix}_{\mathscr{B}_1}$$

Let f(x,y,z) be a function and $\vec{n}$ a unit vector $\vec{n}\begin{vmatrix}\alpha\\\beta\\\gamma\end{vmatrix}_{\mathscr{B}}$ $\overrightarrow{\text{grad}}\,f\begin{vmatrix}\partial_x f\\\partial_y f\\\partial_z f\end{vmatrix}_{\mathscr{B}}$

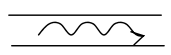
$$\frac{\partial f}{\partial n}=\frac{\partial f}{\partial u}=\frac{\partial f}{\partial x}\underbrace{\frac{\partial x}{\partial u}}_{\alpha}+\frac{\partial f}{\partial y}\underbrace{\frac{\partial y}{\partial u}}_{\beta}+\frac{\partial f}{\partial z}\underbrace{\frac{\partial z}{\partial u}}_{\gamma}$$

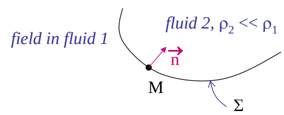
$$\Rightarrow\ \frac{\partial f}{\partial n}=\overrightarrow{\text{grad}}\,f\cdot\vec{n}$$

Usual boundary conditions (linear acoustics), homogeneous fluid medium, independent of time, at rest (2/4)

- Perfectly soft wall

field inside a duct opened on the infinite space

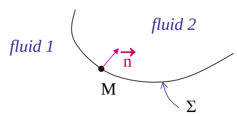




$$p(\vec{r}\in\Sigma;t)=0,\ \forall\ M\in\Sigma,\ \forall\ t$$

Dirichlet condition

- Interface separating two non viscous fluids



equality of the normal components of the velocity:

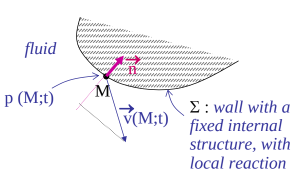
$$\vec{v}_1(\vec{r}\in\Sigma;t)\cdot\vec{n}=\vec{v}_2(\vec{r}\in\Sigma;t)\cdot\vec{n},\ \forall\ M\in\Sigma,\ \forall\ t$$

equality of acoustic pressures:

$$p_1(\vec{r}\in\Sigma;t)=p_2(\vec{r}\in\Sigma;t),\ \forall\ M\in\Sigma,\ \forall\ t$$

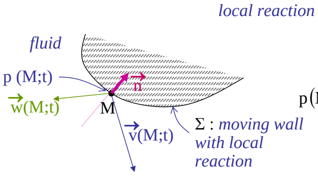
Usual boundary conditions (linear acoustics), homogeneous fluid medium, independent of time, at rest (3/4)

- Wall with local reaction - Wall impedance notion



$$p(M\in\Sigma;t)=\mathscr{T}_Z\left[\vec{v}(M\in\Sigma;t)\cdot\vec{n}\right],\ \forall\ M\in\Sigma,\ \forall\ t$$

Linear operator (reflection function). BUT, *this proportionality parameter between the acoustic pressure and the particle velocity* ***(impedance of the wall), is only defined for one frequency, in the complex domain***

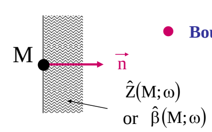


$$p(M\in\Sigma;t)=\mathscr{T}_Z\left\{\left[\vec{v}(M\in\Sigma;t)-\vec{w}(M\in\Sigma;t)\right]\cdot\vec{n}\right\},\ \forall\ M\in\Sigma,\ \forall\ t$$

$$\Rightarrow\ \hat{P}(M\in\Sigma;\omega)=\hat{Z}(M;\omega)\left[\hat{\vec{V}}(M\in\Sigma;\omega)-\hat{\vec{W}}(M\in\Sigma;\omega)\right]\cdot\vec{n},\ \forall\ M\in\Sigma,\ \omega\ \text{fixed}$$

wall impedance

Wall with fixed internal structure, with local reaction



- **Boundary condition** $\hat{P}(M;\omega)=\hat{Z}(M;\omega)\hat{\vec{V}}(M;\omega)\cdot\vec{n}$ **(1)**

$\vec{n}$ *: normal outward the considered medium*
≡ incoming normal for the material

with $\hat{p}(M;t)=\hat{P}(M;\omega)e^{i\omega t}$ *and* $\hat{\vec{v}}(M;t)=\hat{\vec{V}}(M;\omega)e^{i\omega t}$

- **Projection on $\vec{n}$ of Euler equation** $\left[\rho_0\partial_t\hat{\vec{v}}(M;t)\right]\cdot\vec{n}=-\left[\overrightarrow{\text{grad}}\,\hat{p}(M;t)\right]\cdot\vec{n}$

i.e. $\rho_0 i\omega\hat{\vec{V}}(M;\omega)\cdot\vec{n}=-\partial_n\hat{P}(M;\omega)\ \Leftrightarrow\ \hat{\vec{V}}(M;\omega)\cdot\vec{n}=\dfrac{-1}{i\rho_0\omega}\partial_n\hat{P}(M;\omega)$ **(2)**

(1), **(2)** $\Rightarrow\ \partial_n\hat{P}(M;\omega)+i\dfrac{\rho_0\omega}{\hat{Z}(M;\omega)}\hat{P}(M;\omega)=0$ *hence* $\partial_n\hat{P}(M;\omega)+ik_0\hat{\beta}(M;\omega)\hat{P}(M;\omega)=0$

$k_0=\omega/c_0$

with $\hat{\beta}(M;\omega)=\dfrac{\rho_0 c_0}{\hat{Z}(M;\omega)}$ *surface admittance of the material, normalized by the characteristic impedance of the fluid* $\rho_0 c_0$

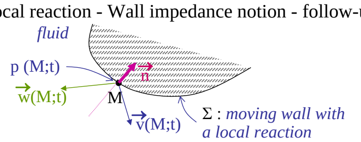
- **Rigid wall (Neuman condition)** $\left.\begin{matrix}\hat{Z}(M;\omega)\to\infty\\\hat{\beta}(M;\omega)=0\end{matrix}\right\}\Rightarrow\ \partial_n\hat{P}(M;\omega)=0$

- **Wall without reaction and without dissipation (Dirichlet condition)** $\left.\begin{matrix}\hat{Z}(M;\omega)=0\\\hat{\beta}(M;\omega)\to\infty\end{matrix}\right\}\Rightarrow\ \hat{P}(M;\omega)=0$

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Usual boundary conditions (linear acoustics), homogeneous fluid medium, independent of time, at rest (4/4)

- Wall with local reaction - Wall impedance notion - follow-up



$$\left[\frac{\partial}{\partial n} + i k_0 \hat{\beta}(M;\omega)\right]\hat{P}(M;\omega) = \hat{U}(M;\omega), \ \forall M \in \Sigma, \ \omega \text{ given}$$

$$\hat{\beta}(M;\omega) = \frac{\rho_0 c_0}{\hat{Z}(M;\omega)}$$ *surface admittance of the material*

$$\hat{U}(M;\omega) = -\rho_0 i\omega \hat{\vec{W}}(M;\omega)\cdot\vec{n}$$

$$k_0 = \omega / c_0$$

in time domain $\left[\partial_n + \widetilde{i k_0 \hat{\beta}(M;t) *}\right] \tilde{p}(M;t) = \hat{u}(M;t) \ ; \ \forall M \in \Sigma \ ; \forall t > t_i$

with $\widetilde{i k_0 \beta(\vec{r};t)} = TF_{/\omega}\left[i k_0 \beta(\vec{r};\omega)\right]$

Well-posed acoustical problem

- Frequential (Fourier) domain

Helmholtz equation $\left(\Delta + k_0^2\right)\hat{P}(M;\omega) = -\hat{F}(M;\omega), \ \forall M \in \mathscr{V}$

Boundary conditions $\left[\frac{\partial}{\partial n} + i k_0 \hat{\beta}(M;\omega)\right]\hat{P}(M;\omega) = \hat{U}(M;\omega), \ \forall M \in \Sigma, \ \omega \text{ given}$

Radiation conditions at infinite (eventually)

- Time domain

Propagation equation $\left(\Delta - \frac{1}{c_0^2}\partial_{tt}^2\right)\tilde{p}(M;t) = -\hat{f}(M;t), \ \forall M \in \mathscr{V}, \ \forall t > t_i$

Boundary conditions $\left[\partial_n + \widetilde{i k_0 \hat{\beta}(M;t) *}\right] \tilde{p}(M;t) = \hat{u}(M;t) \ ; \ \forall M \in \Sigma \ ; \forall t > t_i$

Initial conditions $\partial_t \tilde{p}(M;t_i) = \hat{A}(M;t_i) \ ; \ \tilde{p}(M;t_i) = \hat{B}(M;t_i) \ ; \ \forall M \in \mathscr{V} \ ; \ t = t_i$

functions known in the whole domain $\mathscr{V}$, at initial time $t = t_i$

Total density of instantaneous acoustic energy (1/2)

- Density of energy = energy stored per unit of volume ($E_{\text{volume } d\mathscr{V}} / d\mathscr{V}$)
- Density of total acoustic energy

$$E_a = E_c + E_p$$

density of ***total*** *acoustic energy* — ***kinetic*** *energy density* — ***potential*** *energy density*

- Instantaneous kinetic energy density

$E_c = \frac{1}{2}\rho_{tot}\vec{v}^2$ → *linear acoustics* → $E_c = \frac{1}{2}\rho_0 \vec{v}^2$

Total density of instantaneous acoustic energy (2/2)

- Instantaneous potential energy density

M	M	M
state at rest	*"ongoing" state*	*"current" state*
P_0, ρ_0	$\tilde{P}_{tot} = P_0 + \tilde{p} \ ; \ \tilde{\rho}_{tot} = \rho_0 + \tilde{\rho}$	$P_{tot} = P_0 + p \ ; \ \rho_{tot} = \rho_0 + \rho$

✓ elementary volume: $V_{él} = m_{él} / \tilde{\rho}_{tot}$

✓ elementary work received by the particle (stored energy):
$$\delta W_{él} = -\tilde{p}\, dV_{él} = -\tilde{p}\, d\left(m_{él} / \tilde{\rho}_{tot}\right)$$

potential energy because adiabatic transformations: $dU = \delta W_{él} + \delta Q_{él} = \delta W_{él}$ — *variation of internal energy*

✓ potential energy density:

$$\delta W = \delta W_{él} / V_{él} = -\frac{\tilde{p}\tilde{\rho}_{tot}}{m_{él}} d\left(\frac{m_{él}}{\tilde{\rho}_{tot}}\right) = -\tilde{\rho}_{tot}\tilde{p}\, d\left(1/\tilde{\rho}_{tot}\right) = \frac{1}{\tilde{\rho}_{tot}}\tilde{p}\, d\tilde{\rho}_{tot} = \frac{1}{\rho_0 + \tilde{\rho}}\tilde{p}\, d\left(\rho_0 + \tilde{\rho}\right) = \frac{1}{\rho_0 + \tilde{\rho}}\tilde{p}\, d\tilde{\rho}$$

linear acoustics → $\delta W \approx \frac{1}{\rho_0}\tilde{p}\, d\tilde{\rho}$

\+ *"behavior" law:* $\tilde{p} = c_0^2 \tilde{\rho}$

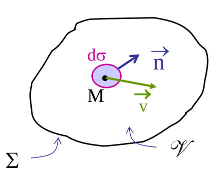
→ $\delta W \approx \frac{1}{\rho_0} c_0^2 \tilde{\rho}\, d\tilde{\rho}$

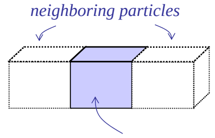
✓ instantaneous potential energy density: $E_p = \frac{c_0^2}{\rho_0}\int_0^{\rho} \tilde{\rho}\, d\tilde{\rho} = \frac{c_0^2 \rho^2}{2\rho_0}$ → $E_p = \frac{p^2}{2\rho_0 c_0^2}$

- Total density of instantaneous acoustic energy $E_a = E_c + E_p = \frac{1}{2}\left(\rho_0 \vec{v}^2 + \frac{p^2}{\rho_0 c_0^2}\right)$

Energy conservation law (1/4)

During acoustic propagation





The acoustic energy E_a which is locally present in the particle ($E_c + E_p$) thus results from an ***input and*** a ***loss of energy***.

↔ energy flux added and removed to the volume, permanently.

$\vec{\mathscr{P}} = p\vec{v}$: instantaneous flow of acoustic energy, per unit of area and per unit of time (instantaneous power crossing the unit surface $d\sigma$, conveyed by the acoustic wave)

$p\vec{v}\cdot\vec{d\sigma}\, dt$: elementary work provided by a particle travail to its neighboring during time dt.

Energy conservation law (2/4)

- Outside sources

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}}\left(E_c + E_p\right) d\mathscr{V} = -\iint_{\Sigma} p\vec{v}\cdot\vec{n}\, d\sigma$$

variation per unit of time of the acoustic energy which is contained in the volume $\mathscr{V}$

opposite of the outward energy flux, per unit of time

$$-\iiint_{\mathscr{V}} \operatorname{div}\left(p\vec{v}\right) d\mathscr{V}$$ *(Ostrogradsky theorem)*

$$\Rightarrow \iiint_{\mathscr{V}}\left[\frac{\partial}{\partial t}\left(E_c + E_p\right) + \operatorname{div}\left(p\vec{v}\right)\right] d\mathscr{V} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(E_c + E_p\right) + \operatorname{div}\left(p\vec{v}\right) = 0$$

Energy conservation law outside sources

with $E_a = E_c + E_p$ and $\vec{\mathscr{P}} = p\vec{v}$

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Energy conservation law (3/4)

- With sources

✓ $\mathrm{div}\,\vec{v} = -\frac{1}{\rho_0}\frac{\partial \rho}{\partial t} + q$ ⟹ $\vec{v}\cdot\overrightarrow{\mathrm{grad}}\,p = \mathrm{div}(p\vec{v}) - p\,\mathrm{div}\,\vec{v} = \mathrm{div}(p\vec{v}) + \frac{p}{\rho_0}\frac{\partial \rho}{\partial t} - pq$

$\frac{\partial \rho}{\partial t} = \frac{1}{c_0^2}\frac{\partial p}{\partial t} - \frac{\alpha \rho_0}{C_p} h$

⟹ $\vec{v}\cdot\overrightarrow{\mathrm{grad}}\,p = \mathrm{div}(p\vec{v}) + \frac{p}{\rho_0 c_0^2}\frac{\partial p}{\partial t} - \frac{\alpha}{C_p} p h - pq$

✓ $\vec{v}\cdot\left(\rho_0 \frac{\partial \vec{v}}{\partial t} + \overrightarrow{\mathrm{grad}}\,p = \rho_0 \vec{F}\right)$ ⟺ $\rho_0 \vec{v}\cdot\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\overrightarrow{\mathrm{grad}}\,p = \rho_0 \vec{v}\cdot\vec{F}$

⟹ $\rho_0 \vec{v}\cdot\frac{\partial \vec{v}}{\partial t} + \mathrm{div}(p\vec{v}) + \frac{p}{\rho_0 c_0^2}\frac{\partial p}{\partial t} - \frac{\alpha}{C_p} p h - pq = \rho_0 \vec{v}\cdot\vec{F}$

⟹ $\frac{1}{2}\rho_0 \frac{\partial \vec{v}^2}{\partial t} + \frac{1}{2\rho_0 c_0^2}\frac{\partial p^2}{\partial t} + \mathrm{div}(p\vec{v}) = \rho_0 \vec{v}\cdot\vec{F} + \frac{\alpha}{C_p} p h + pq$

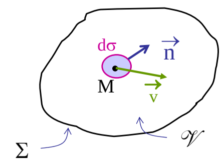
$$\frac{\partial}{\partial t}\Bigg(\underbrace{\frac{1}{2}\rho_0 \vec{v}^2}_{E_c} + \underbrace{\frac{p^2}{\rho_0 c_0^2}}_{E_p}\Bigg) + \mathrm{div}(\underbrace{p\vec{v}}_{\vec{\mathscr{P}}}) = \rho_0 \vec{v}\cdot\vec{F} + \frac{\alpha}{C_p} p h + pq$$

$$\frac{\partial E_a}{\partial t} + \mathrm{div}\left(\vec{\mathscr{P}}\right) = \rho_0 \vec{v}\cdot\vec{F} + \frac{\alpha}{C_p} p h + pq$$

Energy conservation law with sources

Energy conservation law (4/4)

- Integral balance



volume assumed to be ***fixed***

$$\frac{\partial}{\partial t}\iiint_{\mathscr{V}} E_a \, d\mathscr{V} = -\iint_{\Sigma} p\vec{v}\cdot\vec{n}\, d\sigma + \iiint_{\mathscr{V}} \left(\rho_0 \vec{v}\cdot\vec{F} + \frac{\alpha}{C_p} p h + pq\right) d\mathscr{V}$$

variation per unit of time of the energy contained in the domain 𝒱

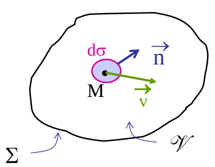
total energy flow incoming in the volume 𝒱 per unit of time

energy brought by the sources, per unit of time

Acoustic intensity vector (1/2)

$$\vec{I} = \overline{\vec{\mathscr{P}}} = \overline{p\vec{v}}$$

REAL quantities



$\vec{I}\cdot\vec{n}\,d\sigma$: average quantity of acoustic energy which travel through a unit area d σ

⟹ $\vec{I}$: density vector of average acoustic power flow, per unit of area(W.m^{-2})

$p = \mathscr{R}e(\hat{p}) = \frac{1}{2}\left(\hat{p} + \hat{p}^*\right)$ *and* $\vec{v} = \mathscr{R}e\left(\hat{\vec{v}}\right) = \frac{1}{2}\left(\hat{\vec{v}} + \hat{\vec{v}}^*\right)$

⟹ $\vec{I} = \frac{1}{4}\overline{\left(\hat{p} + \hat{p}^*\right)\left(\hat{\vec{v}} + \hat{\vec{v}}^*\right)}$

- Monochromatic field: $\hat{p} = \hat{P}\,e^{i\omega t}$ *and* $\hat{\vec{v}} = \hat{\vec{V}}\,e^{i\omega t}$

⟹ $\vec{I} = \frac{1}{4}\left(\hat{P}^*\hat{\vec{V}} + \hat{P}\hat{\vec{V}}^*\right) = \frac{1}{4}\left(\hat{p}^*\hat{\vec{v}} + \hat{p}\hat{\vec{v}}^*\right) = \frac{1}{2}\mathscr{R}e\left(\hat{p}^*\hat{\vec{v}}\right) = \frac{1}{2}\mathscr{R}e\left(\hat{p}\hat{\vec{v}}^*\right)$

Acoustic intensity vector (2/2)

- Outside sources:

✓ $\overline{\frac{\partial E_a}{\partial t}} + \mathrm{div}\,\overline{\vec{\mathscr{P}}} = 0$ *i.e.* $\overline{\frac{\partial E_a}{\partial t}} + \mathrm{div}\,\overline{\vec{\mathscr{P}}} = 0$ *i.e.* $\overline{\frac{\partial E_a}{\partial t}} + \mathrm{div}\,\overline{\vec{\mathscr{P}}} = 0$ ($\vec{I}$)

⟹ $\overline{\frac{\partial E_a}{\partial t}} + \mathrm{div}\,\vec{I} = 0$

$$\overline{\frac{\partial E_a}{\partial t}} = \lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}}\frac{\partial E_a}{\partial t}dt = \lim_{T\to\infty}\frac{1}{T}\left[E_a\left(\frac{T}{2}\right) - E_a\left(-\frac{T}{2}\right)\right] = 0$$

⟹ $\overline{\frac{\partial E_a}{\partial t}} = 0$

⟹ $\mathrm{div}\,\vec{I} = 0$ ***Outside sources, the acoustic intensity vector is of null divergence***

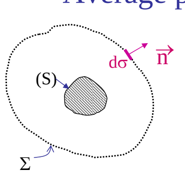
✓ $\rho_0\frac{\partial \vec{v}}{\partial t} + \overrightarrow{\mathrm{grad}}\,p = \vec{0}$ ⟹ amplitude of p proportional to the amplitude of $\vec{v}$

⟹ amplitude of $\vec{\mathscr{P}} = p\vec{v}$ proportional to the square of the amplitude of p

⟹ amplitude I of $\vec{I}$ proportional to p^2_{rms}

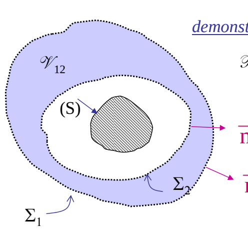
⟹ sound level: $L = 10\log_{10}(I/I_s) = 20\log_{10}(p_{rms}/p_s)$

Average power of a source (1/2)



$$\mathscr{P}_m(S) = \iint_{\Sigma} \vec{I}\cdot\vec{n}\,d\sigma$$

The average power of a source is ***independent of its chosen surrounding surface Σ.***



demonstration:

$\mathscr{P}_m(S/\Sigma_1) = \iint_{\Sigma_1} \vec{I}\cdot\vec{n}_1\,d\sigma$ *et* $\mathscr{P}_m(S/\Sigma_2) = \iint_{\Sigma_2} \vec{I}\cdot\vec{n}_2\,d\sigma$

power flow through (Σ_1+Σ_2) which bounds the closed volume $\mathscr{V}_{12}$:

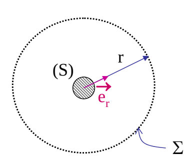
$$\Phi_{12} = \iint_{\Sigma_1} \vec{I}\cdot\vec{n}_1\,d\sigma + \iint_{\Sigma_2} \vec{I}\cdot\left(-\vec{n}_2\right)d\sigma = 0$$

acoustic intensity (of null divergence) outside the sources

⟹ $\mathscr{P}_m(S/\Sigma_1) = \mathscr{P}_m(S/\Sigma_2)$

Average power of a source (2/2)

- Particular case of an omnidirectional source (identical acoustic field in all directions)



$$\vec{I} = I(r)\vec{e}_r$$

$$\mathscr{P}_m(S) = \iint_{\Sigma} I(r)\,r\,d\theta\,r\sin\theta\,d\psi = I(r)r^2\iint_{\Sigma} d\theta\sin\theta\,d\psi$$

⟹ $\mathscr{P}_m(S) = 4\pi r^2 I(r)$

⟹ $I(r) \propto \frac{1}{r^2}$ *and* $p(r) \propto \frac{1}{r}$ (*spherical character*)

Slides based upon

C. POTEL, M. BRUNEAU, *Acoustique Générale - équations différentielles et intégrales, solutions en milieux fluide et solide, applications*, Ed. Ellipse collection Technosup, 352 pages, ISBN 2-7298-2805-2, 2006