C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Université du Maine

Chapter 4

ANALYTICAL FORMULATION OF FUNDAMENTAL LINEAR PROBLEMS OF ACOUSTICS, IN HOMOGENEOUS, TIME INDEPENDENT FLUID MEDIA AT REST: FUNDAMENTAL SOLUTIONS IN CARTESIAN COORDINATES

Solutions of the propagation equation

- Propagation equation

$$\left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(\vec{r};t)=-\hat{f}(\vec{r};t)$$

- Helmholtz equation

$$\left(\Delta+\frac{\omega^2}{c_0^2}\right)\hat{P}(\vec{r};\omega)=-\hat{F}(\vec{r};\omega)$$

✓ No general solution is known, except in the case of one-dimensional propagation

✓ If the boundaries of the domain coincide with surfaces of separable curvilinear coordinates

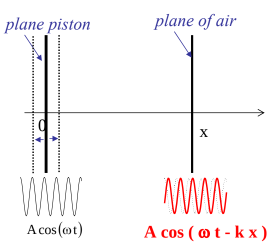
➡ solutions with separable variables

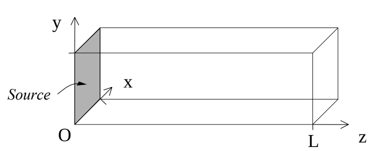
➡ "basis" on which any solution can be expanded

complete family

Choice of the coordinate system

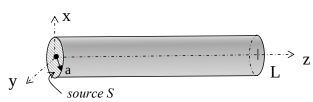
- Cartesian coordinates





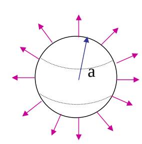
➡ circular functions

- Cylindrical coordinates



➡ Bessel functions

- Spherical coordinates



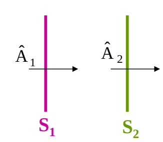
➡ Legendre polynomials

Amplitude of waves in Cartesian, cylindrical and spherical coordinates (1/3)

- Energy flux $\Phi\propto|\hat{A}|^2 S$ *proportional to the crossed surface and to the squared amplitude of the wave*

$\Phi=\text{constant}$ ⟺ $|\hat{A}_1|^2 S_1=|\hat{A}_2|^2 S_2$

- Plane wave



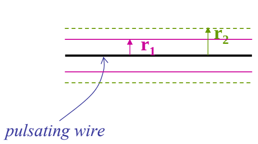
$S_1=S_2$ ➡ $|\hat{A}_1|=|\hat{A}_2|$

Any wave the amplitude of which is independent of the point, in a given space, has a ***plane*** *character in this space.*

Amplitude of waves in Cartesian, cylindrical and spherical coordinates (2/3)

- Cylindrical wave

$\Phi\propto|\hat{A}|^2 S$ with $S\propto 2\pi r$



➡ $\Phi\propto|\hat{A}|^2 r$ and $\Phi=\text{constant}$

➡ $|\hat{A}|^2 r=\text{constant}$ ➡ $|\hat{A}|\propto\frac{1}{\sqrt{r}}$

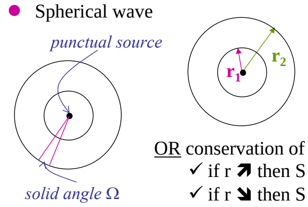
Any wave the amplitude of which decreases such as 1/√r in a given space, has a ***cylindrical*** *character in this space.*

Application: going on holidays using a motorway

As a first approach, the noise emitted by the motorway can be modeled by a field having a cylindrical character

Amplitude of waves in Cartesian, cylindrical and spherical coordinates (3/3)

- Spherical wave



$\Phi\propto|\hat{A}|^2 S$ with $S=4\pi r^2$

➡ $\Phi\propto|\hat{A}|^2 r^2$ and $\Phi=\text{constant}$

➡ $|\hat{A}|^2 r^2=\text{constant}$ ➡ $|\hat{A}|\propto\frac{1}{r}$

OR conservation of the energy flux in a sector of solid angle Ω

✓ if r ↗ then S ↗ and amplitude ↘ : *divergent* wave

✓ if r ↘ then S ↘ and amplitude ↗ : *convergent* wave

Any wave the amplitude of which decreases such as 1/r in a given space, has a ***spherical*** *character in this.*

Application: single car in open country

As a first approach, the car is a punctual source with respect to the house (field with a spherical character)

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One-dimensional problems

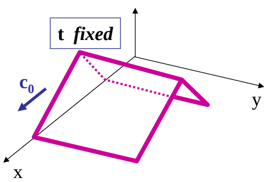
- All the variable of the problem depend on ***only one coordinate***

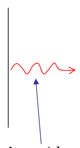
  → $\hat{p}(x;t)$, $\hat{\vec{v}}(x;t)$, $\hat{\varphi}(x;t)$

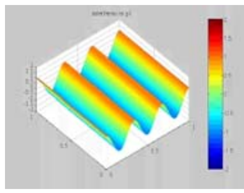
  → fields are uniform in a plane perpendicular to the coordinate

  → fields of plane waves

- Examples







- Good choice of coordinate system $\mathscr{R}=(O,\vec{e}_x,\vec{e}_y,\vec{e}_z)$ → coordinate x

Direction of the velocity vector (1D)

- Good choice of coordinate system $\mathscr{R}=(O,\vec{e}_x,\vec{e}_y,\vec{e}_z)$ → coordinate x

$$\hat{\vec{v}}(x;t)=\hat{v}_x(x;t)\vec{e}_x+\hat{v}_y(x;t)\vec{e}_y+\hat{v}_z(x;t)\vec{e}_z$$

- Euler equation

$$\rho_0\frac{\partial\hat{\vec{v}}}{\partial t}+\overrightarrow{\text{grad}}\,\hat{p}=\vec{0} \;\Rightarrow\; \rho_0\frac{\partial\hat{\vec{v}}(x;t)}{\partial t}+\frac{\partial\hat{p}(x;t)}{\partial x}\vec{e}_x=\vec{0}$$

$$\begin{cases}\rho_0\dfrac{\partial\hat{v}_x(x;t)}{\partial t}+\dfrac{\partial\hat{p}(x;t)}{\partial x}=0\\[2mm] \rho_0\dfrac{\partial\hat{v}_y(x;t)}{\partial t}=0 \;\Rightarrow\; \hat{v}_y(x;t)=\hat{K}_y(x) \;\;=0\\[2mm] \rho_0\dfrac{\partial\hat{v}_z(x;t)}{\partial t}=0 \;\Rightarrow\; \hat{v}_z(x;t)=\hat{K}_z(x) \;\;=0\end{cases}$$

null time averages

$$\Rightarrow \hat{\vec{v}}(x;t)=\hat{v}_x(x;t)\vec{e}_x$$

denoted $\hat{v}(x;t)$

Equations of acoustics for one-dimensional problems

- Fundamental laws

$$\begin{cases}\checkmark\; \rho_0\dfrac{\partial\hat{\vec{v}}}{\partial t}+\overrightarrow{\text{grad}}\,\hat{p}=\vec{0}\\[2mm] \checkmark\; \dfrac{\partial\hat{\rho}}{\partial t}+\rho_0\,\text{div}\,\hat{\vec{v}}=0\\[2mm] \checkmark\; \hat{p}=c_0^2\hat{\rho}\end{cases} \;\Rightarrow\; \begin{cases}\checkmark\; \rho_0\dfrac{\partial\hat{v}(x;t)}{\partial t}+\dfrac{\partial\hat{p}(x;t)}{\partial x}=0\\[2mm] \checkmark\; \dfrac{\partial\hat{\rho}(x;t)}{\partial t}+\rho_0\dfrac{\partial\hat{v}(x;t)}{\partial x}=0\\[2mm] \checkmark\; \hat{p}(x;t)=c_0^2\hat{\rho}(x;t)\end{cases}$$

- Propagation equation

$$\checkmark\; \left(\Delta-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}=0 \;\Rightarrow\; \checkmark\; \left(\frac{\partial^2}{\partial x^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x;t)=0$$

General solution for one-dimensional problems

- Propagation equation $\left(\dfrac{\partial^2}{\partial x^2}-\dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\right)\hat{p}(x;t)=0$

- Change of variables $u=t-\dfrac{x}{c_0}$ *and* $v=t+\dfrac{x}{c_0}$

$$\checkmark\; \frac{\partial\hat{p}}{\partial t}=\frac{\partial\hat{p}}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial\hat{p}}{\partial v}\frac{\partial v}{\partial t}=\frac{\partial\hat{p}}{\partial u}+\frac{\partial\hat{p}}{\partial v} \;\Rightarrow\; \frac{\partial^2\hat{p}}{\partial t^2}=\frac{\partial^2\hat{p}}{\partial u^2}+\frac{\partial^2\hat{p}}{\partial v^2}+2\frac{\partial^2\hat{p}}{\partial u\partial v}$$

$$\checkmark\; \frac{\partial\hat{p}}{\partial x}=\frac{\partial\hat{p}}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\hat{p}}{\partial v}\frac{\partial v}{\partial x}=\frac{1}{c_0}\left(-\frac{\partial\hat{p}}{\partial u}+\frac{\partial\hat{p}}{\partial v}\right) \;\Rightarrow\; \frac{\partial^2\hat{p}}{\partial x^2}=\frac{1}{c_0^2}\left(\frac{\partial^2\hat{p}}{\partial u^2}+\frac{\partial^2\hat{p}}{\partial v^2}-2\frac{\partial^2\hat{p}}{\partial u\partial v}\right)$$

- Substitution into propagation equation

$$\frac{1}{c_0^2}\left(\frac{\partial^2\hat{p}}{\partial u^2}+\frac{\partial^2\hat{p}}{\partial v^2}-2\frac{\partial^2\hat{p}}{\partial u\partial v}\right)-\frac{1}{c_0^2}\left(\frac{\partial^2\hat{p}}{\partial u^2}+\frac{\partial^2\hat{p}}{\partial v^2}+2\frac{\partial^2\hat{p}}{\partial u\partial v}\right)=0$$

$$\Rightarrow \frac{\partial^2\hat{p}}{\partial u\partial v}=0 \;\Rightarrow\; \hat{p}=\hat{f}(u)+\hat{g}(v)$$

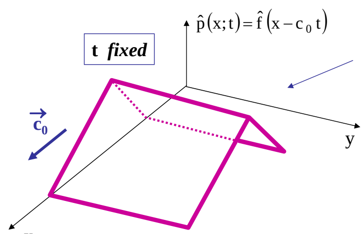
- General solution

$$\hat{p}(x;t)=\hat{f}\left(t-\frac{x}{c_0}\right)+\hat{g}\left(t+\frac{x}{c_0}\right)$$

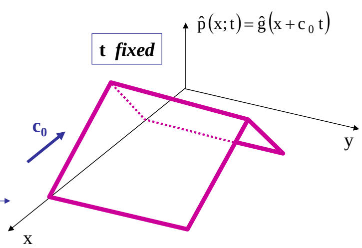
or $\hat{p}(x;t)=\hat{f}(x-c_0t)+\hat{g}(x+c_0t)$

or $\hat{p}(x;t)=\hat{f}[\kappa(x-c_0t)]+\hat{g}[\kappa(x+c_0t)]$

Particular case: progressive plane waves (1D)

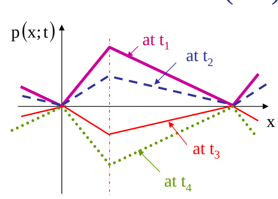


progressive plane wave which propagates following ***increasing*** *values of x*



progressive plane wave which propagates following ***decreasing*** *values of x*

Particular case (1D): standing plane waves (1/2)



$p(x;t)=X(x)T(t)$ (real acoustic pressure)

Succession of nodes (here zeroes of pressure) and of antinodes (here maxima of pressure), which evolves with time following the law imposed by the function T(t).

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p(x;t)=0,\ \forall x,\forall t \;\Rightarrow\; c_0^2\frac{X''(x)}{X(x)}=\frac{T''(t)}{T(t)},\ \forall x,\forall t$$

depends only on x — depends only on t → $-\omega^2$

$$\Rightarrow \begin{cases}T''(t)+\omega^2T(t)=0\\ X''(x)+k^2X(x)=0\end{cases} \quad \textit{with} \quad k=\omega/c_0$$

Separated variable method

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Particular case (1D): standing plane waves (2/2)

- $X''(x)+k^2X(x)=0$ ➡ $X(x)=A\cos(kx)+B\sin(kx)$ *or* $X(x)=\hat{C}e^{ikx}+\hat{D}e^{-ikx}$

- $T''(t)+\omega^2T(t)=0$ ➡ $T(t)=E\cos(\omega t)+F\sin(\omega t)$ *or* $T(t)=\hat{G}e^{i\omega t}+\hat{H}e^{-i\omega t}$

➡ A standing wave is necessarily ***sinusoidal*** with t.

Practically, *choice of a time convention* $e^{i\omega t}$ *or* $e^{-i\omega t}$

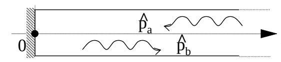
One or the other convention leads to the *same* ***real*** *result*.

Progressive and standing plane waves

- $p(x;t)=A\cos(kx)\cos(\omega t)=\frac{A}{2}\left[\cos(kx-\omega t)+\cos(kx+\omega t)\right]$

  $A\cos(kx)\cos(\omega t)$: standing plane wave; $\cos(kx-\omega t)$: progressive plane wave **+** $\cos(kx+\omega t)$: progressive plane wave

example:



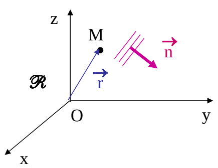
- $p(x;t)=A\cos(kx-\omega t)=A\cos(kx)\cos(\omega t)+A\sin(kx)\sin(\omega t)$

  $A\cos(kx-\omega t)$: progressive plane wave; $A\cos(kx)\cos(\omega t)$: standing plane wave; $A\sin(kx)\sin(\omega t)$: standing plane wave

- ⚠ $e^{i(kx+\omega t)}=e^{ikx}e^{i\omega t}$

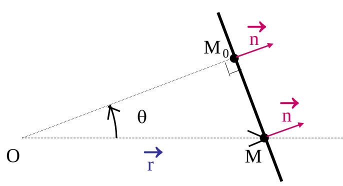
  BUT $\mathcal{R}e\left[e^{i(kx+\omega t)}\right]=\cos(kx+\omega t)\neq\mathcal{R}e\left(e^{ikx}\right)\mathcal{R}e\left(e^{i\omega t}\right)$

Plane waves (1/4)

- 

  $\hat{u}_i(\vec{r};t)=\hat{f}(\vec{n}\cdot\vec{r}-c_0t)+\hat{g}(\vec{n}\cdot\vec{r}+c_0t)$ $\quad\vec{r}=\overrightarrow{OM}$

- $F\left[\kappa\left(c_0t-\vec{n}\cdot\vec{r}\right)\right]$



- *At a given time*, at any given point M such as $\vec{n}\cdot\vec{r}=\text{constant}$

  the value of the field variable (physical quantity) is the same.

  These points are located in the same plane, which is termed ***wave plane*** (plane ***wave surface***), perpendicular to the direction $\vec{n}$:

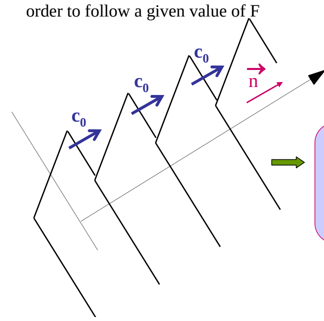
  $$\vec{n}\cdot\vec{r}=\vec{n}\cdot\overrightarrow{OM}=OM\cos\theta=OM_0$$

Plane waves (2/4)

- *When time varies*, follow a given value of ➡ $\left(c_0t-\vec{n}\cdot\vec{r}\right)=\text{constant}$

  i.e. $d\left(c_0t-\vec{n}\cdot\vec{r}\right)=0$ *i.e.* $\vec{n}\cdot\frac{d\vec{r}}{dt}=c_0$

  speed to which a geometrical point M must move in order to follow a given value of F



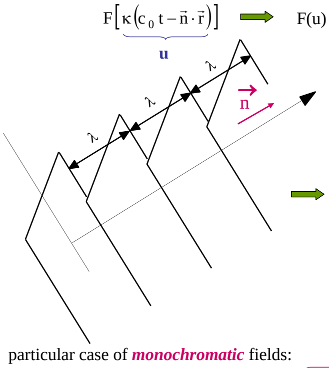
if $\vec{r}\,//\,\vec{n}$ ➡ $\frac{d\vec{r}}{dt}=c_0\vec{n}$

➡ The wave planes which convey a given value of the field variable F, move parallel to themselves, in the direction $\vec{n}$ which is perpendicular to them, and with the propagation velocity c_0.

Plane waves (3/4)

- Particular case of a periodic plane wave: F periodic with period U

  $F\left[\kappa\underbrace{\left(c_0t-\vec{n}\cdot\vec{r}\right)}_{u}\right]$ ➡ F(u) with $u=\kappa\left(c_0t-\vec{n}\cdot\vec{r}\right)$



At a given time t, and for two given values $\vec{r}_1$ and $\vec{r}_2$ of the position vector $\vec{r}$ such that

$$\kappa\vec{n}\cdot\left(\vec{r}_2-\vec{r}_1\right)=U$$

then $\quad F=\text{constant}$

➡ two wave plane which are distant from one to another with a distance equal to **U/κ** convey the same value of the field

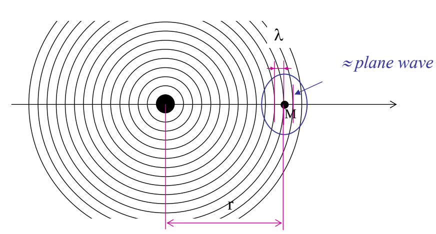
$$\lambda=\frac{U}{\kappa}$$ ***wavelength***

particular case of *monochromatic* fields: $C\cos\left(\omega t-k\vec{n}\cdot\vec{r}+\alpha\right)$ ➡ $U=2\pi$

➡ $\lambda=\frac{2\pi}{k}$

Plane waves (4/4)

- Validity of the plane wave hypothesis



Approximation " quasi plane wave" of a field in the neighboring of an observation point M : case of a spherical field with a far field hypothesis (r >> λ).

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Monochromatic plane waves (1/4)

- Monochromatic wave: propagation at 1 frequency (angular frequency ω given)
  - ➡ frequency driven by a source emitting permanently
  - ➡ ***driven motion, steady state***, linear acoustics
- Any quantity can be written $\hat{u}(x;t)=\hat{U}(x)e^{i\omega t}$ ⚠ *time convention*

➡ $\hat{p}(x;t)=\hat{P}(x)e^{i\omega t}$, $\hat{\vec{v}}(x;t)=\hat{\vec{V}}(x)e^{i\omega t}$, $\hat{\varphi}(x;t)=\hat{\Phi}(x)e^{i\omega t}$, ...

- Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x;t)=0 \;\Rightarrow\; \left[\frac{\partial^2}{\partial x^2}+\left(\frac{\omega}{c_0}\right)^2\right]\hat{P}(x)e^{i\omega t}=0,\ \forall x, \forall t$$

$$\Rightarrow\; \left[\frac{\partial^2}{\partial x^2}+\left(\frac{\omega}{c_0}\right)^2\right]\hat{P}(x)=0,\ \forall x$$

Notation: $k_0=\omega/c_0$

➡ $\left(\partial^2_{xx}+k_0^2\right)\hat{P}(x)=0,\ \forall x$

Monochromatic plane waves (2/4)

- Search for solutions $\left[\frac{\partial^2}{\partial x^2}+\left(\frac{\omega}{c_0}\right)^2\right]\hat{P}(x)=0,\ \forall x$

➡ equation in the form: (factor independent of x) (function of x) $=0,\ \forall x$

➡ $\hat{P}(x)=\hat{A}e^{-ikx}+\hat{B}e^{ikx}$

Substitution into Helmholtz equation: $\left(-k^2+k_0^2\right)\hat{P}(x)=0,\ \forall x$ *with* $k_0=\omega/c_0$

➡ $k^2=k_0^2$ ***dispersion equation*** *i.e.* $k=k_0$ *or* $k=-k_0$

➡ choice: $k=k_0$ — redundant with e^{-ikx} *and* e^{ikx}

➡ $\hat{p}(x;t)=\hat{A}e^{i(-kx+\omega t)}+\hat{B}e^{i(kx+\omega t)}$

progressive plane wave towards x ↗ — progressive plane wave towards x ↘

Monochromatic plane waves (3/4)

- Notation $k_0=\dfrac{\omega}{c_0}$ — ω: *source*; c_0: *propagation medium*

$c_0=\sqrt{\gamma/(\rho_0\chi_T)}$ — ρ_0: *density*; χ_T: *compressibility*

- Dispersion equation in non dissipative fluid: $k=k_0$

⚠ the wave number k is not always equal to k_0

- Example: dissipative term in the form $R\dfrac{\partial\hat{p}}{\partial t}$

$$\Rightarrow\; \left[\frac{\partial^2}{\partial x^2}+R\frac{\partial}{\partial t}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right]\hat{p}(x;t)=0,\ \forall x,\ \forall t$$

$$\hat{P}(x)=\hat{A}e^{-ikx}+\hat{B}e^{ikx} \;\Rightarrow\; \left[\frac{\partial^2}{\partial x^2}+i\omega R+\left(\frac{\omega}{c_0}\right)^2\right]\hat{P}(x)=0,\ \forall x$$

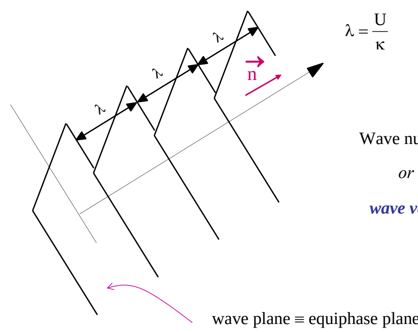
$$\Rightarrow\; \left(-k^2+i\omega R+k_0^2\right)\hat{P}(x)=0,\ \forall x$$

➡ $k^2=k_0^2+i\omega R$ ***dispersion equation***

Monochromatic plane waves (4/4)

$F\left[\kappa\left(c_0t-\vec{n}\cdot\vec{r}\right)\right]$ in the form $C\cos\left(\omega t-k\vec{n}\cdot\vec{r}+\alpha\right)$

periodic function with period $U=2\pi$



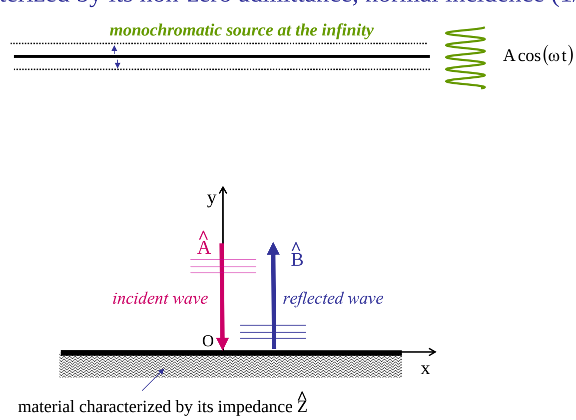
$\lambda=\dfrac{U}{\kappa}$ ➡ $\lambda=\dfrac{2\pi}{k}$ ***wavelength***

Wave number vector $\vec{k}=k\vec{n}$

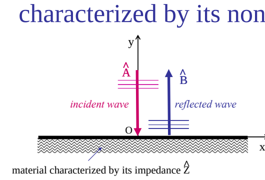
or

wave vector

Interaction of a monochromatic plane wave with a wall characterized by its non-zero admittance, normal incidence (1/5)



Interaction of a monochromatic plane wave with a wall characterized by its non-zero admittance, normal incidence (2/5)



- Propagation equation

$$\left(\frac{\partial^2}{\partial y^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(y;t)=0,\ \forall y\geq 0, \forall t$$

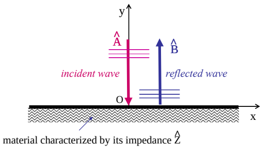
- Incident monochromatic field $\hat{p}(y;t)=\hat{P}(y)e^{i\omega t}$
- Helmholtz equation $\left[\dfrac{\partial^2}{\partial y^2}+\left(\dfrac{\omega}{c_0}\right)^2\right]\hat{P}(y)=0,\ \forall y\geq 0$
- Boundary conditions $\left[\dfrac{\partial}{\partial n}+ik_0\hat{\beta}\right]\hat{P}(0)=0$ *with* $\hat{\beta}=\rho_0c_0/\hat{Z}$ *and* $\vec{n}=-\vec{e}_y$

➡ $\left[-\dfrac{\partial}{\partial y}+ik_0\hat{\beta}\right]\hat{P}(y)=0,\ y=0$

- Return wave $\hat{B}$ which propagates at the infinity

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Interaction of a monochromatic plane wave with a wall characterized by its non-zero admittance, normal incidence (3/5)



- Solutions of the problem
  - $\hat{p}(y;t)=\hat{p}_a(y;t)+\hat{p}_b(y;t)=\underbrace{\left(\hat{A}\,e^{iky}+\hat{B}\,e^{-iky}\right)}_{\hat{P}(y)}e^{i\omega t}$
  - $\dfrac{\partial \hat{P}(y)}{\partial y}=ik\left(\hat{A}\,e^{iky}-\hat{B}\,e^{-iky}\right)$

- Boundary conditions: $\left[-\dfrac{\partial}{\partial y}+ik_0\hat{\beta}\right]\hat{P}(y)=0,\ y=0 \Longrightarrow -ik\left(\hat{A}-\hat{B}\right)+ik_0\hat{\beta}\left(\hat{A}+\hat{B}\right)=0$

  Dispersion equation: $k=k_0 \Longrightarrow \hat{A}\left(1-\hat{\beta}\right)=\hat{B}\left(1+\hat{\beta}\right)$

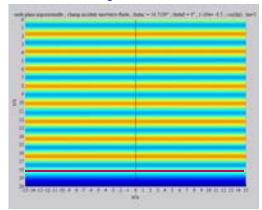
- Reflection coefficient $\hat{\mathscr{R}}_p=\dfrac{\hat{B}}{\hat{A}}=\dfrac{1-\hat{\beta}}{1+\hat{\beta}}$

  If $\hat{\beta}=0$ (perfectly rigid medium): total reflection

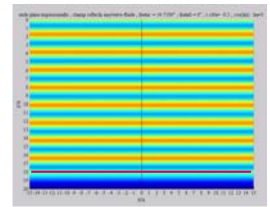
- Acoustic pressure $\hat{P}(y)=\hat{A}\left(e^{iky}+\hat{\mathscr{R}}_p e^{-iky}\right)=\hat{A}\left[2\hat{\mathscr{R}}_p\cos(ky)+\left(1-\hat{\mathscr{R}}_p\right)e^{iky}\right]$

$$\Longrightarrow \hat{p}(y;t)=\hat{A}\left[\underbrace{2\hat{\mathscr{R}}_p\cos(ky)}_{\text{standing part}}+\underbrace{\left(1-\hat{\mathscr{R}}_p\right)e^{iky}}_{\text{propagating part}}\right]e^{i\omega t}$$

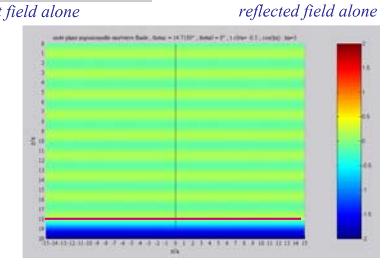
$\mathscr{R}_p$ = 0.9: great standing part (ka = 3)



incident field alone

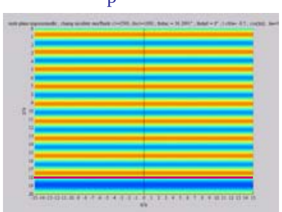


reflected field alone

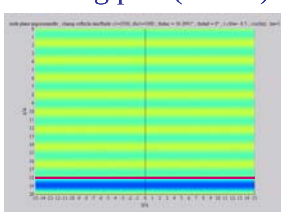


total field

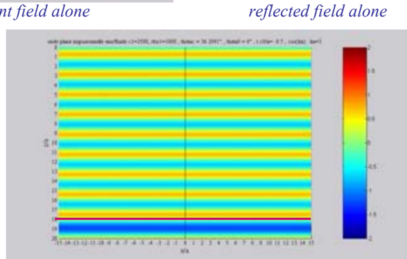
$\mathscr{R}_p$ = 0.26: weak standing part (ka = 3)



incident field alone

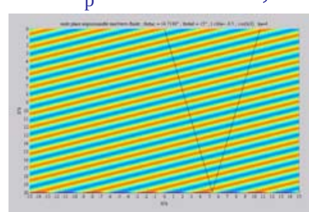


reflected field alone

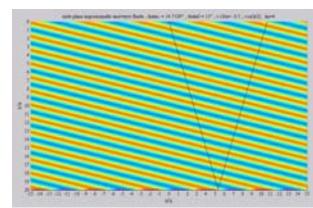


total field

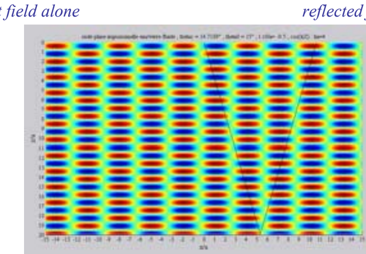
$\mathscr{R}_p$ = 0.99-0.02 i; oblique incidence (ka = 4)



incident field alone

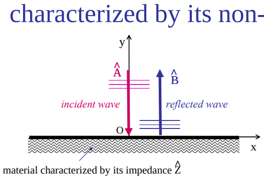


reflected field alone



The whole field has a stationary character along y, but has a propagating character along x

Interaction of a monochromatic plane wave with a wall characterized by its non-zero admittance, normal incidence (4/5)



- Particle velocity
  - Euler equation $\rho_0\dfrac{\partial \hat{\vec{v}}(y;t)}{\partial t}+\overrightarrow{\text{grad}}\,\hat{p}(y;t)=\vec{0}$
  - $\hat{\vec{v}}(y;t)=\hat{\vec{V}}(y)e^{i\omega t} \Longrightarrow \hat{\vec{v}}(y;t)=\dfrac{i}{\rho_0\omega}\overrightarrow{\text{grad}}\,\hat{p}(y;t)$

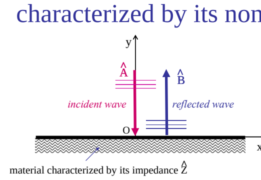
$$\Longrightarrow \hat{v}(y;t)=\frac{i}{\rho_0\omega}\frac{\partial \hat{p}(y;t)}{\partial y}=\frac{i}{\rho_0\omega}\frac{\partial \hat{P}(y)}{\partial y}e^{i\omega t}$$

- Acoustic pressure $\hat{P}(y)=\hat{A}\left(e^{iky}+\hat{\mathscr{R}}_p e^{-iky}\right)$

$$\Longrightarrow \hat{v}(y;t)=\frac{-k\hat{A}}{\rho_0\omega}\left(e^{iky}-\hat{\mathscr{R}}_p e^{-iky}\right)e^{i\omega t}=\frac{-\hat{A}}{\rho_0 c_0}\left(e^{iky}-\hat{\mathscr{R}}_p e^{-iky}\right)e^{i\omega t}$$

dispersion equation: $k=k_0=\omega/c_0$

Interaction of a monochromatic plane wave with a wall characterized by its non-zero admittance, normal incidence (5/5)



- Acoustic intensity $\vec{I}=\dfrac{1}{4}\left(\hat{p}^*\hat{\vec{v}}+\hat{p}\,\hat{\vec{v}}^*\right)$

  here $I=\dfrac{1}{4}\left(\hat{p}^*\hat{v}+\hat{p}\,\hat{v}^*\right)$

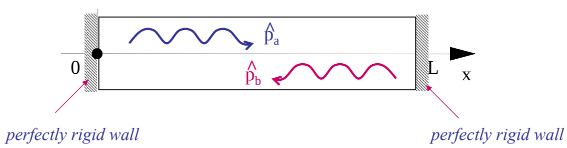
  with $\begin{cases}\hat{p}(y;t)=\hat{A}\left(e^{iky}+\hat{\mathscr{R}}_p e^{-iky}\right)e^{i\omega t}\\ \hat{v}(y;t)=\dfrac{-\hat{A}}{\rho_0 c_0}\left(e^{iky}-\hat{\mathscr{R}}_p e^{-iky}\right)e^{i\omega t}\end{cases}$

$$\Longrightarrow I=\frac{-\hat{A}^*\hat{A}}{4\rho_0 c_0}\left[\left(e^{-iky}+\hat{\mathscr{R}}_p^* e^{iky}\right)\left(e^{iky}-\hat{\mathscr{R}}_p e^{-iky}\right)+\left(e^{iky}+\hat{\mathscr{R}}_p e^{-iky}\right)\left(e^{-iky}-\hat{\mathscr{R}}_p^* e^{iky}\right)\right]$$

$$\Longrightarrow I=\frac{-\left|\hat{A}\right|^2}{2\rho_0 c_0}\left[1-\left|\hat{\mathscr{R}}_p\right|^2\right]$$

If total reflection, $\hat{\mathscr{R}}_p=1 \Longrightarrow I=0$

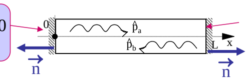
Acoustic field in a finite length waveguide (1D) (1/5)



- $t < 0$: acoustic sources in the waveguide
- $t = 0$: extinction of sources
- the solutions are searched in the form of a ***superposition of monochromatic plane waves*** which satisfy the boundary conditions at the two extremities at x=0 and x=L

Acoustic field in a finite length waveguide (1D) (2/5)

- Propagation equation $\left(\dfrac{\partial^2}{\partial x^2} - \dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\right)\hat{p}(x;t) = 0,\ \forall x \in [0,L],\ \forall t \geq 0$
- Boundary conditions $\hat{v}_x(0;t) = 0\ \ \forall t \geq 0$; $\hat{v}_x(L;t) = 0\ \ \forall t \geq 0$



i.e. $\dfrac{\partial \hat{p}(x;t)}{\partial n} = 0,\ \forall t \geq 0$ *at* $x = 0$ et $x = L$

with $\partial/\partial n = -\partial/\partial x$ T $x = 0$ and $\partial/\partial n = +\partial/\partial x$ T $x = L$

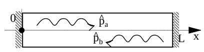
- Source turned off at t = 0

- Searched solution: harmonic character ➡ $\hat{p}(x;t) = \hat{P}(x)e^{i\omega t}$
- Helmholtz equation
- Boundary conditions

$$\begin{cases}\left[\partial_{xx}^2 + (\omega/c_0)^2\right]\hat{P}(x) = 0,\ \forall x \in [0,L],\\ \partial_x \hat{P}(x) = 0,\ x = 0 \text{ et } x = L,\end{cases}$$

Acoustic field in a finite length waveguide (1D) (3/5)

- Solutions of the problem



- $\hat{p}(x;t) = \hat{p}_a(x;t) + \hat{p}_b(x;t) = \underbrace{\left(\hat{A}e^{-ikx} + \hat{B}e^{ikx}\right)}_{\hat{P}(x)}e^{i\omega t},\ \ \forall x \in [0,L]$ *with dispersion equation:* $k = k_0$
- Particle velocity $\hat{v}_x = \dfrac{-1}{i\omega\rho_0}\partial_x \hat{p} = \dfrac{-1}{\rho_0 c_0}\left(-\hat{A}e^{-ikx} + \hat{B}e^{ikx}\right)e^{i\omega t}$
- Boundary conditions

$$\begin{cases}\left[\partial_{xx}^2 + (\omega/c_0)^2\right]\hat{P}(x) = 0,\ \forall x \in [0,L],\\ \partial_x \hat{P}(x) = 0,\ x = 0 \text{ et } x = L,\end{cases} \Rightarrow \begin{cases}\left(-\hat{A} + \hat{B}\right)e^{i\omega t} = 0,\ \forall t,\\ \left(-\hat{A}e^{-ikL} + \hat{B}e^{ikL}\right)e^{i\omega t} = 0,\ \forall t.\end{cases}$$

$$\Rightarrow \begin{cases}\hat{A} = \hat{B},\\ \hat{A}\underbrace{\left(-e^{-ikL} + e^{ikL}\right)}_{2i\sin(kL)} = 0.\end{cases}\quad i.e.\quad 2i\sin(kL) = 0 \quad i.e. \quad kL = m\pi \quad, m \in \mathbb{N}$$

➡ $k_m = \dfrac{m\pi}{L}\ ,\ m \in \mathbb{N}$ ***eigenvalues***

- Trivial solution $\hat{A}_m = 0,\ \forall m$ EXCEPT IF

➡ ***eigen angular frequencies*** $\omega_m = k_m c_0 = \dfrac{m\pi c_0}{L}$ ***; eigenfrequencies*** $f_m = \dfrac{\omega_m}{2\pi} = \dfrac{mc_0}{2L}$

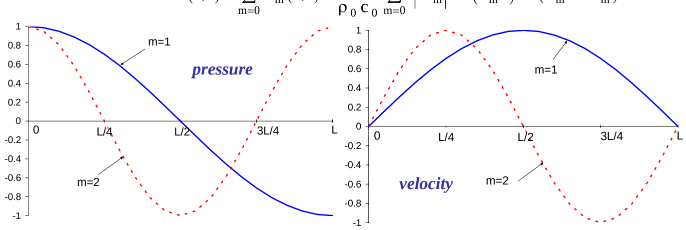
Acoustic field in a finite length waveguide (1D) (4/5)

- Complex field associated to each mode m
  - pressure $\hat{p}_m(x;t) = \hat{A}_m\left(e^{-ik_m x} + e^{ik_m x}\right)e^{i\omega_m t} = 2\hat{A}_m\cos(k_m x)\,e^{i\omega_m t}$
  - velocity $\hat{v}_{x_m}(x;t) = \dfrac{-\hat{A}_m}{\rho_0 c_0}\left(-e^{-ik_m x} + e^{ik_m x}\right)e^{i\omega_m t} = \dfrac{-2i\hat{A}_m}{\rho_0 c_0}\sin(k_m x)e^{i\omega_m t}$
- Total field: superposition of eigenmodes
  - pressure $\hat{p}(x;t) = \sum_{m=0}^{\infty}\hat{p}_m(x;t) = \sum_{m=0}^{\infty}2\hat{A}_m\cos(k_m x)\,e^{i\omega_m t}$
  - velocity $\hat{v}_x(x;t) = \sum_{m=0}^{\infty}v_{x_m}(x;t) = \sum_{m=0}^{\infty}\dfrac{-2i\hat{A}_m}{\rho_0 c_0}\sin(k_m x)e^{i\omega_m t}$

with $\hat{A}_m = \left|\hat{A}_m\right|e^{i\alpha_m}$

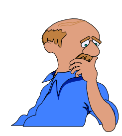
Acoustic field in a finite length waveguide (1D) (5/5)

- Real field associated to each mode m
  - pression $p_m(x;t) = \mathrm{Re}\left[\hat{p}_m(x;t)\right] = 2\left|\hat{A}_m\right|\cos(k_m x)\cos(\omega_m t + \alpha_m)$
  - vitesse $v_m(x;t) = \mathrm{Re}\left[\hat{v}_m(x;t)\right] = \dfrac{2\left|\hat{A}_m\right|}{\rho_0 c_0}\sin(k_m x)\sin(\omega_m t + \alpha_m)$
- Real total field: superposition of eigenmodes
  - pression $p(x;t) = \sum_{m=0}^{\infty}p_m(x;t) = 2\sum_{m=0}^{\infty}\left|\hat{A}_m\right|\cos(k_m x)\cos(\omega_m t + \alpha_m)$
  - vitesse $v(x;t) = \sum_{m=0}^{\infty}v_m(x;t) = \dfrac{2}{\rho_0 c_0}\sum_{m=0}^{\infty}\left|\hat{A}_m\right|\sin(k_m x)\sin(\omega_m t + \alpha_m)$

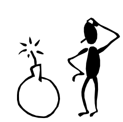


Solutions of 3 dimensional problems (1/5)

- Propagation equation $\left(\Delta - \dfrac{1}{c_0^2}\dfrac{\partial^2}{\partial t^2}\right)\hat{p}(\vec{r};t) = 0,\ \forall \vec{r} \in \mathscr{V},\ \forall t$
- Helmholtz equation $\left[\Delta + \left(\dfrac{\omega}{c_0}\right)^2\right]\hat{P}(\vec{r};\omega) = 0,\ \forall \vec{r} \in \mathscr{V}$



No general solution is known, except in the case of one-dimensional propagation:



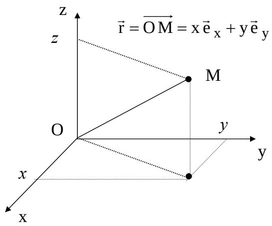
$$\hat{p}(\vec{r};t) = \hat{f}\left(\vec{n}\cdot\vec{r} - c_0 t\right) + \hat{g}\left(\vec{n}\cdot\vec{r} + c_0 t\right)$$

Solutions with separated variables or integral representation

- Cartesian coordinates $\hat{p}(x,y,z;t) = \hat{X}(x)\hat{Y}(y)\hat{Z}(z)\hat{T}(t)$
- Cylindrical coordinates $\hat{p}(r,\psi,z;t) = \hat{R}(r)\hat{\Psi}(\psi)\hat{Z}(z)\hat{T}(t)$
- Spherical coordinates $\hat{p}(r,\theta,\psi;t) = \hat{R}(r)\hat{\Theta}(\theta)\hat{\Psi}(\psi)\hat{T}(t)$

Solution with separated variables ≡ Basis on which any solution of the problem can be projected

Wave equation in Cartesian coordinates



$$\vec{r} = \overrightarrow{OM} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z \;\Rightarrow\; d\vec{r} = d\overrightarrow{OM} = dx\,\vec{e}_x + dy\,\vec{e}_y + dz\,\vec{e}_z$$

$$\vec{A}(x,y,z) = A_x\,\vec{e}_x + A_y\,\vec{e}_y + A_z\,\vec{e}_z$$

- **Usual operators in Cartesian coordinates**

$$\overrightarrow{\text{grad}}\,U = \frac{\partial U}{\partial x}\vec{e}_x + \frac{\partial U}{\partial y}\vec{e}_y + \frac{\partial U}{\partial z}\vec{e}_z$$

$$\text{div}\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\vec{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\vec{e}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\vec{e}_z$$

$$\Delta U = \text{div}\left(\overrightarrow{\text{grad}}\,U\right) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad \textit{and} \quad \Delta\vec{A} = \overrightarrow{\text{grad}}\left(\text{div}\,\vec{A}\right) - \overrightarrow{\text{rot}}\left(\overrightarrow{\text{rot}}\,\vec{A}\right)$$

- **Propagation equation (out sources)**

$$\left(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x,y,z;t) = 0 \;\Rightarrow\; \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x,y,z;t) = 0}$$

Solutions of 3 dimensional problems (2/5)

- **Propagation equation**

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x,y,z;t) = 0\,,\ \forall\,(x,y,z)\in\mathscr{V}\,,\ \forall\,t$$

- **Solutions with separated variables** $\hat{p}(x,y,z;t) = \hat{X}(x)\hat{Y}(y)\hat{Z}(z)\hat{T}(t)$

$$\Rightarrow\; \underbrace{\frac{1}{\hat{X}}\frac{\partial^2\hat{X}}{\partial x^2} + \frac{1}{\hat{Y}}\frac{\partial^2\hat{Y}}{\partial y^2} + \frac{1}{\hat{Z}}\frac{\partial^2\hat{Z}}{\partial z^2}}_{\text{function of x,y,z}} = \underbrace{\frac{1}{\hat{T}}\frac{1}{c_0^2}\frac{\partial^2\hat{T}}{\partial t^2}}_{\text{function of t}}\,,\ \forall\,(x,y,z)\in\mathscr{V}\,,\ \forall\,t$$

$$= -k_0^2 \qquad \textit{let} \quad k_0^2 c_0^2 = \omega^2$$

$$\Rightarrow\; \frac{\partial^2\hat{T}}{\partial t^2} + \omega^2\hat{T} = 0\,,\ \forall\,t \;\Rightarrow\; \left\{\begin{array}{l} e^{i\omega t} \\ e^{-i\omega t}\end{array}\right.$$

choice of a time convention

$$\Rightarrow\; \boxed{\hat{T}(t) = \hat{G}\,e^{i\omega t}}$$

Solutions of 3 dimensional problems (3/5)

- **Solutions with separated variables - solution as a function of x**

$$\underbrace{\frac{1}{\hat{X}}\frac{\partial^2\hat{X}}{\partial x^2}}_{\text{function of x}} = \underbrace{-\frac{1}{\hat{Y}}\frac{\partial^2\hat{Y}}{\partial y^2} - \frac{1}{\hat{Z}}\frac{\partial^2\hat{Z}}{\partial z^2} - k_0^2}_{\text{function of y,z}}\,,\ \forall\,(x,y,z)\in\mathscr{V}$$

$$= -k_x^2$$

$$\Rightarrow\; \frac{\partial^2\hat{X}}{\partial x^2} + k_x^2\hat{X} = 0\,,\ \forall\,x \;\Rightarrow$$

$$\boxed{\begin{array}{l} \hat{X}(x) = \hat{A}\,e^{-ik_x x} + \hat{B}\,e^{ik_x x} \\ \textit{or}\quad \hat{X}(x) = \hat{A}'\cos(k_x x) + \hat{B}'\sin(k_x x) \\ \textit{with}\quad \hat{A}' = \hat{A} + \hat{B}\ \textit{and}\ \hat{B}' = i\left(\hat{B} - \hat{A}\right)\end{array}}$$

Solutions of 3 dimensional problems (4/5)

- **Solutions with separated variables - solution as a function of y**

$$\underbrace{\frac{1}{\hat{Y}}\frac{\partial^2\hat{Y}}{\partial y^2}}_{\text{function of y}} = \underbrace{-\frac{1}{\hat{Z}}\frac{\partial^2\hat{Z}}{\partial z^2} - k_0^2 + k_x^2}_{\text{function of z}}\,,\ \forall\,(y,z)\in\mathscr{V}$$

$$= -k_y^2$$

$$\Rightarrow\; \frac{\partial^2\hat{Y}}{\partial y^2} + k_y^2\hat{Y} = 0\,,\ \forall\,y \;\Rightarrow\; \boxed{\begin{array}{l} \hat{Y}(y) = \hat{C}\,e^{-ik_y y} + \hat{D}\,e^{ik_y y} \\ \textit{or}\quad \hat{Y}(y) = \hat{C}'\cos(k_y y) + \hat{D}'\sin(k_y y)\end{array}}$$

- **Solutions with separated variables - solution as a function of z**

$$\underbrace{\frac{1}{\hat{Z}}\frac{\partial^2\hat{Z}}{\partial z^2}}_{\text{function of z}} = \underbrace{-k_0^2 + k_x^2 + k_y^2}_{\text{constant}}\,,\ \forall\,z\in\mathscr{V}$$

$$= -k_z^2$$

$$\Rightarrow\; \frac{\partial^2\hat{Z}}{\partial z^2} + k_z^2\hat{Z} = 0\,,\ \forall\,z \;\Rightarrow\; \boxed{\begin{array}{l} \hat{Z}(z) = \hat{E}\,e^{-ik_z z} + \hat{F}\,e^{ik_z z} \\ \textit{or}\quad \hat{Z}(z) = \hat{E}'\cos(k_z z) + \hat{F}'\sin(k_z z)\end{array}}$$

Solutions of 3 dimensional problems (5/5)

- **Solutions with separated variables**

$$\boxed{\hat{p}(x,y,z;t) = \underbrace{\left(\hat{A}\,e^{-ik_x x} + \hat{B}\,e^{ik_x x}\right)}_{X(x)}\underbrace{\left(\hat{C}\,e^{-ik_y y} + \hat{D}\,e^{ik_y y}\right)}_{Y(y)}\underbrace{\left(\hat{E}\,e^{-ik_z z} + \hat{F}\,e^{ik_z z}\right)}_{Z(z)}\underbrace{e^{i\omega t}}_{T(t)}}$$

or $$\hat{p}(x,y,z;t) = \left[\hat{A}'\cos(k_x x) + \hat{B}'\sin(k_x x)\right]\left[\hat{C}'\cos(k_y y) + \hat{D}'\sin(k_y y)\right] \cdot \left[\hat{E}'\cos(k_z z) + \hat{F}'\sin(k_z z)\right]\cos(\omega t)\,.$$

or $$\hat{p}(x,y,z;t) = \hat{A}_0\left(e^{-ik_x x} + \hat{\mathscr{R}}_1 e^{ik_x x}\right)\left(e^{-ik_y y} + \hat{\mathscr{R}}_2 e^{ik_y y}\right)\left(e^{-ik_z z} + \hat{\mathscr{R}}_3 e^{ik_z z}\right)e^{i\omega t}$$

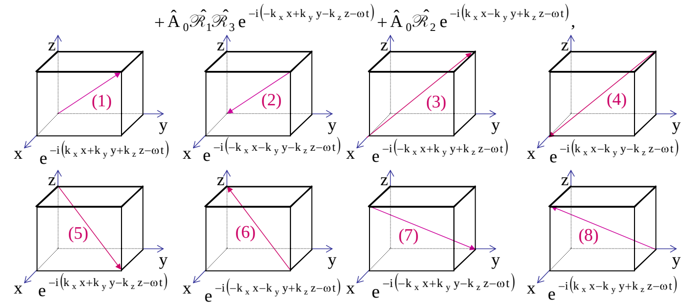
or $$\hat{p}(x,y,z;t) = \hat{A}'_0\left[\cos(k_x x) + \hat{\mathscr{R}}'_1\sin(k_x x)\right]\left[\cos(k_y y) + \hat{\mathscr{R}}'_2\sin(k_y y)\right] \cdot \left[\cos(k_z z) + \hat{\mathscr{R}}'_3\sin(k_z z)\right]\cos(\omega t)\,.$$

only if: $-k_z^2 = -k_0^2 + k_x^2 + k_y^2$ *i.e.* $\boxed{k_x^2 + k_y^2 + k_z^2 = k_0^2 \quad \textit{with}\quad k_0 = \omega/c_0}$

dispersion equation

Waves in 3D infinite space

$$\begin{aligned}\hat{p}(x,y,z;t) = {} & \hat{A}_0\,e^{-i(k_x x + k_y y + k_z z - \omega t)} + + \hat{A}_0\hat{\mathscr{R}}_1\hat{\mathscr{R}}_2\hat{\mathscr{R}}_3\,e^{-i(-k_x x - k_y y - k_z z - \omega t)} \\ & + \hat{A}_0\hat{\mathscr{R}}_1\,e^{-i(-k_x x + k_y y + k_z z - \omega t)} + \hat{A}_0\hat{\mathscr{R}}_2\hat{\mathscr{R}}_3\,e^{-i(k_x x - k_y y - k_z z - \omega t)} \\ & + \hat{A}_0\hat{\mathscr{R}}_3\,e^{-i(k_x x + k_y y - k_z z - \omega t)} + \hat{A}_0\hat{\mathscr{R}}_1\hat{\mathscr{R}}_2\,e^{-i(-k_x x - k_y y + k_z z - \omega t)} \\ & + \hat{A}_0\hat{\mathscr{R}}_1\hat{\mathscr{R}}_3\,e^{-i(-k_x x + k_y y - k_z z - \omega t)} + \hat{A}_0\hat{\mathscr{R}}_2\,e^{-i(k_x x - k_y y + k_z z - \omega t)}\,,\end{aligned}$$

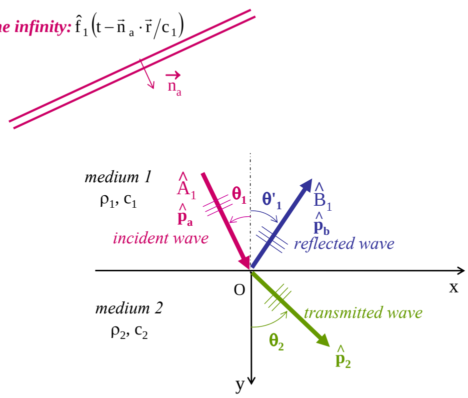


$$\Rightarrow\; \boxed{\hat{p}_i(x,y,z;t) = \hat{A}_i\,e^{-i\left(\vec{k}_i\cdot\overrightarrow{OM} - \omega t\right)}}$$

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Reflection and transmission at the interface between two different fluid media (1/3)

source at the infinity: $\hat{f}_1\left(t-\vec{n}_a\cdot\vec{r}/c_1\right)$



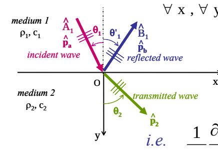
Reflection and transmission at the interface between two different fluid media (2/3)

- Medium 1: well-posed problem
  - Propagation equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}_1(x,y;t)=0\,,\quad \forall\, x\,,\ \forall\, y\le 0,\ \forall\, t$$

- Medium 2: well-posed problem
  - Propagation equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}_2(x,y;t)=0\,,\quad \forall\, x\,,\ \forall\, y\ge 0,\ \forall\, t$$



- Boundary conditions

$$\hat{p}_1(x,y;t)=\hat{p}_2(x,y;t),\ \forall\, x\,,\ y=0,\ \forall\, t$$

$$\hat{\vec{v}}_1(x,y;t)\cdot\vec{n}=\hat{\vec{v}}_2(x,y;t)\cdot\vec{n},\ \forall\, x\,,\ y=0,\ \forall\, t$$

i.e. $\frac{1}{\rho_1}\frac{\partial\hat{p}_1}{\partial y}(x,y;t)=\frac{1}{\rho_2}\frac{\partial\hat{p}_2}{\partial y}(x,y;t),\ \forall\, x\,,\ y=0,\ \forall\, t$

- Driven incident field
  $\hat{f}_1\left(t-\vec{n}_a\cdot\vec{r}/c_1\right)$
- Radiation conditions at infinity

- Medium 1: solution

$$\hat{p}_1(x,y;t)=\hat{f}_1\left(t-\frac{\vec{n}_a\cdot\vec{r}}{c_1}\right)+\hat{g}_1\left(t-\frac{\vec{n}_b\cdot\vec{r}}{c_1}\right)$$

- Medium 2: solution

$$\hat{p}_2(x,y;t)=\hat{f}_2\left(t-\frac{\vec{n}_2\cdot\vec{r}}{c_2}\right)$$

Reflection and transmission at the interface between two different fluid media (3/3)

- Snell-Descartes law
  - Boundary conditions $\quad\hat{p}_1(x,0;t)=\hat{p}_2(x,0;t),\ \underline{\forall\, x}\,,\ y=0,\ \underline{\forall\, t}$

⇒ factorization of a function of x and t

with $\hat{p}_1(x,y;t)=\hat{f}_1\left(t-\frac{\vec{n}_a\cdot\vec{r}}{c_1}\right)+\hat{g}_1\left(t-\frac{\vec{n}_b\cdot\vec{r}}{c_1}\right)$ *and* $\hat{p}_2(x,y;t)=\hat{f}_2\left(t-\frac{\vec{n}_2\cdot\vec{r}}{c_2}\right)$

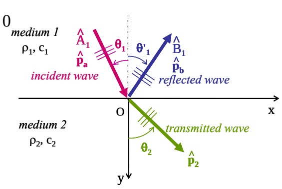
⇒ equal functions: $\hat{f}_1=\hat{g}_1=\hat{f}_2$
and equal arguments: $t-\frac{\vec{n}_a\cdot\vec{r}}{c_1}=t-\frac{\vec{n}_b\cdot\vec{r}}{c_1}=t-\frac{\vec{n}_2\cdot\vec{r}}{c_2}$, $\forall\, x\,,\ y=0,\ \forall\, t$

⇒ $\frac{n_{x_a}x}{c_1}=\frac{n_{x_b}x}{c_1}=\frac{n_{x_2}x}{c_2}$, $\underline{\forall\, x}$, $y=0$

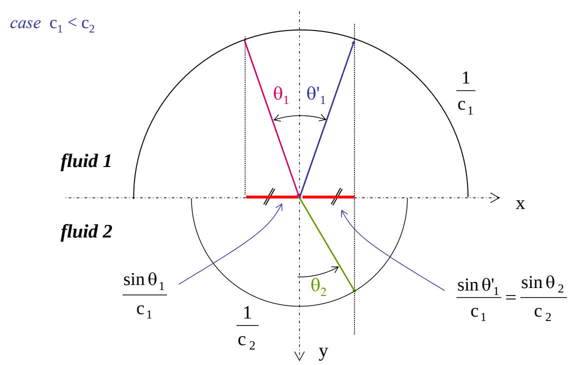
⇒ $\frac{n_{x_a}}{c_1}=\frac{n_{x_b}}{c_1}=\frac{n_{x_2}}{c_2}$

⇒ $\frac{\sin\theta_1}{c_1}=\frac{\sin\theta'_1}{c_1}=\frac{\sin\theta_2}{c_2}$

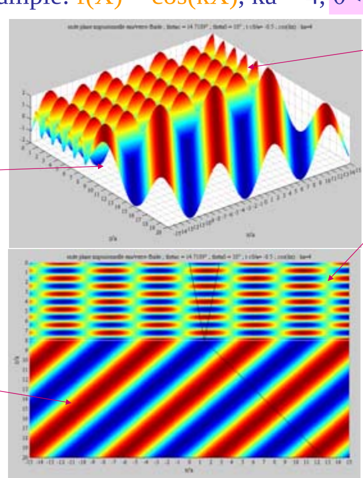
⇒ $\theta_1=\theta'_1$



Slowness surfaces



Example: f(X) = cos(kX), ka = 4, $\theta<\theta_c$



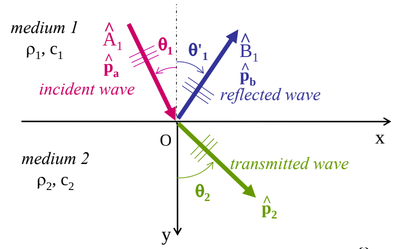
Reflection and transmission at the interface between two different fluid media - harmonic source (1/12)

$$\vec{k}_a=k_1\vec{n}_a=\frac{\omega}{c_1}\vec{n}_a=k_x\vec{e}_x+k_{y_1}\vec{e}_y$$

$$\hat{p}_a(x,y;t)=\hat{A}_1 e^{-i\left(\vec{k}_a\cdot\overrightarrow{OM}-\omega t\right)}$$

$$\vec{k}_b=k_1\vec{n}_b=\frac{\omega}{c_1}\vec{n}_b=k_x\vec{e}_x-k_{y_1}\vec{e}_y$$

$$\hat{p}_b(x,y;t)=\hat{B}_1 e^{-i\left(\vec{k}_b\cdot\overrightarrow{OM}-\omega t\right)}$$



$$\vec{k}_2=k_2\vec{n}_2=\frac{\omega}{c_2}\vec{n}_2=k_x\vec{e}_x+k_{y_2}\vec{e}_y$$

$$\hat{p}_2(x,y;t)=\hat{A}_2 e^{-i\left(\vec{k}_2\cdot\overrightarrow{OM}-\omega t\right)}$$

$\overrightarrow{OM}=x\vec{e}_x+y\vec{e}_y$

angular frequency ω

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Reflection and transmission at the interface between two different fluid media - harmonic source (2/12)

- $\hat{p}_1(x,y;t) = \hat{p}_a(x,y;t) + \hat{p}_b(x,y;t) = \hat{A}_1 e^{-i(k_x x + k_{y_1} y - \omega t)} + \hat{B}_1 e^{-i(k_x x - k_{y_1} y - \omega t)}$

  $\Rightarrow \dfrac{\partial \hat{p}_1}{\partial y}(x,y;t) = i k_{y_1}\left[-\hat{A}_1 e^{-i k_{y_1} y} + \hat{B}_1 e^{i k_{y_1} y}\right] e^{-i(k_x x - \omega t)}$

- $\hat{p}_2(x,y;t) = \hat{A}_2 e^{-i(k_x x + k_{y_2} y - \omega t)} \Rightarrow \dfrac{\partial \hat{p}_2}{\partial y}(x,y;t) = -i k_{y_2} \hat{A}_2 e^{-i k_{y_2} y} e^{-i(k_x x - \omega t)}$

- **Equality of pressures at y = 0**

  $\Rightarrow \hat{A}_1 e^{-i(k_x x - \omega t)} + \hat{B}_1 e^{-i(k_x x - \omega t)} = \hat{A}_2 e^{-i(k_x x - \omega t)}, \ \forall\, x,\ \forall\, t$

  $\Rightarrow \hat{A}_1 + \hat{B}_1 = \hat{A}_2 \Rightarrow -\hat{\mathscr{R}}_p + \hat{\mathscr{T}}_p = 1$ *with* $\hat{\mathscr{R}}_p = \hat{B}_1/\hat{A}_1$ *and* $\hat{\mathscr{T}}_p = \hat{A}_2/\hat{A}_1$

- **Equality of normal velocities at y = 0**

  $\Rightarrow \dfrac{i k_{y_1}}{\rho_1}\left(-\hat{A}_1 + \hat{B}_1\right) e^{-i(k_x x - \omega t)} = \dfrac{-i k_{y_2}}{\rho_2} \hat{A}_2 e^{-i(k_x x - \omega t)}, \ \forall\, x,\ y = 0,\ \forall\, t$

  $\Rightarrow \dfrac{k_{y_1}}{\rho_1}\hat{\mathscr{R}}_p + \dfrac{k_{y_2}}{\rho_2}\hat{\mathscr{T}}_p = \dfrac{k_{y_1}}{\rho_1}$

Reflection and transmission at the interface between two different fluid media - harmonic source (3/12)

- Reflection and transmission coefficients (pressure amplitude)

$$\begin{cases} -\hat{\mathscr{R}}_p + \hat{\mathscr{T}}_p = 1 \\ \dfrac{k_{y_1}}{\rho_1}\hat{\mathscr{R}}_p + \dfrac{k_{y_2}}{\rho_2}\hat{\mathscr{T}}_p = \dfrac{k_{y_1}}{\rho_1} \end{cases} \Rightarrow \begin{cases} \hat{\mathscr{R}}_p = \dfrac{-k_{y_2}/\rho_2 + k_{y_1}/\rho_1}{k_{y_2}/\rho_2 + k_{y_1}/\rho_1} \\ \hat{\mathscr{T}}_p = \dfrac{2 k_{y_1}/\rho_1}{k_{y_2}/\rho_2 + k_{y_1}/\rho_1} \end{cases}$$

or

$$\begin{cases} \hat{\mathscr{R}}_p = \dfrac{-\cos\theta_2/Z_2 + \cos\theta_1/Z_1}{\cos\theta_2/Z_2 + \cos\theta_1/Z_1} \\ \hat{\mathscr{T}}_p = \dfrac{2\cos\theta_1/Z_1}{\cos\theta_2/Z_2 + \cos\theta_1/Z_1} \end{cases}$$

or

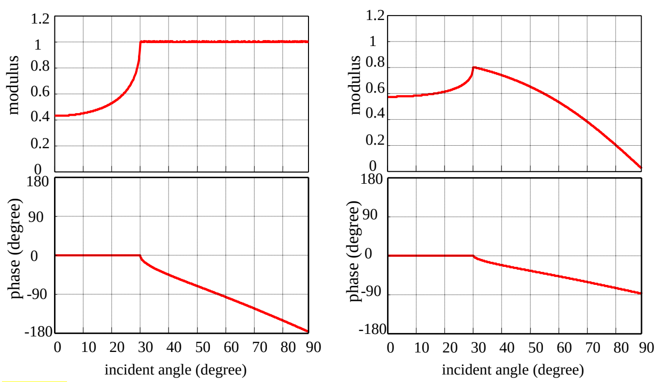
$$\begin{cases} \hat{\mathscr{R}}_p = \dfrac{Z_2/\cos\theta_2 - Z_1/\cos\theta_1}{Z_1/\cos\theta_1 + Z_2/\cos\theta_2} \\ \hat{\mathscr{T}}_p = \dfrac{2 Z_2/\cos\theta_2}{Z_1/\cos\theta_1 + Z_2/\cos\theta_2} \end{cases}$$

with $Z_1 = \rho_1 c_1$; $Z_2 = \rho_2 c_2$ *characteristic impedances*

$k_{y_1} = k_1 \cos\theta_1$; $k_{y_2} = k_2 \cos\theta_2$

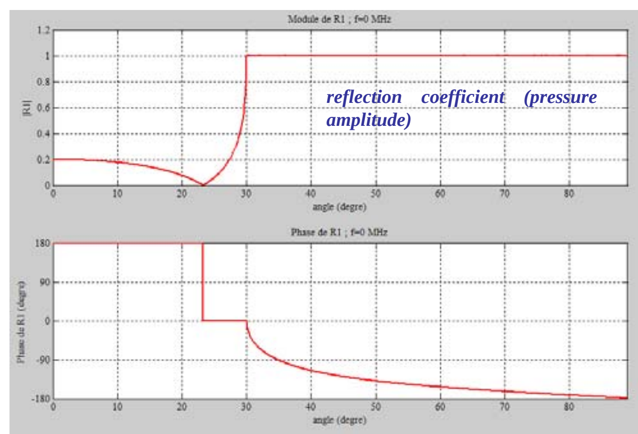
Example: ρ_1=2000 kg/m³, c_1=750 m/s, ρ_2=2500 kg/m³, c_2=1500 m/s

reflection coefficient (pressure amplitude) — *transmission coefficient (pressure amplitude)*



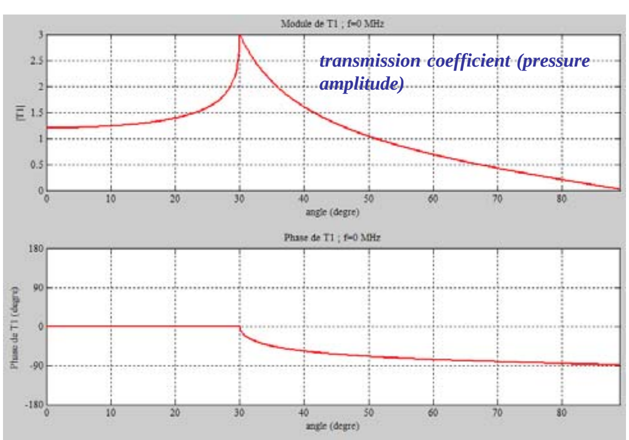
$Z_1 < Z_2$ — *critical angle: 30°*

Example: ρ_1=3000 kg/m³, c_1=750 m/s, ρ_2=1000 kg/m³, c_2=1500 m/s



$Z_1 > Z_2$ — *critical angle: 30°*

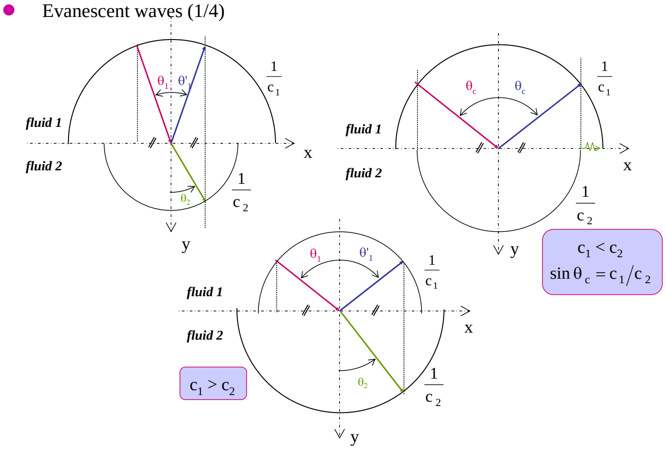
Example: ρ_1=3000 kg/m³, c_1=750 m/s, ρ_2=1000 kg/m³, c_2=1500 m/s



$Z_1 > Z_2$ — *critical angle: 30°*

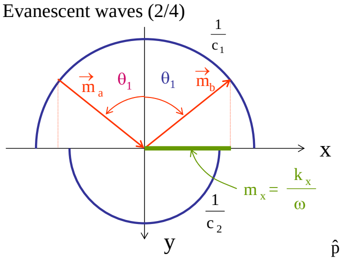
Reflection and transmission at the interface between two different fluid media - harmonic source (4/12)

- Evanescent waves (1/4)



Reflection and transmission at the interface between two different fluid media - harmonic source (5/12)

- Evanescent waves (2/4)



slowness vector: $\vec{m} = \frac{\vec{k}}{\omega}$

$$m_x = \frac{k_x}{\omega} > \frac{1}{c_2} = \frac{k_2}{\omega} \Rightarrow k_x > k_2$$

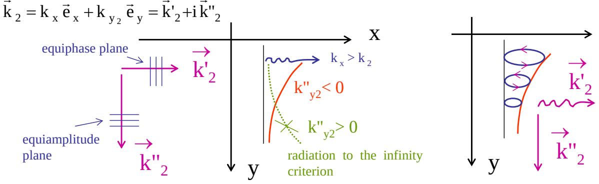
Dispersion relation:

$$k_x^2 + k_{y_2}^2 = k_2^2 \Rightarrow k_{y_2}^2 = k_2^2 - k_x^2 < 0$$

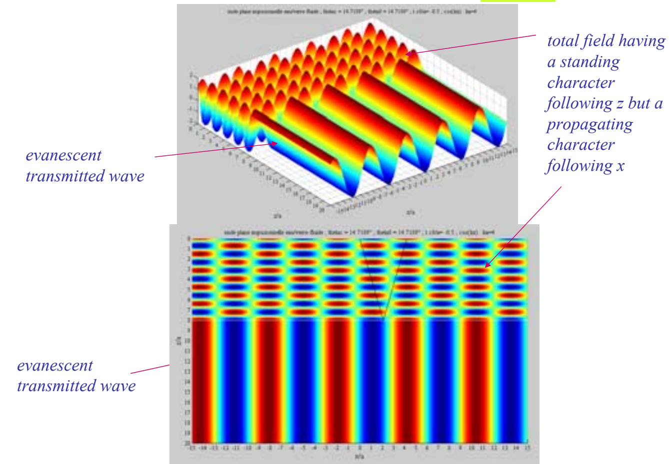
$$\Rightarrow \boxed{k_{y2} = i\,k''_{y2}} \quad k''_{y_2} = -\sqrt{k_x^2 - k_2^2}$$

$$\hat{p}_2(x,y;t) = \hat{A}_1 \hat{\mathcal{T}}_p e^{k''_{y_2} y} e^{-i(k_x x - \omega t)}$$

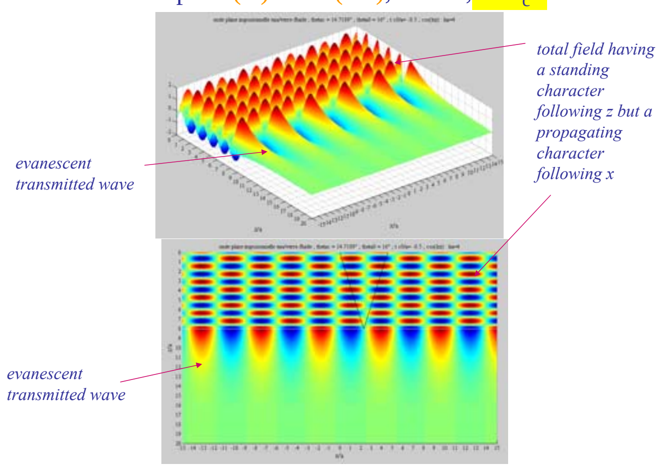
$$\vec{k}_2 = k_x \vec{e}_x + k_{y_2} \vec{e}_y = \vec{k}'_2 + i\vec{k}''_2$$



Example: f(X) = cos(kX), ka = 4, θ=θ$_c$



Example: f(X) = cos(kX), ka = 4, θ>θ$_c$



Reflection and transmission at the interface between two different fluid media - harmonic source (6/12)

- Evanescent waves (3/4)

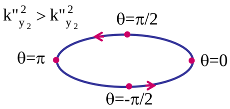
$$\hat{\vec{V}}_2 = \hat{V}_2 \vec{n}_2 = \hat{V}_2\left[(k_x/k_2)\vec{e}_x + \underbrace{(k_{y_2}}_{i\,k''_{y_2}}/k_2)\vec{e}_y\right] = \underbrace{\hat{V}_2}_{|\hat{V}_2|\exp(i\alpha)}\left[(k_x/k_2)\vec{e}_x + i(k''_{y_2}/k_2)\vec{e}_y\right]$$

$$\Rightarrow \hat{\vec{v}}_2(x,y;t) = |\hat{V}_2|\left[(k_x/k_2)\vec{e}_x + i(k''_{y_2}/k_2)\vec{e}_y\right]\exp(k''_{y_2} y)\exp i(-k_x x + \omega t + \alpha)$$

$$\Rightarrow \begin{cases} v_{x_2} = \mathrm{Re}(\hat{v}_{x_2}) = |\hat{V}_2|(k_x/k_2)\exp(k''_{y_2} y)\cos(-k_x x + \omega t + \alpha) \\ v_{y_2} = \mathrm{Re}(\hat{v}_{y_2}) = -|\hat{V}_2|(k''_{y_2}/k_2)\exp(k''_{y_2} y)\sin(-k_x x + \omega t + \alpha) \end{cases}$$

$$\Rightarrow \boxed{\left(\frac{v_{x_2}}{|\hat{V}_2|(k_x/k_2)\exp(k''_{y_2} y)}\right)^2 + \left(\frac{v_{y_2}}{|\hat{V}_2|(k''_{y_2}/k_2)\exp(k''_{y_2} y)}\right)^2 = 1}$$

$$k_x^2 = k_2^2 + k''^2_{y_2} > k''^2_{y_2}$$



$\theta = -k_x x + \omega t + \alpha$

$\theta = 0 \Rightarrow v_{x_2} > 0 \; ; \; v_{y_2} = 0$

$\theta = \pi/2 \Rightarrow v_{x_2} = 0 \; ; \; v_{y_2} > 0$ $\quad (k''_{y_2} < 0)$

Reflection and transmission at the interface between two different fluid media - harmonic source (7/12)

- Evanescent waves (4/4)

reflection coefficient: $\hat{\mathcal{R}}_p = \frac{-k_{y_2}/\rho_2 + k_{y_1}/\rho_1}{k_{y_2}/\rho_2 + k_{y_1}/\rho_1}$ *with* $k_{y_2} = i\,k''_{y_2}$

$$\Rightarrow \hat{\mathcal{R}}_p = \frac{-i\,k''_{y_2}/\rho_2 + k_{y_1}/\rho_1}{i\,k''_{y_2}/\rho_2 + k_{y_1}/\rho_1} \Rightarrow \boxed{|\hat{\mathcal{R}}_p| = 1}$$

total reflection in medium 1

BUT presence of acoustic energy in the transmission medium 2.

Evanescent wave ≡ Accompanying wave in medium 2, which accompanies the total reflection phenomenon in the medium 1, and which has it own energy..

Monochromatic problem ≡ ***steady state*** problem: during the transient state, the energy incoming in the medium 2 is permanently located in this medium, during an infinite time.

Thus, in reality, the monochromatic problem is an asymptotic situation, for a very high time t, which models too strongly the physical reality, which implies the existence of an evanescent wave, which infinite extension in the direction of the interface..

Reflection and transmission at the interface between two different fluid media - harmonic source (8/12)

- **Incident energy flux**

$$\hat{p}_a(x,y;t) = \hat{A}_1 e^{-ik_{y_1}y} e^{-i(k_x x - \omega t)} \Rightarrow \hat{v}_{y_a}(x,y;t) = \frac{i}{\rho_1 \omega}\frac{\partial \hat{p}_a(x,y;t)}{\partial y} = \frac{k_{y_1}}{\rho_1 \omega}\hat{A}_1 e^{-ik_{y_1}y} e^{-i(k_x x - \omega t)}$$

$$\boxed{I_{y_a} = \frac{1}{4}\left(\hat{p}_a^* \hat{v}_{y_a} + \hat{p}_a \hat{v}_{y_a}^*\right) = \frac{1}{2}|\hat{A}_1|^2 \frac{k_{y_1}}{\rho_1 \omega}}$$

- **Reflected energy flux**

$$\hat{p}_b(x,y;t) = \hat{A}_1 \hat{\mathcal{R}}_p e^{ik_{y_1}y} e^{-i(k_x x - \omega t)} \Rightarrow \hat{v}_{y_b}(x,y;t) = \frac{i}{\rho_1 \omega}\frac{\partial \hat{p}_b(x,y;t)}{\partial y} = \frac{-k_{y_1}}{\rho_1 \omega}\hat{A}_1 \hat{\mathcal{R}}_p e^{ik_{y_1}y} e^{-i(k_x x - \omega t)}$$

$$I_{y_b} = \frac{1}{4}\left(\hat{p}_b^* \hat{v}_{y_b} + \hat{p}_b \hat{v}_{y_b}^*\right) = -\frac{1}{2}|\hat{A}_1|^2 |\hat{\mathcal{R}}_p|^2 \frac{k_{y_1}}{\rho_1 \omega} \Rightarrow \boxed{\mathcal{R}^E = (-I_{y_b})/I_{y_a} = |\hat{\mathcal{R}}_p|^2}$$

reflection coefficient (energy)

- **Transmitted energy flux**

$$\hat{p}_2(x,y;t) = \hat{A}_1 \hat{\mathcal{T}}_p e^{-i\hat{k}_{y_2}y} e^{-i(k_x x - \omega t)} \Rightarrow \hat{v}_{y_2}(x,y;t) = \frac{i}{\rho_2 \omega}\frac{\partial \hat{p}_2(x,y;t)}{\partial y} = \frac{\hat{k}_{y_2}}{\rho_2 \omega}\hat{A}_1 \hat{\mathcal{T}}_p e^{-i\hat{k}_{y_2}y} e^{-i(k_x x - \omega t)}$$

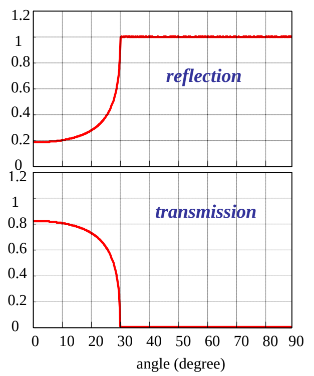
$$I_{y_2} = \frac{1}{4}\left(\hat{p}_2^* \hat{v}_{y_2} + \hat{p}_2 \hat{v}_{y_2}^*\right) = \frac{1}{4}|\hat{A}_1|^2 |\hat{\mathcal{T}}_p|^2 \frac{1}{\rho_2 \omega}\left(\hat{k}_{y_2} + \hat{k}_{y_2}^*\right) e^{-i(\hat{k}_{y_2} - \hat{k}_{y_2}^*)y}$$

$$\Rightarrow \boxed{\mathcal{T}^E = I_{y_2}/I_{y_a} = \frac{Z_1/\cos\theta_1}{Z_2/\cos\theta_2}|\hat{\mathcal{T}}_p|^2} \;\; \theta < \theta_c \qquad \boxed{\mathcal{T}^E = I_{y_2}/I_{y_a} = 0} \;\; \theta > \theta_c$$

transmission coefficient (energy)

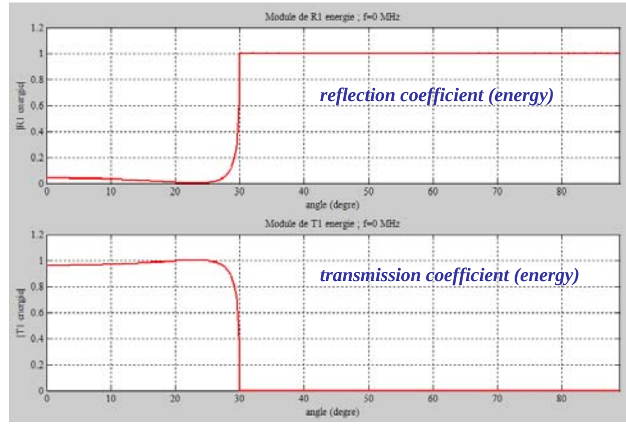
Example: ρ_1=2000 kg/m^3, c_1=750 m/s, ρ_2=2500 kg/m^3, c_2=1500 m/s

coefficients (energy)



$Z_1 < Z_2$ *critical angle: 30°*

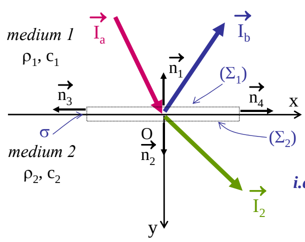
Example: ρ_1=3000 kg/m^3, c_1=750 m/s, ρ_2=1000 kg/m^3, c_2=1500 m/s



$Z_1 > Z_2$ *critical angle: 30°*

Reflection and transmission at the interface between two different fluid media - harmonic source (9/12)

- Conservation of energy



$\mathrm{div}\,\vec{I}=0 \Rightarrow \Phi_{tot}=\iint_S \vec{I}\cdot\vec{n}\,dS=0$

with (S) = $\Sigma_1+\Sigma_2+\sigma$ ($\sigma \to \infty$, $\to 0$)

i.e. $\iint_{\Sigma_1}\vec{I}_1\cdot\vec{n}_1\,d\Sigma_1+\iint_{\Sigma_2}\vec{I}_2\cdot\vec{n}_2\,d\Sigma_2=0$

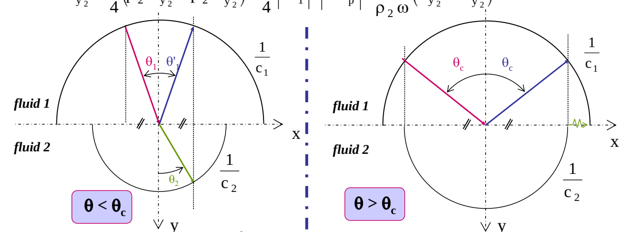
i.e. $\iint_{\Sigma_1}-\left(I_{y_a}+I_{y_b}\right)d\Sigma_1+\iint_{\Sigma_2}I_{y_2}\,d\Sigma_2=0$

$\Rightarrow -I_{y_a}-I_{y_b}+I_{y_2}=0 \Rightarrow \left(-I_{y_b}\right)+I_{y_2}=I_{y_a}$

Reflection and transmission at the interface between two different fluid media - harmonic source (11/12)

- Transmitted energy flux

$I_{y_2}=\frac{1}{4}\left(\hat{p}_2^*\hat{v}_{y_2}+\hat{p}_2\hat{v}_{y_2}^*\right)=\frac{1}{4}\left|\hat{A}_1\right|^2\left|\mathscr{T}_p\right|^2\frac{1}{\rho_2\omega}\left(\hat{k}_{y_2}+\hat{k}_{y_2}^*\right)e^{-i\left(\hat{k}_{y_2}-\hat{k}_{y_2}^*\right)y}$



$\theta < \theta_c$

$I_{y_2}=\frac{1}{2}\left|\hat{A}_1\right|^2\left|\mathscr{T}_p\right|^2\frac{\cos\theta_2}{Z_2}$

$\Rightarrow$ *conservation of energy:*

$1=\left|\mathscr{R}_p\right|^2+\left|\mathscr{T}_p\right|^2\frac{\cos\theta_2}{\cos\theta_1}\frac{k_2}{k_1}\frac{\rho_1}{\rho_2}$

or

$1=\left|\mathscr{R}_p\right|^2+\frac{Z_1/\cos\theta_1}{Z_2/\cos\theta_2}\left|\mathscr{T}_p\right|^2$

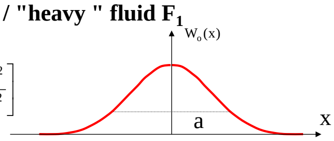
$\theta > \theta_c$

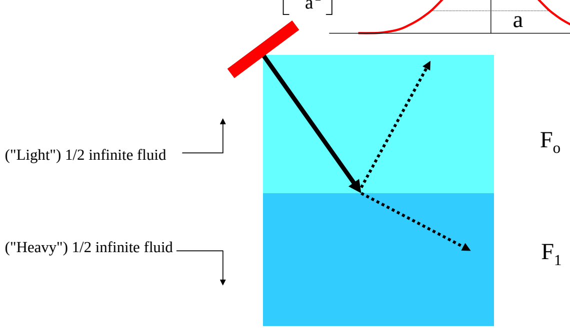
$I_{y_2}=0$

Gaussian beam incident onto an interface

"light" fluid F_o / "heavy " fluid F_1

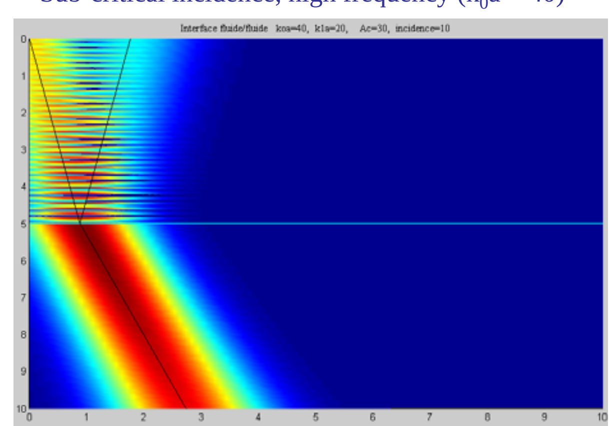
Gaussian emitter: $W_o(x)=W_o\exp\left[-\frac{x^2}{a^2}\right]$





Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

Sub-critical incidence, high frequency (k_0a = 40)



$\rho_0/\rho_1=0{,}5$

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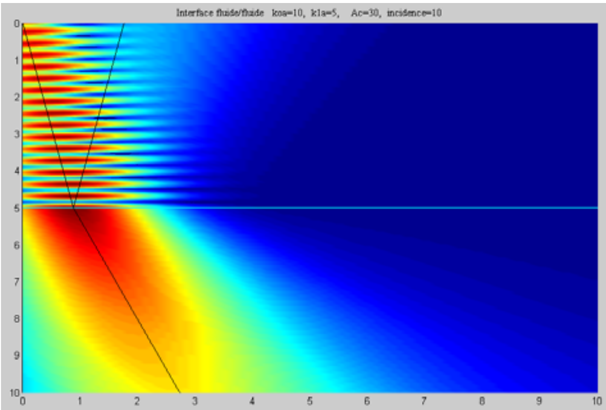
Sub-critical incidence, low frequency (k_0a = 10)



$\rho_0/\rho_1 = 0{,}5$

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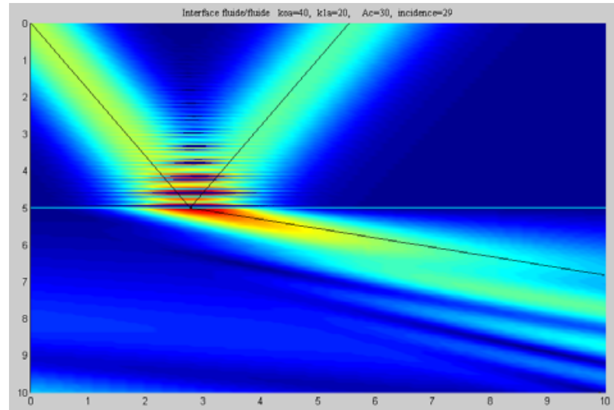
Almost critical incidence, high frequency (k_0a = 40)



$\rho_0/\rho_1 = 0{,}5$

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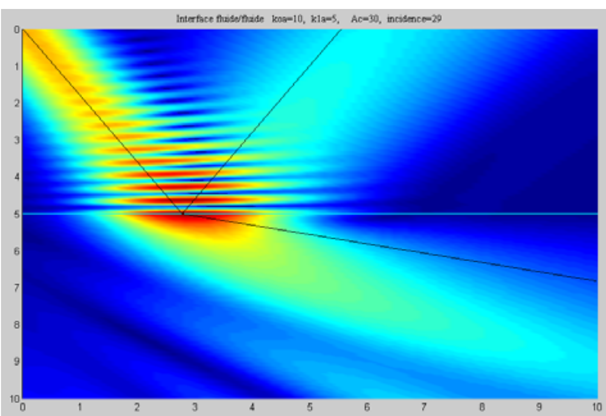
Almost critical incidence, low frequency (k_0a = 10)



$\rho_0/\rho_1 = 0{,}5$

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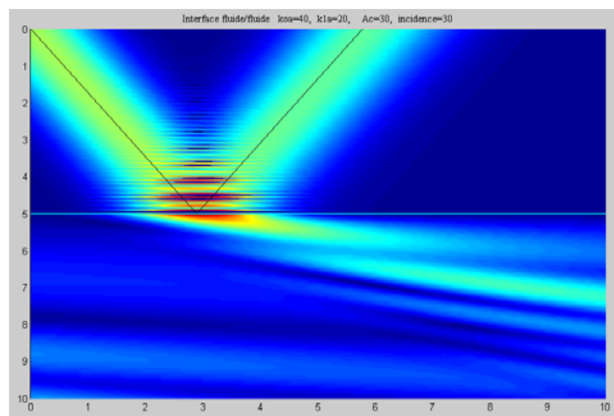
Critical incidence, high frequency (k_0a = 40)



$\rho_0/\rho_1 = 0{,}5$

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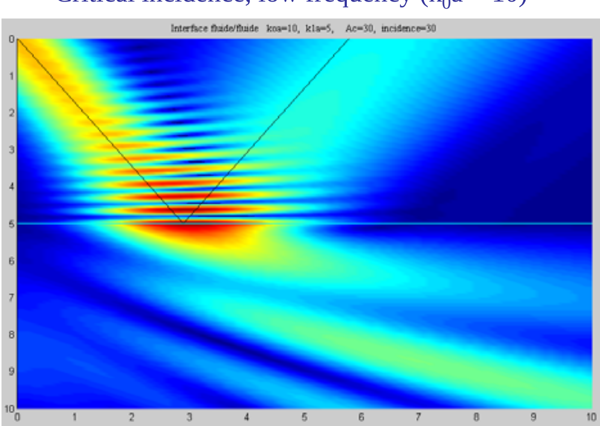
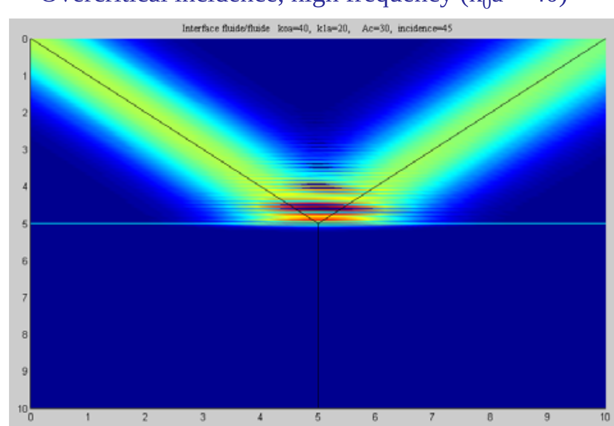
Critical incidence, low frequency (k_0a = 10)



$\rho_0/\rho_1 = 0{,}5$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

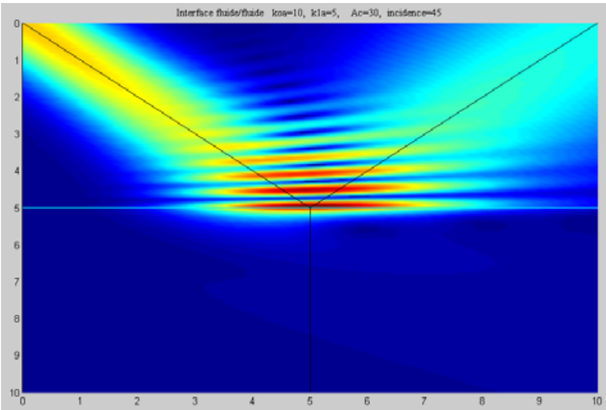
Overcritical incidence, high frequency (k_0a = 40)



$\rho_0/\rho_1 = 0{,}5$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

Overcritical incidence, low frequency ($k_0a = 10$)



$\rho_0/\rho_1 = 0{,}5$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

Reflection and transmission at the interface between two different fluid media - harmonic source (12/12)

- Interface air - water

$Z_1 << Z_2$ ➡

$$\begin{cases} \hat{\mathscr{R}}_p = \dfrac{-\cos\theta_2/Z_2 + \cos\theta_1/Z_1}{\cos\theta_2/Z_2 + \cos\theta_1/Z_1} \Rightarrow \hat{\mathscr{R}}_p \approx 1 \\ \hat{\mathscr{T}}_p = \dfrac{2\cos\theta_1/Z_1}{\cos\theta_2/Z_2 + \cos\theta_1/Z_1} \Rightarrow \hat{\mathscr{T}}_p \approx 2 \end{cases}$$

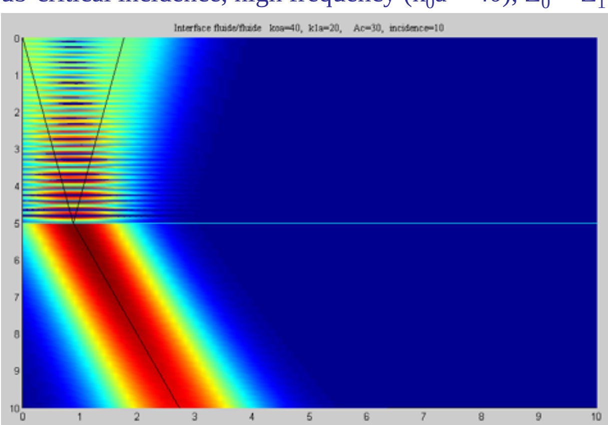
But $\mathscr{T}^E = I_2/I_{y_a} \approx 0$

- Interface water - air

$Z_1 >> Z_2$ ➡

$$\begin{cases} \hat{\mathscr{R}}_p = \dfrac{Z_2/\cos\theta_2 - Z_1/\cos\theta_1}{Z_1/\cos\theta_1 + Z_2/\cos\theta_2} \Rightarrow \hat{\mathscr{R}}_p \approx -1 \\ \hat{\mathscr{T}}_p = \dfrac{2Z_2/\cos\theta_2}{Z_1/\cos\theta_1 + Z_2/\cos\theta_2} \Rightarrow \hat{\mathscr{T}}_p \approx 0 \end{cases}$$

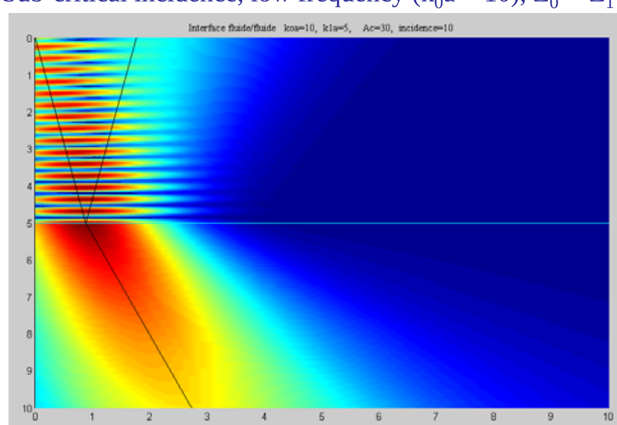
Sub-critical incidence, high frequency ($k_0a = 40$), $Z_0<<Z_1$



$\rho_0/\rho_1 = 0{,}00001$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

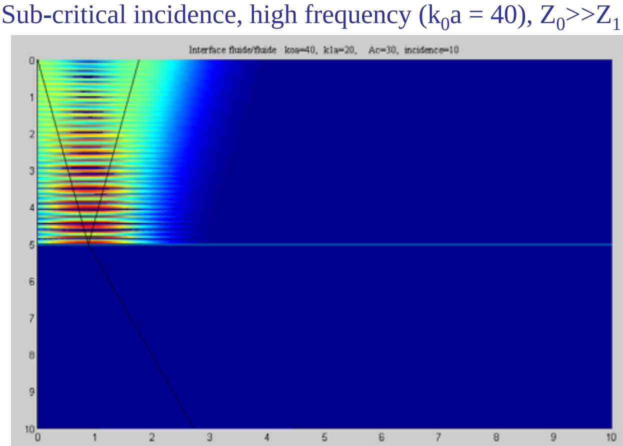
Sub-critical incidence, low frequency ($k_0a = 10$), $Z_0<<Z_1$



$\rho_0/\rho_1 = 0{,}00001$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

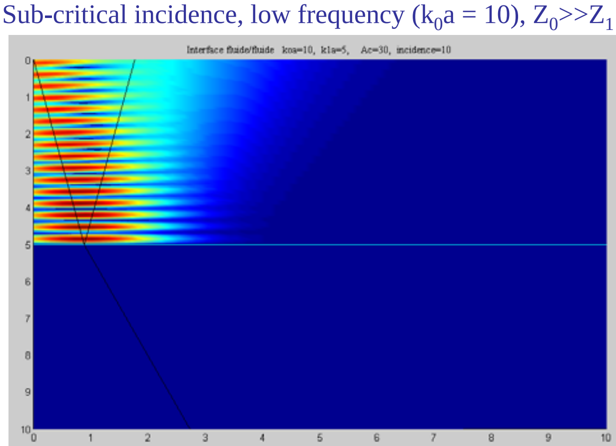
Sub-critical incidence, high frequency ($k_0a = 40$), $Z_0>>Z_1$



$\rho_0/\rho_1 = 500$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

Sub-critical incidence, low frequency ($k_0a = 10$), $Z_0>>Z_1$

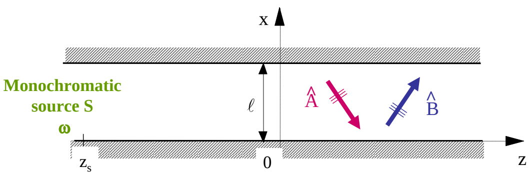


$\rho_0/\rho_1 = 500$

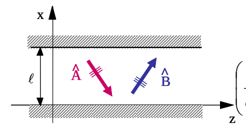
Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

C. Potel, M. Bruneau, "Acoustique générale (...)", Ed. Ellipse, collection Technosup, ISBN 2-7298-2805-2, 2006

Bidimensionnel waveguide (1/8)



Bidimensionnel waveguide (2/8)



- Propagation equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}(x,z;t)=0,\ \forall x\in[0,\ell],\ \forall z\geq 0,\ \forall t$$

- Given incident monochromatic field (not specified) $\hat{p}(x,z;t)=\hat{P}(x,z)e^{i\omega t}$

- Helmholtz equation

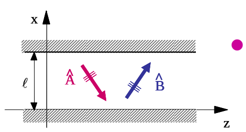
$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+\left(\frac{\omega}{c_0}\right)^2\right]\hat{P}(x,z)=0,\ \forall x\in[0,\ell],\ \forall z\geq 0$$

- Boundary conditions $\dfrac{\partial\hat{p}(x,z;t)}{\partial n}=0,\ \forall z\geq 0,\ \forall t$ *at* $x=0$ *and* $x=\ell$

$$\Rightarrow\ \frac{\partial\hat{P}(x,z)}{\partial x}=0,\ \forall z\geq 0,\ \forall t,\ \text{at } x=0 \text{ and } x=\ell$$

- Field propagating in the direction of increasing z (no return wave)

Bidimensionnel waveguide (3/8)



- **Form of the field**

$$\hat{p}(x,z;t)=\left[\hat{A}\,e^{-i(-k_x x+k_z z)}+\hat{B}\,e^{-i(k_x x+k_z z)}\right]e^{i\omega t}$$

- **Dispersion equation** $k_x^2+k_z^2=k_0^2$ *with* $k_0=\omega/c_0$

- **Boundary conditions** $\dfrac{\partial\hat{P}(x,z)}{\partial x}=0,\ \forall z\geq 0,\ \forall t,\ \text{at } x=0 \text{ and } x=\ell$

with $$\frac{\partial\hat{p}(x,z;t)}{\partial x}=i k_x\left[\hat{A}\,e^{-i(-k_x x+k_z z)}-\hat{B}\,e^{-i(k_x x+k_z z)}\right]e^{i\omega t}$$

$$\begin{cases} i k_x(\hat{A}-\hat{B})e^{-ik_z z}e^{i\omega t}, \text{ en } x=0,\ \forall z\geq 0,\ \forall t \\ i k_x\left(\hat{A}\,e^{ik_x\ell}-\hat{B}\,e^{-ik_x\ell}\right)e^{-ik_z z}e^{i\omega t}, \text{ en } x=\ell,\ \forall z\geq 0,\ \forall t \end{cases}$$

$$\Rightarrow\ \begin{cases}\hat{A}-\hat{B}=0\\ \hat{A}\,e^{ik_x\ell}-\hat{B}\,e^{-ik_x\ell}=0\end{cases}\ \Rightarrow\ \begin{cases}\hat{A}=\hat{B}\\ \hat{A}\underbrace{\left(e^{ik_x\ell}-e^{-ik_x\ell}\right)}_{2i\sin(k_x\ell)}=0\end{cases}\ \Rightarrow\ k_x\ell=m\pi,\ m\in\mathbb{N}$$

Bidimensionnel waveguide (4/8)

- **Solutions of the problem**

$\hat{A}_m=0,\ \forall m$ ***EXCEPT IF*** k_x takes a series of eigenvalues $k_{x_m}=\dfrac{m\pi}{\ell},\ m\in\mathbb{N}$

to which a series of wave numbers k_{z_m} is associated (dispersion equation) such that

$$k_{z_m}^2=k_0^2-k_{x_m}^2 \quad \textit{i.e.} \quad k_{z_m}^2=\left(\frac{\omega}{c_0}\right)^2-\left(\frac{m\pi}{\ell}\right)^2$$

depends on m — *does not depend on* m — *depends on* m

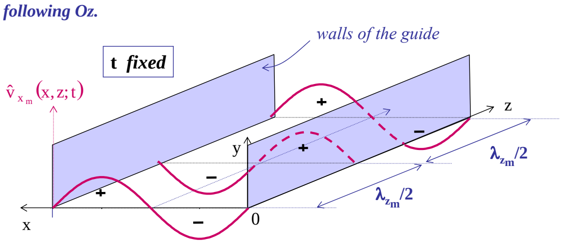
$\Rightarrow$ 2 cases : $k_{z_m}^2>0$ *or* $k_{z_m}^2<0$

Bidimensionnel waveguide (5/8)

- **Pressure associated to each mode m**

$$\hat{p}_m(x,z;t)=\hat{A}_m\left(e^{ik_{x_m}x}+e^{-ik_{x_m}x}\right)e^{-ik_{z_m}z}e^{i\omega t}\ \Rightarrow\ \hat{p}_m(x,z;t)=2\hat{A}_m\cos\left(k_{x_m}x\right)e^{-ik_{z_m}z}e^{i\omega t}$$

standing wave field having a modal character following Ox, and which is moved parallel to Ox, following Oz.



- **Total field** $$\hat{p}(x,z;t)=\sum_{m=0}^{\infty}\hat{p}_m(x,z;t)=2\left(\sum_{m=0}^{\infty}\hat{A}_m\cos\left(k_{x_m}x\right)e^{-ik_{z_m}z}\right)e^{i\omega t}$$

Bidimensionnel waveguide (6/8)

- **Velocity associated to mode m**

$$\hat{\vec{v}}_m(x,z;t)=\frac{i}{\rho_0\omega}\overrightarrow{\text{grad}}\,\hat{p}_m(x,z;t)\ \Rightarrow\ \begin{cases}\hat{v}_{x_m}(x,z;t)=-\dfrac{2ik_{x_m}}{\rho_0\omega}\hat{A}_m\sin\left(k_{x_m}x\right)e^{-ik_{z_m}z}e^{i\omega t}\\ \hat{v}_{z_m}(x,z;t)=\dfrac{2k_{z_m}}{\rho_0\omega}\hat{A}_m\cos\left(k_{x_m}x\right)e^{-ik_{z_m}z}e^{i\omega t}\end{cases}$$

- **Total velocity field**

$$\hat{v}_x(x,z;t)=\sum_{m=0}^{\infty}\hat{v}_{x_m}(x,z;t)=-\frac{2i}{\rho_0\omega}\left(\sum_{m=0}^{\infty}\hat{A}_m k_{x_m}\sin\left(k_{x_m}x\right)e^{-ik_{z_m}z}\right)e^{i\omega t}$$

$$\hat{v}_z(x,z;t)=\sum_{m=0}^{\infty}\hat{v}_{z_m}(x,z;t)=\frac{2}{\rho_0\omega}\left(\sum_{m=0}^{\infty}\hat{A}_m k_{z_m}\cos\left(k_{x_m}x\right)e^{-ik_{z_m}z}\right)e^{i\omega t}$$

Bidimensionnel waveguide (7/8)

- **Propagating and evanescent modes**

$$k_{x_m} = \frac{m\pi}{\ell}, \; m \in \mathbb{N} \quad \textit{with} \quad k_{z_m}^2 = \left(\frac{\omega}{c_0}\right)^2 - \left(\frac{m\pi}{\ell}\right)^2$$

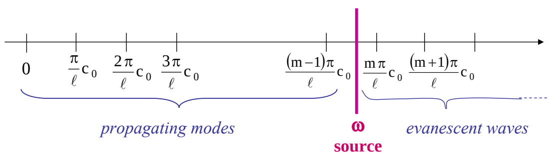
- modes m such that $k_{z_m}^2 > 0$ *i.e.* $\omega > \frac{m\pi}{\ell} c_0$ ➡ ***propagating*** modes m

and $p_m(x,z;t) = \mathrm{Re}\left[\hat{p}_m(x,z;t)\right] = 2\left|\hat{A}_m\right| \cos(k_{x_m} x) \cos(\omega t - k_{z_m} z + \alpha_m)$ *with* $\hat{A}_m = \left|\hat{A}_m\right| e^{i\alpha_m}$

- modes m such that $k_{z_m}^2 < 0$ *i.e.* $\omega < \frac{m\pi}{\ell} c_0$ ➡ ***evanescent*** modes m

and $p_m(x,z;t) = \mathrm{Re}\left[\hat{p}_m(x,z;t)\right] = 2\left|\hat{A}_m\right| e^{k''_{z_m} z} \cos(k_{x_m} x) \cos(\omega t + \alpha_m)$

with $k_{z_m} = i k''_{z_m} = -i\sqrt{(m\pi/\ell)^2 - (\omega/c_0)^2}$



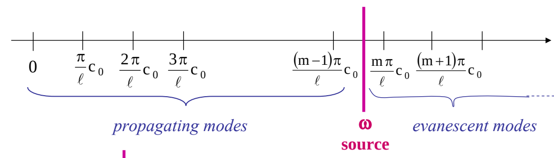
Bidimensionnel waveguide (8/8)

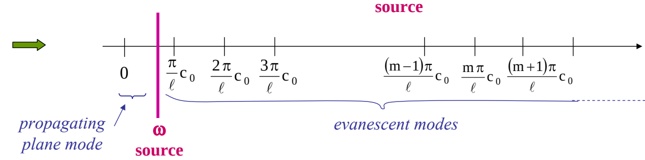
- **Plane mode**

$k_{x_{m=0}} = 0$

$k_{z_{m=0}} = \frac{\omega}{c_0}$ ➡ $p_{m=0}(x,z;t) = 2\left|\hat{A}_0\right| \cos(\omega t - k_{z_0} z + \alpha_0)$

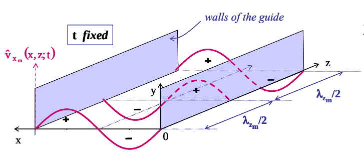
Propagating plane wave in the direction of increasing z





Phase velocity (propagating modes): 1/4

$$p_m(x,z;t) = \mathrm{Re}\left[\hat{p}_m(x,z;t)\right] = 2\left|\hat{A}_m\right| \cos(k_{x_m} x) \cos(\omega t - k_{z_m} z + \alpha_m)$$

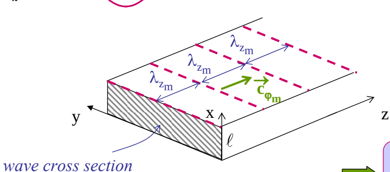


$$\lambda_{z_m} = 2\pi / k_{z_m} = 2\pi \Big/ \sqrt{(\omega/c_0)^2 - (m\pi/\ell)^2}$$

- displacement velocity of a particular observer for whom the wave appears to be fixed (observer linked to a "wave cross section") :

$$\omega t - k_{z_m} z + \alpha_m = \text{constante}$$

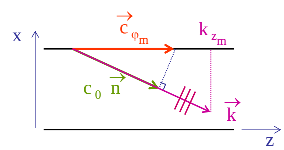
➡ $\frac{dz}{dt} = \frac{\omega}{k_{z_m}}$ (c_{φ_m})

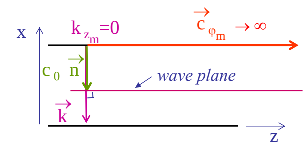


➡ $$c_{\varphi_m} = \frac{\omega}{k_{z_m}} = \omega \Big/ \sqrt{(\omega/c_0)^2 - (m\pi/\ell)^2}$$

Phase velocity (propagating modes): 2/4

$$c_{\varphi_m} = \frac{\omega}{k_{z_m}} = \omega \Big/ \sqrt{(\omega/c_0)^2 - (m\pi/\ell)^2}$$

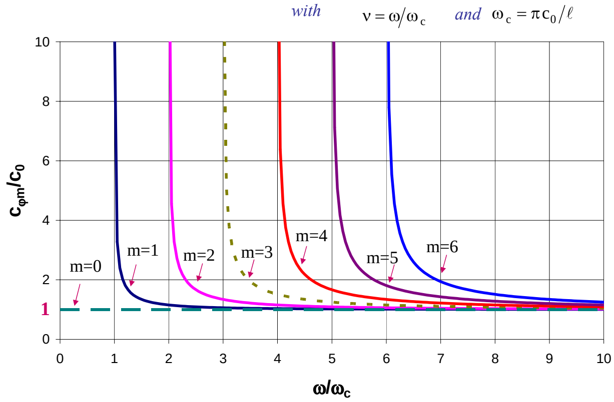




Phase velocity (propagating modes): 3/4

$c_{\varphi_m} = \frac{\omega}{k_{z_m}} = \omega \Big/ \sqrt{(\omega/c_0)^2 - (m\pi/\ell)^2}$ *or else* $c_{\varphi_m}/c_0 = \nu \Big/ \sqrt{\nu^2 - m^2}$

with $\nu = \omega/\omega_c$ *and* $\omega_c = \pi c_0/\ell$



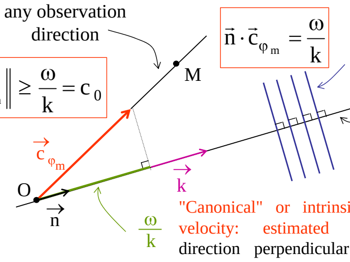
Phase velocity (propagating modes): 4/4

- Generalization

any observation direction

$\vec{n}\cdot\vec{c}_{\varphi_m} = \frac{\omega}{k}$

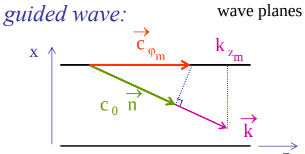
$\left\|\vec{c}_{\varphi_m}\right\| \geq \frac{\omega}{k} = c_0$



"Canonical" or intrinsic phase velocity: estimated in the direction perpendicular to the wave planes

$\vec{c}_{\varphi\,\text{cano}} = \frac{\omega}{k}\vec{n} = c_0\vec{n}$

guided wave:



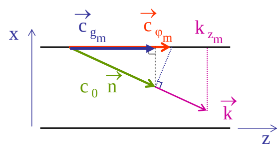
Phase velocity estimated parallel to the guide

$c_{\varphi_m} = \frac{\omega}{k_{z_m}}$

Group velocity (1/2)

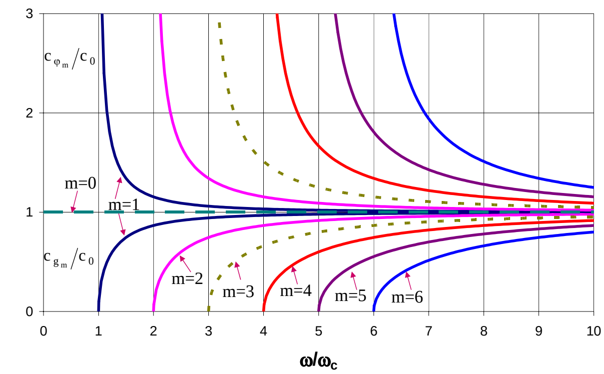
$$c_{g_m} = c_0\,\vec{n}\cdot\vec{e}_z = c_0\,\frac{k_{z_m}}{k} \;\Rightarrow\; c_{g_m} = \frac{c_0^2}{\omega}\sqrt{(\omega/c_0)^2-(m\pi/\ell)^2}$$

and $c_{g_m} = \dfrac{\partial\omega}{\partial k_{z_m}} = 1\Big/\left(\dfrac{\partial k_{z_m}}{\partial\omega}\right)$



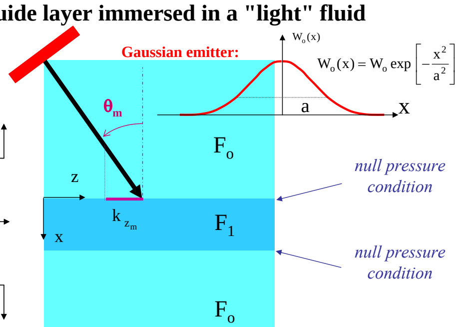
Group velocity (2/2)

$c_{g_m} = \dfrac{c_0^2}{\omega}\sqrt{(\omega/c_0)^2-(m\pi/\ell)^2}$ *or else* $c_{g_m}/c_0 = \sqrt{\nu^2-m^2}/\nu$ *with* $\nu=\omega/\omega_c$ *and* $\omega_c = \pi c_0/\ell$



Gaussian beam incident onto a fluid layer

("heavy") fluide layer immersed in a "light" fluid



1/2 infinite ("light") fluid

("Heavy") fluid layer

1/2 infinite ("light") fluid

null pressure condition

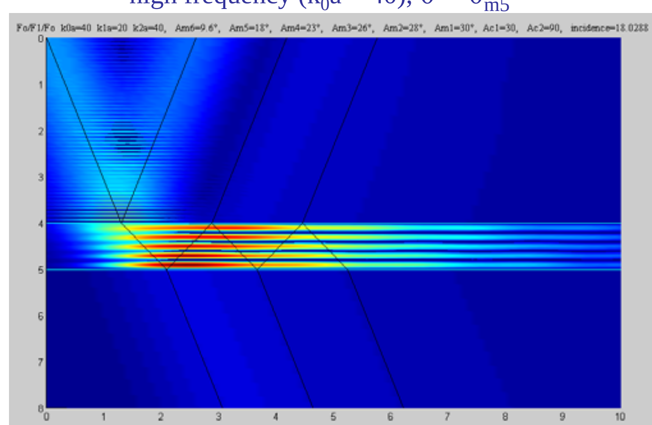
null pressure condition

$$k_{x_m} = \frac{m\pi}{\ell},\; m\in\mathbb{N} \;\Rightarrow\; k_{z_m} = \sqrt{\left(\frac{\omega}{c_1}\right)^2-\left(\frac{m\pi}{\ell}\right)^2} \;\Rightarrow\; \sin\theta_m = \frac{k_{z_m}}{k_0} = \frac{1}{k_0}\sqrt{\left(\frac{\omega}{c_0}\right)^2-\left(\frac{m\pi}{\ell}\right)^2}$$

the more m increases, the more θ_m decreases *propagating modes*

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

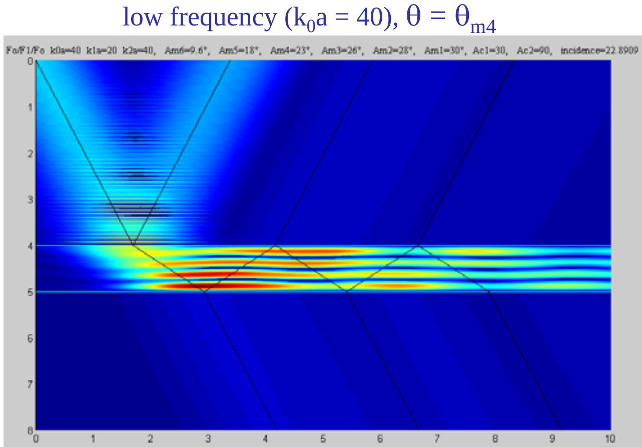
high frequency ($k_0a = 40$), $\theta = \theta_{m5}$



$\rho_0/\rho_1 = 0{,}2$ $\rho_2/\rho_1 = 0{,}2$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

low frequency ($k_0a = 40$), $\theta = \theta_{m4}$



$\rho_0/\rho_1 = 0{,}2$ $\rho_2/\rho_1 = 0{,}2$

Softwares realized by Ph. Gatignol, Pr., Université de Technologie de Compiègne - France

Horns (1/2)

- Phonographs

Edison Home Phonograph (1910)

http://perso.wanadoo.fr/jlf/phonos.htm

Gramophone N° 9 (1904)

http://perso.wanadoo.fr/jlf/phonos.htm

- Megaphones

Horns (2/2)

- Acoustic cornets

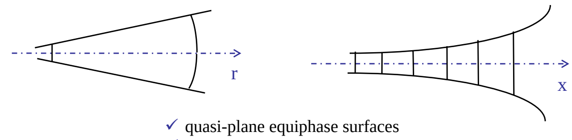
http://www.inrp.fr/she/instruments/lyc_bdb/acoustique.htm

Experiments made in 1826 on the Geneva Lake by the physicists Colladon and Sturm

http://www.beethoven-france.org/Beethoven/Luwig-van-Beethoven.html

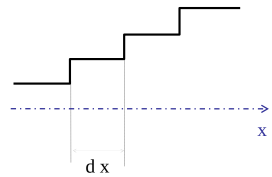
Propagation in horns, 1 parameter theory (1/6)

- Hypotheses



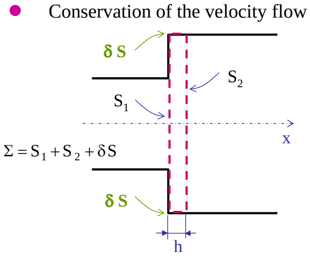
✓ quasi-plane equiphase surfaces
✓ slow variation of the radius as a function of x
✓ particle velocity orientated following x

- Series of discrete elementary cylindrical waveguide with perfectly rigid walls



Propagation in horns, 1 parameter theory (2/6)

- Conservation of the velocity flow at the discontinuity



✓ conservation of mass law:

$$\iiint_{\mathscr{V}} \frac{\partial\rho}{\partial t}\,d\mathscr{V} = -\iint_{\Sigma} \rho\,\vec{v}\cdot\vec{n}\,d\sigma$$

$$\approx h S_2 \frac{\partial\rho}{\partial t} \to 0 \quad \textit{with } h \to 0$$

$$\iint_{\Sigma} \rho\,\vec{v}\cdot\vec{n}\,d\sigma = \rho_0 \underbrace{\iint_{S_1} \vec{v}\cdot\vec{n}\,d\sigma}_{\approx -v_1 S_1} + \rho_0 \underbrace{\iint_{S_2} \vec{v}\cdot\vec{n}\,d\sigma}_{\approx +v_2 S_2} + \rho_0 \underbrace{\iint_{\delta S} \vec{v}\cdot\vec{n}\,d\sigma}_{\to 0}$$

$$\approx \rho_0\left(-v_1 S_1 + v_2 S_2\right)$$

$$\Rightarrow \iint_{\Sigma} \vec{v}\cdot\overrightarrow{dS} = 0 \quad \textit{i.e.} \quad \boxed{v_1 S_1 = v_2 S_2} \quad \textit{i.e.} \quad \boxed{v\,S = \text{constant}}$$

Propagation in horns, 1 parameter theory (3/6)

- Propagation equation

✓ Conservation of the velocity flow at the discontinuity

$$v\,S = \text{constant} \Rightarrow \frac{dv}{v} = -\frac{dS}{S} \quad \textit{i.e.} \quad \boxed{dv = -\frac{v}{S}\frac{dS}{dx}dx}$$

✓ Use of the velocity potential $\quad v = \dfrac{\partial\varphi}{\partial x}$

$$d\left(\frac{\partial\varphi}{\partial x}\right) = -\frac{dS/dx}{S}\left(\frac{\partial\varphi}{\partial x}\right)dx$$

Let $\varphi'_1 = \dfrac{\partial\varphi}{\partial x}$ *and* $S' = \dfrac{dS}{dx}$ $\Rightarrow$ $\boxed{d\varphi'_1 = -\dfrac{S'}{S}\varphi' dx}$

"responsibility" of the section change for the elementary variation of the derivative $d\varphi'_1$ *of the velocity potential*

✓ Between two changes of section

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} \quad \textit{i.e.} \quad \boxed{d\varphi'_2 = \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2}dx} \quad \textit{with} \quad \varphi'_2 = \frac{\partial\varphi}{\partial x}$$

Propagation in horns, 1 parameter theory (4/6)

- Propagation equation - follow-up

✓ Total variation $d\varphi'$ along the length dx

$$d\varphi' = d\varphi'_1 + d\varphi'_2 = \left[-\frac{S'}{S}\varphi' + \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2}\right]dx$$

$$\Rightarrow \boxed{\varphi'' + \frac{S'}{S}\varphi' - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = 0} \;\xrightarrow{\partial/\partial t}\; \frac{\partial}{\partial t}\frac{\partial^2\varphi}{\partial x^2} + \frac{S'}{S}\frac{\partial}{\partial t}\frac{\partial\varphi}{\partial x} - \frac{1}{c_0^2}\frac{\partial^3\varphi}{\partial t^3} = 0 \quad \times\left(-1/\rho_0\right)$$

$$p = -\rho_0\frac{\partial\varphi}{\partial t} \Rightarrow \boxed{\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\partial \ln S}{\partial x}\frac{\partial p}{\partial x} = 0}$$

Equation known as Webster's equation (suggested first by Lagrange and Bernoulli)

Propagation in horns, 1 parameter theory (5/6)

- Solutions for infinite exponential horns $\quad \boxed{S(x) = S_0\, e^{2\alpha x}}$

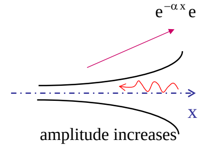
✓ Form of solutions: $\quad e^{\pm ikx}\, e^{i\omega t}$

✓ Substitution in the propagation equation: $\quad \varphi'' + \dfrac{S'}{S}\varphi' - \dfrac{1}{c_0^2}\dfrac{\partial^2\varphi}{\partial t^2} = 0$

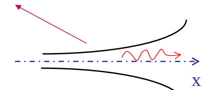
$$\Rightarrow k^2 \mp 2i\alpha k - \frac{\omega^2}{c_0^2} = 0 \Rightarrow \boxed{k = \pm i\alpha \pm \sqrt{\frac{\omega^2}{c_0^2} - \alpha^2}}$$

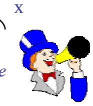
✓ Non divergent physic solutions of the problem

$$e^{-\alpha x}\, e^{i\sqrt{\frac{\omega^2}{c_0^2}-\alpha^2}\,x}\, e^{i\omega t} \quad \textit{and} \quad e^{-\alpha x}\, e^{-i\sqrt{\frac{\omega^2}{c_0^2}-\alpha^2}\,x}\, e^{i\omega t}$$



acoustic cornets





megaphone

Propagation in horns, 1 parameter theory (6/6)

- Solutions for infinite exponential horns $S(x) = S_0 \, e^{2\alpha x}$
  - ✓ Phase velocity: $c_\varphi = \dfrac{\omega}{\sqrt{\dfrac{\omega^2}{c_0^2} - \alpha^2}} = \dfrac{c_0}{\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}}$

    cut frequency: $f_c = \dfrac{\alpha c_0}{2\pi}$

  - ✓ Group velocity: $c_g = \dfrac{\partial \omega}{\partial k} = c_0 \sqrt{1-\left(\dfrac{f_c}{f}\right)^2}$
  - ✓ Acoustic intensity: $I = \dfrac{\hat{p}\hat{p}^*}{4\rho_0 c_0}\left\{\left(\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}\right)^* + \sqrt{1-\left(\dfrac{f_c}{f}\right)^2}\right\}$

    ➡ If $f \le f_c$: $I = 0$

    ➡ If $f > f_c$: $I = \dfrac{e^{-2\alpha x}}{2\rho_0 c_0}\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}$

    ➡ energy density: $E = \dfrac{\rho_0}{4}\hat{v}\hat{v}^* + \dfrac{\hat{p}\hat{p}^*}{4\rho_0 c_0^2} = \dfrac{e^{-2\alpha x}}{2\rho_0 c_0^2}$

    ➡ $\dfrac{I}{E} = c_0 \sqrt{1-(f_c/f)^2} = c_g$

Example of a "fold-up" horn (1/2)




Example of a "fold-up" horn (2/2)

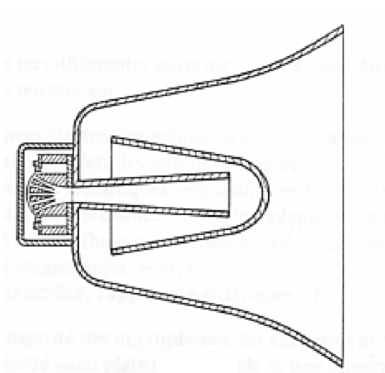


Figure from Mario Rossi, *Traité d'électricité, Volume XXI, Electroacoustique*, Presses Polytechniques Romandees, 1986

Example of tuba

Antonín Dvořák Symphony of new world, beginning of the 2nd movement (Largo)

http://www.gleblanc.com/

Example of Alp-horn


Slides based upon

C. POTEL, M. BRUNEAU, *Acoustique Générale - équations différentielles et intégrales, solutions en milieux fluide et solide, applications*, Ed. Ellipse collection Technosup, 352 pages, ISBN 2-7298-2805-2, 2006