Propagation and Non Destructive Testing in solids AC03

Master Sciences et Technologie
mention " **A**coustics and **M**echanics"
research specialities "acoustics" and "materials and acoustics"

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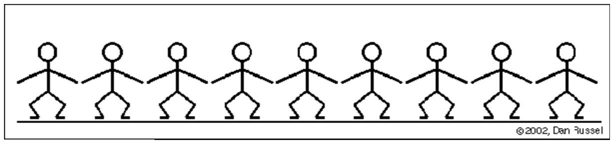
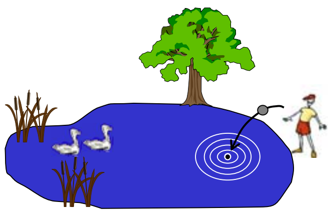
Mechanical wave (1/5)

- A mechanical wave is an oscillatory motion which is gradually transmitted in a material medium, by vicinity, like information, a change of position which one transmits to his neighbor.



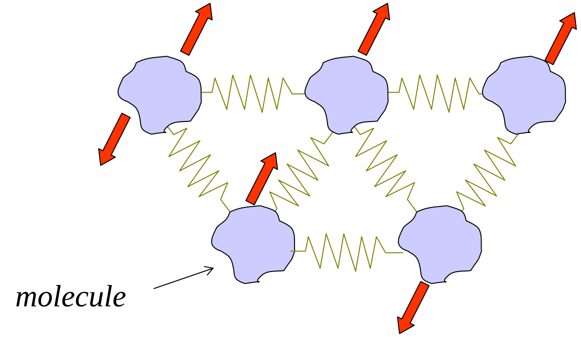
http://www.kettering.edu/~drussell

Animation courtesy of Dr. Dan Russell, Kettering University



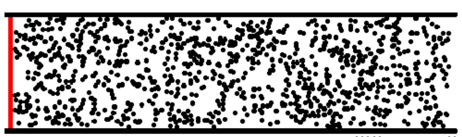
The water particle in the center moves and transmits its movement to the others

Mechanical wave (2/5)



Diagrammatic representation of matter made up of molecules (of given masses) in elastic interactions.

Mechanical wave: pressure wave (3/5)



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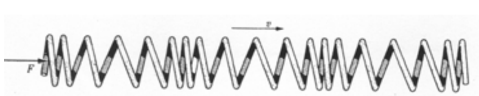
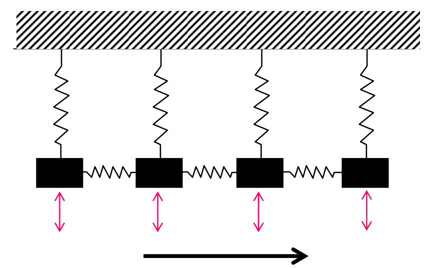
Animation courtesy of Dr. Dan Russell, Kettering University

in a gas



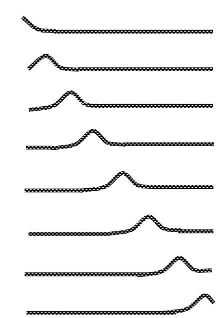
in a spring

Mechanical wave: shear wave (4/5)

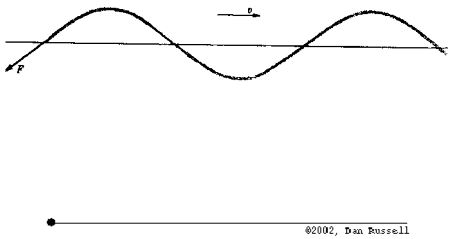


discrete system

continuous system: propagation of an impulse along a spring. The sections of the spring move from top to bottom as pulse moves from left to right



Mechanical wave: flexural wave (5/5)



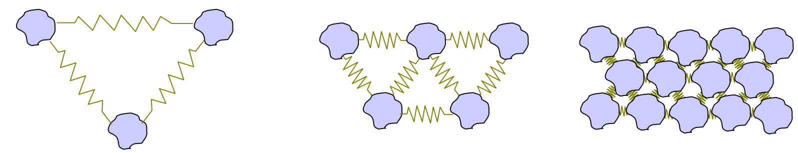
http://www.kettering.edu/~drussell

Animation courtesy of Dr. Dan Russell, Kettering University

flexure waves in a vibrating string

Propagation velocity

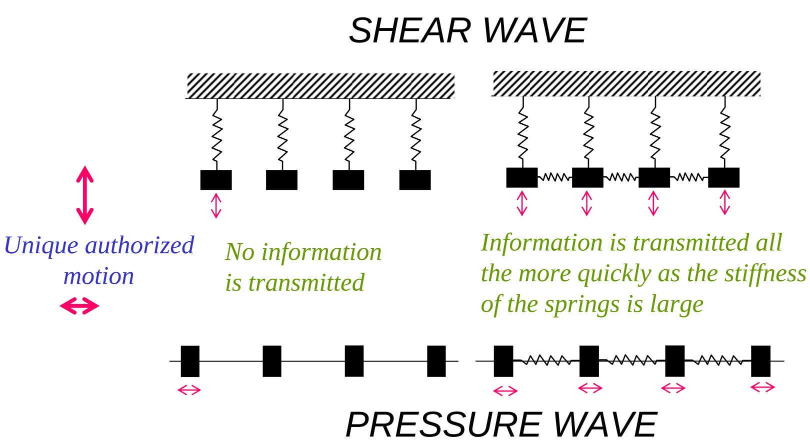
GAS *LIQUID* *SOLID*



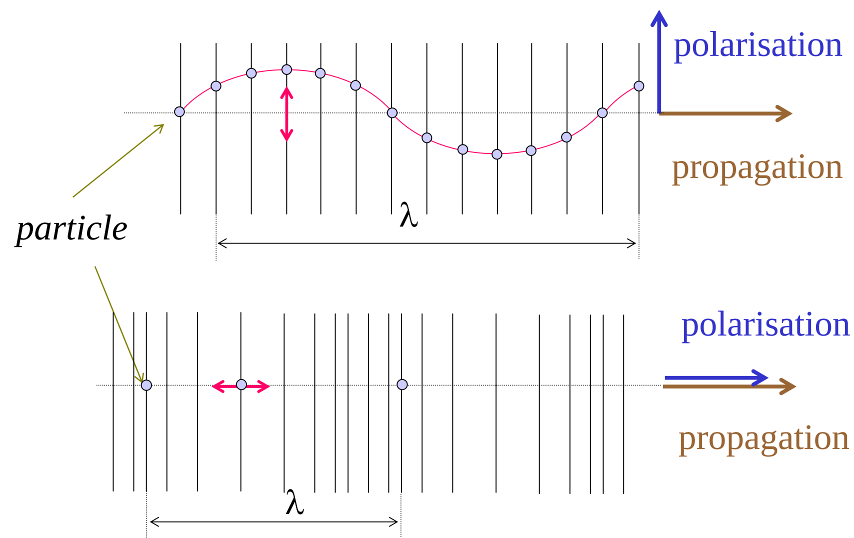
$V_{air} = 340$ m/s $V_{water} = 1500$ m/s $V_{metal} \cong 6000$ m/s

Diagrammatic aspect of the three fundamental states of the matter and order of magnitude of the compression propagation velocity for each one of them

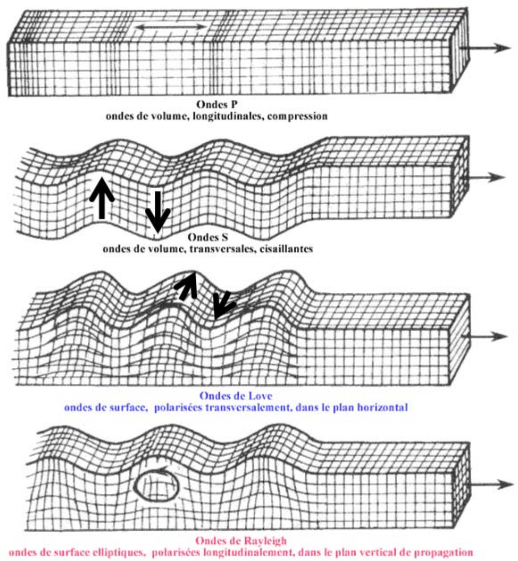
From discontinuous medium...



... to continuous medium

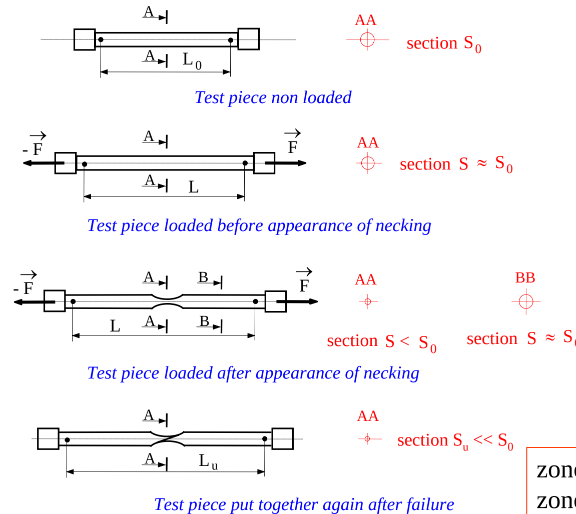


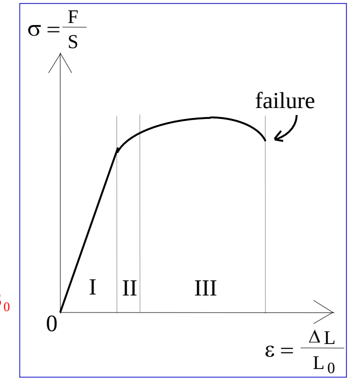
Different types of waves (examples)



http://www.ens-lyon.fr/Planet-Terre/Infosciences/Geodynamique/Structure-interne/Sismologie/pendulum.html

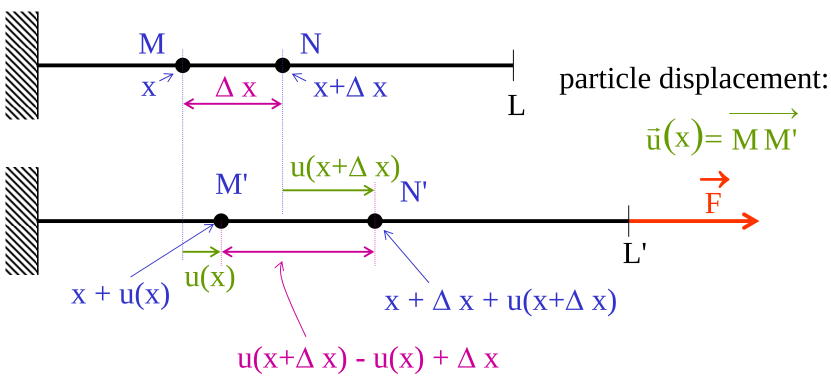
Traction test





zone I	Linear elasticity
zone II	Non linear elasticity
zone III	Plasticity

Elongation of an extensible line



particle displacement:

$\vec{u}(x) = \overrightarrow{MM'}$

relative variation length of the little piece MN:

$$\frac{[u(x+\Delta x)-u(x)+\Delta x]-\Delta x}{\Delta x} = \frac{\Delta u}{\Delta x}$$

$\Delta u = u(x+\Delta x) - u(x)$

$\Delta u = 0$ simple translation

$\Delta u \neq 0$ deformation

Strain tensor $\overline{\overline{S}}$

$$d\vec{u} = \frac{\partial \vec{u}}{\partial x_1} dx_1 + \frac{\partial \vec{u}}{\partial x_2} dx_2 + \frac{\partial \vec{u}}{\partial x_3} dx_3 = \left(\overline{\overline{\text{grad}\,\vec{u}}}\right)\cdot d\vec{x} \qquad \text{with} \qquad \overline{\overline{\text{grad}\,\vec{u}}} = \overline{\overline{S}} + \overline{\overline{\Omega}}$$

$$\overline{\overline{S}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

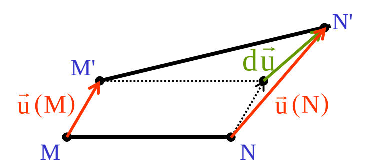
$$\overline{\overline{\Omega}} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}-\frac{\partial u_2}{\partial x_1}\right) & 0 & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) \\ -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1}\right) & -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}-\frac{\partial u_3}{\partial x_2}\right) & 0 \end{bmatrix}$$

symmetric

antisymmetric

$$\vec{u}(N) = \vec{u}(M) + d\vec{u} \quad \Longleftrightarrow \quad \vec{u}(N) = \vec{u}(M) + \overline{\overline{\Omega}}\cdot d\overrightarrow{OM} + \overline{\overline{S}}\cdot d\overrightarrow{OM}$$

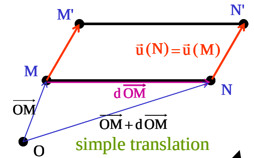
$$u_i(x_j + dx_j) = u_i(x_j) + S_{ij}\,dx_j + \Omega_{ij}\,dx_j$$



$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

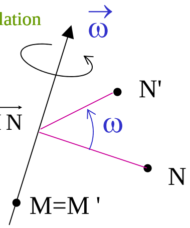
Interpretation (1/3)

$\vec{u}(N) = \vec{u}(M) + \overline{\overline{\Omega}} \cdot d\overrightarrow{OM} + \overline{\overline{S}} \cdot d\overrightarrow{OM}$



- If $\overline{\overline{\Omega}} = \overline{\overline{0}}$ and $\overline{\overline{S}} = \overline{\overline{0}}$ then $\vec{u}(N) = \vec{u}(M)$
- If $\vec{u}(M) = \vec{0}$ and $\overline{\overline{S}} = \overline{\overline{0}}$ then $\vec{u}(N) = \overline{\overline{\Omega}} \cdot d\overrightarrow{OM}$

$$\vec{u}(N) = \begin{vmatrix} du_1 \\ du_2 \\ du_3 \end{vmatrix}_{\mathscr{B}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}_{\mathscr{B}} \begin{vmatrix} dx_1 \\ dx_2 \\ dx_3 \end{vmatrix} = \begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix}_{\mathscr{B}} \wedge \begin{vmatrix} dx_1 \\ dx_2 \\ dx_3 \end{vmatrix}_{\mathscr{B}} = \vec{\omega} \wedge \overrightarrow{MN}$$



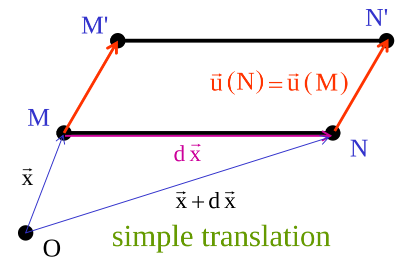
⇒ $\vec{u}(M) + \overline{\overline{\Omega}} \cdot d\overrightarrow{OM}$: displacement of the solid (mechanical meaning) **simple rotation**

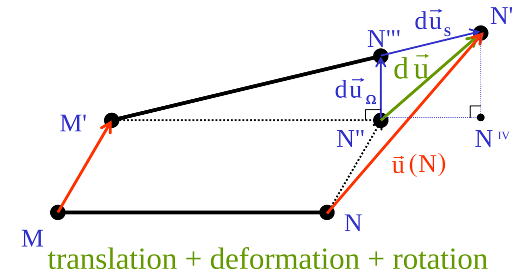
- If $\vec{u}(M) = \vec{0}$ and $\overline{\overline{\Omega}} = \overline{\overline{0}}$ then $\vec{u}(N) = \overline{\overline{S}} \cdot d\overrightarrow{OM}$ **deformation**

$$\vec{u}(N) = \vec{u}(M) + \overline{\overline{\Omega}} \cdot d\overrightarrow{OM} + \overline{\overline{S}} \cdot d\overrightarrow{OM}$$

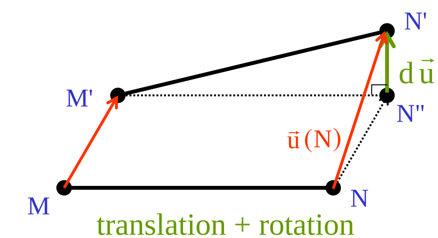
translation *rotation* (*mechanics*) *pure deformation*

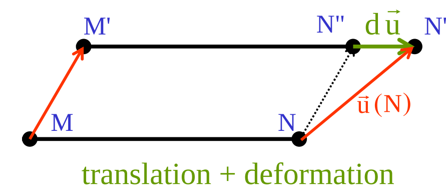
Interpretation (2/3): local displacement of two points





$\vec{u}(N) = \vec{u}(M) + \underbrace{d\vec{u}_s + d\vec{u}_\Omega}_{d\vec{u}}$



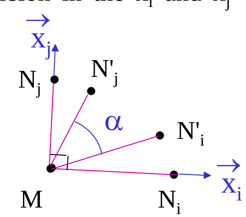


Interpretation (3/3)

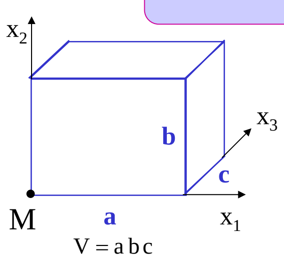
$$\overline{\overline{S}} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

S_{ii} : deformation in the x_i -direction

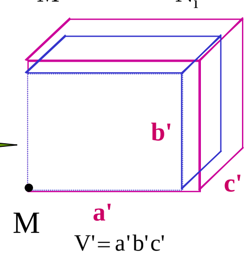
S_{ij} : demi distorsion in the x_i and x_j -directions



$$\gamma(M, \vec{n}_1, \vec{n}_2) = \lim_{\substack{dx_1 \to 0 \\ dx_2 \to 0}} \left(\frac{\pi}{2} - \alpha\right) = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$



deformation assumed without shearing

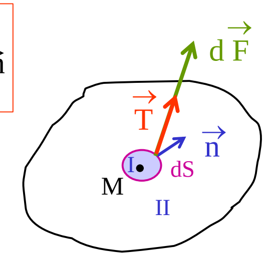


$V' \approx V\left(1 + \partial u_1/\partial x_1 + \partial u_2/\partial x_2 + \partial u_3/\partial x_3\right)$ ⇒ $(V' - V)/V \approx \operatorname{div} \vec{u} = \operatorname{trace} \overline{\overline{S}}$

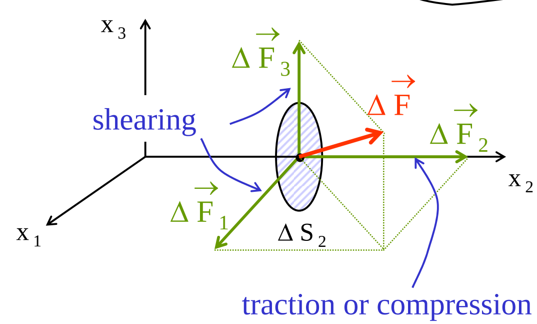
Stress tensor $\overline{\overline{T}}$ (1/2)

$$\vec{T}(M, \vec{n}) = \lim_{dS \to 0} \frac{d\vec{F}}{dS} = \overline{\overline{T}} . \vec{n}$$

$\mathscr{T}_i = T_{ik} n_k$



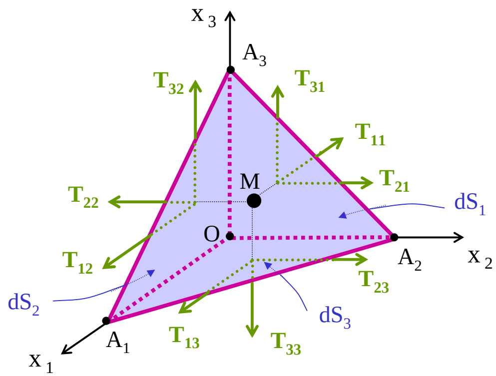
$$\overline{\overline{T}} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$



$$T_{ik} = \lim_{\Delta s_k \to 0} \frac{\Delta F_i}{\Delta S_k} = T_{ki}$$

$$\vec{T}(M, \vec{e}_{x_2}) = \overline{\overline{T}} \cdot \vec{e}_{x_2} = \begin{vmatrix} T_{12} \\ T_{22} \\ T_{32} \end{vmatrix}_{\mathscr{B}}$$

Stress tensor $\overline{\overline{T}}$ (2/2)



Hooke law

- Small deformations hypothesis :

Hooke law $T_{ij} = c_{ijk\ell}\ S_{k\ell}$

elastic rigidity

- Matrix notation : $c_{\alpha\beta} = c_{ijk\ell}$

$\alpha \leftrightarrow (i\,j)$

$\beta \leftrightarrow (k\,\ell)$

$(11)\leftrightarrow 1 \quad (22)\leftrightarrow 2 \quad (33)\leftrightarrow 3$

$(32)=(23)\leftrightarrow 4 \quad (31)=(13)\leftrightarrow 5 \quad (12)=(21)\leftrightarrow 6$

Hooke law

$T_\alpha = c_{\alpha\beta}\ S_\beta$ with $S_1 = S_{11} \quad S_2 = S_{22} \quad S_3 = S_{33}$
$S_4 = 2S_{23} \quad S_5 = 2S_{13} \quad S_6 = 2S_{12}$

and $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Relations in isotropic solid

	E, ν	E, μ	λ, μ	c_{11}, c_{12}
λ	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{\mu(E-2\mu)}{3\mu - E}$	λ	c_{12}
μ	$\frac{E}{2(1+\nu)}$	μ	μ	$\frac{c_{11}-c_{12}}{2}$
E	E	E	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$c_{11} - 2\frac{c_{12}^2}{c_{11}+c_{12}}$
B	$\frac{E}{3(1-2\nu)}$	$\frac{\mu E}{3(3\mu - E)}$	$\lambda + \frac{2}{3}\mu$	$\frac{c_{11}+2c_{12}}{3}$
ν	ν	$\frac{E-2\mu}{2\mu}$	$\frac{\lambda}{2(\lambda+\mu)}$	$\frac{c_{12}}{c_{11}+c_{12}}$

c_{11} , c_{12} : Rigidity constants (Pa)

E : Young modulus (Pa)

ν : Poisson ratio (no unit)

B : Voluminal elasticity modulus (Pa/m²)

λ , μ : Lamé coefficients (Pa)

Cristal symmetries

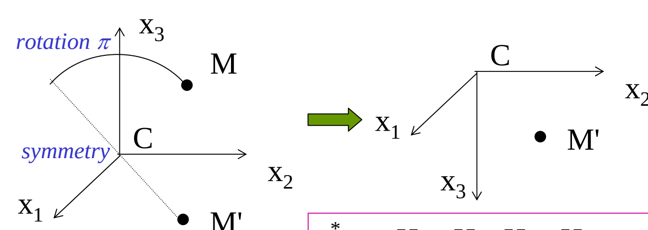
- Direct symmetry : **A_n**, order n rotation axis

rotation angle **$2\pi / n$** ➡ unchanged properties

- Inverse symmetry: **$\overline{A}_n$**, inverse order n rotation axis

rotation angle **$2\pi / n$**
symmetry with respect to a center C } ➡ unchanged properties

- Mirror : inverse order 2 axis: **$\overline{A}_2 = M$**



Change-of-basis matrix

$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$c^*_{ijk\ell} = H_{mi}\, H_{nj}\, H_{pk}\, H_{q\ell}\, c_{mnpq} = c_{ijk\ell}$

Elastic rigidity constants

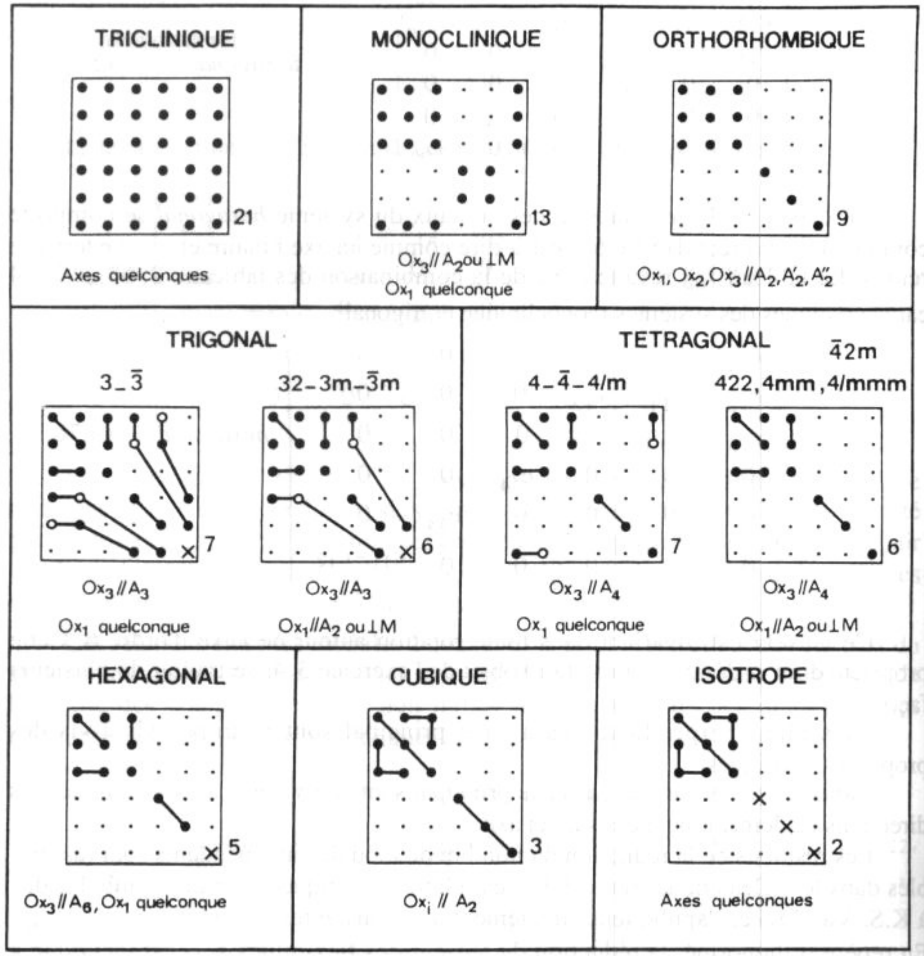


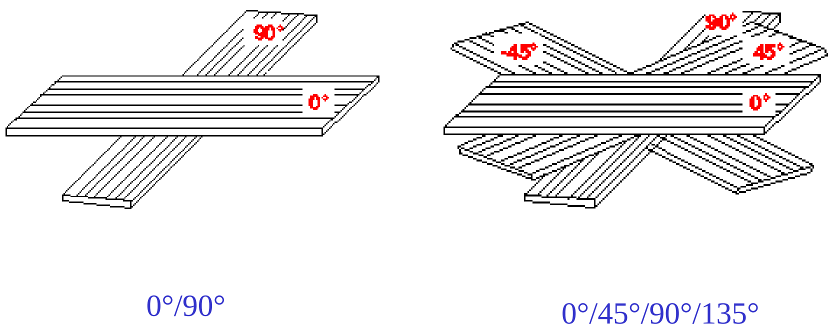
FIG. 3. 9. *Composantes $c_{\alpha\beta}$ du tenseurs des rigidités élastiques suivant les systèmes de symétrie avec les axes de référence de la figure 2. 22.*

composante – *non nulle* : ● ○ – *nulle* : ·
composantes – *égales* : ●—● – *opposées* : ●—○
– *égales à* $(c_{11} - c_{12})/2$: ×

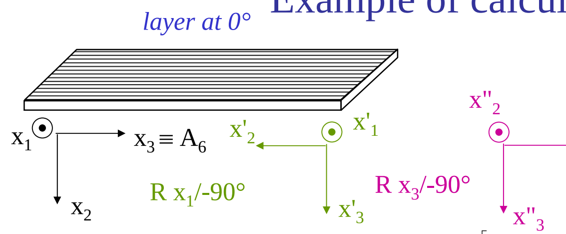
La symétrie par rapport à la diagonale principale n'est pas mentionnée. Le nombre de composantes indépendantes est indiqué, en bas, à droite de chaque ensemble.

from D. Royer and E. Dieulesaint, "Ondes élastiques dans les solides", tome 1 : propagation libre et guidée, Masson, (1996)

Composites materials: example of carbone-epoxy type composites



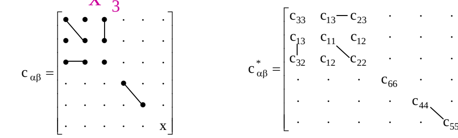
Example of calculation



$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Change-of-basis matrix

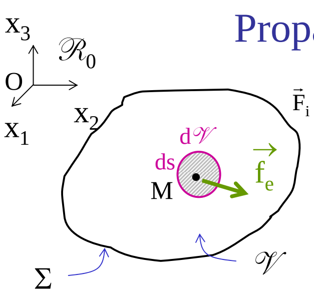
$$c^*_{ijk\ell} = H_{mi} H_{nj} H_{pk} H_{q\ell}\, c_{mnpq} = c_{ijk\ell}$$



- $c^*_{11} = c^*_{1111} = H_{m1} H_{n1} H_{p1} H_{q1}\, c_{mnpq}$ — only $H_{31} \neq 0$ ➡ m=n=p=q=3 ➡ $c^*_{11} = c_{3333} = c_{33}$
- $c^*_{12} = c^*_{1122} = H_{m1} H_{n1} H_{p2} H_{q2}\, c_{mnpq}$ — only $H_{31} \neq 0$ and $H_{12} \neq 0$ ➡ m=n=3 ; p=q=1 ➡ $c^*_{12} = c_{31}$
- $c^*_{13} = c^*_{1133} = H_{m1} H_{n1} H_{p3} H_{q3}\, c_{mnpq}$ — only $H_{31} \neq 0$ and $H_{23} \neq 0$ ➡ m=n=3 ; p=q=2 ➡ $c^*_{13} = c_{32}$

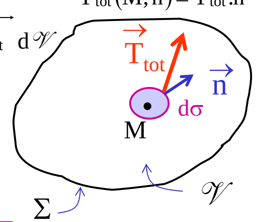
etc.

Propagation equation (1/3)



$$\vec{F}_i = \oiint_\Sigma \overline{\overline{T}}_{tot}.\vec{n}\, d\sigma = \iiint_{\mathscr{V}} \overrightarrow{\operatorname{div} \overline{\overline{T}}_{tot}}\, d\mathscr{V}$$

$$\vec{T}_{tot}(M,\vec{n}) = \overline{\overline{T}}_{tot}.\vec{n}$$



$$\vec{F}_e = \iiint_{\mathscr{V}} \vec{f}_e\, d\mathscr{V}$$

with $\vec{f}_e = \vec{f}_{e_0} + \delta\vec{f}_e$

- Dynamic resultant $\vec{d}(\mathscr{V}/\mathscr{R}_0) = \iiint_{\mathscr{V}} \rho \dfrac{\partial^2 \vec{u}}{\partial t^2}\, d\mathscr{V}$
- Resultant of external forces $\vec{\mathscr{R}}(ext \to \mathscr{V}) = \iiint_{\mathscr{V}} \left(\vec{f}_e + \overrightarrow{\operatorname{div} \overline{\overline{T}}}\right) d\mathscr{V}$

with

$$\overrightarrow{\operatorname{div} \overline{\overline{T}}_{tot}} = \begin{vmatrix} \operatorname{div}\left(\overline{\overline{T}}^t_{tot}.\vec{e}_{x_1}\right) \\ \operatorname{div}\left(\overline{\overline{T}}^t_{tot}.\vec{e}_{x_2}\right) \\ \operatorname{div}\left(\overline{\overline{T}}^t_{tot}.\vec{e}_{x_3}\right) \end{vmatrix}_{\mathscr{B}_0} = \begin{vmatrix} \dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} \\ \dfrac{\partial T_{21}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{23}}{\partial x_3} \\ \dfrac{\partial T_{31}}{\partial x_1} + \dfrac{\partial T_{32}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} \end{vmatrix}_{\mathscr{B}_0} = \begin{vmatrix} \sum_{j=1}^{3} \dfrac{\partial T_{1j}}{\partial x_j} \\ \sum_{j=1}^{3} \dfrac{\partial T_{2j}}{\partial x_j} \\ \sum_{j=1}^{3} \dfrac{\partial T_{3j}}{\partial x_j} \end{vmatrix}_{\mathscr{B}_0} = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} \right) \vec{e}_{x_i}$$

Propagation equation (2/3)

- Fundamental Relation of Dynamics for resultants: $\vec{\mathscr{R}}(ext \to \mathscr{V}) = \vec{d}(\mathscr{V}/\mathscr{R}_0),\ \forall\ \mathscr{V}$

$$\iiint_{\mathscr{V}} \left(\vec{f}_e + \overrightarrow{\operatorname{div} \overline{\overline{T}}_{tot}}\right) d\mathscr{V} = \iiint_{\mathscr{V}} \rho \frac{\partial^2 \vec{u}}{\partial t^2}\, d\mathscr{V},\ \forall\ \mathscr{V} \quad \Rightarrow \quad \vec{f}_e + \overrightarrow{\operatorname{div} \overline{\overline{T}}_{tot}} = \rho \frac{\partial^2 \vec{u}}{\partial t^2}$$

with $\vec{f}_e = \vec{f}_{e_0} + \delta\vec{f}_e$

- At equilibrium $\vec{f}_{e_0} + \overrightarrow{\operatorname{div} \overline{\overline{T}}_0} = \vec{0}$ change of variable: $\overline{\overline{T}} = \overline{\overline{T}}_{tot} - \overline{\overline{T}}_0$

stress variation about equilibrium position

- Substituting in propagation equation $\vec{f}_{e_0} + \delta\vec{f}_e + \overrightarrow{\operatorname{div}\left(\overline{\overline{T}} + \overline{\overline{T}}_0\right)} = \rho \dfrac{\partial^2 \vec{u}}{\partial t^2}$ ($\vec{0}$)

➡ $\delta\vec{f}_e + \overrightarrow{\operatorname{div} \overline{\overline{T}}} = \rho \dfrac{\partial^2 \vec{u}}{\partial t^2}$ *with sources*

and $\overrightarrow{\operatorname{div} \overline{\overline{T}}} = \rho \dfrac{\partial^2 \vec{u}}{\partial t^2}$ *ouside sources*

using $\overrightarrow{\operatorname{div} \overline{\overline{T}}} = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \dfrac{\partial T_{ij}}{\partial x_j} \vec{e}_{x_i} \right)$ ➡

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j}, \quad i = 1,2,3$$

Propagation equation (3/3)

- Hooke law $T_{ij} = c_{ijk\ell} S_{k\ell}$ *with* $S_{k\ell} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\right)$

$$\Rightarrow T_{ij} = \frac{1}{2} c_{ijk\ell} \frac{\partial u_k}{\partial x_\ell} + \frac{1}{2} c_{ijk\ell} \frac{\partial u_\ell}{\partial x_k} = \frac{1}{2}\left(c_{ij\ell k} + c_{ijk\ell}\right)\frac{\partial u_\ell}{\partial x_k}$$

$$\Rightarrow \boxed{T_{ij} = c_{ijk\ell} \frac{\partial u_\ell}{\partial x_k}}$$

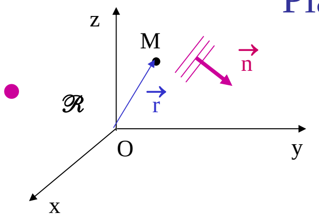
- Derivative of stresses $\frac{\partial T_{ij}}{\partial x_j} = c_{ijk\ell} \frac{\partial^2 u_\ell}{\partial x_j \partial x_k}$

- Propagation equation

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial T_{ij}}{\partial x_j} \,,\quad i = 1,2,3 \;\Rightarrow\; \boxed{\rho \frac{\partial^2 u_i}{\partial t^2} = c_{ijk\ell} \frac{\partial^2 u_\ell}{\partial x_j \partial x_k} \,,\quad i = 1,2,3}$$

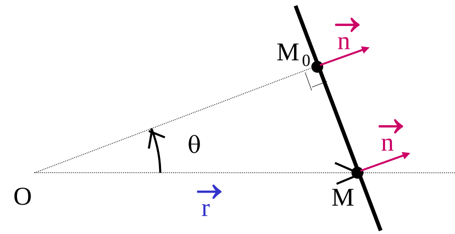
Remark: no plane wave hypothesis

Plane waves (1/2)



$$\boxed{\hat{u}_i(\vec{r};t) = \hat{f}(\vec{n}\cdot\vec{r} - c_0 t) + \hat{g}(\vec{n}\cdot\vec{r} + c_0 t)} \qquad \vec{r} = \overrightarrow{OM}$$

- $F\left[\kappa\left(c_0 t - \vec{n}\cdot\vec{r}\right)\right]$



- ***At a given time***, at any given point M such as

$$\boxed{\vec{n}\cdot\vec{r} = \text{constant}}$$

the value of the field variable (physical quantity) is the same.

These points are located in the same plane, which is termed ***wave plane*** (plane ***wave surface***), perpendicular to the direction $\vec{n}$:

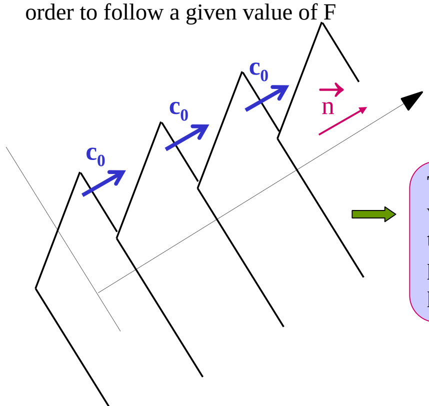
$$\vec{n}\cdot\vec{r} = \vec{n}\cdot\overrightarrow{OM} = OM\cos\theta = OM_0$$

Plane waves (2/2)

- ***When time varies***, follow a given value of $\Rightarrow$ $\boxed{\left(c_0 t - \vec{n}\cdot\vec{r}\right) = \text{constant}}$

i.e. $d\left(c_0 t - \vec{n}\cdot\vec{r}\right) = 0$ *i.e.* $\vec{n}\cdot\frac{d\vec{r}}{dt} = c_0$

speed to which a geometrical point M must move in order to follow a given value of F



if $\vec{r} \,//\, \vec{n}$ $\Rightarrow$ $\frac{d\vec{r}}{dt} = c_0 \vec{n}$

$\Rightarrow$ The wave planes which convey a given value of the field variable F, move parallel to themselves, in the direction $\vec{n}$ which is perpendicular to them, and with the propagation velocity c_0.

Solution for plane waves (1/2)

- Propagation equation $\rho \frac{\partial^2 u_i}{\partial t^2} = c_{ijk\ell} \frac{\partial^2 u_\ell}{\partial x_j \partial x_k} \,,\quad i = 1,2,3$ **(1)**

- Particular form of solutions $u_i = P_i F\left(t - \frac{\vec{n}\cdot\vec{r}}{V}\right) = P_i F\left(t - \frac{n_j x_j}{V}\right)$

$$\frac{\partial^2 u_i}{\partial t^2} = P_i F''\left(t - \frac{n_j x_j}{V}\right) \quad \textbf{(2)}$$

$$\frac{\partial u_\ell}{\partial x_j} = -P_\ell \frac{n_j}{V} F'\left(t - \frac{n_j x_j}{V}\right)$$

$$\frac{\partial^2 u_\ell}{\partial x_j \partial x_k} = -P_\ell \frac{n_j}{V}\left(-\frac{n_k}{V}\right) F''\left(t - \frac{n_j x_j}{V}\right) = P_\ell \frac{n_j n_k}{V^2} F''\left(t - \frac{n_j x_j}{V}\right) \quad \textbf{(3)}$$

(2) and (3) in (1) $\Rightarrow$ $\rho P_i F''\left(t - \frac{n_j x_j}{V}\right) = c_{ijk\ell} P_\ell \frac{n_j n_k}{V^2} F''\left(t - \frac{n_j x_j}{V}\right) \,,\ \forall \vec{r} \,,\ \forall t$

$$\Rightarrow \rho P_i = c_{ijk\ell} \frac{n_j n_k}{V^2} P_\ell$$

$$\Rightarrow \boxed{\rho V^2 P_i = c_{ijk\ell}\, n_j n_k P_\ell}$$

$\Gamma_{i\ell}$ *: Christoffel tensor*

Solution for plane waves (2/2): Christoffel tensor

$$\Gamma_{i\ell} = c_{ijk\ell} n_j n_k$$
$$= c_{i11\ell} n_1^2 + c_{i22\ell} n_2^2 + c_{i33\ell} n_3^2 + (c_{i12\ell} + c_{i21\ell}) n_1 n_2 + (c_{i13\ell} + c_{i31\ell}) n_1 n_3 + (c_{i23\ell} + c_{i32\ell}) n_2 n_3$$

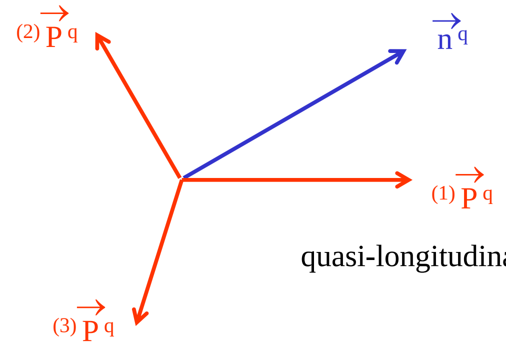
$$\Gamma_{11} = c_{11} n_1^2 + c_{66} n_2^2 + c_{55} n_3^2 + 2c_{16} n_1 n_2 + 2c_{15} n_1 n_3 + 2c_{56} n_2 n_3$$
$$\Gamma_{12} = c_{16} n_1^2 + c_{26} n_2^2 + c_{45} n_3^2 + (c_{12} + c_{66}) n_1 n_2 + (c_{14} + c_{56}) n_1 n_3 + (c_{46} + c_{25}) n_2 n_3$$
$$\Gamma_{13} = c_{15} n_1^2 + c_{46} n_2^2 + c_{35} n_3^2 + (c_{14} + c_{56}) n_1 n_2 + (c_{13} + c_{55}) n_1 n_3 + (c_{36} + c_{45}) n_2 n_3$$
$$\Gamma_{22} = c_{66} n_1^2 + c_{22} n_2^2 + c_{44} n_3^2 + 2c_{26} n_1 n_2 + 2c_{46} n_1 n_3 + 2c_{24} n_2 n_3$$
$$\Gamma_{23} = c_{56} n_1^2 + c_{24} n_2^2 + c_{34} n_3^2 + (c_{46} + c_{25}) n_1 n_2 + (c_{36} + c_{45}) n_1 n_3 + (c_{23} + c_{44}) n_2 n_3$$
$$\Gamma_{33} = c_{55} n_1^2 + c_{44} n_2^2 + c_{33} n_3^2 + 2c_{45} n_1 n_2 + 2c_{35} n_1 n_3 + 2c_{34} n_2 n_3$$
$$\Gamma_{21} = \Gamma_{12} \qquad \Gamma_{31} = \Gamma_{13} \qquad \Gamma_{32} = \Gamma_{23}$$

Propagation equation: $\Gamma_{i\ell} P_\ell = \rho V^2 P_i$

P_i : eigenvector of $\Gamma_{i\ell}$; ρV^2 : eigenvalue of $\Gamma_{i\ell}$

Propagation in a material q

quasi-transversal wave (QT1)



quasi-longitudinal wave (QL)

quasi-transversal wave (QT2)

$\vec{n}^q$ propagation direction

$(\eta)\,\vec{P}^q$ polarisation vector of the wave (η)

Propagation following directions linked to symmetry elements, in the x_3 -direction

- Propagation direction: $\vec{n} = \vec{e}_{x_3}$
- Christoffel tensor: $\Gamma_{i\ell} = c_{ijk\ell} n_j n_k = \begin{bmatrix} c_{55} & c_{45} & c_{35} \\ c_{45} & c_{44} & c_{34} \\ c_{35} & c_{34} & c_{33} \end{bmatrix}$
- Monoclininc symmetry system: $\Gamma_{i\ell} = \begin{bmatrix} c_{55} & c_{45} & 0 \\ c_{45} & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$

➡ eigenvector $^{(1)}\vec{P} = \vec{e}_{x_3}$ eigenvalue $^{(1)}V = \sqrt{\dfrac{c_{33}}{\rho}}$

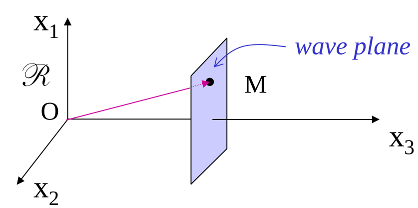
➡ 1 longitudinal wave + 2 shear waves

- Solid with p-order axis (p>2): $\Gamma_{i\ell} = \begin{bmatrix} c_{44} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{33} \end{bmatrix}$

➡ two identical eigenvalues

➡ two degenerated shear waves; Ox_3 = acoustical axis

Isotropic solid (1/5): decoupling of the propagation equation



$$\vec{u}(\vec{x};t) = \vec{u}(x_3;t) \;\Rightarrow\; \begin{cases} \partial u_i / \partial x_1 = 0 \\ \partial u_i / \partial x_2 = 0 \end{cases}$$

- Hooke Law

$$T_{ii} = \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \right) + 2\mu \frac{\partial u_i}{\partial x_i} \qquad \begin{cases} T_{11} = \lambda\, \partial u_3 / \partial x_3 \\ T_{22} = \lambda\, \partial u_3 / \partial x_3 \\ T_{33} = (\lambda + 2\mu)\, \partial u_3 / \partial x_3 \end{cases}$$ *normal stresses*

$$T_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \qquad \begin{cases} T_{12} = 0 \\ T_{13} = \mu\, \partial u_1 / \partial x_3 \\ T_{23} = \mu\, \partial u_2 / \partial x_3 \end{cases}$$ *tangential stresses*

Isotropic solid (2/5) : decoupling of the propagation equation

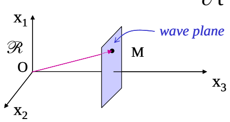
- i = 3 $\rho\dfrac{\partial^2 u_3}{\partial t^2}=\sum_{j=1}^{3}\dfrac{\partial T_{3j}}{\partial x_j}$ ➡ $\dfrac{\partial^2 u_3}{\partial t^2}=\left(\dfrac{\lambda+2\mu}{\rho}\right)\dfrac{\partial^2 u_3}{\partial x_3^2}$ *homogeneous to L^2T^{-2}*

let $V_L=\sqrt{\dfrac{\lambda+2\mu}{\rho}}$ ➡ $\dfrac{\partial^2 u_3}{\partial x_3^2}-\dfrac{1}{V_L^2}\dfrac{\partial^2 u_3}{\partial t^2}=0$ *pressure waves (longitudinal)*

- i = 1 $\rho\dfrac{\partial^2 u_1}{\partial t^2}=\sum_{j=1}^{3}\dfrac{\partial T_{1j}}{\partial x_j}$ ➡ $\dfrac{\partial^2 u_1}{\partial t^2}=\dfrac{\mu}{\rho}\dfrac{\partial^2 u_1}{\partial x_3^2}$ *homogeneous to L^2T^{-2}*

let $V_T=\sqrt{\dfrac{\mu}{\rho}}$ ➡ $\dfrac{\partial^2 u_1}{\partial x_3^2}-\dfrac{1}{V_T^2}\dfrac{\partial^2 u_1}{\partial t^2}=0$ *shear waves (transversal)*

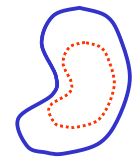
- i = 2 $\rho\dfrac{\partial^2 u_2}{\partial t^2}=\sum_{j=1}^{3}\dfrac{\partial T_{2j}}{\partial x_j}$ ➡ $\dfrac{\partial^2 u_2}{\partial t^2}=\dfrac{\mu}{\rho}\dfrac{\partial^2 u_2}{\partial x_3^2}$



➡ $\dfrac{\partial^2 u_2}{\partial x_3^2}-\dfrac{1}{V_T^2}\dfrac{\partial^2 u_2}{\partial t^2}=0$ *shear waves (transversal)*

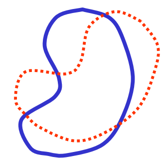
Isotropic solid (3/5): deformations

Without rotation, but with a variation of volume $\overrightarrow{\text{rot}}\,\vec{u}_L=\vec{0}$



scalar potential ψ : $\vec{u}_L=\overrightarrow{\text{grad}}\,\psi$

With rotation, but without variation of volume $\text{div}\,\vec{u}_T=0$



vectorial potential $\vec{\chi}$: $\vec{u}_T=\overrightarrow{\text{rot}}\,\vec{\chi}$

➡ particle displacement : $\vec{u}=\overrightarrow{\text{grad}}\,\psi+\overrightarrow{\text{rot}}\,\vec{\chi}=\vec{\nabla}\psi+\vec{\nabla}\wedge\vec{\chi}$

Isotropic solid (4/5) : any waves

- **Propagation equation:** $\rho\dfrac{\partial^2\vec{u}}{\partial t^2}=(\lambda+\mu)\overrightarrow{\text{grad}}(\text{div}\,\vec{u})+\mu\Delta\vec{u}$

  with $\vec{u}=\overrightarrow{\text{grad}}\,\Psi+\overrightarrow{\text{rot}}\,\vec{\chi}=\vec{u}_L+\vec{u}_T$

- **Decoupling of the propagation equation:**

  pressure wave propagating with the velocity $V_L=\sqrt{\dfrac{\lambda+2\mu}{\rho}}=\sqrt{\dfrac{c_{11}}{\rho}}$

  $$\frac{\partial^2\vec{u}_L}{\partial t^2}-V_L^2\Delta\vec{u}_L=\vec{0}$$

  shear wave propagating with the velocity $V_T=\sqrt{\dfrac{\mu}{\rho}}=\sqrt{\dfrac{c_{11}-c_{12}}{2\rho}}$

  $$\frac{\partial^2\vec{u}_T}{\partial t^2}-V_T^2\Delta\vec{u}_T=\vec{0}$$

- **Case of plane waves,** in connection with a propagation direction $\vec{n}$

  pressure wave = longitudinal wave
  shear wave = transversale wave

Isotropic solid (5/5): monochromatic waves

- Wave equations $\Delta\psi-\dfrac{1}{V_L^2}\dfrac{\partial^2\psi}{\partial t^2}=0$ and $\Delta\vec{\chi}-\dfrac{1}{V_T^2}\dfrac{\partial^2\vec{\chi}}{\partial t^2}=\vec{0}$

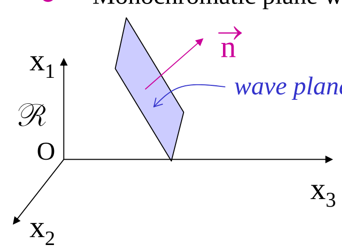
- $\hat{\psi}(\vec{x};t)=\hat{\Psi}(\vec{x})e^{+i\omega t}$ and $\hat{\vec{\chi}}(\vec{x};t)=\hat{\vec{X}}(\vec{x})e^{+i\omega t}$

  ➡ $\Delta\hat{\psi}-\dfrac{1}{V_L^2}\dfrac{\partial^2\hat{\psi}}{\partial t^2}=0$ and $\Delta\hat{\vec{\chi}}-\dfrac{1}{V_T^2}\dfrac{\partial^2\hat{\vec{\chi}}}{\partial t^2}=\vec{0}$

- Helmholtz equations $\Delta\hat{\Psi}+k_L^2\hat{\Psi}=0$ and $\Delta\hat{\vec{X}}+k_T^2\hat{\vec{X}}=\vec{0}$

  with $k_L=\omega/V_L$ and $k_T=\omega/V_T$

- Monochromatic plane waves

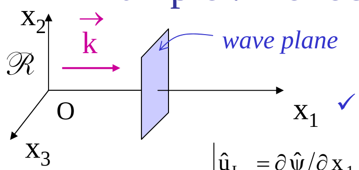


$\vec{k}_L=k_L\vec{n}$ and $\vec{k}_T=k_T\vec{n}$

$$\hat{\psi}(\vec{x})=\hat{A}_L e^{-i(\vec{k}_L\cdot\vec{x}-\omega t)}$$

$$\hat{\vec{\chi}}(\vec{x})=\hat{\vec{A}}_T e^{-i(\vec{k}_T\cdot\vec{x}-\omega t)}$$

Example : monochromatic plane waves (1/2)



- Longitudinal waves

✓ $\hat{\vec u}_L(x_1;t)=\overrightarrow{\text{grad}}\,\hat\psi(x_1;t)$ *with* $\hat\psi(x_1;t)=\hat A_L e^{-i(k_L x_1-\omega t)}$

⇒ $\hat{\vec u}_L(x_1;t)=\left|\begin{array}{l}\hat u_{L_1}=\partial\hat\psi/\partial x_1=-i k_L \hat A_L e^{-i(k_L x_1-\omega t)}\\ \hat u_{L_2}=\partial\hat\psi/\partial x_2=0\\ \hat u_{L_3}=\partial\hat\psi/\partial x_3=0\end{array}\right._{\mathscr B}$

⇒ $\vec u_L(x_1;t)=\mathscr{R}e\left[\hat{\vec u}_L(x_1;t)\right]=k_L\left|\hat A_L\right|\sin(\omega t-k_L x_1+\alpha_L)\vec e_{x_1}$ *with* $\hat A_L=\left|\hat A_L\right|e^{i\alpha_L}$

✓ $\hat{\vec T}_L=\hat{\bar{\bar T}}\cdot\vec e_{x_1}=\hat T_{11}\vec e_{x_1}+\hat T_{21}\vec e_{x_2}+\hat T_{31}\vec e_{x_3}$

with $\hat T_{11}=(\lambda+2\mu)\partial\hat u_{L_1}/\partial x_1=-(\lambda+2\mu)k_L^2\hat A_L e^{-i(k_L x_1-\omega t)}$

$\hat T_{21}=\mu\,\partial\hat u_{L_2}/\partial x_1=0$ *and* $\hat T_{31}=\mu\,\partial\hat u_{L_3}/\partial x_1=0$

⇒ $\hat{\vec T}_L=\hat T_{11}\vec e_{x_1}=-(\lambda+2\mu)k_L^2\hat A_L e^{-i(k_L x_1-\omega t)}\vec e_{x_1}$

⇒ $\vec T_L=\mathscr{R}e\left[\hat{\vec T}_L\right]=-(\lambda+2\mu)k_L^2\left|\hat A_L\right|\cos(\omega t-k_L x_1+\alpha_L)\vec e_{x_1}$ // $\vec n=\vec e_{x_1}$

⇒ **compression / expansion motion**

Example : monochromatic plane waves (2/2)

- Shear waves

✓ $\hat{\vec u}_T(x_1;t)=\overrightarrow{\text{rot}}\,\hat{\vec\chi}(x_1;t)$ *with* $\hat{\vec\chi}(x_1;t)=\hat{\vec A}_T e^{-i(k_T x_1-\omega t)}$

⇒ $\hat{\vec u}_T(x_1;t)=\left|\begin{array}{l}\partial/\partial x_1\\ \partial/\partial x_2\\ \partial/\partial x_3\end{array}\right._{\mathscr B}\wedge\left|\begin{array}{l}\hat\chi_1\\ \hat\chi_2\\ \hat\chi_3\end{array}\right._{\mathscr B}=\left|\begin{array}{l}\partial\hat\chi_3/\partial x_2-\partial\hat\chi_2/\partial x_3\\ \partial\hat\chi_1/\partial x_3-\partial\hat\chi_3/\partial x_1\\ \partial\hat\chi_2/\partial x_1-\partial\hat\chi_1/\partial x_2\end{array}\right._{\mathscr B}=\left|\begin{array}{l}0\\ -\partial\hat\chi_3/\partial x_1\\ \partial\hat\chi_2/\partial x_1\end{array}\right._{\mathscr B}=\left|\begin{array}{l}\hat u_{T_1}=0\\ \hat u_{T_2}=i k_T\hat A_{T_3}e^{-i(k_T x_1-\omega t)}\\ \hat u_{T_3}=-i k_T\hat A_{T_2}e^{-i(k_T x_1-\omega t)}\end{array}\right._{\mathscr B}$

⇒ $\vec u_T(x_1;t)=\mathscr{R}e\left[\hat{\vec u}_T(x_1;t)\right]=-k_T\left|\hat A_{T_3}\right|\sin(\omega t-k_T x_1+\alpha_{T_3})\vec e_{x_2}+k_T\left|\hat A_{T_2}\right|\sin(\omega t-k_T x_1+\alpha_{T_2})\vec e_{x_3}$ *with* $\hat A_{T_2}=\left|\hat A_{T_2}\right|e^{i\alpha_{T_2}}$, $\hat A_{T_3}=\left|\hat A_{T_3}\right|e^{i\alpha_{T_3}}$

✓ $\hat{\vec T}_T=\hat{\bar{\bar T}}\cdot\vec e_{x_1}=\hat T_{11}\vec e_{x_1}+\hat T_{21}\vec e_{x_2}+\hat T_{31}\vec e_{x_3}$

with $\hat T_{11}=(\lambda+2\mu)\partial\hat u_{T_1}/\partial x_1=0$; $\hat T_{21}=\mu\,\partial\hat u_{T_2}/\partial x_1=\mu k_T^2\hat A_{T_3}e^{-i(k_T x_1-\omega t)}$

and $\hat T_{31}=\mu\,\partial\hat u_{T_3}/\partial x_1=-\mu k_T^2\hat A_{T_2}e^{-i(k_T x_1-\omega t)}$

⇒ $\hat{\vec T}_T=\hat T_{21}\vec e_{x_2}+\hat T_{31}\vec e_{x_3}=\mu k_T^2\hat A_{T_3}e^{-i(k_T x_1-\omega t)}\vec e_{x_2}-\mu k_T^2\hat A_{T_2}e^{-i(k_T x_1-\omega t)}\vec e_{x_3}$

⇒ $\vec T_T=\mathscr{R}e\left[\hat{\vec T}_T\right]=\mu k_T^2\left|\hat A_{T_3}\right|\cos(\omega t-k_L x_1+\alpha_{T_3})\vec e_{x_2}-\mu k_T^2\left|\hat A_{T_2}\right|\cos(\omega t-k_L x_1+\alpha_{T_2})\vec e_{x_3}$

// $(O\,x_2\,x_3)$ ⊥ $\vec n=\vec e_{x_1}$ ⇒ **shear motion**

Acoustical energy conservation (1/3)

- Integration of the propagation equation

$$\rho\frac{\partial u_i}{\partial t}\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial u_i}{\partial t}\frac{\partial T_{ij}}{\partial x_j}$$

with $T_{ij}=c_{ijk\ell}S_{k\ell}$ *Hooke law*

$$\frac{\partial\mathscr E_c}{\partial t}$$

with $\mathscr E_c=\frac{1}{2}\rho\left(\frac{\partial u_i}{\partial t}\right)^2$ *voluminal density of kinetic energy*

$$\frac{\partial}{\partial x_j}\left(T_{ij}\frac{\partial u_i}{\partial t}\right)-T_{ij}\frac{\partial^2 u_i}{\partial x_j\partial t}$$

$$T_{ij}\frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_j}\right)=T_{ij}\frac{\partial}{\partial t}(S_{ij})$$

$$\frac{\partial\mathscr E_p}{\partial t}$$

with $\mathscr E_p=\frac{1}{2}c_{ijk\ell}S_{k\ell}S_{ij}$ *voluminal density of potential energy*

$$\text{div}\left(T_{ij}\frac{\partial u_i}{\partial t}\right)=-\text{div}(\vec{\mathscr P})$$

with $\vec{\mathscr P}=-\bar{\bar T}\cdot\frac{\partial\vec u}{\partial t}$ *Poynting vector*

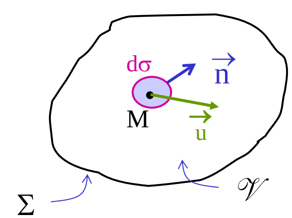
- Local equation of energy $\frac{\partial\mathscr E}{\partial t}+\text{div}(\vec{\mathscr P})=0$

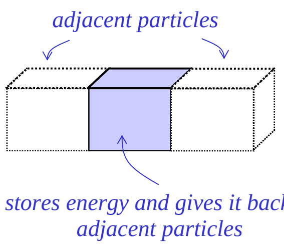
with $\mathscr E=\mathscr E_c+\mathscr E_p$ *densité volumique d'énergie acoustique totale*

- Nota Bene: energy density = energy stored per volume unit ($\mathscr E_{\text{volume } d\mathscr V}/ d\mathscr V$)

Acoustical energy conservation (2/3)

During the acoustic propagation





The acoustical energy $\mathscr E$ which is locally presentin the particle ($\mathscr E_c$ + $\mathscr E_p$) thus results from a ***supply and*** from a ***loss of energy***.

⇔ Energy flux, provided and withdrawn continuously to the volume.

$\vec{\mathscr P}=-\bar{\bar T}\cdot\frac{\partial\vec u}{\partial t}$: instantaneous acoustic energy flux, brought back to unit area and to unit time (instantaneous power crossing the unit of area dσ, transported by the acoustic wave ; analogue with $\vec{\mathscr P}=p\vec v$ in fluid

$\left(-\bar{\bar T}\cdot\frac{\partial\vec u}{\partial t}\right)\cdot\overrightarrow{d\sigma}\,dt$: elementary work provided by a particle to its environment during time dt.

Acoustical energy conservation (3/3)

- Integral balance $\frac{\partial}{\partial t}\iiint_{\mathscr{V}}\left(\mathscr{E}_c+\mathscr{E}_p\right)d\mathscr{V} = -\iiint_{\mathscr{V}} \operatorname{div}\left(\vec{\mathscr{P}}\right)d\mathscr{V}$

variation per unit time of the acoustical energy contained in a volume $\mathscr{V}$

opposite of the outgoing energy, per unit time

(Ostrogradsky Theorem)

$-\iint_{\Sigma}\vec{\mathscr{P}}\cdot\vec{n}\,d\sigma$ *total flux of energy incoming into the volume $\mathscr{V}$ per unit time*

$\Rightarrow \iiint_{\mathscr{V}}\left[\frac{\partial}{\partial t}\left(\mathscr{E}_c+\mathscr{E}_p\right)+\operatorname{div}\left(\vec{\mathscr{P}}\right)\right]d\mathscr{V} = 0$

- Poynting vector $\vec{\mathscr{P}} = -\bar{\bar{T}}\cdot\frac{\partial\vec{u}}{\partial t}$ gives the ***direction of the propagation of the energy***

- Energy velocity $\vec{V}^e = \frac{\vec{\mathscr{P}}}{\mathscr{E}}$ *with* $\mathscr{E}=\mathscr{E}_c+\mathscr{E}_p=\frac{1}{2}\rho\left(\frac{\partial u_i}{\partial t}\right)^2+\frac{1}{2}c_{ijk\ell}S_{k\ell}S_{ij}$

Energy velocity for a plane wave

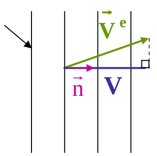
- Energy velocity $\vec{V}^e = \frac{\vec{\mathscr{P}}}{\mathscr{E}}$ *with* $\begin{cases}\mathscr{P}_i = -T_{ij}\frac{\partial u_j}{\partial t}\\ \mathscr{E}=\mathscr{E}_c+\mathscr{E}_p=\frac{1}{2}\rho\left(\frac{\partial u_i}{\partial t}\right)^2+\frac{1}{2}c_{ijk\ell}S_{k\ell}S_{ij}\\ u_i = P_i F\left(t-\frac{n_j x_j}{V}\right)\end{cases}$

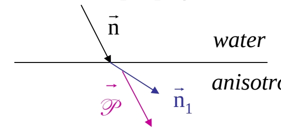
- $\mathscr{P}_i = c_{ijk\ell}P_\ell P_j n_k \frac{F'^2}{V}$

- $\mathscr{E}_c = \frac{1}{2}\rho P_i P_i F'^2 = \mathscr{E}_p$ $\Rightarrow$ during the propagation, energy is distributed in an equal way between kinetic energy and potential energy

- $V_i^e = \frac{c_{ijk\ell}P_\ell P_j n_k}{\rho P_m P_m V}$

- Projection of the energy velocity on the propagation direction $\vec{V}^e\cdot\vec{n} = V$



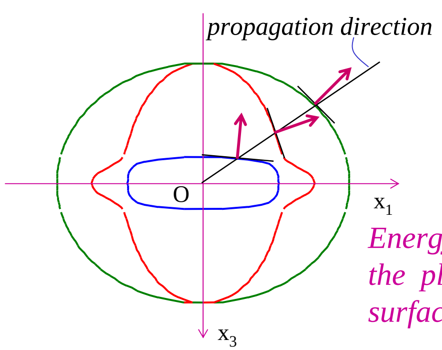


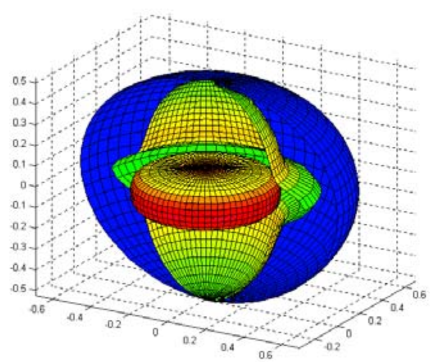
the vibration propagates in the direction of $\vec{\mathscr{P}}$

Slowness surface (1/3)

Location of the ends of the slowness vector $\vec{m}$, drawn from a fixed point O, when the propagation $\vec{n}$ varies.

Slowness vector : $\vec{m} = \frac{\vec{n}}{V}$

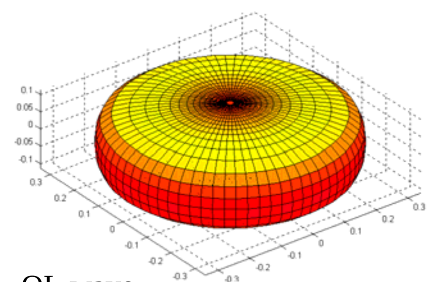




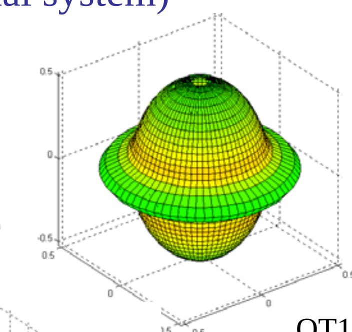
Energy velocity perpendicular to the plane tangent to the slowness surface

Carbon/Epoxy, hexagonal, axis A_6 // x_1

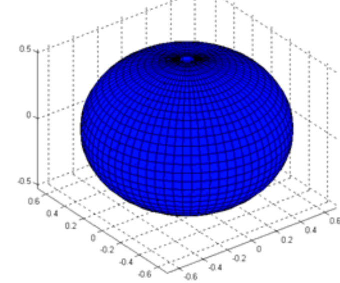
Slowness surface (2/3): example of carbone/epoxy (hexagonal system)



QL wave



QT1 wave



QT2 wave

Slowness surface (3/3)

- Direction of the tangent plane: $d\vec{m} = \frac{\partial \vec{m}}{\partial n_i} dn_i$ *i.e.* $dm_j = \frac{\partial m_j}{\partial n_i} dn_i$

Using $m_j = \frac{n_j}{V}$ ⟹ $\frac{\partial m_j}{\partial n_i} = \frac{\delta_{ij}}{V} - \frac{1}{V^2}\frac{\partial V}{\partial n_i} n_j$

- Energy velocity: $\vec{V}^e \cdot \vec{n} = V$ ⟹ $V_i^e = \frac{\partial V}{\partial n_i}$

⟹ $\frac{\partial m_j}{\partial n_i} = \frac{\delta_{ij}}{V} - \frac{1}{V^2} V_i^e n_j$

⟹ $dm_j = \left(\frac{\delta_{ij}}{V} - \frac{1}{V^2} V_i^e n_j\right) dn_i$

- $\vec{V}^e$ perpendicular to the tangent plane ⟺ $d\vec{m} \cdot \vec{V}^e = 0$

Demontration:

$$d\vec{m} \cdot \vec{V}^e = dm_j V_j^e = \left(\frac{\delta_{ij}}{V} - \frac{1}{V^2} V_i^e n_j\right) V_j^e dn_i = \left(\underbrace{\frac{\delta_{ij}}{V} V_j^e}_{\frac{V_i^e}{V}} - \frac{1}{V^2} V_i^e \underbrace{V_j^e n_j}_{\mathbf{V}}\right) dn_i$$

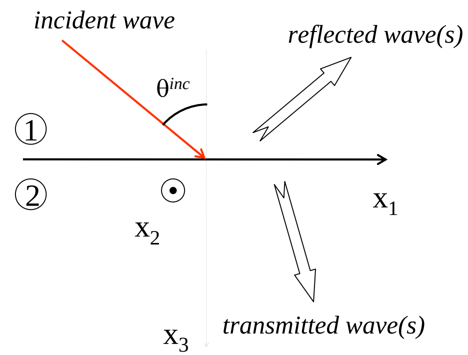
Using $\vec{V}^e \cdot \vec{n} = V \Rightarrow V_j^e n_j = V$

⟹ $d\vec{m} \cdot \vec{V}^e = \left(\frac{V_i^e}{V} - \frac{1}{V^2} V_i^e V\right) dn_i = 0$

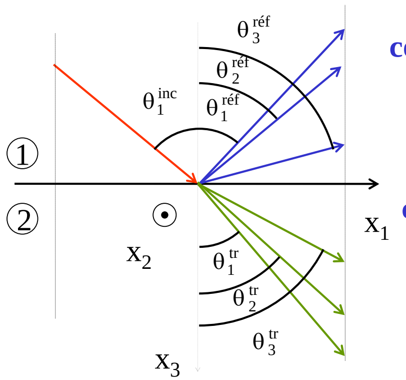
II. REFLECTION AND REFRACTION OF MONOCHROMATIC PLANE WAVES

1 Boundary conditions (perfect bonding)
2 Conservation of the frequency and of the projection of the wave vectors on the interface
3 Graphical construction: use of slowness surfaces
4 Critical angles - evanescent waves
5 Reflection and transmission coefficients

Reflection and refraction (1/2)



Reflection and refraction (2/2)



continuity of displacements at $x_3 = 0$:

$$u_i^{inc} + \sum_{réf} u_i^{réf} = \sum_{tr} u_i^{tr}$$

continuity of stress at $x_3 = 0$:

$$T_{i3}^{inc} + \sum_{réf} T_{i3}^{réf} = \sum_{tr} T_{i3}^{tr}$$

stresses acting on a surface element at the interface

$$\vec{T} = \overline{\overline{T}} \cdot \vec{e}_{x_3} = \mathscr{B} \begin{vmatrix} T_{13} = T_5 \\ T_{23} = T_4 \\ T_{33} = T_3 \end{vmatrix}$$

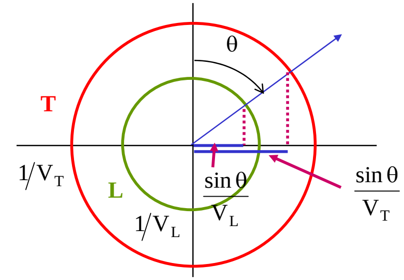
Visualisation of Snell-Descartes' law: slowness surface

Location of the ends of the slowness vector $\vec{m}$, drawn from a fixed point O, when the propagation $\vec{n}$ varies.

Slowness vector : $\vec{m} = \dfrac{\vec{n}}{V}$

Isotropic medium : 2 velocities for a given propagation direction
same velocities in any direction

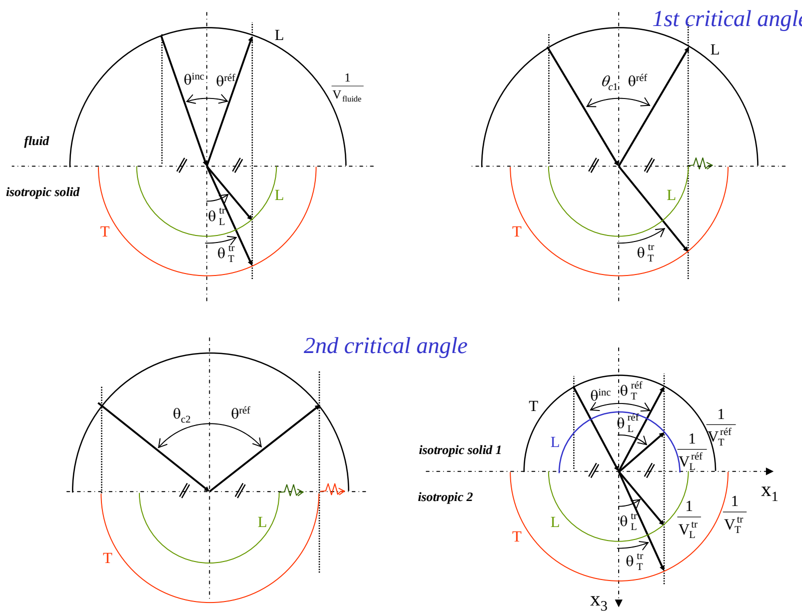
slowness surfaces = spheres



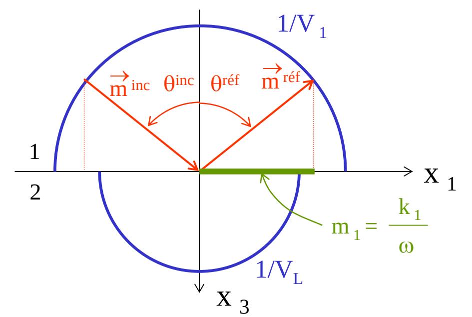
Snell-Descartes' law :

$$\frac{\sin\theta_1}{V_1} = \frac{\sin\theta_2}{V_2}$$

Isotropic media: critical angles -evanescent waves



Isotropic medium: evanescent waves



slowness vector: $\vec{m} = \frac{\vec{k}}{\omega}$

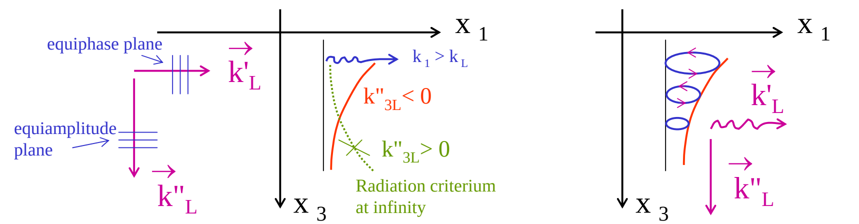
$m_1 = \frac{k_1}{\omega} > \frac{1}{V_L} = \frac{k_L}{\omega} \Rightarrow k_1 > k_L$

Dispersion relation:

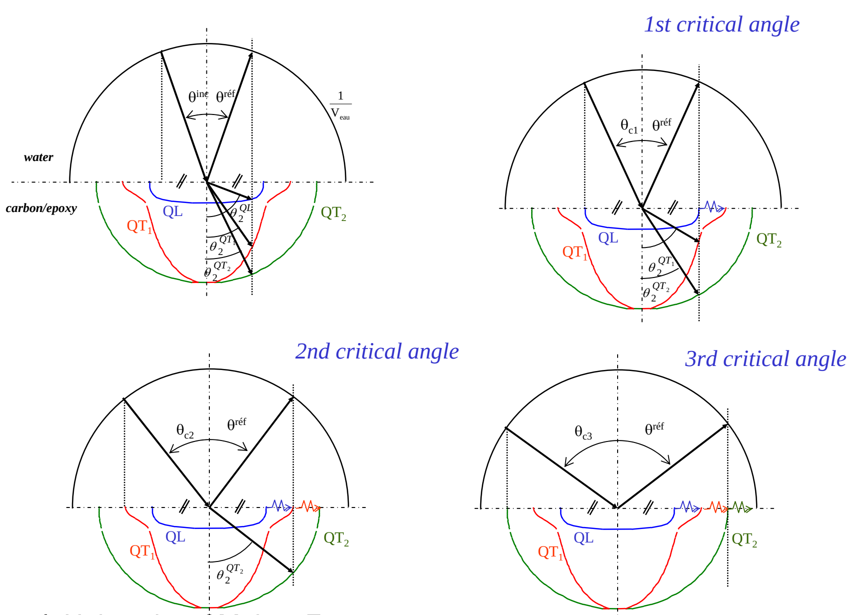
$k_1^2 + k_{3L}^2 = k_L^2 \Rightarrow k_{3L}^2 = k_L^2 - k_1^2 < 0$

$k_{3L} = i\, k''_{3L}$ $\quad k''_{3L} = \pm\sqrt{k_1^2 - k_L^2}$

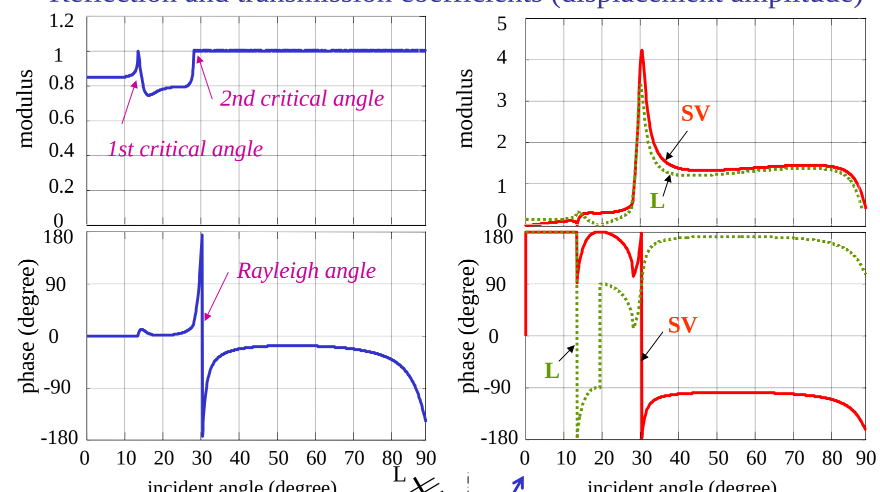
$\hat{\psi} = \hat{A}_L\, e^{k''_{3L} x_3}\, e^{-i(k_1 x - \omega t)}$



Anisotropic media: critical angles - evanescent waves



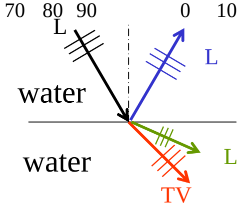
Reflection and transmission coefficients (displacement amplitude)



Water

$\rho = 1000$ kg/m³

$V_L = 1480$ m/s



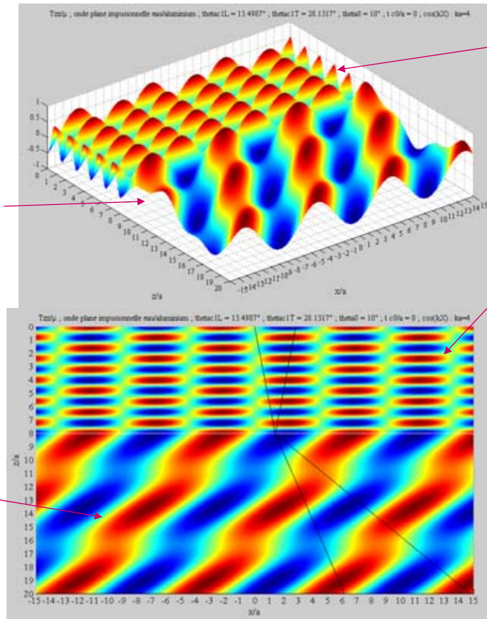
Aluminium

$\rho = 2786$ kg/m³

$V_L = 6650$ m/s

$V_T = 3447$ m/s

Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta<\theta_{cL}$



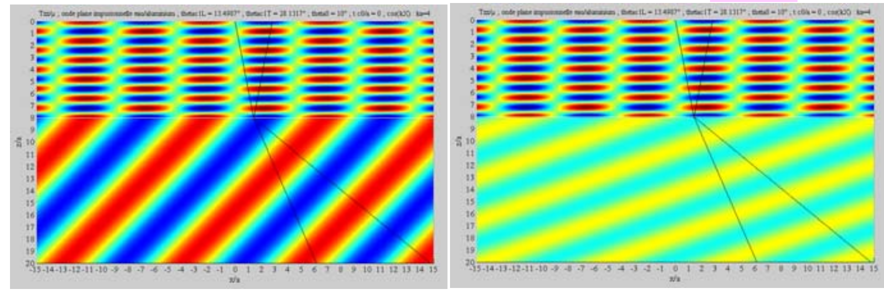
total field with a stationary character along z, and a propagative character along x

transmitted waves

transmitted waves

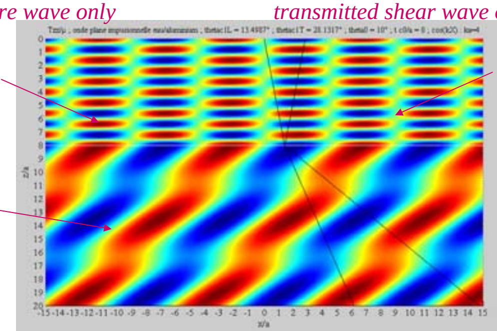
isotropic

Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta<\theta_{cL}$



transmitted pressure wave only

transmitted shear wave only



reflected wave

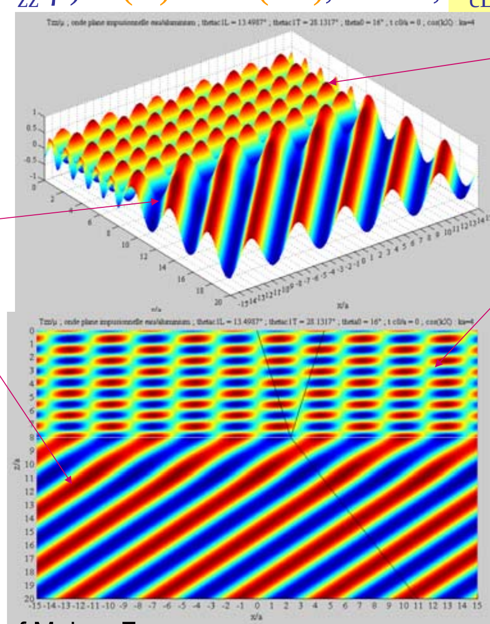
incident wave

transmitted waves

summation of pressure and shear fields

isotropic

Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta_{cL}< \theta <\theta_{cT}$

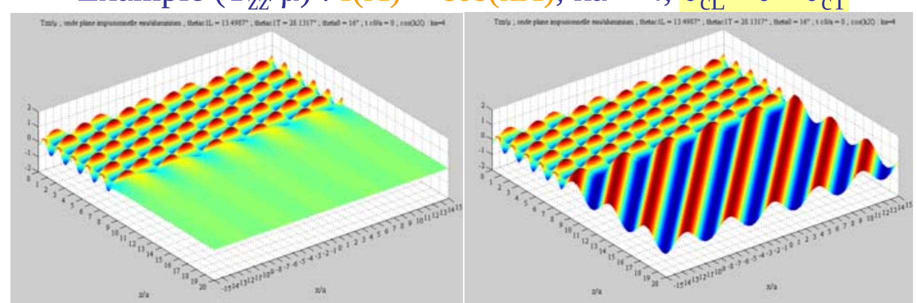


total field with a stationary character along z, and a propagative character along x

transmitted evanescent pressure wave and propagative shear waves

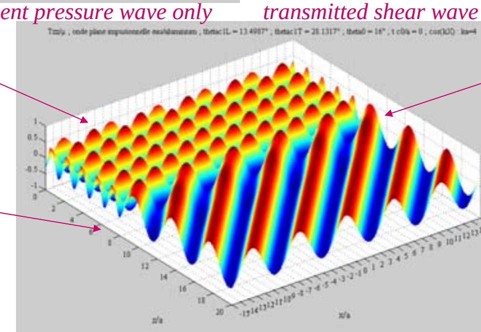
isotropic

Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta_{cL}< \theta <\theta_{cT}$



transmitted evanescent pressure wave only

transmitted shear wave only



reflected wave

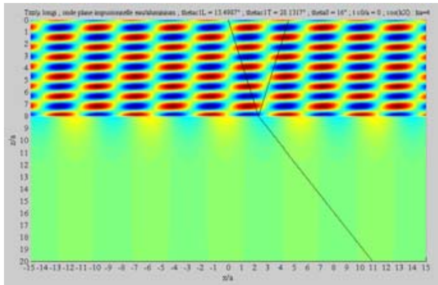
incident wave

transmitted waves

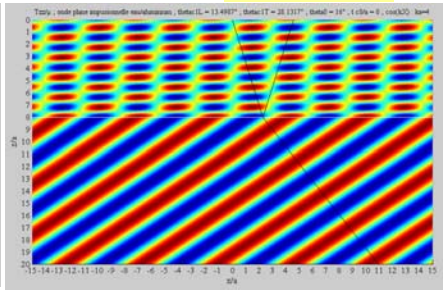
summation of pressure and shear fields

isotropic

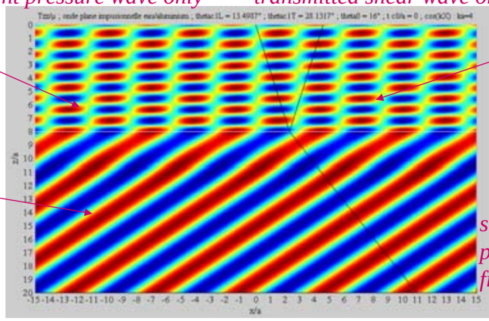
Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta_{cL} < \theta < \theta_{cT}$



transmitted evanescent pressure wave only

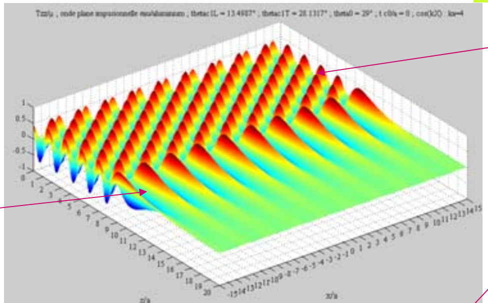


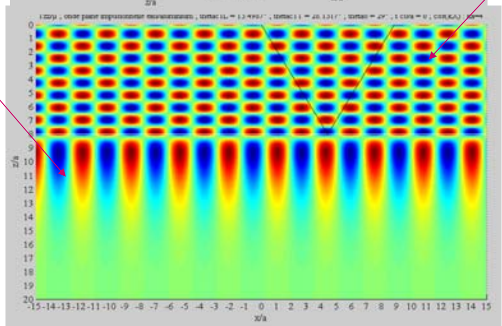
transmitted shear wave only



isotropic

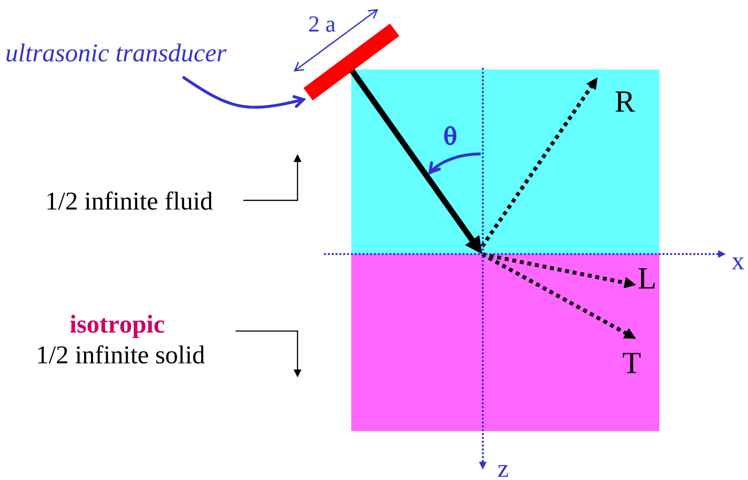
Example (T_{zz}/μ) : $f(X) = \cos(kX)$, ka = 4, $\theta > \theta_{cT}$





isotropic

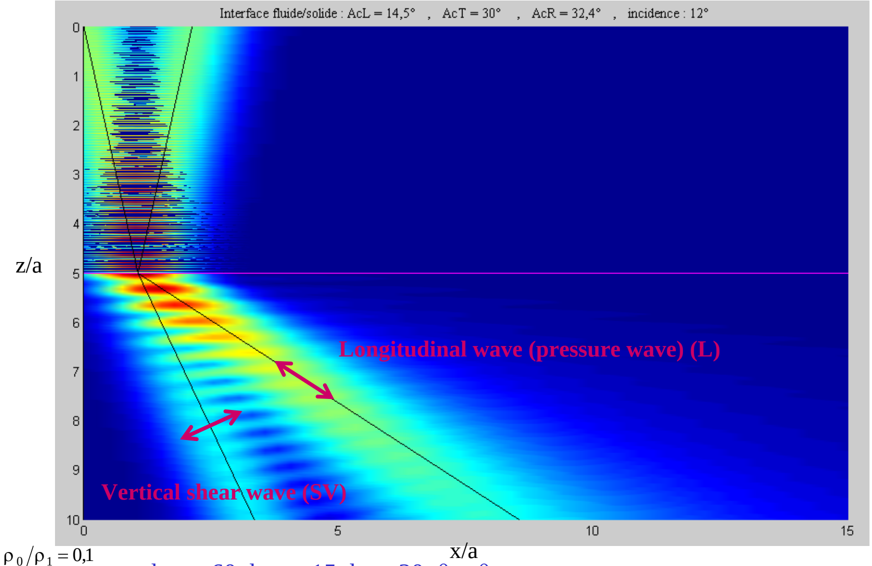
Gaussian incident beam onto an interface fluid/solid



Modulus of the stress

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne

Before the 1st critical angle

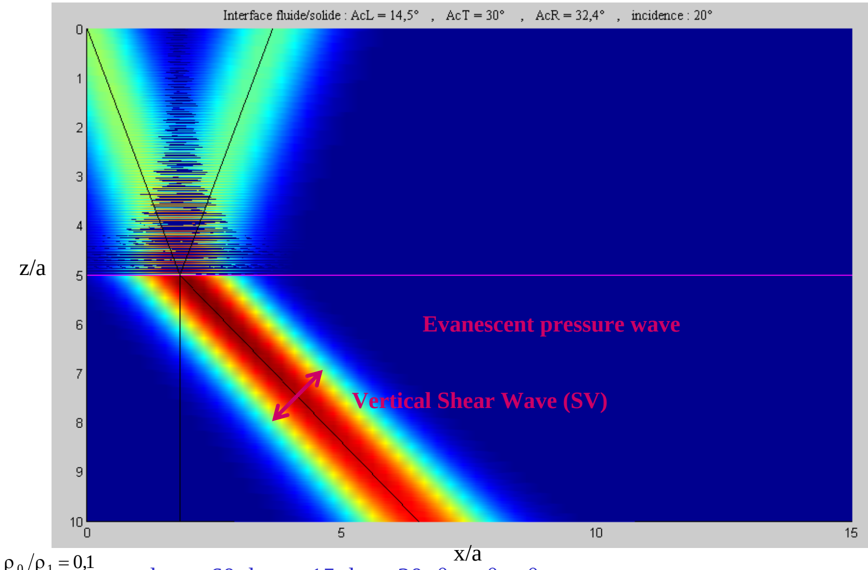


$\rho_0/\rho_1 = 0{,}1$

$k_0a = 60$, $k_La = 15$, $k_Ta=30$, $\theta < \theta_{c1}$

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne

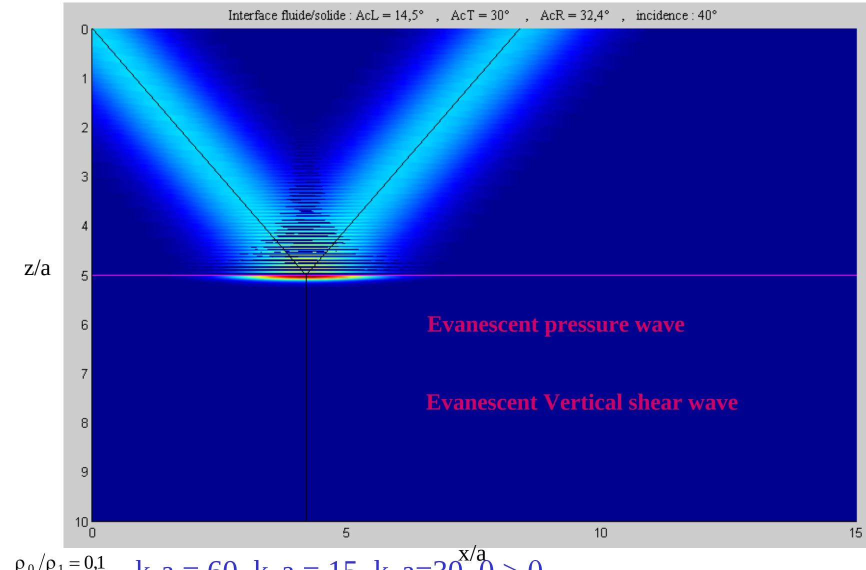
Between the two critical angles



$k_0a = 60$, $k_La = 15$, $k_Ta=30$, $\theta_{c1} < \theta < \theta_{c2}$

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne

After the two critical angles

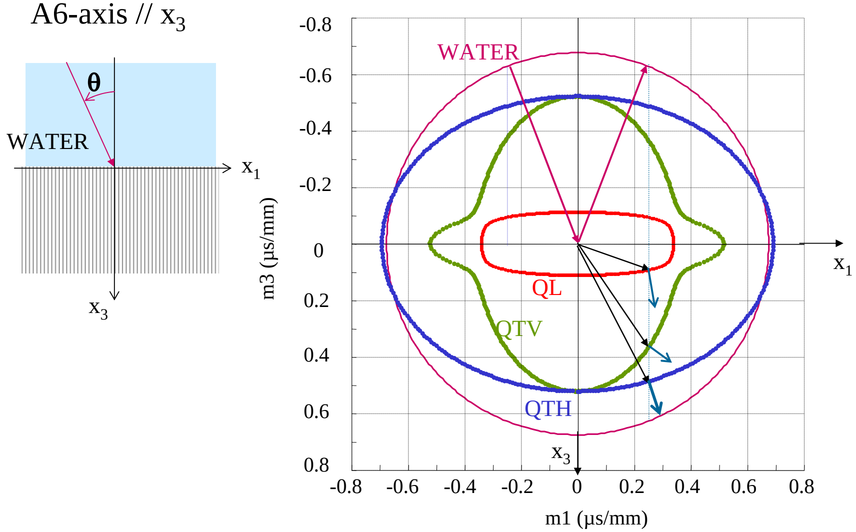


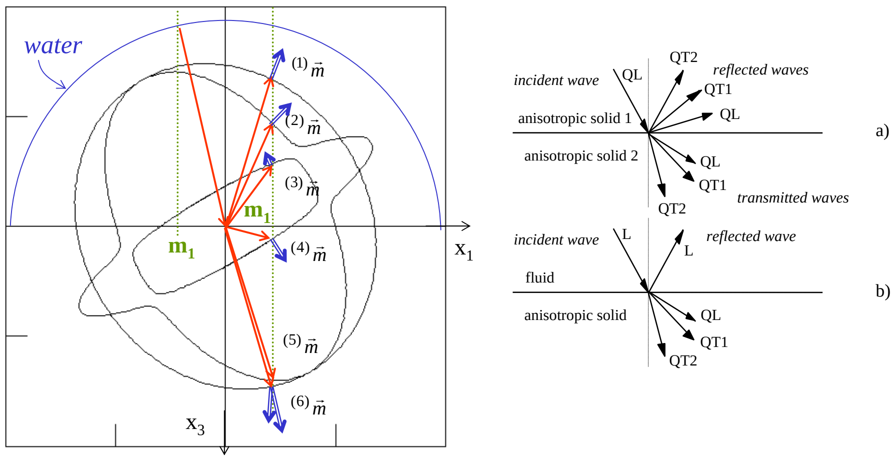
$k_0a = 60$, $k_La = 15$, $k_Ta=30$, $\theta > \theta_{c2}$

Programs realised by Ph. Gatignol, Pr., Université de Technologie de Compiègne

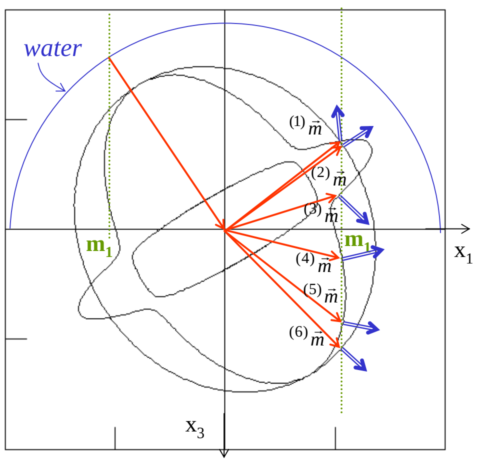
Interface water / unidirectional composite

A6-axis // x_3





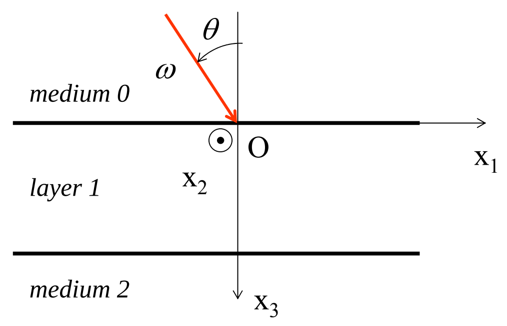
Cross section of the slowness surface (carbon/epoxy)



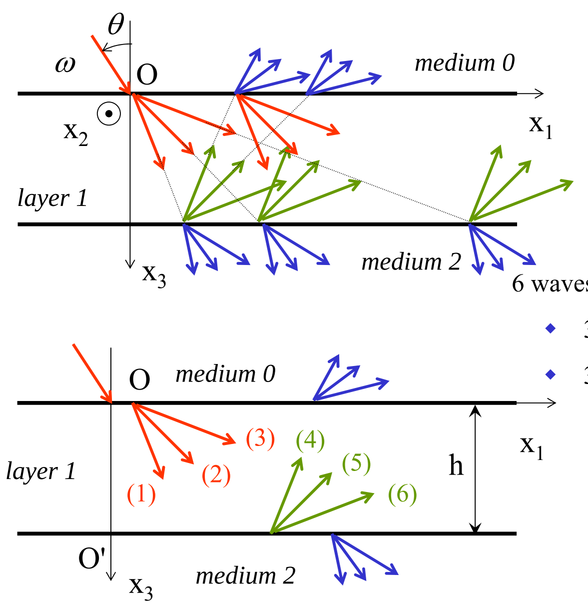
Cross section of the slowness surface (carbon/epoxy)

III. PROPAGATION IN A SINGLE LAYER

Reflection - transmission in a layer q (1/4)



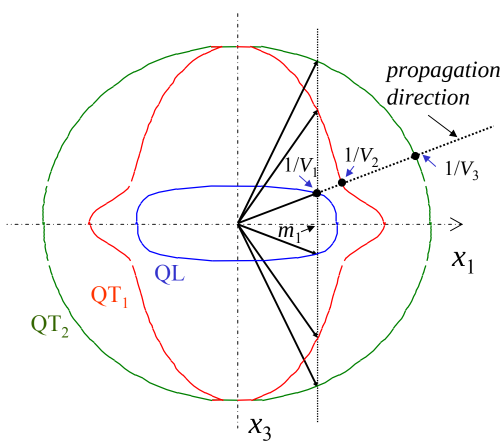
Reflection - transmission in a layer q (2/4)



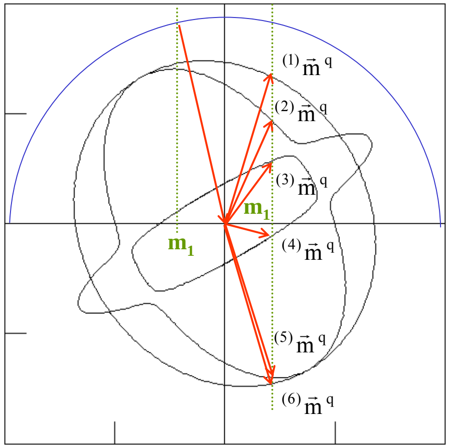
6 waves in each layer:

- 3 in the direction $x_3 > 0$
- 3 in the direction $x_3 < 0$

Reflection - transmission in a layer q (3/4)



Reflection - transmission in a layer q (4/4)



Cross section of the slowness surface (carbon/epoxy)

Slowness vector:

$$^{(\eta)}\vec{m}^{\,q} = \frac{^{(\eta)}\vec{n}^{\,q}}{^{(\eta)}V^{q}} = \frac{^{(\eta)}\vec{k}^{\,q}}{\omega}$$

Propagation equation

$$\left(c_{ijkl}\,{}^{(\eta)}m_j\,{}^{(\eta)}m_l - \rho\,\delta_{ik}\right){}^{(\eta)}P_k = 0 \qquad \textbf{(1)}$$

Données : m_1, $\quad {}^{(\eta)}m_2 = 0$ *(choice of the sagittal plane)*

Unknowns: ${}^{(\eta)}m_3$, ${}^{(\eta)}P_k \quad k = 1,2,3$

Solving: $\det\left(c_{ijkl}\,{}^{(\eta)}m_j\,{}^{(\eta)}m_l - \rho\,\delta_{ik}\right) = 0$

(2) → 6-th degree equation in ${}^{(\eta)}m_3$

→ 6 slowness vectors ${}^{(\eta)}\vec{m} \qquad \eta = 1,\ldots,6$

(3) → 6 polarisation vectors ${}^{(\eta)}\vec{P}$

Particle displacement vector

- Monochromatique plane wave (η) $\;{}^{(\eta)}\vec{u}(\vec{x};t) = {}^{(\eta)}a\,{}^{(\eta)}\vec{P}\,e^{i\left(\omega t - {}^{(\eta)}\vec{k}.\vec{x}\right)}$

→ $${}^{(\eta)}\vec{u}(x_1,x_2,x_3;t) = {}^{(\eta)}a\,{}^{(\eta)}\vec{P}\,e^{i\omega\left(t - m_1 x_1 - {}^{(\eta)}m_3 x_3\right)}$$

amplitude *polarisation vector*

- Total particle displacement

$$\vec{u}(x_1,x_2,x_3;t) = e^{i\omega\left(t - m_1 x_1\right)} \sum_{\eta=1}^{6} {}^{(\eta)}a\,{}^{(\eta)}\vec{P}\,e^{-i\omega\,{}^{(\eta)}m_3 x_3}$$

Hooke law (reminder)

- Small deformations hypothesis :

Hooke law $\quad T_{ij} = c_{ijk\ell}\, S_{k\ell}$

elastic rigidity

- Matrix notation : $\quad c_{\alpha\beta} = c_{ijk\ell}$

$\alpha \leftrightarrow (i\,j)$

$\beta \leftrightarrow (k\,\ell)$

$(11)\leftrightarrow 1 \quad (22)\leftrightarrow 2 \quad (33)\leftrightarrow 3$

$(32)=(23)\leftrightarrow 4 \quad (31)=(13)\leftrightarrow 5 \quad (12)=(21)\leftrightarrow 6$

Hooke law

$T_\alpha = c_{\alpha\beta}\, S_\beta$ with $\quad S_1 = S_{11} \quad S_2 = S_{22} \quad S_3 = S_{33}$; $\quad S_4 = 2S_{23} \quad S_5 = 2S_{13} \quad S_6 = 2S_{12}$

and $\quad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

Stress vector

- normal to interfaces: $\vec{e}_{x_3}$ ⇒ $\vec{T}\left(M,\vec{e}_{x_3}\right)=\begin{vmatrix}T_{13}\\T_{23}\\T_{33}\end{vmatrix}_{\mathscr{B}}=\begin{vmatrix}T_5\\T_4\\T_3\end{vmatrix}_{\mathscr{B}}$ — T_5, T_4: *tangential*; T_3: *normal*

⇒

$$T_\alpha=-i\omega e^{i\omega\left(t-m_1x_1\right)}\sum_{\eta=1}^{6}{}^{(\eta)}a\,e^{-i\omega\,{}^{(\eta)}m_3x_3}\Big[c_{\alpha1}m_1{}^{(\eta)}P_1+c_{\alpha3}{}^{(\eta)}m_3{}^{(\eta)}P_3+c_{\alpha4}{}^{(\eta)}m_3{}^{(\eta)}P_2+c_{\alpha6}m_1{}^{(\eta)}P_2+c_{\alpha5}\left({}^{(\eta)}m_3{}^{(\eta)}P_1+m_1{}^{(\eta)}P_3\right)\Big],\ \alpha=3,4,5.$$

Displacements-stresses vector

- Displacement-stress vector $\mathscr{W}\left(x_3\right)=\left\langle u_1,u_2,u_3,T_{33},T_{23},T_{13}\right\rangle^T,\ 0\le x_3\le h$
- Amplitude vector $\mathscr{A}=\left\langle {}^{(1)}a,{}^{(2)}a,{}^{(3)}a,{}^{(4)}a,{}^{(5)}a,{}^{(6)}a\right\rangle^T$

⇒ $\mathscr{W}\left(x_3\right)=\Omega A\,\mathscr{H}\left(x_3\right)\mathscr{A}\,e^{i\omega\left(t-m_1x_1\right)}$

$$\Omega=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-i\omega&0&0\\0&0&0&0&-i\omega&0\\0&0&0&0&0&-i\omega\end{bmatrix}$$

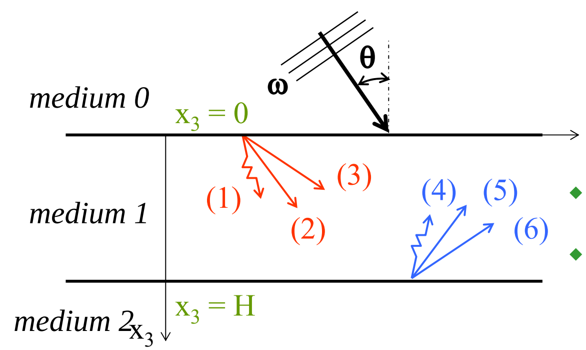
$$\mathscr{H}\left(x_3\right)=\begin{bmatrix}e^{-i\omega{}^{(1)}m_3x_3}&0&0&0&0&0\\0&e^{-i\omega{}^{(2)}m_3x_3}&0&0&0&0\\0&0&e^{-i\omega{}^{(3)}m_3x_3}&0&0&0\\0&0&0&e^{-i\omega{}^{(4)}m_3x_3}&0&0\\0&0&0&0&e^{-i\omega{}^{(5)}m_3x_3}&0\\0&0&0&0&0&e^{-i\omega{}^{(6)}m_3x_3}\end{bmatrix}$$

Propagation matrix of a layer q

η-th column given by:

$$\left(A^q\right)_\eta=\begin{Bmatrix}{}^{(\eta)}P_1^q\\{}^{(\eta)}P_2^q\\{}^{(\eta)}P_3^q\\c_{31}^q m_1{}^{(\eta)}P_1^q+c_{33}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_3^q+c_{34}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_2^q+c_{36}^q m_1{}^{(\eta)}P_2^q+c_{35}^q\left({}^{(\eta)}m_3^q{}^{(\eta)}P_1^q+m_1{}^{(\eta)}P_3^q\right)\\c_{41}^q m_1{}^{(\eta)}P_1^q+c_{43}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_3^q+c_{44}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_2^q+c_{46}^q m_1{}^{(\eta)}P_2^q+c_{45}^q\left({}^{(\eta)}m_3^q{}^{(\eta)}P_1^q+m_1{}^{(\eta)}P_3^q\right)\\c_{51}^q m_1{}^{(\eta)}P_1^q+c_{53}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_3^q+c_{54}^q{}^{(\eta)}m_3^q{}^{(\eta)}P_2^q+c_{56}^q m_1{}^{(\eta)}P_2^q+c_{55}^q\left({}^{(\eta)}m_3^q{}^{(\eta)}P_1^q+m_1{}^{(\eta)}P_3^q\right)\end{Bmatrix}$$

Numerical problems



- wave (1): inhomogeneous
- wave (4): inhomogeneoux

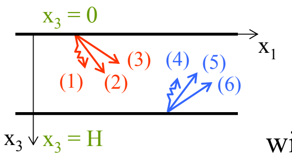
exponentiel propagation factor:

$$e^{-ik_3H}=e^{-ik'_3H}\,e^{k''_3H}$$

- Solution: reference wave (1) at $x_3=0$ and wave (4) at $x_3=H$

Change of reference

- Displacements and stresses in the layer, of the form:



$$\sum_{\eta=1}^{3} \alpha_\eta \, e^{-i\omega\, {}^{(\eta)}m_3 (x_3 - 0)} + \sum_{\eta=4}^{6} \alpha_\eta \, e^{-i\omega\, {}^{(\eta)}m_3 (x_3 - H)}$$

with $\quad {}^{(\eta)}m_3 = {}^{(\eta)}m'_3 + i\, {}^{(\eta)}m''_3$

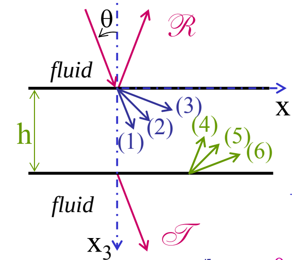
- At $x_3 = 0$

$$\sum_{\eta=1}^{3} \alpha_\eta + \underbrace{\sum_{\eta=4}^{6} \alpha_\eta \, e^{+i\omega\, {}^{(\eta)}m'_3 H} e^{-\omega\, {}^{(\eta)}m''_3 H}}_{\to 0 \ \text{if} \ \eta = 4}$$

- At $x_3 = H$

$$\underbrace{\sum_{\eta=1}^{3} \alpha_\eta e^{-i\omega\, {}^{(\eta)}m'_3 H} e^{+\omega\, {}^{(\eta)}m''_3 H}}_{\to 0 \ \text{if} \ \eta = 1} + \sum_{\eta=4}^{6} \alpha_\eta$$

Case of an anisotropic layer immersed in a fluid



Boundary conditions	Unknowns
$x_3 = 0$: equality u_3, T_{13}, T_{23}, T_{33}	ℛ, 𝒯
$x_3 = h$: equality u_3, T_{13}, T_{23}, T_{33}	${}^{(\eta)}a$, $\eta = 1,\ldots,6$
8 equations	*8 unknowns*

$$\vec{P}^{inc} = \begin{vmatrix} \sin\theta \\ 0 \\ \mathscr{B} \ \cos\theta \end{vmatrix} = \vec{P}^{tr}$$

$$\vec{P}^{réf} = \begin{vmatrix} \sin\theta \\ 0 \\ \mathscr{B} \ -\cos\theta \end{vmatrix}$$

$m_1 = \sin\theta / V_f$

$m_3^{inc} = \cos\theta / V_f = m_3^{tr}$

$m_3^{réf} = -\cos\theta / V_f$

$x_3 = 0$

$$u_3 : a^{inc} P_3^{inc} + a^{réf} P_3^{réf} = \sum_{\eta=1}^{6} {}^{(\eta)}a \, \underbrace{{}^{(\eta)}P_3}_{A_{3\eta}}$$

$$T_{31} : 0 = \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{6\eta}$$

$$T_{32} : 0 = \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{5\eta}$$

$$T_{33} : a^{inc} \rho_f V_f + a^{réf} \rho_f V_f = \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{4\eta}$$

$x_3 = h$

$$u_3 : \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{3\eta} \mathscr{H}_{\eta\eta}(h) = a^{tr} P_3^{tr}$$

$$T_{31} : \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{6\eta} \mathscr{H}_{\eta\eta}(h) = 0$$

$$T_{32} : \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{5\eta} \mathscr{H}_{\eta\eta}(h) = 0$$

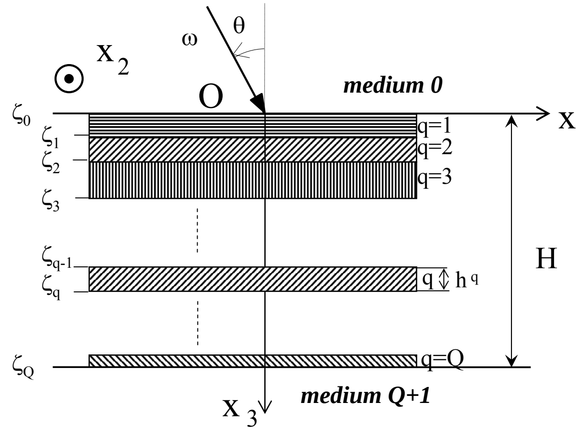
$$T_{33} : \sum_{\eta=1}^{6} {}^{(\eta)}a \, A_{4\eta} \mathscr{H}_{\eta\eta}(h) = a^{tr} \rho_f V_f$$

IV. PROPAGATION IN A MULTILAYERED MEDIUM

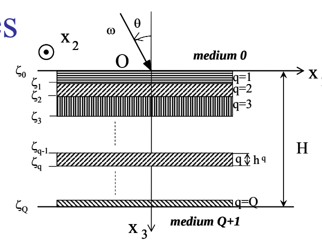
1 Transfer matrix of a layer q

2 Boundary conditions at the extreme interfaces

Multilayered medium



Transfer matrices



- In a layer q , $\zeta_{q-1} \le x_3 \le \zeta_q$

$$\mathscr{W}^q(x_3) = B^q \mathscr{H}^q(x_3) \mathscr{A}^q \exp i\omega(t - m_1 x_1)$$

with $B^q = \Omega A^q$ *and* $\mathscr{H}^q = \mathscr{H}^q(h^q)$

- At $x_3 = \zeta_{q-1}$

$$\mathscr{W}^q(\zeta_{q-1}) = B^q \mathscr{A}^q e^{i\omega(t - m_1 x_1)}$$

- At $x_3 = \zeta_q$

$$\mathscr{W}^q(\zeta_q) = B^q \mathscr{H}^q \mathscr{A}^q e^{i\omega(t - m_1 x_1)}$$

$$\mathscr{W}^q(\zeta_q) = \underbrace{B^q \mathscr{H}^q (B^q)^{-1}}_{} \mathscr{W}^q(\zeta_{q-1})$$

τ^q *: tranfer matrix of the layer q*

- Equality of displacements and stresses at $x_3 = \zeta_q$

$$\mathscr{W}^q(\zeta_q) = \mathscr{W}^{q+1}(\zeta_q) \Rightarrow \mathscr{W}^Q(\zeta_Q) = \underbrace{\tau^Q \tau^{Q-1} \cdots \tau^1}_{} \mathscr{W}^1(\zeta_0)$$

or (equivalent method)

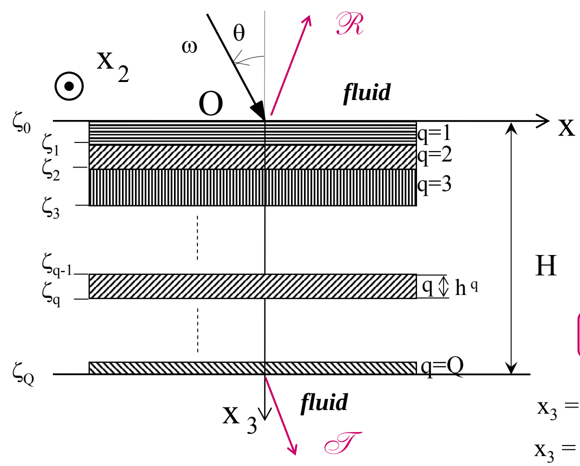
τ *: transfer matrix of the multilayered medium*

$$B^q \mathscr{H}^q \mathscr{A}^q e^{i\omega(t - m_1 x_1)} = B^{q+1} \mathscr{H}^{q+1} \mathscr{A}^{q+1} e^{i\omega(t - m_1 x_1)}$$

$$\Rightarrow \mathscr{A}^{q+1} = (B^{q+1})^{-1} B^q \mathscr{H}^q \mathscr{A}^q = (A^{q+1})^{-1} A^q \mathscr{H}^q \mathscr{A}^q$$

$$\Rightarrow \mathscr{A}^Q = (B^Q)^{-1} \left(\prod_{q=Q-1}^{2} B^q \mathscr{H}^q (B^q)^{-1} \right) B^1 \mathscr{H}^1 \mathscr{A}^1$$

Case of a multilayered medium immersed in a fluid



Unknowns

$\mathscr{R}$, $\mathscr{T}$

${}^{(\eta)}a^q$, η=1,...,6 ; q=1, ...,Q

(6 Q + 2) *unknowns*

Boundary conditions

$x_3 = 0$: equality u_3 , T_{13} , T_{23} , T_{33}

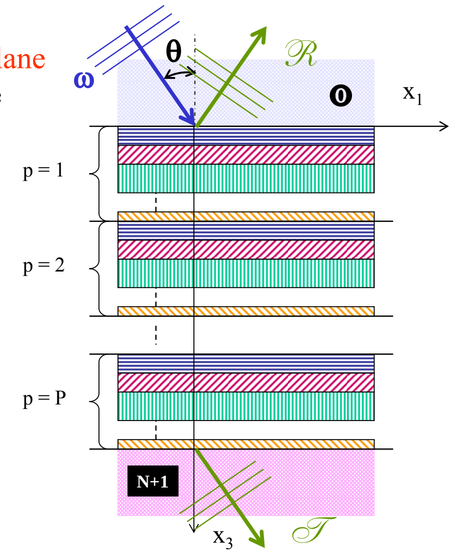
$x_3 = \zeta_q$: equality $\vec{u}$, $\vec{T}$; q=1,...,Q-1

$x_3 = H$: equality u_3 , T_{13} , T_{23} , T_{33}

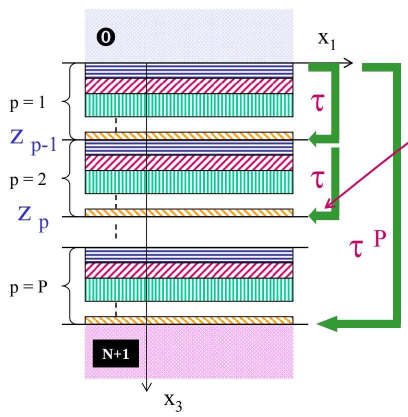
4+6(Q-1)+4=6Q+2 *equations*

Periodically multilayered medium

monochromatic plane
incident wave



Transfer matrix of a period



- Displacement - stress vector:

$$\mathscr{W}(z_p) = \begin{Bmatrix} \vec{u}_p \\ \vec{T}_p \end{Bmatrix}$$

- Boundary conditions:
  - equality of displacements $\vec{u}_p$
  - equality of stresses $\vec{T}_p$

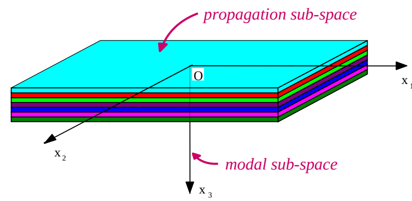
➡ $\mathscr{W}(z_p) = \boldsymbol{\tau}\, \mathscr{W}(z_{p-1})$

➡ $\boldsymbol{\tau}$: transfer matrix of a period, 6th order

V. MODAL WAVES: PARTICULAR CASE OF LAMB WAVES

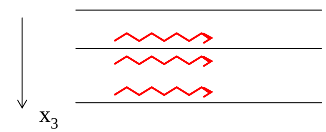
Modal waves (1/4)

Mechanical waves → local waves / modal waves



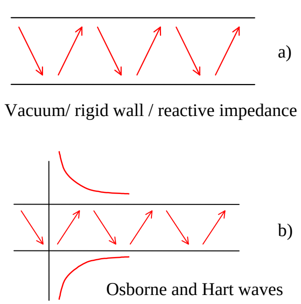
- guided waves
- surface waves de
- interface waves

acoustical energy:
- propagates along layers
- is bounded following x_3

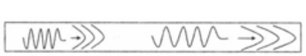


Modal waves (2/4)

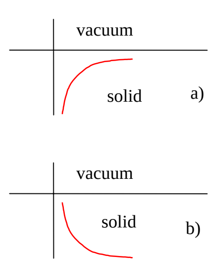
guided waves



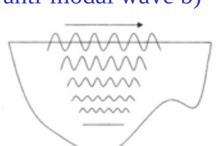
Lamb wave



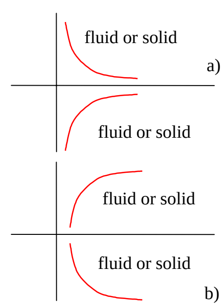
surface waves



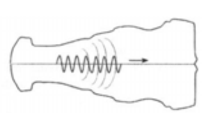
- Rayleigh wave a)
- anti-modal wave b)



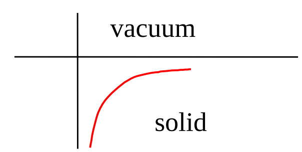
interface waves

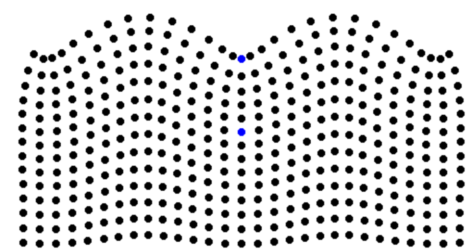


- Scholte wave, Stoneley wave, Rayleigh-Cezawa wave, etc... a)
- anti-modal wave b)



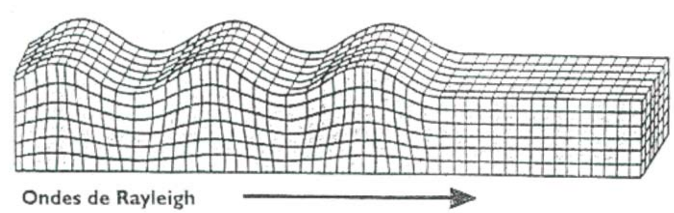
Modal waves (3/4): Rayleigh wave



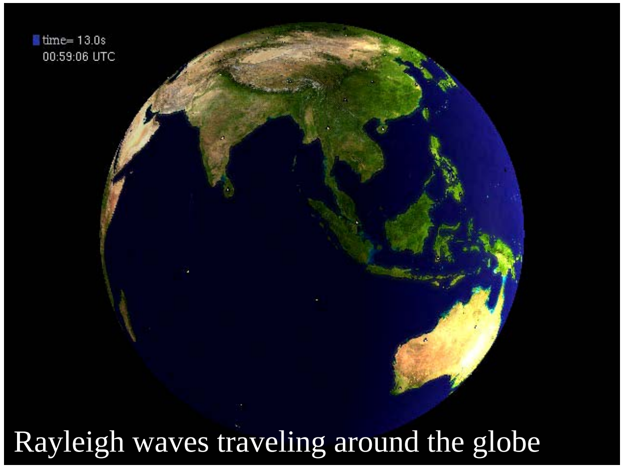


http://www.kettering.edu/~drussell

Animation courtesy of Dr. Dan Russell, Kettering University



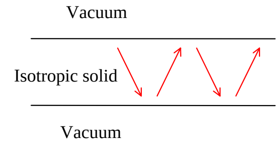
Earth-quake Sumatra-Andama (2004)



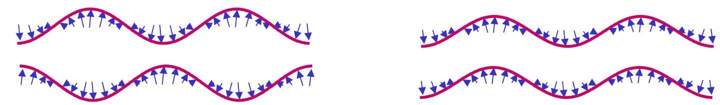
Rayleigh waves traveling around the globe

Courtesy http://www.sciencemag.org/cgi/content/full/308/5725/1133/DC1
Charles J. Ammon, Chen Ji, Hong-Kie Thio, David Robinson, Sidao Ni, Vala Hjorleifsdottir, Hiroo Kanamori, Thorne Lay, Shamita Das, Don Helmberger, Gene Ichinose, Jascha Polet, David Wald ; The animation was made with the help of Santiago Lombeyda at the Center for Advanced Computing Research, Caltech

Modal waves (4/4): Lamb waves

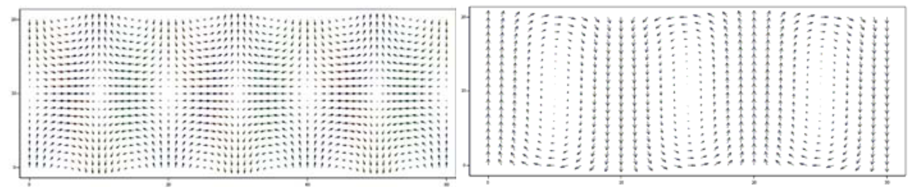


guided waves, combinations of pressure and shear waves



pressure waves (symmetric mode) — *flexure waves (antisymmetric mode)*

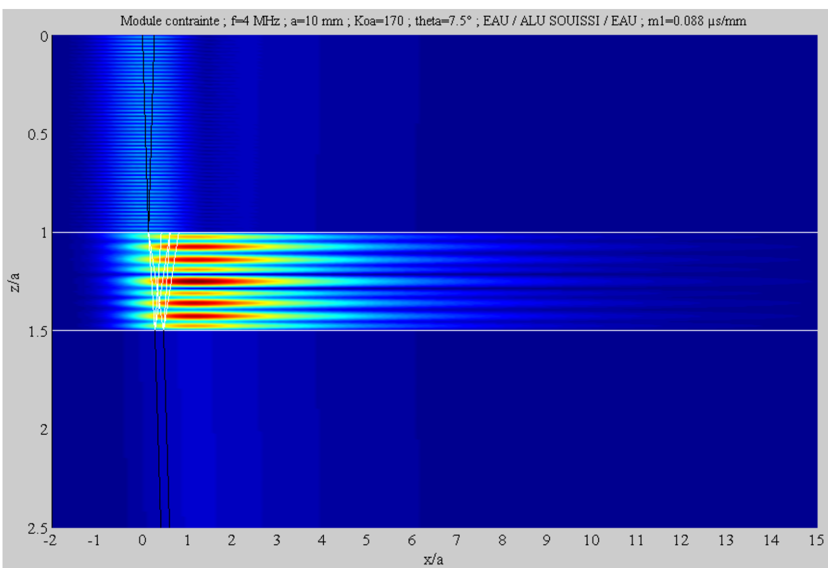
vectorial particle displacement field at the surface of the plate; its effect on the shape of the plate



mode S_0 — mode A_0

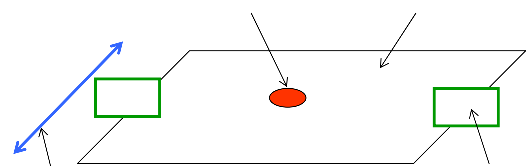
animations realised by Patrick Lanceleur, Université de Technologie de Compiègne
http://www.utc.fr/~lanceleu/links_CT04.html

Water / Aluminium / Water ; ka=170 ; H=5 mm ; before the 1st critical angle



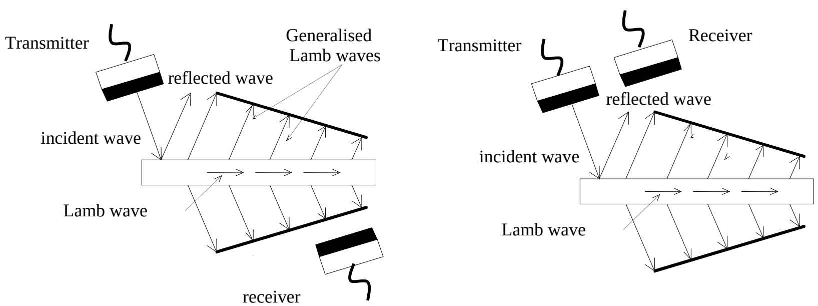
Lamb mode

Lamb waves: interest in NDT

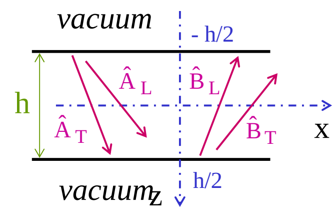


defect → perturbation of the wave propagation

Cartographies using Lamb waves



Displacements-stresses in an isotropic medium



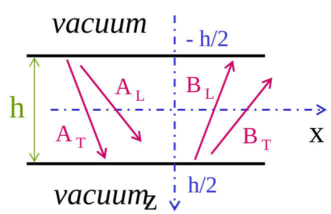
$$k_L = \frac{\omega}{V_L}\ ;\ k_L = \frac{\omega}{V_L}\ ;\ \left(\frac{k_L}{k_T}\right)^2 = \frac{\mu}{\lambda + 2\mu}\ ;\ k_x > k_T > k_L$$

$$k_{z_L} = \sqrt{k_x^2 - k_L^2} \qquad k_{z_T} = \sqrt{k_x^2 - k_T^2}$$

$$\begin{Bmatrix} \hat{u}_x \\ \hat{u}_z \\ \hat{T}_{xz} \\ \hat{T}_{zz} \end{Bmatrix} = \begin{bmatrix} k_x/k_L & k_x/k_L & -k_{z_T}/k_T & -k_{z_T}/k_T \\ k_{z_L}/k_L & -k_{z_L}/k_L & k_x/k_T & -k_x/k_T \\ -2i\mu k_x k_{z_L}/k_L & +2i\mu k_x k_{z_L}/k_L & -i\mu\left(2k_x^2 - k_T^2\right)/k_T & +i\mu\left(2k_x^2 - k_T^2\right)/k_T \\ +i\mu\left(2k_x^2 - k_T^2\right)/k_L & +i\mu\left(2k_x^2 - k_T^2\right)/k_L & -2i\mu k_x k_{z_T}/k_T & -2i\mu k_x k_{z_T}/k_T \end{bmatrix} \begin{Bmatrix} \hat{A}_L e^{-ik_{z_L} z} \\ \hat{B}_L e^{+ik_{z_L} z} \\ \hat{A}_T e^{-ik_{z_T} z} \\ \hat{B}_T e^{+ik_{z_T} z} \end{Bmatrix}$$

omitting exp[-i ($k_x x$ - ωt)] *factor*

Lamb waves in isotropic media



Boundary conditions:

$$\begin{cases} T_{xz}\left(-\frac{h}{2}\right) = 0 \\ T_{zz}\left(-\frac{h}{2}\right) = 0 \end{cases} \qquad \begin{cases} T_{xz}\left(\frac{h}{2}\right) = 0 \\ T_{zz}\left(\frac{h}{2}\right) = 0 \end{cases}$$

homogeneous 4th order system

$$\begin{cases} (\quad)A_L + (\quad)A_T + (\quad)B_L + (\quad)B_T = 0 \\ (\quad)A_L + (\quad)A_T + (\quad)B_L + (\quad)B_T = 0 \\ (\quad)A_L + (\quad)A_T + (\quad)B_L + (\quad)B_T = 0 \\ (\quad)A_L + (\quad)A_T + (\quad)B_L + (\quad)B_T = 0 \end{cases}$$

⇒ determinant (4 × 4) = 0

$$\left[4k_x^2 k_{z_L} k_{z_T} \cos\left(k_{z_L}\frac{h}{2}\right)\sin\left(k_{z_T}\frac{h}{2}\right) + \left(2k_x^2 - k_T^2\right)^2 \cos\left(k_{z_T}\frac{h}{2}\right)\sin\left(k_{z_L}\frac{h}{2}\right)\right] \bullet$$

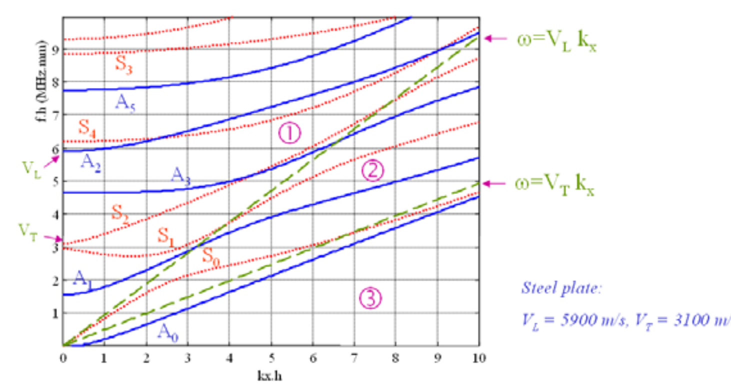
antisymmetric modes

⇒

$$\left[4k_x^2 k_{z_L} k_{z_T} \cos\left(k_{z_T}\frac{h}{2}\right)\sin\left(k_{z_L}\frac{h}{2}\right) + \left(2k_x^2 - k_T^2\right)^2 \cos\left(k_{z_L}\frac{h}{2}\right)\sin\left(k_{z_T}\frac{h}{2}\right)\right] = 0$$

symmetric modes

Dispersion curves for Lamb waves plane (k_x.h , f.h) or (k_x.h/(2π) , ωh/(2π))

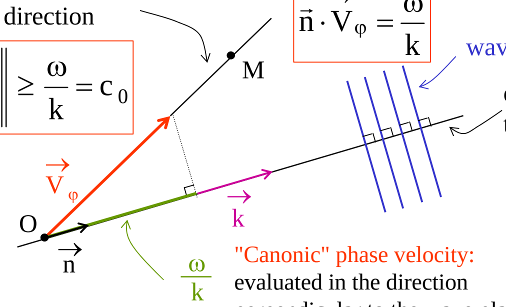


① k_{zL} and k_{zT} real *propagative OL and OT*

② k_{zL} pure imaginary and k_{zT} real *evanescent OL and propagative OT*

③ k_{zL} and k_{zT} pure imaginaries *evanescentes OL and OT*

Phase velocity



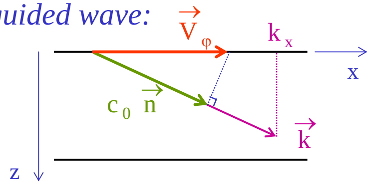
$$\left\| \vec{V}_\varphi \right\| \geq \frac{\omega}{k} = c_0$$

$$\vec{n} \cdot \vec{V}_\varphi = \frac{\omega}{k}$$

"Canonic" phase velocity: evaluated in the direction perpendicular to the wave planes

$$\vec{V}_{\varphi\,cano} = \frac{\omega}{k}\,\vec{n} = c_0\,\vec{n}$$

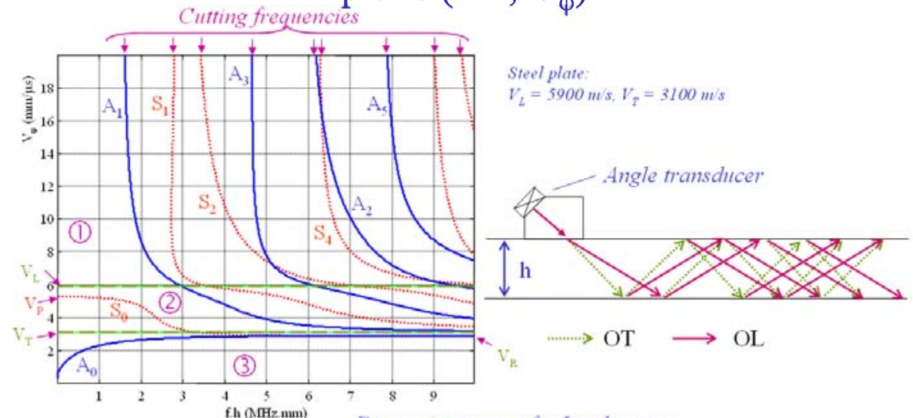
guided wave:



Phase velocity evaluated parallely to the guide

$$V_\varphi = \frac{\omega}{k_x}$$

Dispersion curves for Lamb waves plane (f h , V_φ)



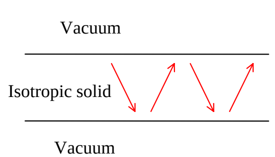
Dispersion curves for Lamb waves

① k_{zL} and k_{zT} real *propagative OL and OT*

② k_{zL} pure imaginary and k_{zT} real *evanescent OL and propagative OT*

③ k_{zL} and k_{zT} pure imaginaries *evanescentes OL and OT*

Displacements of Lamb waves (1/2)



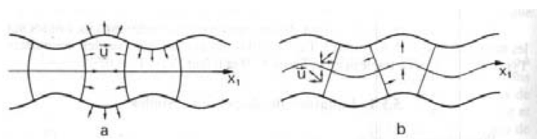
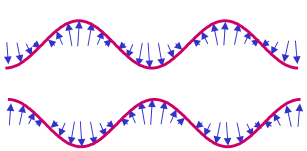
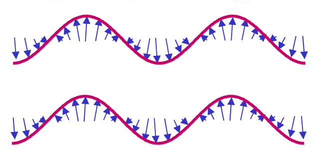


FIG. 5.36. *Onde de Lamb. a) symétrique : de part et d'autre du plan médian, les composantes longitudinales sont égales et les composantes transversales sont opposées. b) antisymétrique : de part et d'autre du plan médian, les composantes transversales sont égales et les composantes longitudinales sont opposées.*



pressure waves (symmetric mode)

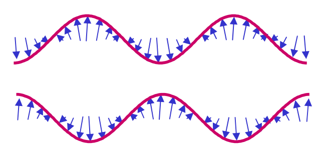


flexure waves (antisymmetric mode)

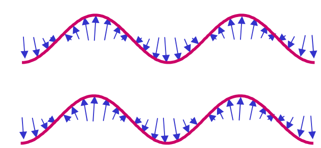
vectorial particle displacement field at the surface of the plate ;
its effect on the shape of the plate

figures from:
D. Royer et E. Dieulesaint, "Ondes élastiques dans les solides", tome 1 : propagation libre et guidée, Masson, (1996)
J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Displacements of Lamb waves (2/2)

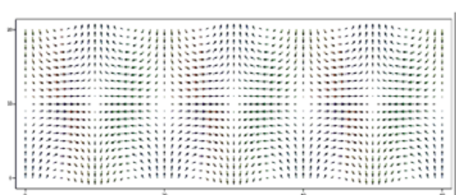


pressure waves (symmetric mode)

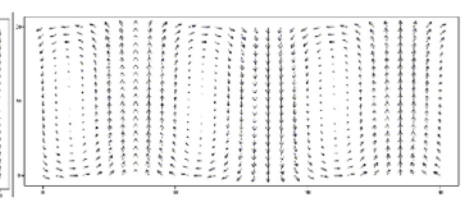


flexure waves (antisymmetric mode)

vectorial particle displacement field at the surface of the plate ;
its effect on the shape of the plate



mode S_0



mode A_0

animations realised by Patrick Lanceleur, Université de Technologie de Compiègne
http://www.utc.fr/~lanceleu/links_CT04.html

Lamb modes when k h <<1

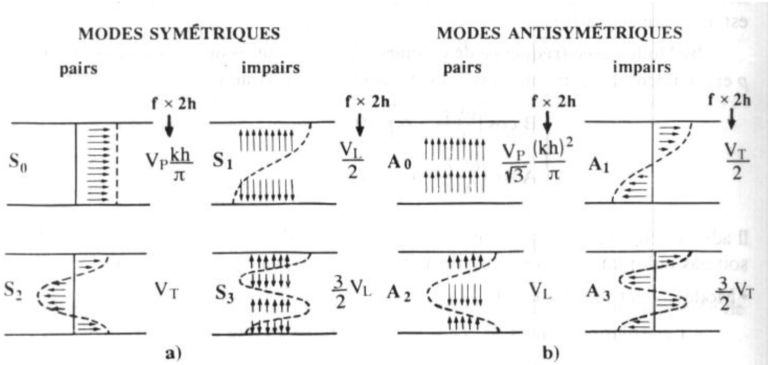
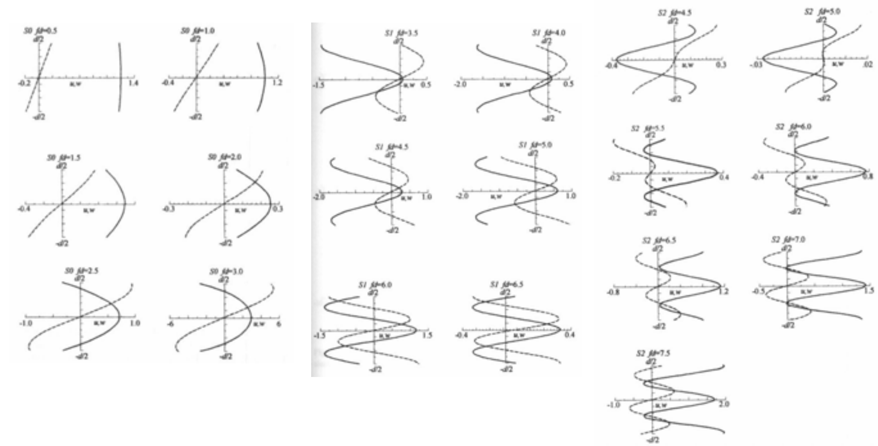


FIG. 5. 38. *Onde de Lamb dans une plaque isotrope. Déplacement mécanique des premiers modes pour k ≅ 0 (kh << 1). a) Modes symétriques → modes de dilatation. b) Modes antisymétriques → modes de flexion.*

figures from:
D. Royer et E. Dieulesaint, "Ondes élastiques dans les solides", tome 1 : propagation libre et guidée, Masson, (1996)

Particle displacements - symmetric modes

S_0 S_1 S_2

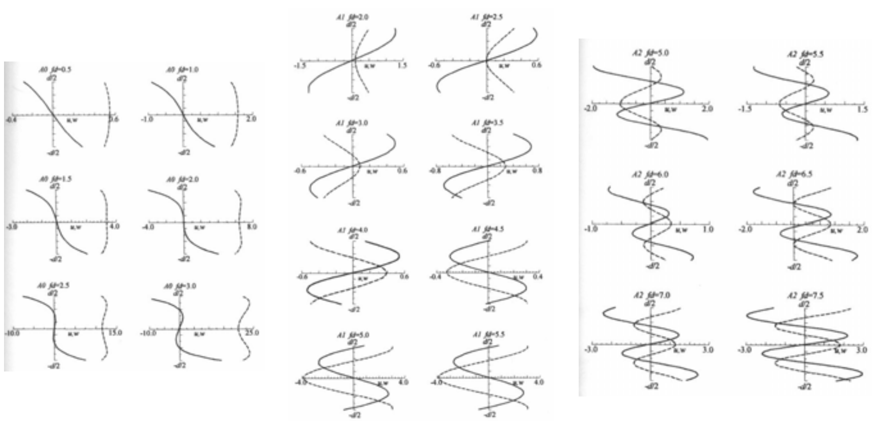


Aluminium plate, thickness d, V_L = 6300 m/s ; V_T = 3100 m/s ; u = u_1 (solid line) et w = u_3 (dotted line)

figures from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Particle displacements - antisymmetric modes

A_0 A_1 A_2



Aluminium plate, thickness d, V_L = 6300 m/s ; V_T = 3100 m/s ; u = u_1 (solid line) et w = u_3 (dotted line)

figures from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Dominating "in-plane" displacements

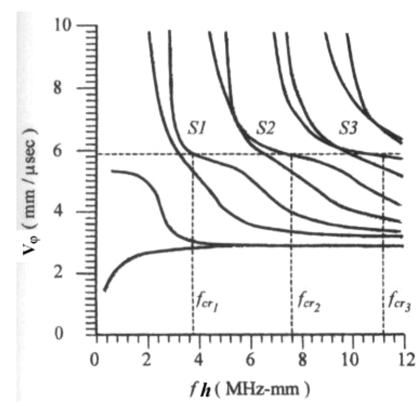


figure from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Lamé mode

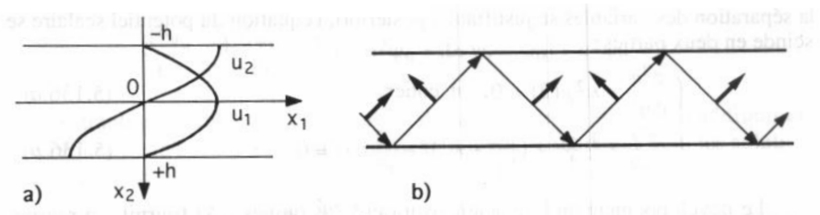
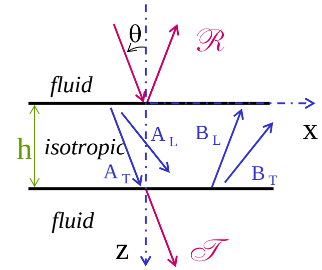


FIG. 5.41. Cas particulier $k^2 = q^2$. a) Composantes du déplacement du mode S_0. b) L'onde de Lamb est une onde transversale (mode de Lamé) se propageant à 45° de l'axe de la plaque.

$$k_x = k_{z_T} \quad \Rightarrow \quad V_\varphi = \sqrt{2}\ V_T$$

figures from:
D. Royer et E. Dieulesaint, "Ondes élastiques dans les solides", tome 1 : propagation libre et guidée, Masson, (1996)

Generalised Lamb modes



Bondary conditions	Unknowns
z = 0 : equality u_z, T_{xz}, T_{zz}	ℛ, 𝒯
z = h : equality u_z, T_{xz}, T_{zz}	A_T, A_L, B_L, B_T
6 equations	*6 unknowns*

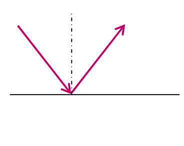
$$\mathcal{R} = \frac{C_s C_a - \tau^2}{(C_s - i\tau)(C_a + i\tau)} \qquad \mathcal{T} = i\tau \frac{C_s + C_a}{(C_s - i\tau)(C_a + i\tau)} \qquad \text{avec} \qquad \tau = \frac{\rho_f}{\rho_s}\frac{V_f}{V_L}\frac{\cos\theta_L}{\cos\theta}$$

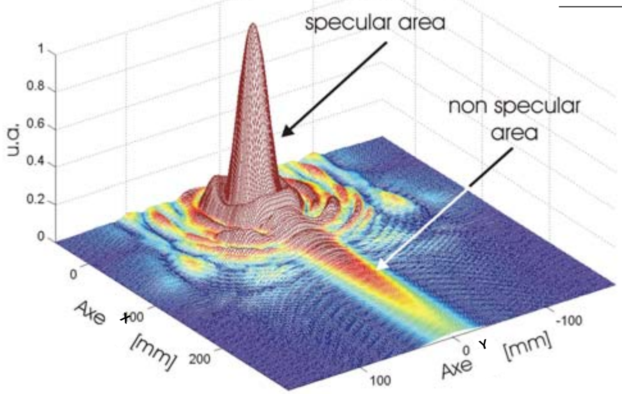
$C_s = 0$: symmetric Lamb modes ; $C_a = 0$: antisymmetric Lamb modes

If $\frac{\rho_f}{\rho_s} \ll 1$: Lamb modes ⟺ $\mathcal{R} = 0$

⇒ **minimum of | ℛ |**

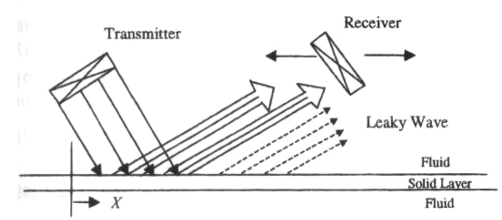
Reflected field

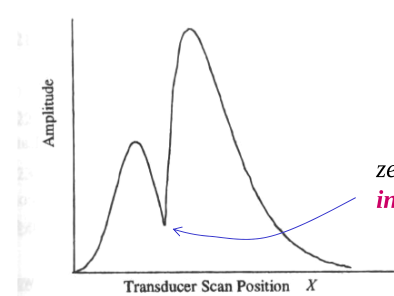




unidirectional carbon/epoxy plate
θ = 9.8°, f = 1.35 MHz ; e = 0.59 mm

Non specular reflection



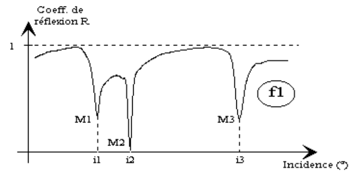


zero associated to an ***interference phenomenon***

figures from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Lamb modes in anisotropic medium (1/2)

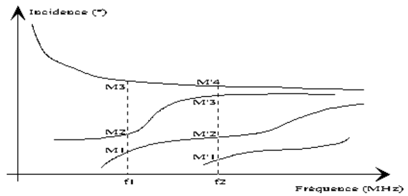
isotropic plates: analytical formulae
anisotropic plates
anisotropic multilayered media → model



f1 fixed

↓

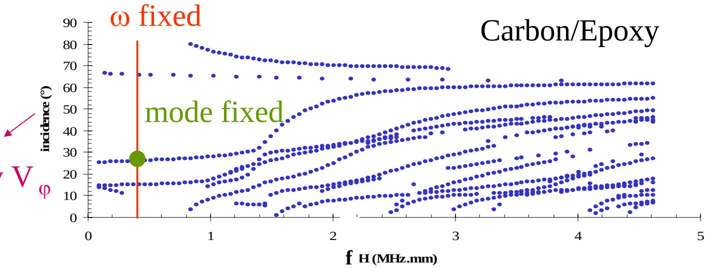
3 minima of $\mathscr{R}$



→ Dispersion curves

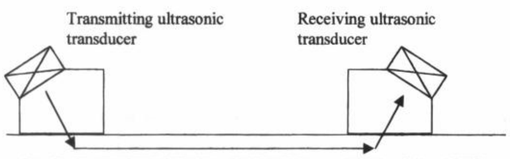
Lamb modes in anisotropic medium (2/2)

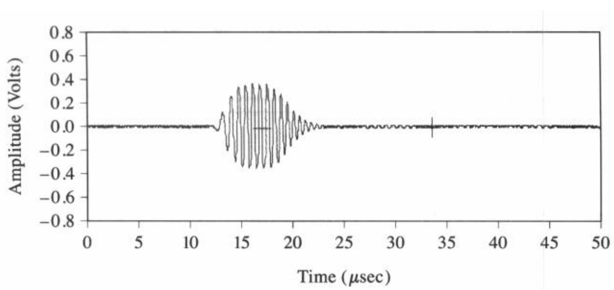
Velocity V_φ



$$m_x = \frac{\sin\theta}{V_{eau}} \Rightarrow V_\varphi = \frac{1}{m_x} = \frac{V_{eau}}{\sin\theta}$$

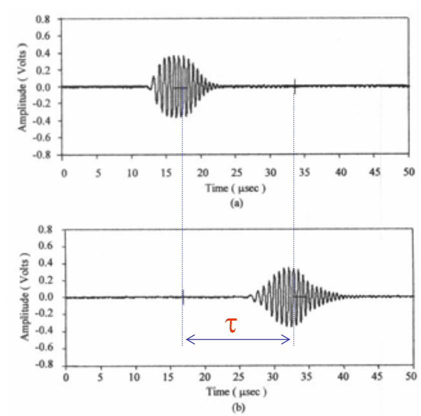
Experimental set-up

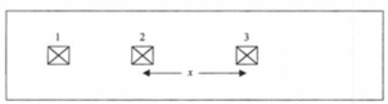




figures from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

Measure of the group velocity





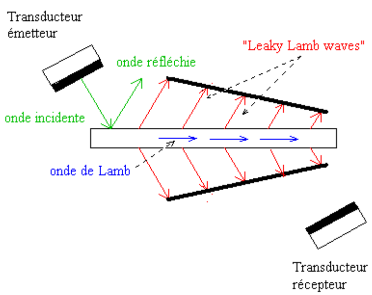
Aluminium plate

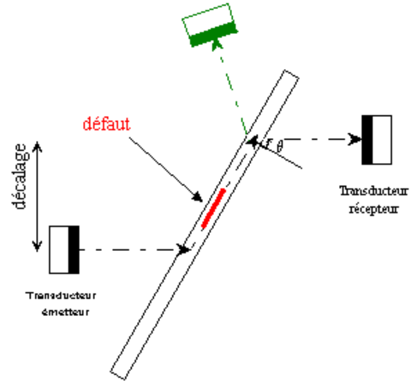
Mode S_0, fh=1.434 MHz.mm

x = 76.2 mm
τ = 15.875 µs ==> V_g = 4800 m/s

figures from: J.L. Rose, "Ultrasonic waves in solid media",Cambridge Univ. Press, 1999

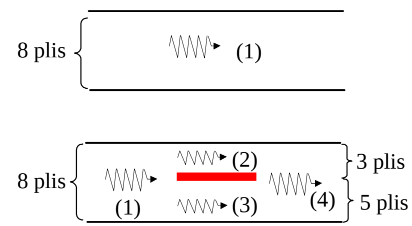
Transmission or Reflection set-up



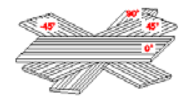


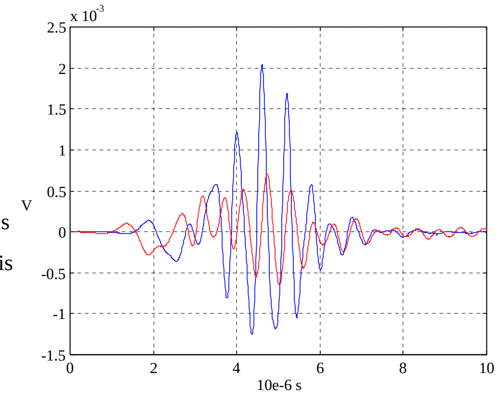
defect ➡ propagation of the perturbed wave

Detection of a defect using Lamb waves (1/6)



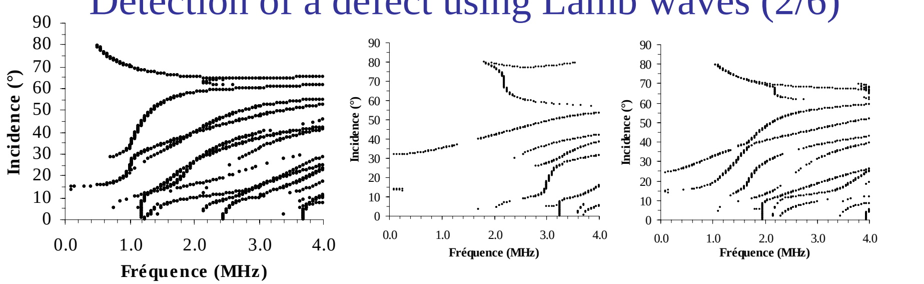
plaques en carbone/époxyde : défaut entre le 3ème et le 4ème pli



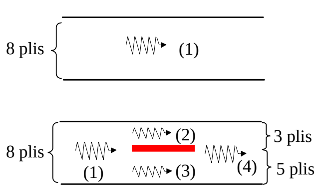


{0°/45°/90°/135°}$_{0°}$ en carbone/époxyde, comportant 8 plis en symétrie miroir. Incidence de 10,3°, fréquence de 2 MHz.
bleu : plaque saine
rouge : plaque avec défaut

Detection of a defect using Lamb waves (2/6)



8 layers [0°/45°/90°/135°]$_{2s}$ *3 layers 0°/45°/90°* *5 layers 135°/135°/90°/45°/0°*



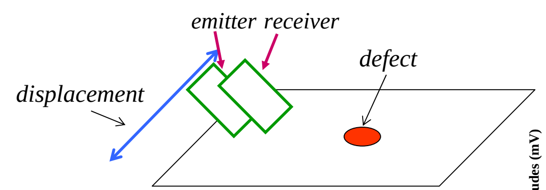
Conversion of mode (1) into (2) and (3) then (4)

If mode (2) or mode (3) near of mode (1)
then mode (4) ≈ mode (1) ➡ non detected defect

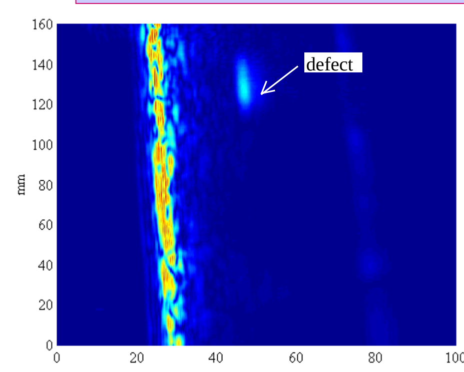
If mode (2) or mode (3) differents from mode (1)
then mode (4) ≠ mode (1) ➡ detected defect

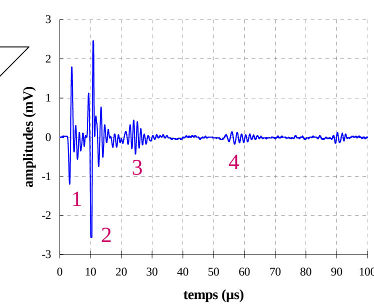
Defect between 3rd and 4th layer

Detection of a defect using Lamb waves (3/6) : cartography



Echography experimental set-up

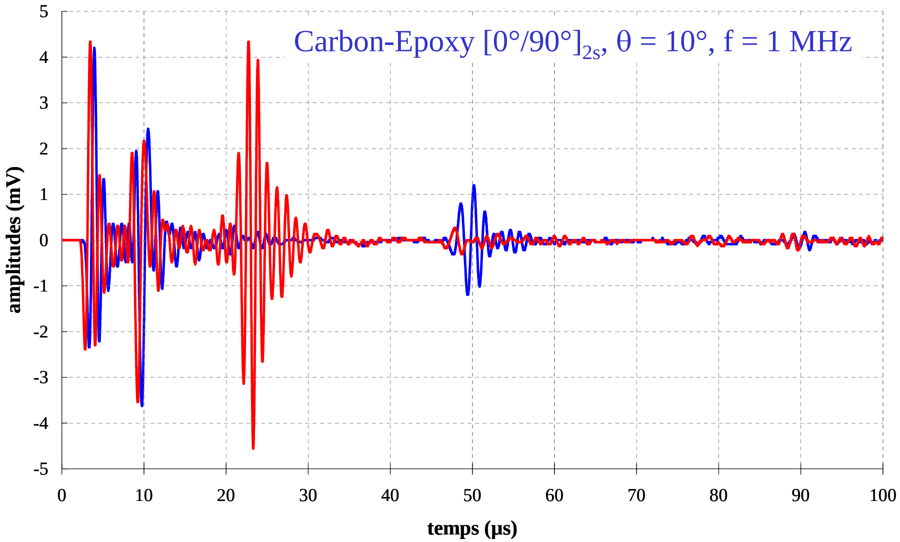




1 : surface echo
2 : radiation of Lamb wave
3 : defect echo
4 : edge plate echo

0°/90° mirror - 8 layers
f=1 MHz, θ=10°

Detection of a defect using Lamb waves (4/6)



blue : plate without defect red : plate without defect

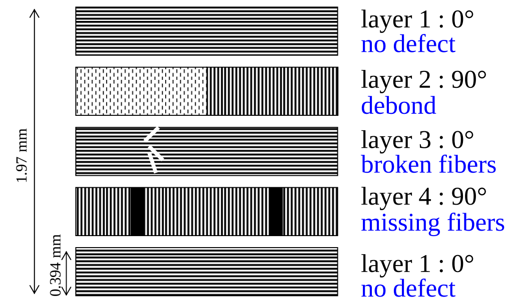
Detection of defect (5/6): cartography using Lamb waves

5 layers 0°/90° mirror

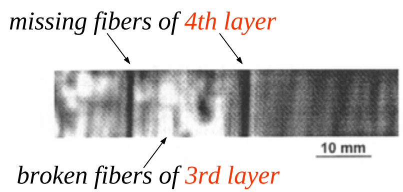
SCS-6

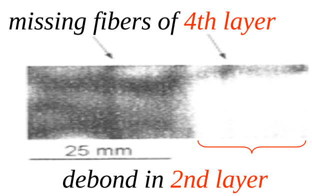
matrix Ti - 6 Al - 4 V

T. Kundu et al., Ultrasonics, 1996 and 1997



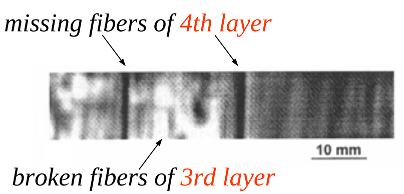
L-Scan

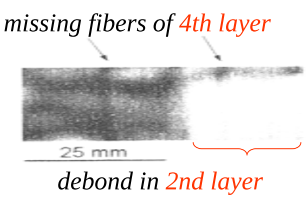




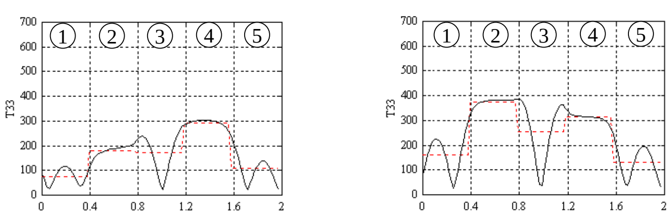
θ = 20° ; f = 5.05 MHz θ = 21° ; f = 5.15 MHz

Detection of defect (6/6): cartography using Lamb waves





θ = 20° ; f = 5.05 MHz θ = 21° ; f = 5.15 MHz

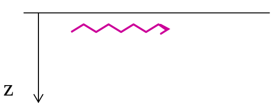


repartition of normal stress as a function of thickness

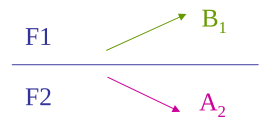
VI. MODAL WAVES: PARTICULAR CASE OF RAYLEIGH WAVES

Existence condition of a surface wave

- Can waves propagating along an interface exist, without any permanent energy contribution (modal wave)?

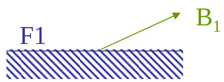


- Fluid 1/Fluid 2

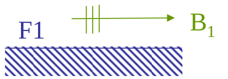


***No**: B_1 and A_2 move energy away from the interface, though there is no energy contribution (no incident wave)*

- Fluid 2/Mirror

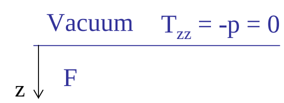


***No**: the energy moves awy with energy contribution*



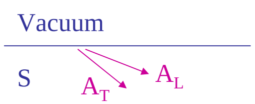
***Yes**: 1/2 plane wave which propagates parallely to the interface*

- Vacuum/Fluid

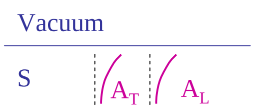


***No**: p=0 imposed at z=0, thus also elsewhere in the fluid medium ==> no acoustics*

- Vacuum/Solid

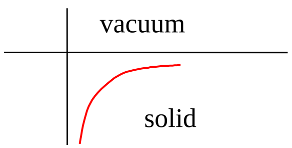


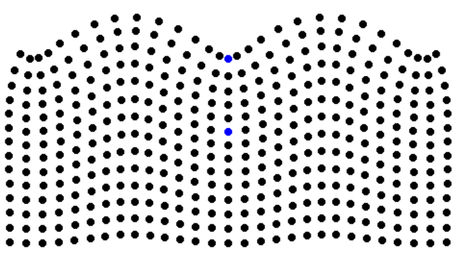
***No**: A_L and A_T move energy away from the interface, though there is no energy contribution (no incident wave)*



***Yes**: evanescent waves ==> energy carried along the interface.*

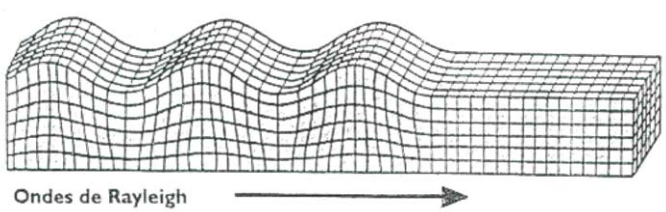
Rayleigh wave



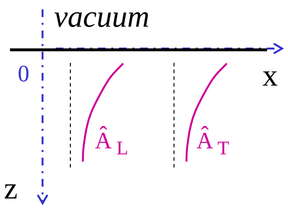


http://www.kettering.edu/~drussell

Animation courtesy of Dr. Dan Russell, Kettering University



Displacements-stresses in an isotropic medium



$$k_L = \frac{\omega}{V_L}\ ;\ k_L = \frac{\omega}{V_L}\ ;\ \left(\frac{k_L}{k_T}\right)^2 = \frac{\mu}{\lambda + 2\mu}\ ;\ k_x > k_T > k_L$$

$$k_{z_L} = \sqrt{k_L^2 - k_x^2} = -i k''_{z_L} \quad \textit{with}\quad k''_{z_L} = \sqrt{k_x^2 - k_L^2} > 0$$

$$k_{z_T} = \sqrt{k_T^2 - k_x^2} = -i k''_{z_T} \quad \textit{with}\quad k''_{z_T} = \sqrt{k_x^2 - k_T^2} > 0$$

$$\begin{Bmatrix} \hat{u}_x \\ \hat{u}_z \\ \hat{T}_{xz} \\ \hat{T}_{zz} \end{Bmatrix} = \begin{bmatrix} k_x/k_L & -k_{z_T}/k_T \\ k_{z_L}/k_L & k_x/k_T \\ -2i\mu k_x k_{z_L}/k_L & -i\mu\left(2k_x^2 - k_T^2\right)/k_T \\ i\mu\left(2k_x^2 - k_T^2\right)/k_L & -2i\mu k_x k_{z_T}/k_T \end{bmatrix} \begin{Bmatrix} \hat{A}_L e^{-ik_{z_L} z} \\ \hat{A}_T e^{-ik_{z_T} z} \end{Bmatrix}$$

omitting $\exp[-i\,(k_x x - \omega t)\,]$

$$= \begin{bmatrix} k_x/k_L & ik''_{z_T}/k_T \\ -ik''_{z_L}/k_L & k_x/k_T \\ -2\mu k_x k''_{z_L}/k_L & -i\mu\left(2k_x^2 - k_T^2\right)/k_T \\ i\mu\left(2k_x^2 - k_T^2\right)/k_L & -2\mu k_x k''_{z_T}/k_T \end{bmatrix} \begin{Bmatrix} \hat{A}_L e^{-k''_{z_L} z} \\ \hat{A}_T e^{-k''_{z_T} z} \end{Bmatrix}$$

Displacements in isotropic media (1/2): longitudinal waves

- Displacement vector: longitudinal waves

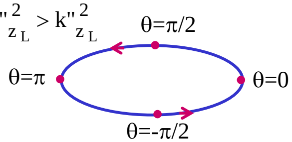
$$\hat{u}_L = \underbrace{\hat{A}_L}_{|\hat{A}_L|\exp(i\alpha_L)} \left[(k_x/k_L)\vec{e}_x - (i k''_{z_L}/k_L)\vec{e}_z \right] e^{-k''_{z_L} z} e^{i(-k_x x+\omega t)}$$

$$\Rightarrow \quad \hat{u}_L = |\hat{A}_L| \left[(k_x/k_L)\vec{e}_x - (i k''_{z_L}/k_L)\vec{e}_z \right] e^{-k''_{z_L} z} e^{i(-k_x x+\omega t+\alpha_L)}$$

$$\Rightarrow \quad \begin{cases} u_{x_L} = \mathcal{R}e(\hat{u}_{x_L}) = |\hat{A}_L| (k_x/k_L) e^{-k''_{z_L} z} \cos(-k_x x + \omega t + \alpha_L) \\ u_{z_L} = \mathcal{R}e(\hat{u}_{z_L}) = |\hat{A}_L| (k''_{z_L}/k_L) e^{-k''_{z_L} z} \sin(-k_x x + \omega t + \alpha_L) \end{cases}$$

$$\Rightarrow \quad \left(\frac{u_{x_L}}{|\hat{A}_L| (k_x/k_L) e^{-k''_{z_L} z}} \right)^2 + \left(\frac{u_{z_L}}{|\hat{A}_L| (k''_{z_L}/k_L) e^{-k''_{z_L} z}} \right)^2 = 1$$

$k_x^2 = k_L^2 + k''^2_{z_L} > k''^2_{z_L}$



$\theta = -k_x x + \omega t + \alpha_L$

$\theta = 0 \Rightarrow u_{x_L} > 0 \; ; \; u_{z_L} = 0$

$\theta = \pi/2 \Rightarrow u_{x_L} = 0 \; ; \; u_{z_L} > 0 \quad (k''_{z_L} > 0)$

Displacements in isotropic media (2/2): shear waves

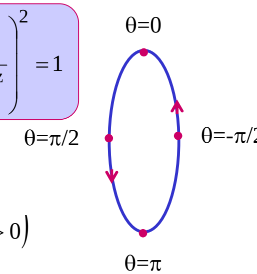
- Displacement vector: shear (transversal) waves

$$\hat{u}_T = \underbrace{\hat{A}_T}_{|\hat{A}_T|\exp(i\alpha_T)} \left[(i k''_{z_T}/k_T)\vec{e}_x + (k_x/k_T)\vec{e}_z \right] e^{-k''_{z_T} z} e^{i(-k_x x+\omega t)}$$

$$\Rightarrow \quad \hat{u}_T = |\hat{A}_T| \left[(i k''_{z_T}/k_T)\vec{e}_x + (k_x/k_T)\vec{e}_z \right] e^{-k''_{z_T} z} e^{i(-k_x x+\omega t+\alpha_T)}$$

$$\Rightarrow \quad \begin{cases} u_{x_T} = \mathcal{R}e(\hat{u}_{x_T}) = -|\hat{A}_T| (k''_{z_T}/k_T) e^{-k''_{z_T} z} \sin(-k_x x + \omega t + \alpha_T) \\ u_{z_T} = \mathcal{R}e(\hat{u}_{z_T}) = |\hat{A}_T| (k_x/k_T) e^{-k''_{z_T} z} \cos(-k_x x + \omega t + \alpha_T) \end{cases}$$

$$\Rightarrow \quad \left(\frac{u_{x_T}}{|\hat{A}_T| (k''_{z_T}/k_T) e^{-k''_{z_T} z}} \right)^2 + \left(\frac{u_{z_T}}{|\hat{A}_T| (k_x/k_T) e^{-k''_{z_T} z}} \right)^2 = 1$$



$k_x^2 = k_T^2 + k''^2_{z_T} > k''^2_{z_T}$

$\theta = -k_x x + \omega t + \alpha_T$

$\theta = 0 \Rightarrow u_{x_T} = 0 \; ; \; u_{z_T} > 0$

$\theta = \pi/2 \Rightarrow u_{x_L} < 0 \; ; \; u_{z_T} = 0 \quad (k''_{z_T} > 0)$

Stresses in isotropic media (1/2): longitudinal waves

- Stress vector: longitudinal waves

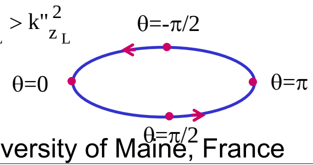
$$\hat{\vec{T}}_L = \underbrace{\hat{A}_L}_{|\hat{A}_L|\exp(i\alpha_L)} \left[-2\mu k_x k''_{z_L}/k_L \vec{e}_x + i\mu(2k_x^2 - k_T^2)/k_L \vec{e}_y \right] e^{-k''_{z_L} z} e^{i(-k_x x+\omega t)}$$

$$\Rightarrow \quad \hat{\vec{T}}_L = |\hat{A}_L| \left[-2\mu k_x k''_{z_L}/k_L \vec{e}_x + i\mu(2k_x^2 - k_T^2)/k_L \vec{e}_z \right] e^{-k''_{z_L} z} e^{i(-k_x x+\omega t+\alpha_L)}$$

$$\Rightarrow \quad \begin{cases} T_{x_L} = \mathcal{R}e(\hat{T}_{x_L}) = -|\hat{A}_L| (2\mu k_x k''_{z_L}/k_L) e^{-k''_{z_L} z} \cos(-k_x x + \omega t + \alpha_L) \\ T_{z_L} = \mathcal{R}e(\hat{T}_{z_L}) = -|\hat{A}_L| \left[\mu(2k_x^2 - k_T^2)/k_L \right] e^{-k''_{z_L} z} \sin(-k_x x + \omega t + \alpha_L) \end{cases}$$

$$\Rightarrow \quad \left(\frac{T_{x_L}}{|\hat{A}_L| (2\mu k_x k''_{z_L}/k_L) e^{-k''_{z_L} z}} \right)^2 + \left(\frac{T_{z_L}}{|\hat{A}_L| \mu(2k_x^2 - k_T^2)/k_L e^{-k''_{z_L} z}} \right)^2 = 1$$

$k_x^2 = k_L^2 + k''^2_{z_L} > k''^2_{z_L}$



$\theta = -k_x x + \omega t + \alpha_L$

$\theta = 0 \Rightarrow T_{x_L} < 0 \; ; \; T_{z_L} = 0$

$\theta = \pi/2 \Rightarrow T_{x_L} = 0 \; ; \; T_{z_L} < 0 \quad (k''_{z_L} > 0)$

Stresses in isotropic media (2/2): shear waves

- Stress vector: shear (transversal) waves

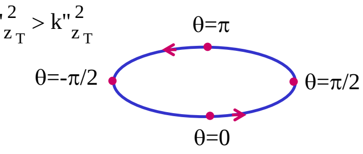
$$\hat{\vec{T}}_T = \underbrace{\hat{A}_T}_{|\hat{A}_T|\exp(i\alpha_T)} \left[-(i\mu(2k_x^2 - k_T^2)/k_T)\vec{e}_x - (2\mu k_x k''_{z_T}/k_T)\vec{e}_z \right] e^{-k''_{z_T} z} e^{i(-k_x x+\omega t)}$$

$$\Rightarrow \quad \hat{\vec{T}}_T = |\hat{A}_T| \left[-(i\mu(2k_x^2 - k_T^2)/k_T)\vec{e}_x - (2\mu k_x k''_{z_T}/k_T)\vec{e}_z \right] e^{-k''_{z_T} z} e^{i(-k_x x+\omega t+\alpha_T)}$$

$$\Rightarrow \quad \begin{cases} T_{x_T} = \mathcal{R}e(\hat{T}_{x_T}) = |\hat{A}_T| \left[\mu(2k_x^2 - k_T^2)/k_T \right] e^{-k''_{z_T} z} \sin(-k_x x + \omega t + \alpha_T) \\ T_{z_T} = \mathcal{R}e(\hat{T}_{z_T}) = -|\hat{A}_T| (2\mu k_x k''_{z_T}/k_T) e^{-k''_{z_T} z} \cos(-k_x x + \omega t + \alpha_T) \end{cases}$$

$$\Rightarrow \quad \left(\frac{T_{x_T}}{|\hat{A}_T| (\mu(2k_x^2 - k_T^2)/k_T) e^{-k''_{z_T} z}} \right)^2 + \left(\frac{T_{z_T}}{|\hat{A}_T| (2\mu k_x k''_{z_T}/k_T) e^{-k''_{z_T} z}} \right)^2 = 1$$

$k_x^2 = k_T^2 + k''^2_{z_T} > k''^2_{z_T}$



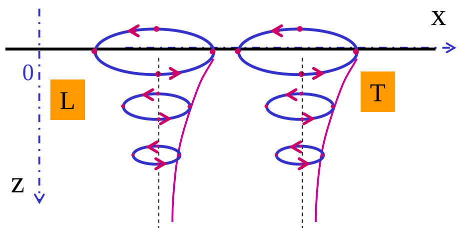
$\theta = -k_x x + \omega t + \alpha_T$

$\theta = 0 \Rightarrow T_{x_T} = 0 \; ; \; T_{z_T} < 0$

$\theta = \pi/2 \Rightarrow T_{x_L} > 0 \; ; \; T_{z_T} = 0 \quad (k''_{z_T} > 0)$

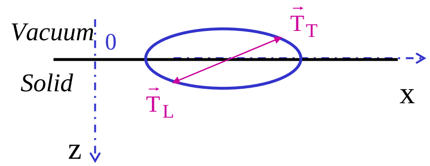
Boundary conditions: geometrical method

- Ellipses described by the particles



- Boundary conditions

$\vec{T} = \vec{T}_L + \vec{T}_T = \vec{0}$ on the surface



The ***larger axis of the ellipses*** described by the displacement vectors of the longitudinal (L) and transversal (T) waves have the ***same orientation***.

➡ In order to satisfy the boundary conditions, the amplitudes A_L et A_T just have to be adjusted
- in module in order to equalize the two ellipses
- in phase so that the two stress vectors are opposite (and remain opposite at every time).

➡ Same shape ratio of the two ellipses: $E_L = E_T$

Rayleigh wave velocity using a geometrical reasoning (1/2)

- Shape ratio of the longitudinal ellipse: $E_L = b_L/a_L$

$$\begin{cases} T_{x_L} = \mathscr{R}e\left(\hat{T}_{x_L}\right) = -\underbrace{\left|\hat{A}_L\right|\left(2\mu k_x k''_{z_L}/k_L\right)e^{-k''_{z_L}z}}_{a_L}\cos\left(-k_x x + \omega t + \alpha_L\right) \\ T_{z_L} = \mathscr{R}e\left(\hat{T}_{z_L}\right) = -\underbrace{\left|\hat{A}_L\right|\left[\mu\left(2k_x^2 - k_T^2\right)/k_L\right]e^{-k''_{z_L}z}}_{b_L}\sin\left(-k_x x + \omega t + \alpha_L\right) \end{cases}$$

$$E_L = \frac{b_L}{a_L} = \frac{2k_x^2 - k_T^2}{2k_x k''_{z_L}}$$

- Shape ratio of the transversal ellipse: $E_T = b_T/a_T$

$$\begin{cases} T_{x_T} = \mathscr{R}e\left(\hat{T}_{x_T}\right) = \underbrace{\left|\hat{A}_T\right|\left[\mu\left(2k_x^2 - k_T^2\right)/k_T\right]e^{-k''_{z_T}z}}_{a_T}\cos\left(-k_x x + \omega t + \alpha_T - \pi/2\right) \\ T_{z_T} = \mathscr{R}e\left(\hat{T}_{z_T}\right) = \underbrace{\left|\hat{A}_T\right|\left(2\mu k_x k''_{z_T}/k_T\right)e^{-k''_{z_T}z}}_{b_T}\sin\left(-k_x x + \omega t + \alpha_T - \pi/2\right) \end{cases}$$

$$E_T = \frac{b_T}{a_T} = \frac{2k_x k''_{z_T}}{2k_x^2 - k_T^2}$$

Rayleigh wave velocity using a geometrical reasoning (2/2)

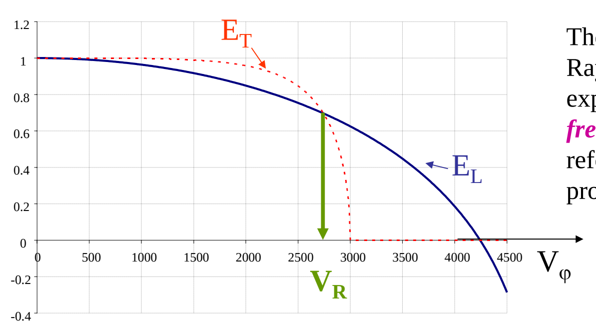
- Equality of the shape ratio of the ellipses

$$E_L = \frac{b_L}{a_L} = \frac{2k_x^2 - k_T^2}{2k_x k''_{z_L}} = \frac{V_L}{2V_T^2}\frac{2V_T^2 - V_\varphi^2}{\sqrt{V_L^2 - V_\varphi^2}}$$

$$E_T = \frac{b_T}{a_T} = \frac{2k_x k''_{z_T}}{2k_x^2 - k_T^2} = \frac{2V_T\sqrt{V_T^2 - V_\varphi^2}}{2V_T^2 - V_\varphi^2}$$

with

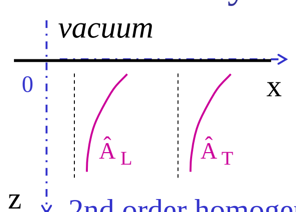
$$\begin{cases} k''_{z_L} = \sqrt{k_x^2 - k_L^2} > 0 \\ k''_{z_T} = \sqrt{k_x^2 - k_T^2} > 0 \\ k_x = \omega/V_\varphi \\ k_T = \omega/V_T \quad k_L = \omega/V_L \end{cases}$$



V_L = 5000 m/s ; V_T = 3000 m/s

The velocity V_R of the Rayleigh wave, is, as expected, ***independent of the frequency***, since there is no reference length in the problem.

Boundary conditions: analytical method (1/2)



$$\begin{cases} \hat{T}_{xz}(x, z=0; t) = 0 \ , \ \forall\ x,\ z=0,\ \forall\ t \\ \hat{T}_{zz}(x, z=0; t) = 0 \ , \ \forall\ x,\ z=0,\ \forall\ t \end{cases}$$

2nd order homogeneous system ➡ determinant (2 × 2) = 0

$$\begin{vmatrix} -2\mu k_x k''_{z_L}/k_L & -i\mu\left(2k_x^2 - k_T^2\right)/k_T \\ i\mu\left(2k_x^2 - k_T^2\right)/k_L & -2\mu k_x k''_{z_T}/k_T \end{vmatrix} = 0$$

➡

$$\left(2k_x^2 - k_T^2\right)^2 - 4k_x^2 k''_{z_L} k''_{z_T} = 0$$

or else

$$\left[2 - \left(\frac{V_\varphi}{V_T}\right)^2\right]^4 = 16\left[1 - \left(\frac{V_\varphi}{V_T}\right)^2\right]\left[1 - \left(\frac{V_\varphi}{V_L}\right)^2\right]$$

with $V_\varphi = k_x/\omega$ — *Rayleigh equation*

The solution $V_\varphi = V_R$ (or k_x) of this equation permits to obtain the velocity of the Rayleigh wave

Boundary conditions: analytical method (2/2)

- Approached formula

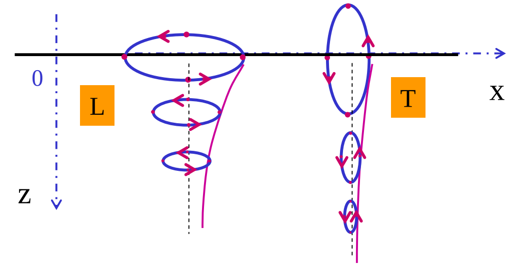
Poisson ratio: $\nu = \dfrac{c_{12}}{c_{11}+c_{12}} = \dfrac{V_L^2 - 2V_T^2}{2\left(V_L^2 - V_T^2\right)}$

if $0 < \nu < 0.5$: $V_R \approx V_T \dfrac{0.87 + 1.13\nu}{1+\nu}$

	V_L (m/s)	V_T (m/s)	V_R (m/s)
Steel	5970	3100	2883
Nickel	6040	3040	2827
Aluminum	6380	3100	2900
Copper	4700	2260	2114

Back on displacements: geometrical reasoning

- Ellipses described by the particles



The larger axes of the ellipses described by the displacement vectors of longitudinal (L) and transversal (T) waves ***do not have the same orientation***.

➡ These two ellipses will never be identical in order than the two vectors $\vec{u}_L$ and $\vec{u}_T$ be opposite (and remain opposite at every time).

➡ Rigid solid / Elastic solid

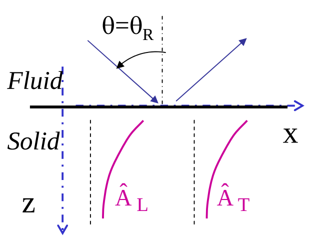
Boundary conditions:

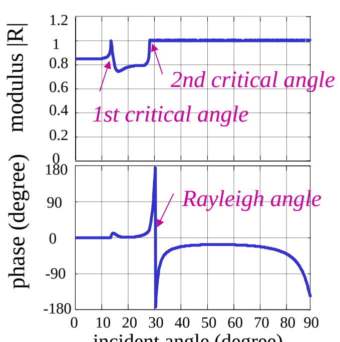
$\vec{u} = \vec{u}_L + \vec{u}_T = \vec{0}$ on the surface

➡ no possibility of a surface wave

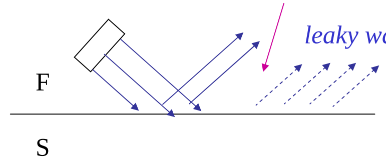
Leaky Rayleigh wave

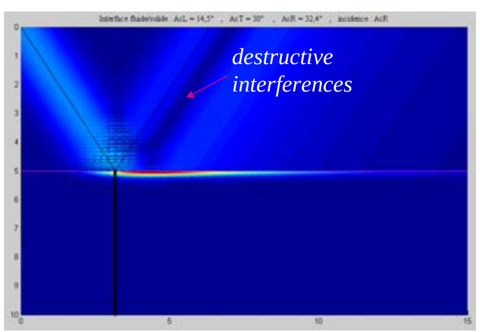
- Plane waves





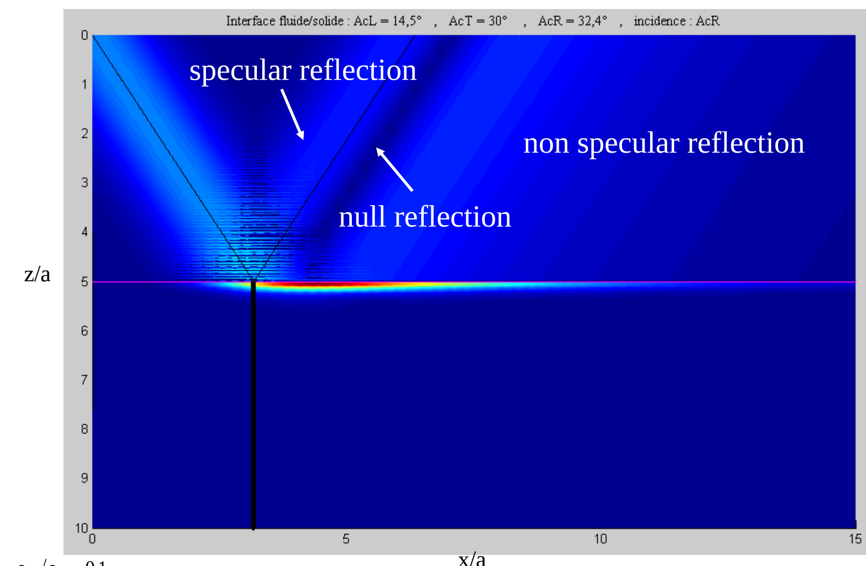
- Bounded beam





$\rho_0/\rho_1 = 0{,}1$

$k_0a = 60$, $k_La = 15$, $k_Ta=30$, $\theta = \theta_{Rayleigh}$



$\rho_0/\rho_1 = 0{,}1$

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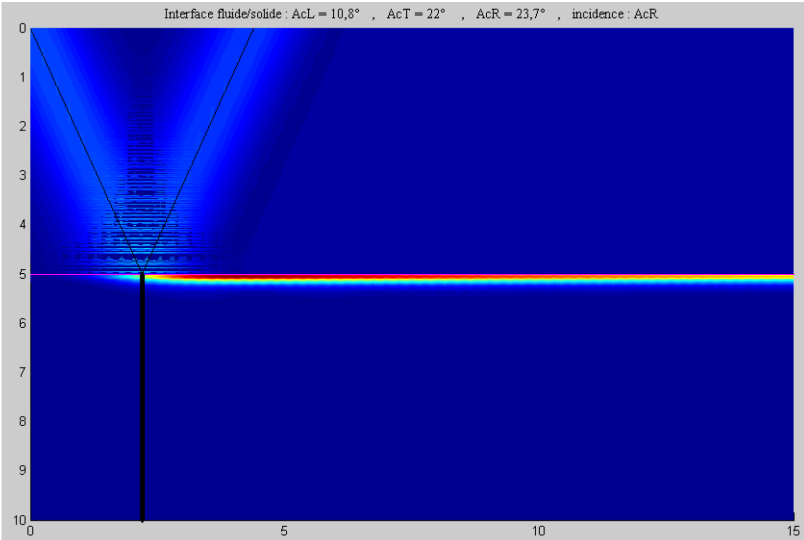
$k_0a = 80$, $k_La = 15$, $k_Ta=30$, $\theta = \theta_{Rayleigh}$



$\rho_0/\rho_1 = 0{,}02$

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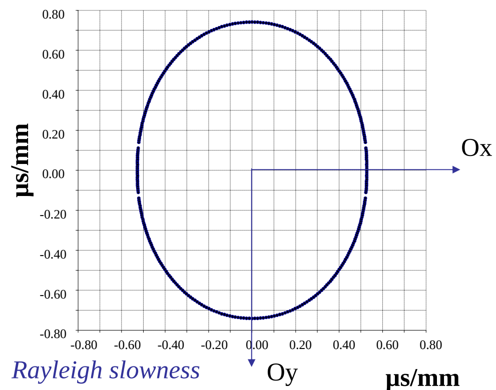
Rayleigh wave: generalization (1/3)

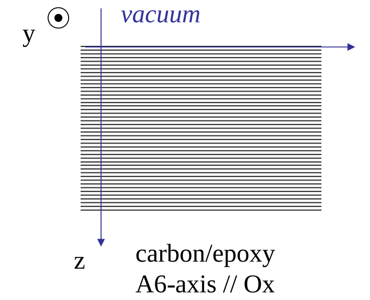
- Anisotropic media

The velocity of Rayleigh waves does not depend on frequency, but depends on the orientation of the medium

➡ i.e. no frequencial dispersion, but ***angular dispersion***



Rayleigh slowness curves (µs/mm)

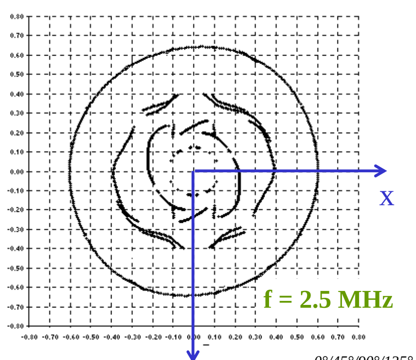


Rayleigh wave: generalization (2/3)

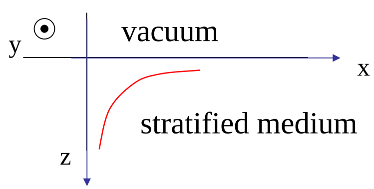
- Anisotropic multilayered media

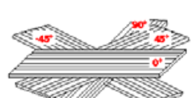
The velocity of Rayleigh waves depends on the frequency (presence of a length reference) and on the orientation of the medium

➡ i.e. ***frequencial*** AND ***angular dispersion***

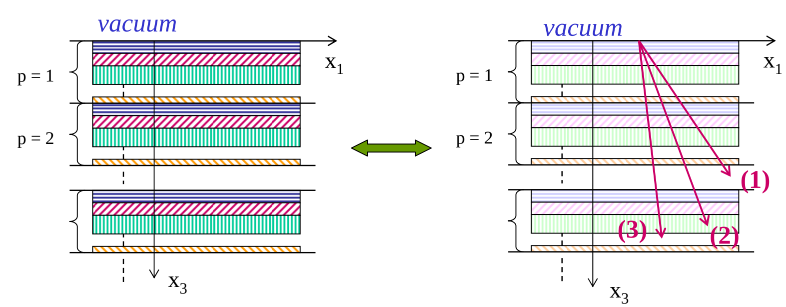


Rayleigh slowness curves (µs/mm)





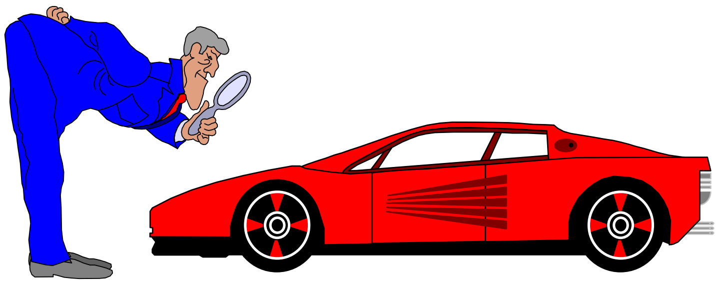
Rayleigh wave: generalization (3/3)

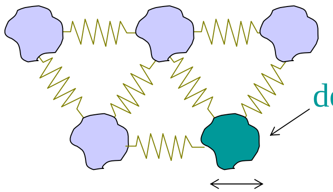


- **Floquet waves**
  - ✓ propagation modes of the infinite periodically multilayered medium
  - ✓ independent solutions linked to the eigenvalues and eigenvectors de of the transfer matrix τ of one period
- **Multilayered Rayleigh wave**
  - ✓ Modale surface wave
  - ✓ Linear combination of 3 inhomogeneous Floquet waves (for an anisotropic multilayered medium)
  - ✓ Dispersive wave

VII. INTRODUCTION TO ULTRASONIC NDT

Non Destructive Testing (NDT) by ultrasounds





$$\lambda = \frac{V}{f}$$

The more the defect is small, the more the frequency has to be high

NDT and NDE

NON DESTRUCTIVE TESTING

Presence or not of **defects**

Understanding of propagation phenomena

Determination of elastic (or viscoelastic) **properties**

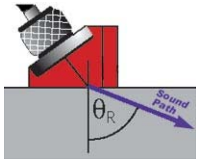
NON DESTRUCTIVE EVALUATION

Transducer types

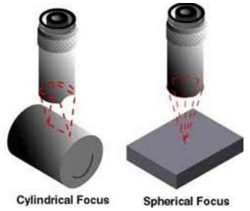
- **Transducers**

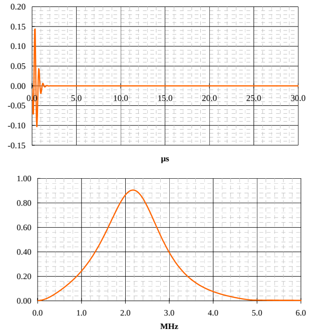
http://www.ndt-ed.org

- **Angle beam transducers**

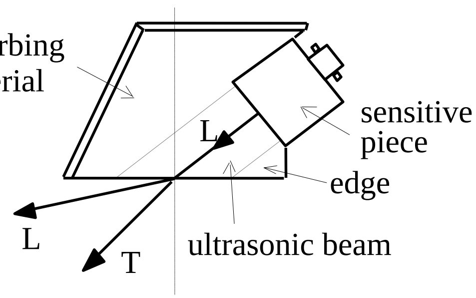


- **Immersion transducters**

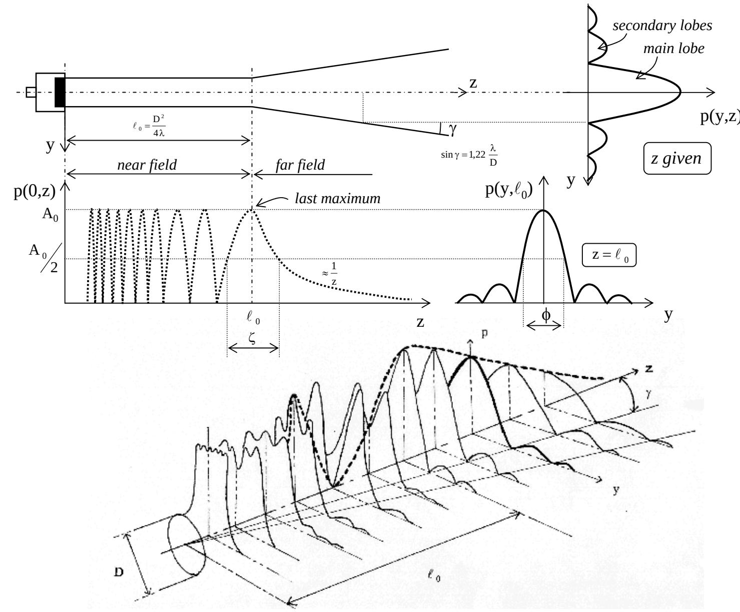




Angle beam transducer

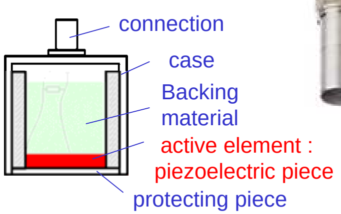


Ultrasonic field generated by a plane ultrasonic transducer



NDT by ultrasounds : principle

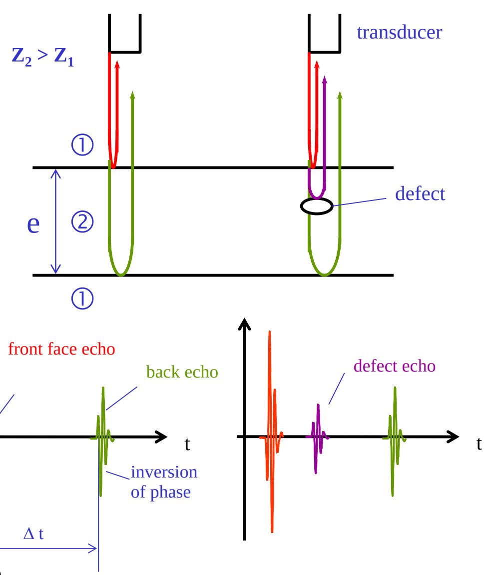
- **Transducer**



Transforms an electrical signal into a mechanical vibration and conversely

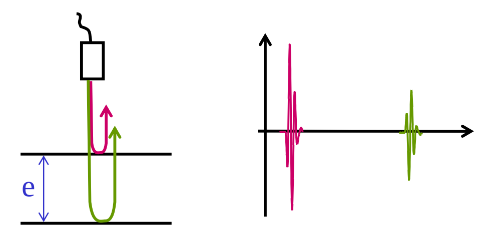
$$\Delta t = \frac{2e}{V}$$

- **Ultrasonic echography**

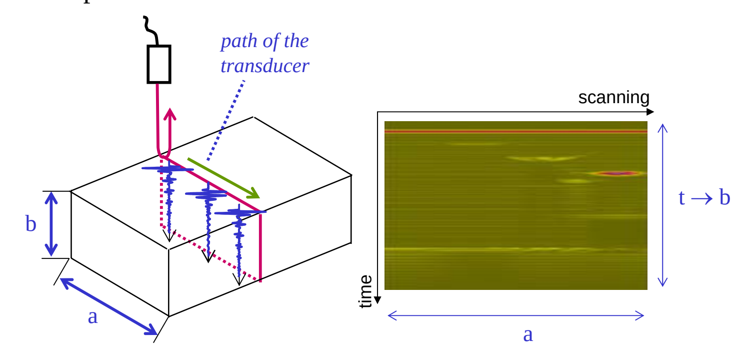


A-scan ; B-scan

- A-Scan

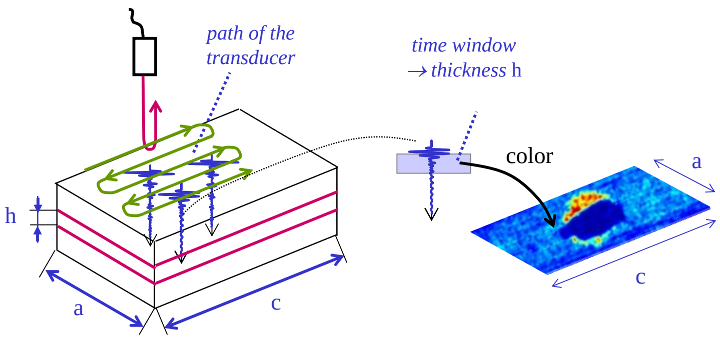


- B-Scan : corresponds to a **cut of the material**



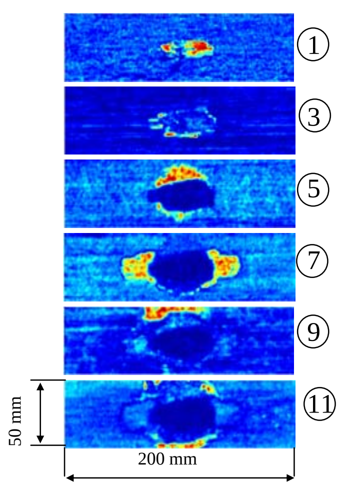
C-Scan

- C-scan : corresponds to a **representation of a material slice**

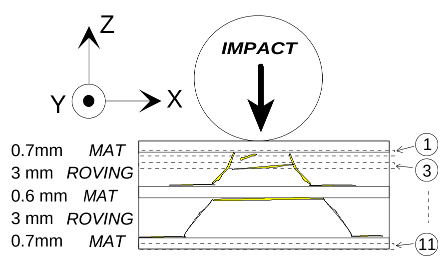


C-scan on a coin of currency

C-scan on an impacted beam

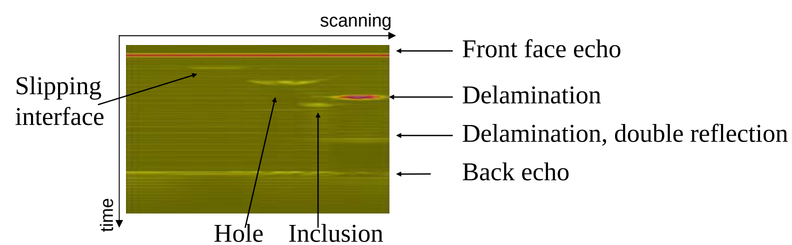


Impact on a pultruded beam



Hybrid model

Defects of different kind taken into account – **Example** (simulated Bscan)

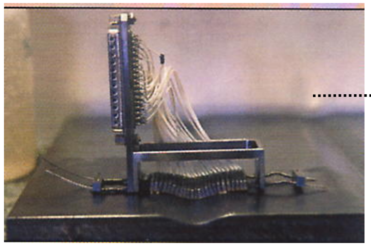


Comparison simulation / experiments

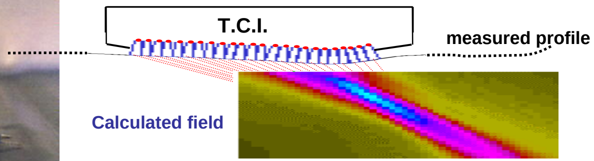
CEA courtesy

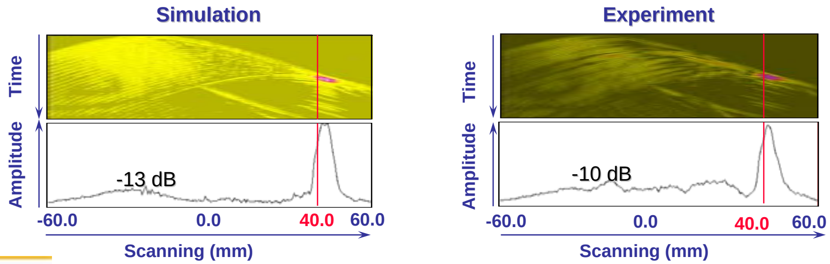
DETECS / Service simulation et systèmes pour la Surveillance et le Contrôle

B-scan with a multi-element transducer



Contact flexible multi-element transducer (T.C.I., CEA)



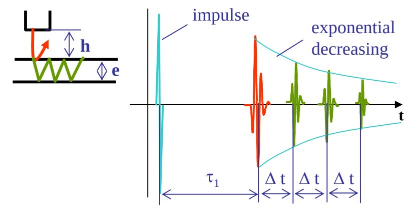


CEA courtesy

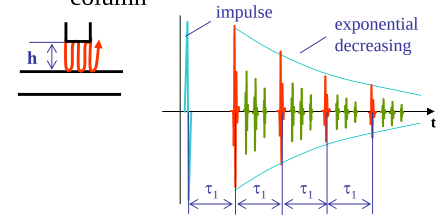
DETECS / Service simulation et systèmes pour la Surveillance et le Contrôle

Precautions of adjustment (1/2)

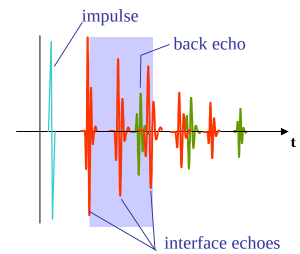
- Multiples reflections in a plate



- Multiples reflections in the water column



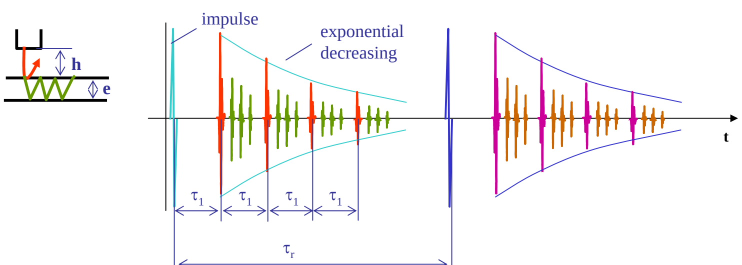
- Conquences of a bad adjustment of the water column



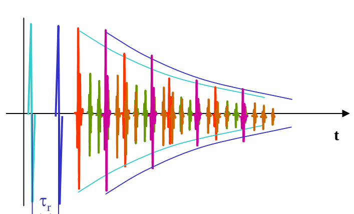
Insertion of a back echo between the interface echos. This back echo could be mixed up with a defect echo.

Precautions of adjustment (2/2)

- Adjustement of the pulse repetition frequency

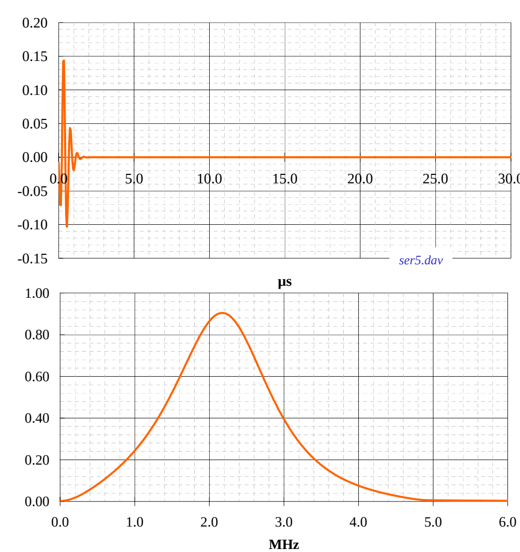


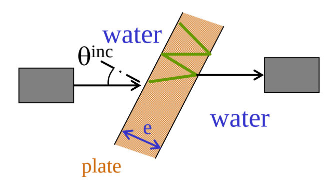
- Consequences of a bad adjustment of the pulse repetition frequency

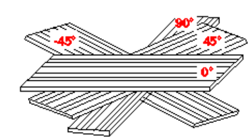


Insertion of interface or back echoes of the 2nd repetition between interface echoes of the 1st recurrence, which could be mixed up with a defect echo.

Example of input signal

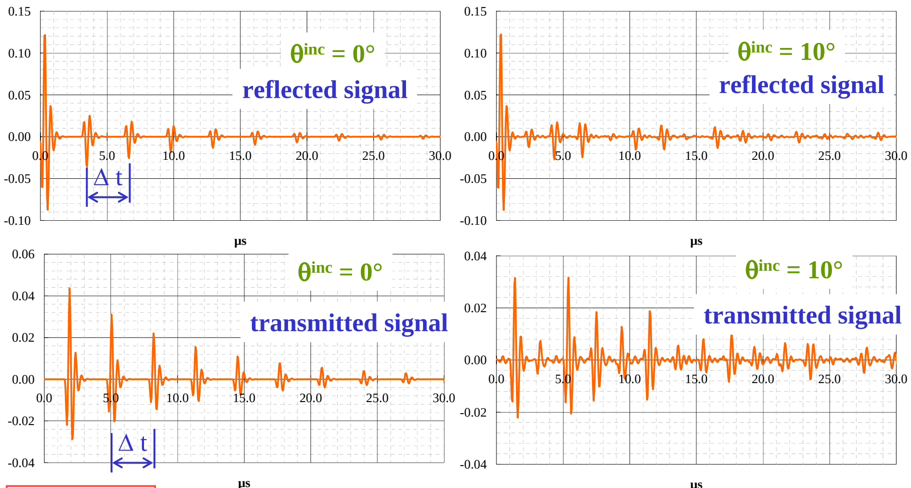






0°/45°/90°/135° carbon-epoxy plate

Aluminium plate immersed in water (plane wave simulation)



$$V_L = \frac{2\,e}{\Delta t}$$

$e = 10$ mm ; $\lambda_L = 2.8$ mm ; $\lambda_T = 1.4$ mm

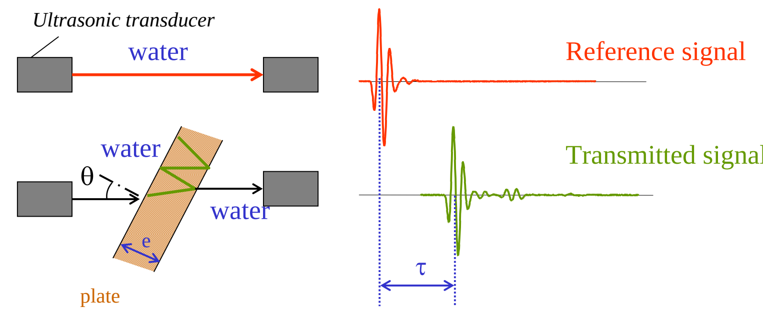
$V_L = 6340.4$ m/s

$V_T = 3138.9$ m/s

1st critical angle = 13.5°

2nd critical angle = 28°

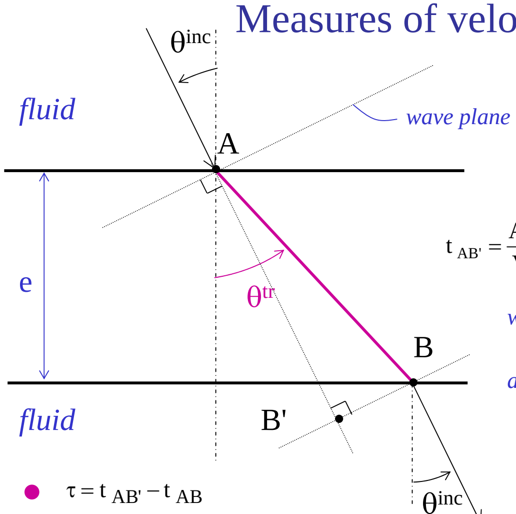
Measures of velocities (1/2)



Propagation velocity of a wave in the medium, for a given direction

$$V = \frac{V_{eau}}{\sqrt{1 + \frac{\tau V_{eau}}{e}\left(\frac{\tau V_{eau}}{e} - 2\cos\theta\right)}}$$ → **Real elastic constants**

Measures of velocities (2/2)



- Path in the plate

$$t_{AB} = \frac{AB}{V} = \frac{e}{V\cos\theta^{tr}}$$

- Path in the fluid

$$t_{AB'} = \frac{AB'}{V_0} = \frac{AB\cos\left(\theta^{tr} - \theta^{inc}\right)}{V_0} = \frac{e\cos\left(\theta^{tr} - \theta^{inc}\right)}{V_0\cos\theta^{inc}}$$

with $n = \frac{\sin\theta^{inc}}{\sin\theta^{tr}} = \frac{V_0}{V}$ → $V = \frac{V_0}{n}$

and $\sin\theta^{tr} = \frac{1}{n}\sin\theta^{inc}$, $\cos\theta^{tr} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta^{inc}}$

- $\tau = t_{AB'} - t_{AB}$

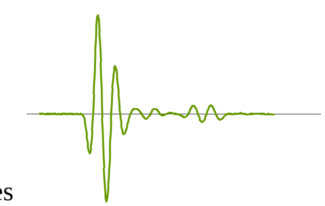
$$= \frac{e}{V\cos\theta^{tr}}\left[\cos\left(\theta^{tr} - \theta^{inc}\right) - \frac{V_0}{V}\right] = \frac{e}{V\cos\theta^{tr}}\left[\cos\theta^{tr}\cos\theta^{inc} + \sin\theta^{tr}\sin\theta^{inc} - n\right]$$

→ $\left(\frac{\tau V_0}{e} - \cos\theta^{inc}\right)^2 = n^2 - \sin^2\theta^{inc}$ → $V = \frac{V_0}{n} = V_0 \Big/ \sqrt{1 + \frac{\tau V_0}{e}\left(\frac{\tau V_0}{e} - 2\cos\theta^{inc}\right)}$

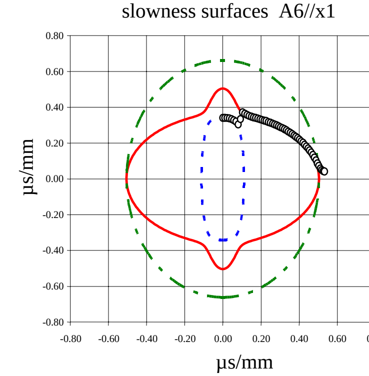
BUT...

- Wavelength >> e_{total}
  → difficult to separate the different echoes
- Deformation of echoes
  → difficult evaluation of times
- Influence of the anisotropy
- Inverse problem

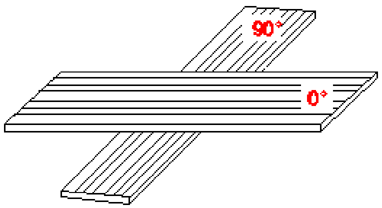
time of flight
↓
velocity
↓
inverse of the velocity (slowness)
↓
elastic constants



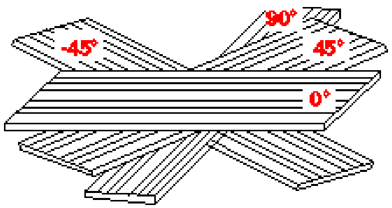
slowness surfaces A6//x1



Composite materials (1/2) : example of carbon/epoxy composites

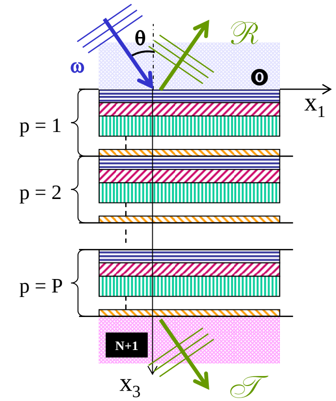


0°/90°

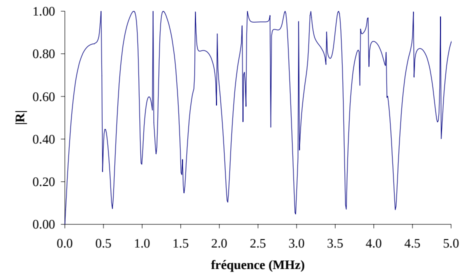


0°/45°/90°/135°

Composite materials (2/2) : example of carbon/epoxy composites

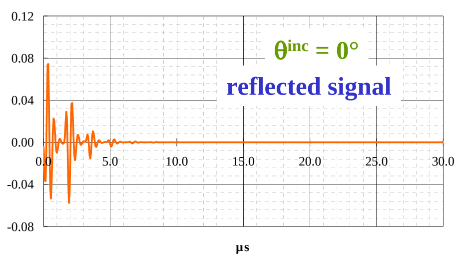


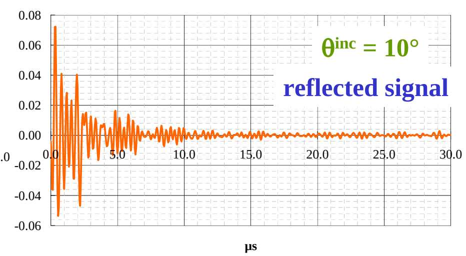
- Reflection coefficient(s) $\mathscr{R}$
- Transmission coefficient(s) $\mathscr{T}$
- characteristics of all the waves

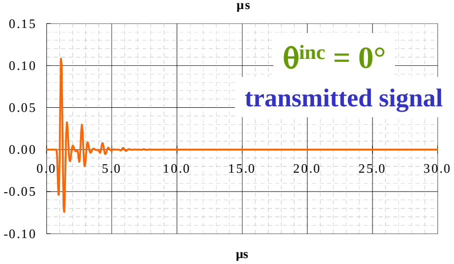


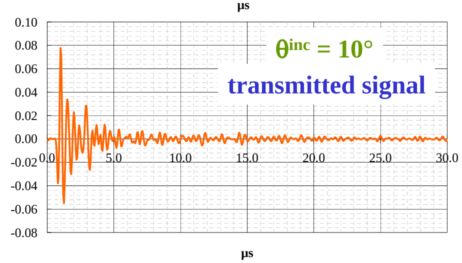
0°/45°/90°/135°
P=5 ; θ = 10°

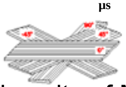
Composite plate immersed in water (Plane wave simulation)











0°/45°/90°/135° ; 5 periods (20 layers) ; carbon/epoxy plate