Chapter 8
ELASTIC WAVES IN ISOTROPIC SOLIDS
APPLICATION TO ULTRASONIC NON DESTRUCTIVE TESTING

Mechanical wave (1/5)
- A mechanical wave is an oscillatory motion which is gradually transmitted in a material medium, by vicinity, like information, a change of position which one transmits to his neighbor.

Mechanical wave (2/5)

Mechanical wave: pressure wave (3/5)
in a gas

Mechanical wave: shear wave (4/5)

Mechanical wave: flexure wave (5/5)
flexure waves in a vibrating string

Schematic representation of matter made up of molecules (of given masses) with elastic interactions.

Continuous system: propagation of a pulse along a spring. The sections of the spring move from top to bottom as the pulse moves from left to right.

http://www.kettering.edu/~drussell
Animation courtesy of Dr. Dan Russell, Kettering University

The water particle at the centre moves and transmits its motion to the others

Propagation velocity

GAS  LIQUID  SOLID

\[ V_{\text{air}} = 340 \text{ m/s} \quad V_{\text{water}} = 1500 \text{ m/s} \quad V_{\text{metal}} \cong 6000 \text{ m/s} \]

Schematic look of the three fundamental states of the matter, and order of magnitude of the propagation velocity of pressure waves for each one of them.

From discontinuous matter...

SHEAR WAVE

Unique authorized motion

No information is transmitted

Informations are transmitted all the more quickly as the stiffness of the springs is great.

PRESSURE WAVE

... to continuous matter

Different types of waves

Elongation of an extensible line

Particle displacement vector

Relative variation length of the little piece MN:

\[
\frac{[u(x+\Delta x) - u(x)] - \Delta x}{\Delta x} = \frac{\Delta u}{\Delta x}
\]

\[
\Delta u = u(x + \Delta x) - u(x)
\]

\[
\Delta u = 0 \quad \text{simple translation}
\]

\[
\Delta u \neq 0 \quad \text{deformation}
\]


http://www.ens-lyon.fr/Planet-Terre/Infosciences/Geodynamique/Structure-interne/Sismologie/pendulum.html
Displacement

\[ \ddot{u}(M) = MM' \]

\[ \ddot{u}(N) = NN' \]

Strain tensor \( \mathbf{S} \)

\[ \ddot{u} = \frac{\partial \ddot{u}}{\partial x_1} dx_1 + \frac{\partial \ddot{u}}{\partial x_2} dx_2 + \frac{\partial \ddot{u}}{\partial x_3} dx_3 = \left( \text{grad} \ddot{u} \right) dx \]

\[ \nabla \ddot{u} = S \]

Interpretation (1/3)

- If \( \mathbf{S} = 0 \) and \( \mathbf{S} = \mathbf{0} \) then \( \ddot{u}(N) = \ddot{u}(M) \)
- If \( \ddot{u}(M) = 0 \) and \( \mathbf{S} = \mathbf{0} \) then \( \ddot{u}(N) = \mathbf{0} \cdot d\mathbf{O}M \)
- If \( \ddot{u}(M) = 0 \) and \( \mathbf{S} = \mathbf{0} \) then \( \ddot{u}(N) = \mathbf{S} \cdot d\mathbf{O}M \)

Interpretation (2/3): local displacement of two points

- Translation + deformation
- Translation + rotation
- Pure deformation

Interpretation (3/3)

\[ S_{ij} \text{: deformation in the } x_i \text{-direction} \]

\[ S_{ij} \text{: demi distorsion in the } x_i \text{ and } x_j \text{-directions} \]

\[ S = \begin{pmatrix}
0 & S_{12} & S_{13} \\
S_{12} & 0 & S_{23} \\
S_{13} & S_{23} & 0
\end{pmatrix} \]

\[ \gamma(M, \ddot{u}, \ddot{u}_z) = \lim_{d \rightarrow 0} \frac{\pi}{2} \alpha \left[ \frac{\partial \ddot{u}_2}{\partial x_1} \ddot{u}_1 + \frac{\partial \ddot{u}_1}{\partial x_2} \ddot{u}_2 \right] \]

\[ \mathbf{V} = \mathbf{V}' + \nabla \ddot{u} = \text{trace} \mathbf{S} \]
Stress tensor $\mathbf{T}$ (1/2)

$$\mathbf{T}(\mathbf{M}, \mathbf{n}) = \lim_{dS \to 0} \frac{dF}{dS} = \mathbf{n}$$

$T_{ik} = \lim_{\Delta x_k \to 0} \frac{\Delta F_i}{\Delta S_k} = T_{ki}$

$$\mathbf{T}(\mathbf{M}, \mathbf{\tilde{e}}_{x_2}) = \mathbf{T} \cdot \mathbf{\tilde{e}}_{x_2} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Expansion or compression

Expansion or compression

Hooke's law for isotropic media

- Normal stresses:
  $$T_{ii} = \lambda (S_{11} + S_{22} + S_{33}) + 2 \mu S_{ii} \quad i \text{ given}$$
  $$\lambda, \mu: \text{Lamé's coefficients}$$
  $$S_{11} + S_{22} + S_{33} = \text{div} \mathbf{u} = \frac{\Delta V}{V} \quad \lambda, \mu: \text{Lamé's coefficients}$$

- Tangential stresses:
  $$T_{ij} = 2 \mu S_{ij} \quad i \neq j$$

Relations between Young's modulus, Poisson's ratio and Lamé's coefficients

- Hooke's law for normal stresses
  $$T_{22} = \lambda (S_{11} + S_{22} + S_{33}) + 2 \mu S_{22}$$
  $$\frac{\lambda}{E} - \frac{2\mu}{E} T_{11} - \frac{2\mu}{E} T_{11}$$
  $$\frac{\lambda}{E} = \frac{1}{2(\lambda + \mu)}$$

- Tangential stresses:
  $$T_{ii} = \lambda (S_{11} + S_{22} + S_{33}) + 2 \mu S_{ii}$$
  $$T_{ii} = \lambda (1 - 2v) T_{11} + 2\mu T_{11}$$
  $$E = (1 - 2v) + 2\mu$$

Traction test

- Test piece non loaded
  $$\sigma = F / S$$
  - zone I: Linear elasticity
  - zone II: Non linear elasticity
  - zone III: Plasticity
Relations in isotropic solids

<table>
<thead>
<tr>
<th></th>
<th>( E, \nu )</th>
<th>( E, \mu )</th>
<th>( \lambda, \mu )</th>
<th>( c_{11}, c_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>( E ) ( (1+\nu)(1-2\nu) )</td>
<td>( \mu(1-2\nu) )</td>
<td>( \lambda )</td>
<td>( c_{12} )</td>
</tr>
<tr>
<td>( \mu )</td>
<td>( E ) ( 2(1+\nu) )</td>
<td>( \mu )</td>
<td>( \mu )</td>
<td>( c_{11} - c_{12} )</td>
</tr>
<tr>
<td>( E )</td>
<td>( E ) ( (1-\nu)(1-2\nu) )</td>
<td>( \mu(1-2\nu) )</td>
<td>( \lambda + \mu )</td>
<td>( 2c_{12} )</td>
</tr>
<tr>
<td>( B )</td>
<td>( E ) ( 3(1-2\nu) )</td>
<td>( \mu E ) ( 3(1-2\nu) )</td>
<td>( \lambda ) ( 2/3 )</td>
<td>( c_{11} + 2c_{12} )</td>
</tr>
<tr>
<td>( \nu )</td>
<td>( E ) ( 2\mu )</td>
<td>( 2(\lambda + \mu) )</td>
<td>( c_{12} )</td>
<td>( c_{11} + c_{12} )</td>
</tr>
</tbody>
</table>

\( c_{11}, c_{12} \): Rigidity constants (Pa)

\( E \): Young modulus (Pa)

\( \nu \): Poisson ratio (no unit)

\( B \): Voluminal elasticity modulus (Pa/m²)

\( \lambda, \mu \): Lamé coefficients (Pa)

Propagation equation

- Fundamental Relation of Dynamics for resultants: \( \ddot{u} (\text{ext} \rightarrow \mathcal{Y}) = \ddot{u} (\mathcal{Y} \rightarrow \mathcal{R}_z) \)

- At equilibrium: \( \dot{r}_s + \dot{r} T = 0 \)

- Stress variation about equilibrium position: \( \dot{r} + \dot{r} T = \mathcal{T}_{\mathcal{R}z} - \mathcal{T}_0 \)

- Substituting in propagation equation

\[ \dot{r} + \dot{r} T = \frac{\partial^2 \dot{u}}{\partial t^2} \]

with sources

\[ \text{and} \quad \frac{\partial^2 \dot{u}}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} + \sigma_{ij} \frac{\partial T_{ij}}{\partial x_j} \]

Isotropic solid (1/5):

**Decoupling of the propagation equation**

\[ \sigma_{ij} = \begin{cases} \sigma_{ii} & \text{normal stresses} \\ \sigma_{ij} & \text{tangential stresses} \end{cases} \]

\[ T_{ij} = \begin{cases} \lambda \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) & \text{normal stresses} \\ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) & \text{tangential stresses} \end{cases} \]

Isotropic solid (2/5):

**Decoupling of the propagation equation**

\[ \begin{align*}
\text{homogeneous to } L^2 T^{-2}, \quad & \text{with variation of volume} \\
\text{homogeneous to } L^2 T^{-2}, \quad & \text{without rotation}
\end{align*} \]

**Scalar potential** \( \psi \): \( \nabla \psi = \text{grad} \psi \)

**Vectorial potential** \( \chi \): \( \text{rot} \chi = \frac{\partial \psi}{\partial t} + \nabla \cdot \chi \)

Particle displacement:

\[ \dot{u} = \text{grad} \psi + \text{rot} \chi \]

Isotropic solid (3/5):

**Deformations**

**Without rotation, but with a variation of volume**

\[ \nabla \psi = \text{grad} \psi \]

**With rotation, but without variation of volume**

\[ \text{div} \dot{u} = 0 \]
Isotropic solid (4/5): any waves

- Propagation equation: 
  \[ \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \text{grad}(\text{div}\vec{u}) + \mu \Delta \vec{u} \]
  with \( \vec{u} = \text{grad} \Psi + \text{rot} \vec{z} = \vec{u}_L + \vec{u}_T \)

- Decoupling of the propagation equation:
  - pressure wave propagating with the velocity \( V_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{\sigma_{11} - \sigma_{12} - \sigma_{13}}{2\rho}} \)
  - shear wave propagating with the velocity \( V_T = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{\sigma_{12} + \sigma_{13}}{2\rho}} \)

- Case of plane waves, in connection with a propagation direction \( \vec{n} \)
  - pressure wave = longitudinal wave
  - shear wave = transverse wave

Example: monochromatic plane waves (1/2)

- Longitudinal waves
  \[ \vec{u}_L(x_1;t) = \text{grad} \Psi(x_1;t) \quad \text{with} \quad \Psi(x_1;t) = A_L e^{-i(k_1 x_1 + \omega t)} \]
  \[ \vec{u}_T(x_1;t) = \text{rot} \vec{z}(x_1;t) \quad \text{with} \quad \vec{z}(x_1;t) = A_T e^{-i(k_2 x_1 + \omega t)} \]

Example: monochromatic plane waves (2/2)

- Shear waves
  \[ \vec{u}_L(x_1,t) = \text{rot} \vec{z}(x_1,t) = \frac{\partial \vec{z}(x_1,t)}{\partial x_2} = A_L e^{-i(k_1 x_1 + \omega t)} \]
  \[ \vec{u}_T(x_1,t) = \text{rot} \vec{z}(x_1,t) = \frac{\partial \vec{z}(x_1,t)}{\partial x_3} = A_T e^{-i(k_2 x_1 + \omega t)} \]

Reflection and refraction (1/3)

1. incident wave
2. reflected wave(s)
3. transmitted wave(s)

Reflection and refraction (2/3)
Reflection and refraction (3/3)

Continuity of displacements at \( x_3 = 0 \):
\[
\mathbf{u}^{\text{inc}} + \sum_{\text{ref}} \mathbf{u}^{\text{ref}} = \sum_{\text{tr}} \mathbf{u}^{\text{tr}}
\]

Continuity of stress at \( x_3 = 0 \):
\[
\mathbf{T}^{\text{inc}} + \sum_{\text{ref}} \mathbf{T}^{\text{ref}} = \sum_{\text{tr}} \mathbf{T}^{\text{tr}}
\]

Stresses acting on a surface element at the interface
\[
\mathbf{T} = \mathbf{T} \cdot \mathbf{e}_{x_3} = \mathbf{T}_{13} = T_5
\]
\[
\mathbf{T}_{23} = T_4
\]
\[
\mathbf{T}_{33} = T_3
\]

Visualisation of Snell-Descartes' law:

Slowness surface

Location of the ends of the slowness vector \( \mathbf{m} \), drawn from a fixed point \( O \), when the propagation \( n \) varies.

Slowness vector:
\[
\mathbf{m} = \frac{k}{\omega}
\]

Isotropic medium: 2 velocities for a given propagation direction

Same velocities in any direction

Slowness surfaces = spheres

Snell-Descartes' law:
\[
\frac{1}{V_1} \sin \theta_1 = \frac{1}{V_2} \sin \theta_2
\]

Isotropic media: critical angles - evanescent waves

Isotropic medium: evanescent waves

Anisotropic media: critical angles - evanescent waves

Reflection and transmission coefficients (displacement amplitude)

<table>
<thead>
<tr>
<th>Water</th>
<th>Aluminium</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho = 1000 \text{ kg/m}^3 )</td>
<td>( \rho = 2786 \text{ kg/m}^3 )</td>
</tr>
<tr>
<td>( V_L = 1480 \text{ m/s} )</td>
<td>( V_L = 6650 \text{ m/s} )</td>
</tr>
<tr>
<td>( V_T = 3447 \text{ m/s} )</td>
<td>( V_T = 3447 \text{ m/s} )</td>
</tr>
</tbody>
</table>

Reflection and Transmission Coefficients (Displacement Amplitude)
Example (T_{zz}/\mu): f(X) = \cos(kX), \, ka = 4, \, \theta < \theta_{cL}

- Transmitted waves
- Total field with a stationary character along z, and a propagative character along x

- Transmitted waves
- Summation of pressure and shear fields

- Transmitted evanescent pressure wave only
- Transmitted shear wave only

- Incident wave
- Reflected wave

Example (T_{zz}/\mu): f(X) = \cos(kX), \, ka = 4, \, \theta < \theta_{cT}

- Transmitted evanescent pressure wave and propagative shear waves
- Total field with a stationary character along z, and a propagative character along x

- Transmitted waves
- Summation of pressure and shear fields

- Incident wave
- Reflected wave

Example (T_{zz}/\mu): f(X) = \cos(kX), \, ka = 4, \, \theta > \theta_{cT}

- Transmitted evanescent pressure and shear waves
- Total field with a stationary character along z, and a propagative character along x

- Transmitted waves
- Summation of pressure and shear fields

- Incident wave
- Reflected wave

Gaussian incident beam onto an interface fluid/solid

Before the 1st critical angle

Between the two critical angles

After the two critical angles

Modal waves (1/3)

Modal waves (2/3): Rayleigh wave

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne


Animation courtesy of Dr. Dan Russell, Kettering University
Earthquake Sumatra-Andama (2004)

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne

Modal waves (3/3): Lamb waves

Testing of material

- **Aim:** Detect defects in a piece

- **2 types:**
  - Destructive Testing \(\Rightarrow\) non-reusable piece
  - Non Destructive Testing (NDT) \(\Rightarrow\) reusable piece

- **Different NDT:**
  - Eddy currents
  - X-Rays, γ-Rays
  - Ultrasounds
    - not very expensive
    - access to one side of the piece
    - detection in defects located deeply in the piece
The more the defect is small, the more the frequency has to be high.

\[ \lambda = \frac{V}{f} \]

Non Destructive Testing (NDT) by ultrasounds

NDT and NDE

- NON DESTRUCTIVE TESTING
  - Presence or not of defects
  - Understanding of propagation phenomena
  - Determination of elastic (or viscoelastic) properties

NON DESTRUCTIVE EVALUATION

Transducer types

- Transducers
- Immersion transductors
- Angle beam transducers

Angle beam transducer

- Absorbing material
- Sensitive piece
- Edge
- Ultrasonic beam

Ultrasonic field generated by a plane ultrasonic transducer

- Near field
- Far field
- Main lobe
- Secondary lobes

- Transducer by ultrasounds: principle
- Ultrasonic echography

- Transducer connection
- Piezoelectric piece
- Backing material

- Defect detection
- Front face echo
- Back echo
- Defect echo

\[ \Delta t = \frac{2e}{V} \]
A-scan; B-scan

- A-Scan

- B-Scan: corresponds to a cut of the material

Identification of defects

- Front echo
- Delamination
- Delamination, double reflection
- Back echo

Comparison modeling / experiment (CIVA from CEA)

C-Scan

- C-scan: corresponds to a representation of a material slice

C-scan on a coin of currency

C-scan on an impacted beam

Hybrid model

Defects of different kind taken into account – Example (simulated Bscan)

Comparison simulation / experiments

CEA courtesy.

Detecs / Service simulation et systèmes pour la Surveillance et le Contrôle
**B-scan with a multi-element transducer**

**Contact flexible multi-element transducer (T.C.I., CEA)**

**Precautions of adjustment (1/2)**

- Multiples reflections in a plate
- Multiples reflections in the water column

**Simulations**

**Experiment**

**Consequences of a bad adjustment of the water column**

Insertion of a back echo between the interface echoes. This back echo could be mixed up with a defect echo.

**Precautions of adjustment (2/2)**

- Adjustment of the pulse repetition frequency
- Consequences of a bad adjustment of the pulse repetition frequency

**Example of input signal**

**Aluminium plate immersed in water (plane wave simulation)**

**Measures of velocities (1/2)**

**Example of input signal**

**Carbon-epoxy plate**

**Carbon-epoxy plate**

**Ultrasonic transducer**

**Reference signal**

**Transmitted signal**

**Propagation velocity of a wave in the medium, for a given direction**

**Real elastic constants**

\[ V = \sqrt{\frac{\rho V_m}{\rho e - 2\cos \theta}} \]
Measures of velocities (2/2)

- Path in the plate
  \[ t_{\text{AB}} = \frac{e}{V \cos \theta} \]
- Path in the fluid
  \[ t_{\text{AB}} = \frac{AB}{V_0} = \frac{e}{V_0 \cos \theta} \]

with
\[
\tau = t_{\text{AB}} - t_{\text{AB}}^0
\]

and
\[
\tau = t_{\text{AB}} - t_{\text{AB}}^0 = \frac{e}{V_0 \cos \theta} \left[ \cos \left( \theta - \theta^\text{inc} \right) \right] - n \frac{e}{V_0 \cos \theta} \left[ \cos \left( \theta - \theta^\text{inc} \right) + \sin \left( \theta - \theta^\text{inc} \right) \right]
\]

\[
V = \frac{V_0}{n} = V_0 \sqrt{1 + \frac{\tau V_0}{e} - \frac{\tau V_0}{e} - 2 \cos \theta^\text{inc}}
\]

Composite materials (1/2): example of carbon/epoxy composites

- Reflection coefficient(s) \( R \)
- Transmission coefficient(s) \( T \)
- Characteristics of all the waves

Composite materials (2/2): example of carbon/epoxy composites

- Pressure waves (symmetric mode)
- Flexure waves (antisymmetric mode)

Lamb waves

Vacuum

Isotropic solid

Vacuum

Water / Aluminium / Water; \( k_a = 170 \); \( H = 5 \) mm; before the 1st critical angle

Lamb waves: interest in NDT

Cartographies using Lamb waves

Lamb waves in isotropic media

Dispersion curves for Lamb waves plane \((k_x, h, f)\) or \((k_x, h/(2\pi), \omega h/(2\pi))\)

Phase velocity

Dispersion curves for Lamb waves plane \((f, h, V_\phi)\)

Displacements of Lamb waves (1/2)

Isotropic solid

Vacuum

Displacements of Lamb waves (2/2)

pressure waves (symmetric mode)

flexure waves (antisymmetric mode)

vectorial particle displacement field at the surface of the plate:
its effect on the shape of the plate

mode \( S_0 \)

mode \( A_0 \)

animations realised by Patrick Lanceleur, Université de Technologie de Compiègne
http://www.utc.fr/~lanceleu/links_CT04.html

Experiment set-up

Aluminium plate
Mode \( S_{0.9} \), \( f h = 1.434 \text{ MHz.mm} \)
\( x = 76.2 \text{ mm} \)
\( \tau = 15.875 \mu s \Rightarrow V_g = 4800 \text{ m/s} \)

unidirectional carbon/epoxy plate
\( \theta = 9.8^\circ, f = 1.35 \text{ MHz} ; e = 0.59 \text{ mm} \)

Measure of the group velocity

Non specular reflection

zero associated to an interference phenomenon

figures from: J.L. Rose, "Ultrasonic waves in solid media", Cambridge Univ. Press, 1999

Detection of a defect using Lamb waves (1/6)

8 plis

8 plis

plaques en carbone/époxyde : défaut entre le 3ème et le 4ème pli

Detection of a defect using Lamb waves (2/6)

Conversion of mode (1) into (2) and (3) then (4)

If mode (2) or mode (3) different from mode (1) then mode (4) ≠ mode (1) detected defect

Detection of a defect using Lamb waves (3/6): cartography

Echography experimental set-up

Detection of a defect using Lamb waves (4/6)

Carbon-Epoxy [0°/90°]_2s, θ = 10°, f = 1 MHz

Detection of defect (5/6): cartography using Lamb waves

5 layers 0°/90° mirror

Detection of defect (6/6): cartography using Lamb waves

missing fibers of 4th layer

broken fibers of 3rd layer

θ = 20° ; f = 5.05 MHz

θ = 21° ; f = 5.15 MHz

repartition of normal stress as a function of thickness
Slides based upon