Amplification and saturation of the thermoacoustic instability in a standing-wave prime mover.

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Framework

1. **Experiments**
   - description of the thermoacoustic device
   - transient regimes measurements
2. **Theory**
   - description of acoustic propagation using transfer matrices
   - amplification/attenuation of the acoustic wave
   - determination of the onset threshold
   - transient regime
3. **Simulations of transient regimes**
4. **Conclusion**

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Experiments

description of the thermoacoustic device

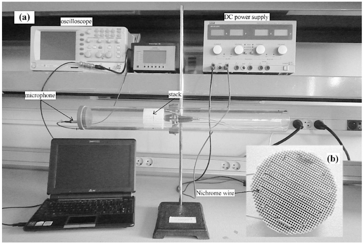


Figure: (a) Photograph of the experimental apparatus. (b) Photograph of the hot end of the stack.

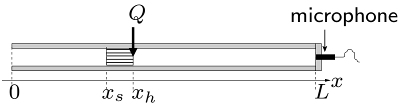


Figure: Schematic drawing of the standing-wave prime mover.

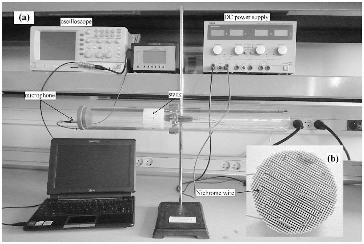
glass tube		600 CPSI ceramic stack	
L	$59\ cm$	l_s	$4.8\ cm$
R	$2.6\ cm$	r_s	$0.45\ mm$

Microphone
Bruel & Kjaer – $1/4$ inch

Acquisition
PC SoundCard

Experiments

description of the thermoacoustic device



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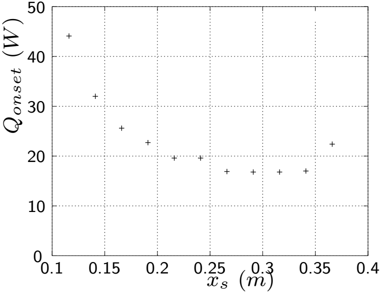


Figure: Stability curve as function of the location x_s of the stack.

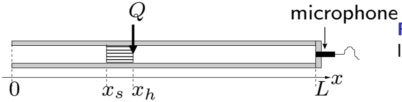


Figure: Schematic drawing of the standing-wave prime mover.

Experiments

transient regimes measurements

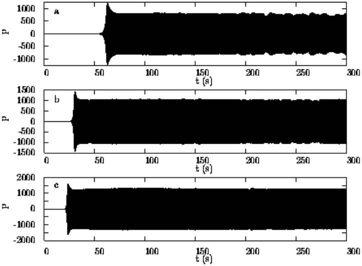


Figure: $Q(t=0) = 16W$ (slightly below $Q_{onset} = 16.9W$). (a) $\Delta Q/Q = 16\%$, (b) $\Delta Q/Q = 34\%$,(c) $\Delta Q/Q = 53\%$.

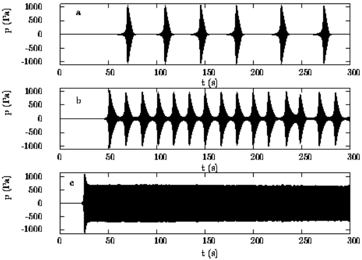
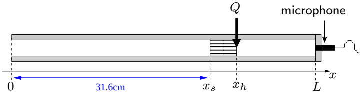
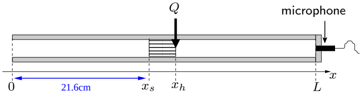


Figure: $Q(t=0) = 18W$ (slightly below $Q_{onset} = 19.6W$). (a) $\Delta Q/Q = 16\%$, (b) $\Delta Q/Q = 24\%$,(c) $\Delta Q/Q = 30\%$.





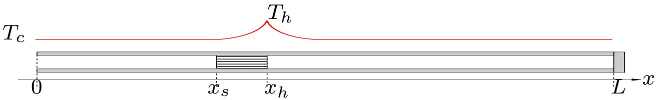
Theory

description of acoustic propagation using transfer matrices

Harmonic plane wave assumption

$$p_1(x,t) = \Re\left\{\tilde{p}_1(x)e^{-i\omega t}\right\} \quad \text{and} \quad \xi_1(x,y,t) = \Re\left\{\tilde{\xi}_1(x,y)e^{-i\omega t}\right\}, \qquad (1)$$

where $\xi_1 = v_{1,x}, \rho_1, \tau_1, s_1$.



$$\begin{pmatrix} \tilde{p}_1(L) \\ \tilde{u}_{1,x}(L) \end{pmatrix} = \mathbf{M_w} \times \mathbf{M_s} \times \mathbf{M_1} \times \begin{pmatrix} \tilde{p}_1(0) \\ \tilde{u}_{1,x}(0) \end{pmatrix}$$

- $\mathbf{M_1} = \begin{pmatrix} \cos(kx_s) & iZ_c\sin(kx_s) \\ iZ_c^{-1}\sin(kx_s) & \cos(kx_s) \end{pmatrix}$
- $\mathbf{M_s}$ and $\mathbf{M_w}$ derived from the linear thermoacoustic propagation equation transformed into a Volterra integral equation of the second kind [Penelet *et al.*, *Acust. Acta Acust.* (2005)].

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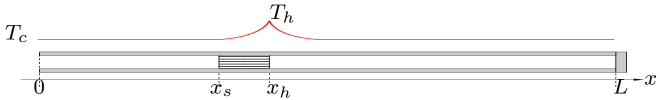

$$\begin{pmatrix} \tilde{p}_1(L) \\ \tilde{u}_{1,x}(L) \end{pmatrix} = \begin{pmatrix} \mathbf{M_{pp}}(\omega, T(x)) & \mathbf{M_{pu}}(\omega, T(x)) \\ \mathbf{M_{up}}(\omega, T(x)) & \mathbf{M_{uu}}(\omega, T(x)) \end{pmatrix} \times \begin{pmatrix} \tilde{p}_1(0) \\ \tilde{u}_{1,x}(0) \end{pmatrix}$$

Appropriate boundary conditions

- rigid wall : $\tilde{u}_{1,x}(L) = 0$
- no radiation : $\tilde{p}_1(0) = 0$

$$\Longrightarrow \qquad \mathbf{M_{uu}}(\omega, T(x)) = 0.$$

Theory

amplification/attenuation of the acoustic wave

$$\mathbf{M_{uu}}\left(\omega, T(x)\right) = 0. \quad (2)$$

A solution (ω, T) of Eq. (2) represents an operating point of the system.
In the Fourier domain ($\omega \in \mathbb{R}$), it describes an equilibrium point :

- either unstable (onset threshold),
- or stable (steady state),
- corresponding to an acoustic wave which is neither amplified, nor attenuated in both cases.

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"quasi-steady" state assumption

- $\omega = \Omega + i\epsilon_g \quad \Rightarrow \quad p_1(x,t) = e^{\epsilon_g t}\Re\left\{\tilde{p}_1(x)e^{-i\Omega t}\right\}$,
- $\epsilon_g << \Omega$ on the time scale of few acoustic periods.

For a fixed temperature distribution $T(x)$, the solution of

$$\mathbf{M_{uu}}\left(\Omega, \epsilon_g\right) = 0$$

gives the angular frequency of the oscillations and the amplification rate.

Theory

determination of the onset threshold

$$\epsilon_g\left(T(x)\right) = 0. \tag{3}$$

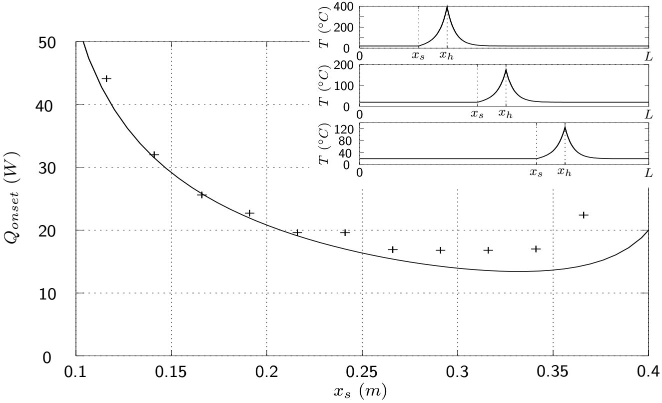

Figure: Stability curve as function of the location x_s of the stack.

Theory
transient regime

Ordinary differential equation for the acoustic pressure amplitude

$$\frac{dP_1}{dt} - \epsilon_g\left(T(x,t)\right)P_1(t) = 0, \quad \text{with} \quad P_1(t) = |p_1(L,t)|. \tag{4}$$

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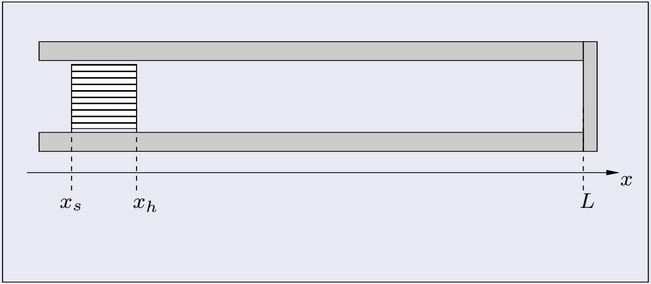
one-dimensional heat diffusion in the prime mover



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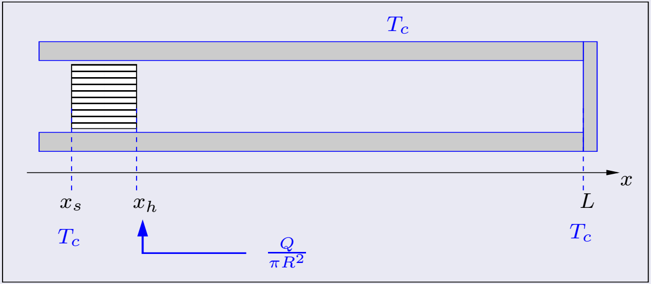

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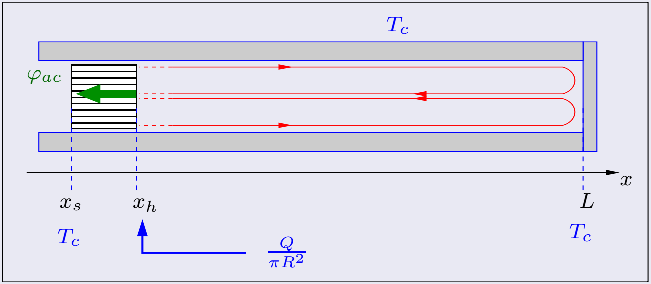


Theory

transient regime

thermoacoustic heat flux along the stack

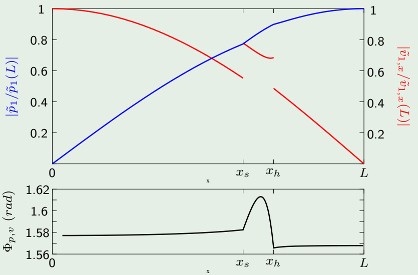


Figure: Acoustic fields in the thermoacoustic prime mover at onset threshold, for $x_s = 30\ cm$ and for an arbitrary acoustic pressure amplitude $|\tilde{p}_1(L)| = 1\ Pa$.

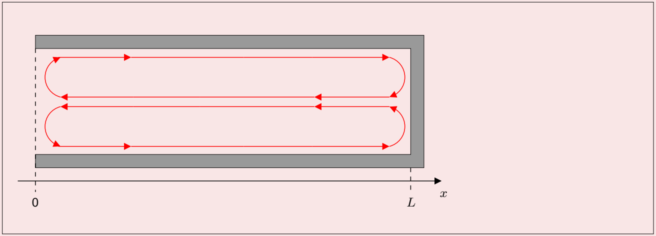
$$\varphi_{ac} = \frac{1}{2}\rho_0 c_0 \Re\left\{\langle \tilde{s}_1 \tilde{v}_{1,x}^* \rangle\right\},$$

oscillating part of entropy :

$$\begin{aligned}\tilde{s}_1 &= \frac{-\tilde{p}_1}{\rho_0 T} F_\kappa(y) \\ &- i\frac{C_p}{\omega}\frac{\partial_x T}{T}\frac{\langle \tilde{v}_{1,x}\rangle}{1-f_\nu} \\ &\times \left(1 - \frac{\sigma F_\nu(y) - F_\kappa(y)}{\sigma - 1}\right)\end{aligned}$$

Theory
transient regime

acoustic streaming



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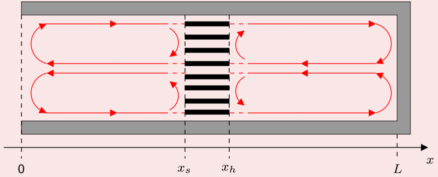

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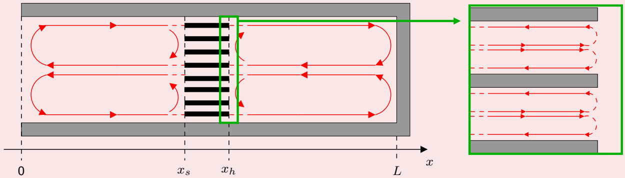

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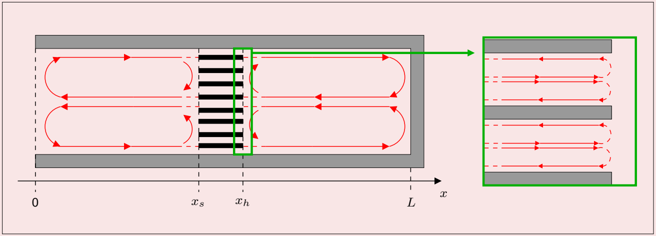
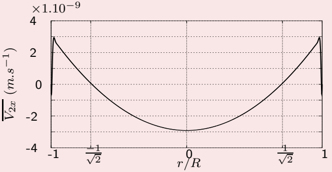


Axial component of the streaming velocity :

$$\overline{V_{2,x}} = \overline{v_{2,x}} + \frac{\overline{\rho_1 v_{1,x}}}{\rho_0},$$

where $\overline{v_{2,x}}$ is the second-order time-averaged Eulerian velocity [Bailliet *et al.*, *J. Acoust. Soc. Am.* (2001)].



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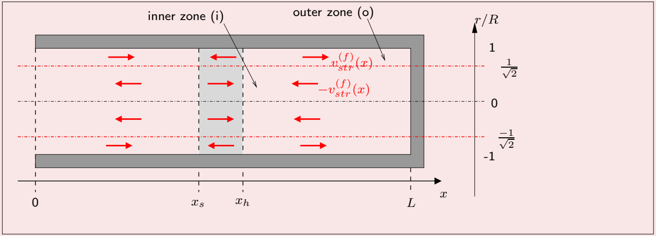
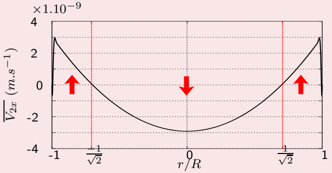


Averaged streaming velocity :

$$v_{str}^{(f)}(x) = \frac{2\pi}{\pi R^2} \int_0^R \left|\overline{V_{2,x}}\right| r dr,$$

$$v_{str}^{(s)}(x) = \frac{2\pi}{\pi r_s^2} \int_0^{r_s} \left|\overline{V_{2,x}}\right| r dr.$$



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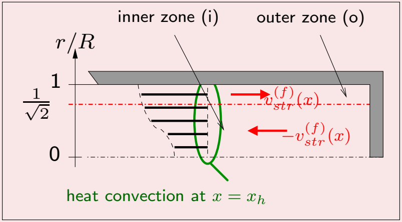

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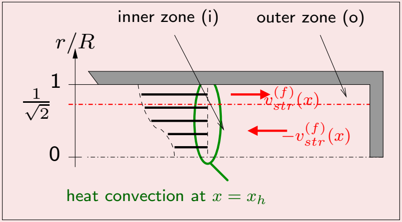

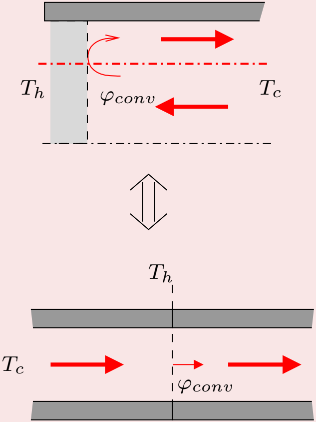

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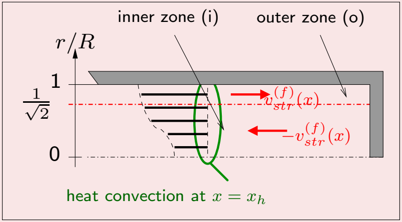
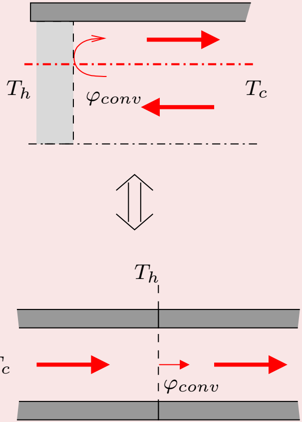


Estimation of the convection flux taken away from the hot end stack by the mass flow :

$$\varphi_{conv}(x_h) \simeq \rho_0 C_p v_{str}^{(f)}(x_h)\,(T(x_h) - T_c)$$



Theory

transient regime

acoustic streaming

Computation of the temporal evolution of acoustic streaming in the resonator :

1. First estimation of averaged streaming velocity

$$\Gamma_v^{(f)}(x) = v_{str}^{(f)}(P_1 = 1\ Pa, T_{onset}(x)),$$

2. First order differential equation for the acoustic streaming :

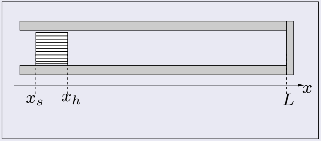
$$\tau^{(f)} \frac{dv_{str}^{(f)}}{dt} + v_{str}^{(f)} = \Gamma_v^{(f)} P_1^2,$$

   where $\tau^{(f)} = \frac{4R^2}{\pi^2 \nu}$ is a characteristic time for stabilization of acoustic streaming [Amari *et al.*, *Acust. Acta Acust.* (2003)].

φ_{conv} and $\tau^{(f)}$ are estimated in a very simplified way and can constitute adjusting parameters for the model.

Theory
transient regime

one-dimensional heat diffusion in the prime mover



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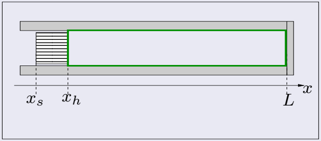

- $$(\rho_0 C_p)^{(i,o)} \frac{DT^{(i,o)}}{Dt} = \frac{\partial}{\partial x}\left(\lambda_0^{(i,o)} \frac{\partial T^{(i,o)}}{\partial x}\right) - h^{(i,o)}\left(T^{(i,o)} - T_c\right),$$

with $\frac{DT^{(i,o)}}{Dt} = \frac{\partial T^{(i,o)}}{\partial t} \mp v_{str}^{(f)} \frac{\partial T^{(i,o)}}{\partial x}$,

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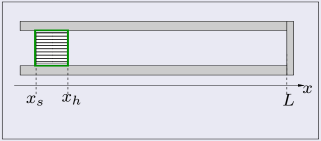

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Theory
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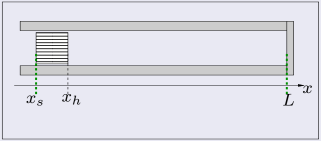

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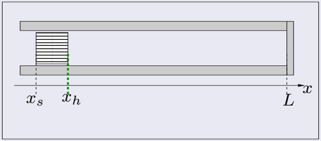

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- $T(x_s) = T^{(i)}(L) = T^{(o)}(L) = T_c,$

- $T^{(i)}(x_h) - T^{(o)}(x_h) = 0,$

- $$\lambda_s \left.\partial_x T\right|_{x_h^+} - \lambda_f \left.\partial_x T\right|_{x_h^-} + \varphi_{conv}(x_h) - \varphi_{ac}(x_h) = \frac{Q}{\pi R^2},$$

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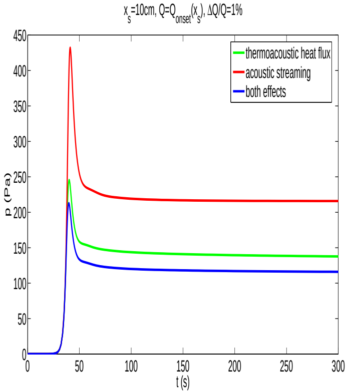
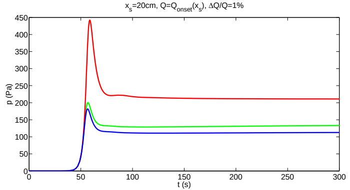
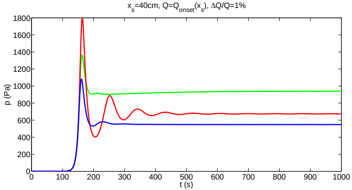
Simulations of transient regimes







Simulations of transient regimes

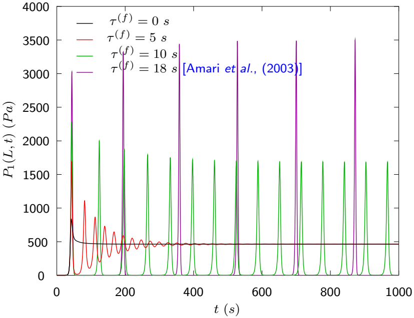


Figure: At $t = 0$, $Q = Q_{onset}$, $\frac{\Delta Q}{Q} = 1\%$. $x_s = 10\ cm$ and $\varphi_{ac} = 0$.

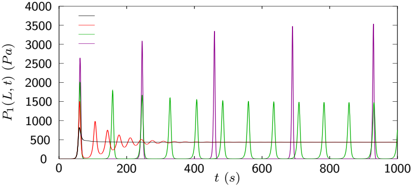


Figure: $x_s = 20\ cm$.

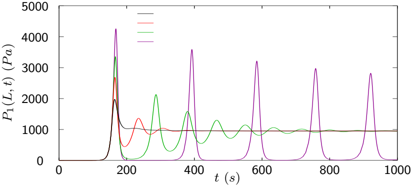


Figure: $x_s = 40\ cm$.

1. Experiments
   - description of the thermoacoustic device
   - transient regimes measurements
2. Theory
   - description of acoustic propagation using transfer matrices
   - amplification/attenuation of the acoustic wave
   - determination of the onset threshold
   - transient regime
3. Simulations of transient regimes
4. **Conclusion**

Conclusion

- A **quarter-wavelength thermoacoustic prime mover** has been studied and measurements of transient regimes show an **"overshoot" before stabilization** and **periodic "on-off"** of the acoustic wave for particular exciting conditions.
- A model describing amplification and saturation of the acoustic wave has been developped and applied to this prime mover, including **two non-linear effects** : **thermoacoustic heat flux** in the stack, and **acoustic streaming** in the heated part of the resonator.
- The **"overshoot"** is reproduced by **both effects**.
- **Saturation** seems to be mainly due to **thermoacoustic heat pumping** in the stack
- Complete switch-off of the wave would be linked to **different time scales** (**stabilization time of acoustic streaming** seems to play an important role).

Appendix

other experimental results

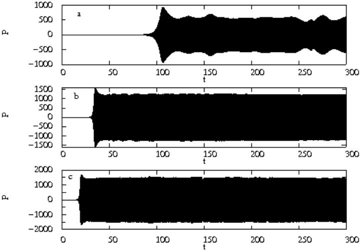


Figure: $x_s = 34.1\ cm,\ Q_{onset} = 17\ W$

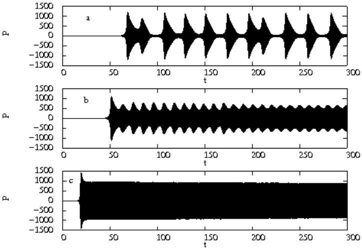


Figure: $x_s = 26.6\ cm,\ Q_{onset} = 16.9\ W$

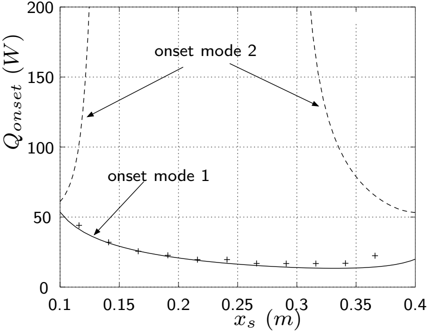


Figure: Stability curve as function of the location x_s of the stack.

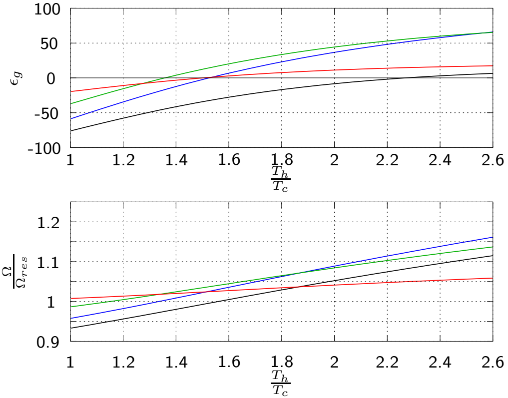
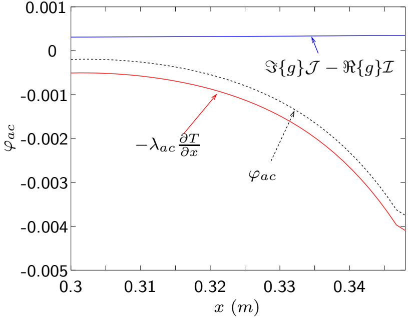


Figure: Amplification rate ϵ_g and normalised frequency $\frac{\Omega}{\Omega_{res}}$ in function of temperature for 4 locations of the stack.



$$\varphi_{ac} = \Im\{g\}\mathcal{J} - \Re\{g\}\mathcal{I} - \lambda_{ac}\frac{\partial T}{\partial x},$$

with

$$\mathcal{I} = \frac{1}{2}\Re\{\tilde{p}_1 \langle \tilde{v}^*_{1,x}\rangle\}$$

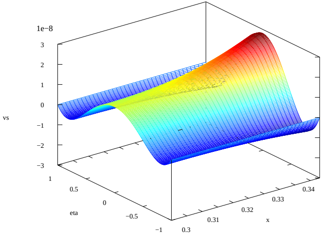
$$\mathcal{J} = \frac{1}{2}\Im\{\tilde{p}_1 \langle \tilde{v}^*_{1,x}\rangle\}$$

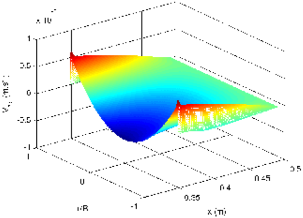
$$g = \frac{f_\kappa - f^*_\nu}{(\sigma+1)(1-f_\nu)}$$

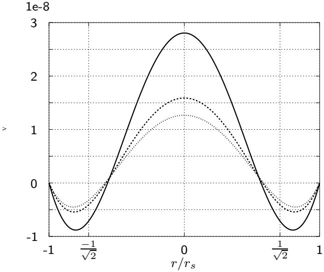
$$\lambda_{ac} = \rho_0 C_p \frac{\Im\{\sigma f_\nu - f_\kappa\}}{(\sigma^2-1)|1-f_\nu|^2}\frac{|\langle \tilde{v}_{1,x}\rangle|^2}{2\omega}$$

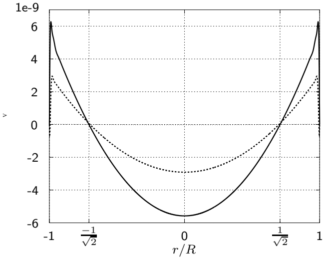
Appendix

spatial distributions of acoustic streaming







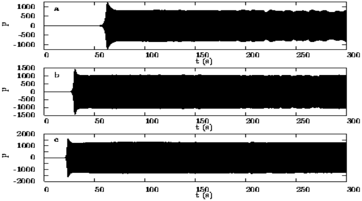


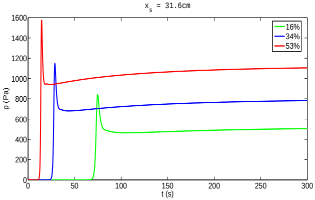
Appendix

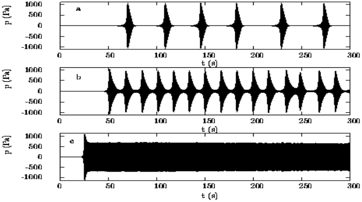
taking into account heat convection at interface

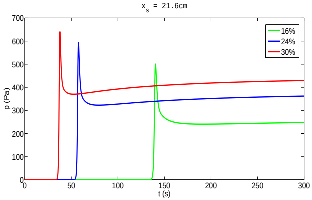
Appendix

other simulations of transient regimes









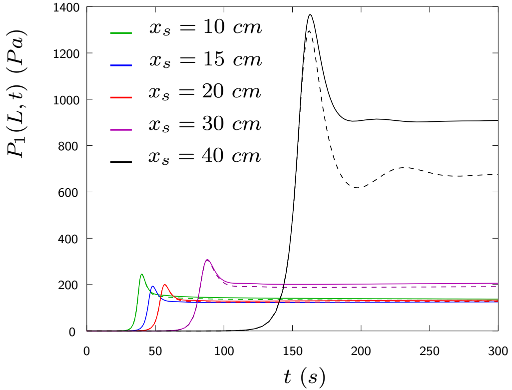


Figure: At $t = 0$, $Q = Q_{onset}$, $\frac{\Delta Q}{Q} = 1\%$.
— : $v_{str}^{(f)} = 0$.
$--$: both non-linear effects are taken into account.