Amplification and saturation of the thermoacoustic instability in a standing-wave prime mover.

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1 Experiments
   - description of the thermoacoustic device
   - transient regimes measurements

2 Theory
   - description of acoustic propagation using transfer matrices
   - amplification/attenuation of the acoustic wave
   - determination of the onset threshold
   - transient regime

3 Simulations of transient regimes

4 Conclusion
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Experiments

description of the thermoacoustic device

Figure: (a) Photograph of the experimental apparatus. (b) Photograph of the hot end of the stack.

Figure: Schematic drawing of the standing-wave prime mover.

<table>
<thead>
<tr>
<th>glass tube</th>
<th>600 CPSI</th>
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<tbody>
<tr>
<td>L</td>
<td>59 cm</td>
</tr>
<tr>
<td>R</td>
<td>2.6 cm</td>
</tr>
<tr>
<td>l_s</td>
<td>4.8 cm</td>
</tr>
<tr>
<td>r_s</td>
<td>0.45 mm</td>
</tr>
</tbody>
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Microphone
Brul & Kjaer – 1/4 inch

Acquisition
PC SoundCard
Experiments
description of the thermoacoustic device

Figure: (a) Photograph of the experimental apparatus. (b) Photograph of the hot end of the stack.

Figure: Schematic drawing of the standing-wave prime mover.

Figure: Stability curve as function of the location $x_s$ of the stack.
Experiments

transient regimes measurements

Figure: \( Q(t = 0) = 16W \) (slightly below \( Q_{\text{onset}} = 16.9W \) ). (a) \( \Delta Q/Q = 16\% \), (b) \( \Delta Q/Q = 34\% \), (c) \( \Delta Q/Q = 53\% \).

Figure: \( Q(t = 0) = 18W \) (slightly below \( Q_{\text{onset}} = 19.6W \) ). (a) \( \Delta Q/Q = 16\% \), (b) \( \Delta Q/Q = 24\% \), (c) \( \Delta Q/Q = 30\% \).
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Harmonic plane wave assumption

\[ p_1(x, t) = \Re \{ \tilde{p}_1(x)e^{-i\omega t} \} \quad \text{and} \quad \xi_1(x, y, t) = \Re \{ \tilde{\xi}_1(x, y)e^{-i\omega t} \}, \quad (1) \]

where \( \xi_1 = v_{1,x}, \rho_1, \tau_1, s_1 \).

\[
\begin{pmatrix}
\tilde{p}_1(L) \\
\tilde{u}_{1,x}(L)
\end{pmatrix}
= M_W \times M_s \times M_1 \times 
\begin{pmatrix}
\tilde{p}_1(0) \\
\tilde{u}_{1,x}(0)
\end{pmatrix}
\]

\[
M_1 = 
\begin{pmatrix}
\cos(kx_s) & iZ_C \sin(kx_s) \\
iZ_C^{-1} \sin(kx_s) & \cos(kx_s)
\end{pmatrix}
\]

\( M_s \) and \( M_W \) derived from the linear thermoacoustic propagation equation transformed into a Volterra integral equation of the second kind [Penelet et al., Acust. Acta Acust. (2005)].
Harmonic plane wave assumption

\[ p_1(x, t) = \Re \left\{ \tilde{p}_1(x)e^{-i\omega t} \right\} \quad \text{and} \quad \xi_1(x, y, t) = \Re \left\{ \tilde{\xi}_1(x, y)e^{-i\omega t} \right\}, \quad (1) \]

where \( \xi_1 = v_1, x; \rho_1, \tau_1, s_1. \)

Appropriate boundary conditions

- rigid wall: \( \tilde{u}_{1, x}(L) = 0 \)
- no radiation: \( \tilde{p}_1(0) = 0 \)

\[ \begin{pmatrix} \tilde{p}_1(L) \\ \tilde{u}_{1, x}(L) \end{pmatrix} = \begin{pmatrix} M_{pp}(\omega, T(x)) & M_{pu}(\omega, T(x)) \\ M_{up}(\omega, T(x)) & M_{uu}(\omega, T(x)) \end{pmatrix} \times \begin{pmatrix} \tilde{p}_1(0) \\ \tilde{u}_{1, x}(0) \end{pmatrix} = 0. \]
Theory
amplification/attenuation of the acoustic wave

\[ M_{uu}(\omega, T(x)) = 0. \quad (2) \]

A solution \((\omega, T)\) of Eq. (2) represents an operating point of the system. In the Fourier domain \((\omega \in \mathbb{R})\), it describes an equilibrium point:

- either unstable (onset threshold),
- or stable (steady state),
- corresponding to an acoustic wave which is neither amplified, nor attenuated in both cases.
A solution \((\omega, T')\) of Eq. (2) represents an operating point of the system. In the Fourier domain \((\omega \in \mathbb{R})\), it describes an equilibrium point:

- either unstable (onset threshold),
- or stable (steady state),
- corresponding to an acoustic wave which is neither amplified, nor attenuated in both cases.

**“quasi-steady” state assumption**

\[
\omega = \Omega + i\epsilon_g \implies p_1(x, t) = e^{\epsilon_g t} \Re \{ \tilde{p}_1(x)e^{-i\Omega t} \},
\]

\(\epsilon_g \ll \Omega\) on the time scale of few acoustic periods.

For a fixed temperature distribution \(T(x)\), the solution of

\[
M_{uu}(\omega, T(x)) = 0.
\]

gives the angular frequency of the oscillations and the amplification rate.
Theory
determination of the onset threshold

\[ \epsilon_g(T(x)) = 0. \] (3)

**Figure:** Stability curve as function of the location \( x_s \) of the stack.

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Ordinary differential equation for the acoustic pressure amplitude

\[
\frac{dP_1}{dt} - \epsilon_g (T(x, t)) P_1(t) = 0, \quad \text{with} \quad P_1(t) = |p_1(L, t)|.
\]  

(4)
Ordinary differential equation for the acoustic pressure amplitude

\[ \frac{dP_1}{dt} - \epsilon_g(T(x, t)) P_1(t) = 0, \quad \text{with} \quad P_1(t) = |p_1(L, t)|. \] (4)

One-dimensional heat diffusion in the prime mover

\[ \frac{d}{dx} \left( \frac{1}{\rho C_p} \frac{dT}{dx} \right) = -Q, \]
Theory

Ordinary differential equation for the acoustic pressure amplitude

\[
\frac{dP_1}{dt} - \epsilon_g (T(x, t)) P_1(t) = 0, \quad \text{with} \quad P_1(t) = |p_1(L, t)|. \tag{4}
\]

one-dimensional heat diffusion in the prime mover

\[x_s \quad x_h \quad x \quad L \quad T_c \]

\[T_c \quad \frac{Q}{\pi R^2} \quad T_c\]
Ordinary differential equation for the acoustic pressure amplitude

\[
\frac{dP_1}{dt} - \epsilon_g(T(x, t)) P_1(t) = 0, \quad \text{with} \quad P_1(t) = |p_1(L, t)|.
\]  

(one-dimensional heat diffusion in the prime mover)
transient regime

**Description of acoustic propagation using transfer matrices**

- Amplification/attenuation of the acoustic wave
- Determination of the onset threshold

**Simulations of transient regimes**

**Conclusion**

### Theory

**Thermoacoustic heat flux along the stack**

![Graph showing thermoacoustic heat flux along the stack](image)

\[
\varphi_{ac} = \frac{1}{2} \rho_0 c_0 \Re \{ \langle \tilde{s}_1 \tilde{v}_1, x \rangle \},
\]

**Oscillating part of entropy**

\[
\tilde{s}_1 = \frac{-\tilde{p}_1}{\rho_0 T} F_\kappa(y)
- i \frac{C_p}{\omega} \frac{\partial x T}{T} \langle \tilde{v}_1, x \rangle
\times \left( 1 - \frac{\sigma F_\nu(y) - F_\kappa(y)}{\sigma - 1} \right)
\]

**Figure:** Acoustic fields in the thermoacoustic prime mover at onset threshold, for \( x_s = 30 \text{ cm} \) and for an arbitrary acoustic pressure amplitude \( |\tilde{p}_1(L)| = 1 \text{ Pa} \).
Theory

transient regime

acoustic streaming

Matthieu Guédra et al. Thermoacoustic instability in a standing-wave engine
Theory

transient regime

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acoustic streaming

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acoustic streaming

0  \ x_s  \ x_h  \ L  \ x

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Acoustic streaming

Axial component of the streaming velocity:

$$
\overline{V_{2,x}} = \overline{v_{2,x}} + \frac{\rho_1 \overline{v_{1,x}}}{\rho_0},
$$

where $\overline{v_{2,x}}$ is the second-order time-averaged Eulerian velocity [Bailliet et al., J. Acoust. Soc. Am. (2001)].
Theory

transient regime

acoustic streaming

\[ v^{(f)}_{str}(x) = \frac{2\pi}{\pi R^2} \int_0^R \sqrt{2, x} r \, dr, \]
\[ v^{(s)}_{str}(x) = \frac{2\pi}{\pi r_s^2} \int_0^{r_s} \sqrt{2, x} r \, dr. \]
acoustic streaming

\[ r/R \]
\[ 1 \]
\[ \frac{1}{\sqrt{2}} \]
\[ 0 \]

inner zone (i) outer zone (o)

heat convection at \( x = x_h \)

\[ v_{str}(x) \]
\[ v_{str}^{(f)}(x) \]

\[ v_{str}^{(f)}(x) \]

transient regime

Mathieu Guédra et al.

Thermoacoustic instability in a standing-wave engine
acoustic streaming

\[ v^{(f)}_{str}(x) \]

heat convection at \( x = x_h \)
Theory

### transient regime

**acoustic streaming**

- inner zone (i)
- outer zone (o)

\[
\frac{1}{\sqrt{2}} \quad 1 \quad 0
\]

- heat convection at \(x = x_h\)

Estimation of the convection flux taken away from the hot end stack by the mass flow:

\[
\varphi_{\text{conv}}(x_h) \simeq \rho_0 C_p v_{\text{str}}(x_h) (T(x_h) - T_c)
\]
acoustic streaming

Computation of the temporal evolution of acoustic streaming in the resonator:

1. First estimation of averaged streaming velocity

\[ \Gamma_v^{(f)}(x) = v_{str}^{(f)}(P_1 = 1 \, \text{Pa}, T_{onset}(x)), \]

2. First order differential equation for the acoustic streaming:

\[ \tau^{(f)} \frac{dv_{str}^{(f)}}{dt} + v_{str}^{(f)} = \Gamma_v^{(f)} P_1^2, \]

where \( \tau^{(f)} = \frac{4R^2}{\pi^2 \nu} \) is a characteristic time for stabilization of acoustic streaming [Amari et al., Acust. Acta Acust. (2003)].

\( \varphi_{conv} \) and \( \tau^{(f)} \) are estimated in a very simplified way and can constitute adjusting parameters for the model.
one-dimensional heat diffusion in the prime mover
one-dimensional heat diffusion in the prime mover

\begin{align*}
\rho_0 C_p (i,o) \frac{DT(i,o)}{Dt} &= \frac{\partial}{\partial x} \left( \lambda_0(i,o) \frac{\partial T(i,o)}{\partial x} \right) \\
&- h(i,o) \left( T(i,o) - T_c \right),
\end{align*}

with \( \frac{DT(i,o)}{Dt} = \frac{\partial T(i,o)}{\partial t} + v_{str} \frac{\partial T(i,o)}{\partial x} \),
one-dimensional heat diffusion in the prime mover

\[ \begin{align*}
\text{\textbullet} & \quad (\rho_0 C_p)^{(i,o)} \frac{DT}{Dt} = \frac{\partial}{\partial x} \left( \lambda_0^{(i,o)} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
& \quad - h^{(i,o)} \left( T^{(i,o)} - T_c \right), \\
\text{\textbullet} & \quad \frac{(\rho_s C_s)^{(i,o)}}{D_t} = \frac{\partial}{\partial x} \left( \frac{\lambda_s^{(i,o)}}{\partial x} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
& \quad - h_s^{(i,o)} \left( T^{(i,o)} - T_c \right) \\
& \quad - \frac{\partial \varphi_{ac}}{\partial x},
\end{align*} \]

with \[ \begin{align*}
\frac{DT}{Dt} &= \frac{\partial T}{\partial t} \pm v_{str}^{(s)} \frac{\partial T^{(i,o)}}{\partial x},
\end{align*} \]
The transient regime

One-dimensional heat diffusion in the prime mover

\[
\begin{align*}
\bullet \quad \frac{(\rho_0 C_p)(i,o)}{D_t} \frac{DT^{(i,o)}}{Dt} &= \frac{\partial}{\partial x} \left( \frac{\lambda^{(i,o)}}{0} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
&\quad - h^{(i,o)} \left( T^{(i,o)} - T_c \right) \\
\text{with} \quad \frac{DT^{(i,o)}}{Dt} &= \frac{\partial T^{(i,o)}}{\partial t} \mp v_{str}^{(f)} \frac{\partial T^{(i,o)}}{\partial x},
\end{align*}
\]

\[
\begin{align*}
\bullet \quad (\rho_s C_s)(i,o) \frac{DT^{(i,o)}}{Dt} &= \frac{\partial}{\partial x} \left( \frac{\lambda_s^{(i,o)}}{s} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
&\quad - h_s^{(i,o)} \left( T^{(i,o)} - T_c \right) \\
&\quad - \frac{\partial \varphi_{ac}}{\partial x},
\end{align*}
\]

with \( DT^{(i,o)} = \frac{\partial T^{(i,o)}}{\partial t} \pm v_{str}^{(s)} \frac{\partial T^{(i,o)}}{\partial x} \)

\[
\begin{align*}
\bullet \quad T(x_s) = T^{(i)}(L) = T^{(o)}(L) = T_c,
\end{align*}
\]
one-dimensional heat diffusion in the prime mover

\[ \begin{align*}
\frac{(\rho_0 C_p)(i,o)}{Dt} DT^{(i,o)} &= \frac{\partial}{\partial x} \left( \lambda_0^{(i,o)} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
&- h^{(i,o)} \left( T^{(i,o)} - T_c \right),
\end{align*} \]

with \[ \frac{DT^{(i,o)}}{Dt} = \frac{\partial T^{(i,o)}}{\partial t} + v_{str}^{(s)} \frac{\partial T^{(i,o)}}{\partial x}, \]

\[ (\rho_s C_s)^{(i,o)} \frac{DT^{(i,o)}}{Dt} = \frac{\partial}{\partial x} \left( \lambda_s^{(i,o)} \frac{\partial T^{(i,o)}}{\partial x} \right) \\
- h_s^{(i,o)} \left( T^{(i,o)} - T_c \right) \\
- \frac{\partial \varphi_{ac}}{\partial x}, \]

with \[ \frac{DT^{(i,o)}}{Dt} = \frac{\partial T^{(i,o)}}{\partial t} \pm v_{str}^{(s)} \frac{\partial T^{(i,o)}}{\partial x} \]

\[ T(x_s) = T^{(i)}(L) = T^{(o)}(L) = T_c, \]

\[ T^{(i)}(x_h) - T^{(o)}(x_h) = 0, \]

\[ \lambda_s \frac{\partial x T}{x_h} - \lambda_f \frac{\partial x T}{x_h} = \frac{Q}{\pi R^2}, \]

\[ + \varphi_{conv}(x_h) - \varphi_{ac}(x_h) = \frac{Q}{\pi R^2}, \]
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Simulations of transient regimes

$x_s = 10\text{cm}, Q = Q_{\text{onset}}(x_s), \Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$

$\Delta Q/Q = 1\%$
Simulations of transient regimes

\[ P_1(L, t) (\text{Pa}) = \begin{cases} 
\tau(f) = 0 \text{ s} \\
\tau(f) = 5 \text{ s} \\
\tau(f) = 10 \text{ s} \\
\tau(f) = 18 \text{ s} \text{ [Amari et al., (2003)]}
\end{cases} \]

**Figure:** $x_s = 10 \ cm$ and $Q_{onset}$, $\Delta Q/Q = 1\%$.

**Figure:** $x_s = 20 \ cm$.

**Figure:** $x_s = 40 \ cm$. 

At $t = 0$, $Q = Q_{onset}$, $\Delta Q/Q = 1\%$. 

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3. **Simulations of transient regimes**

4. **Conclusion**
A quarter-wavelength thermoacoustic prime mover has been studied and measurements of transient regimes show an “overshoot” before stabilization and periodic “on-off” of the acoustic wave for particular exciting conditions.

A model describing amplification and saturation of the acoustic wave has been developed and applied to this prime mover, including two non-linear effects: thermoacoustic heat flux in the stack, and acoustic streaming in the heated part of the resonator.

The “overshoot” is reproduced by both effects.

Saturation seems to be mainly due to thermoacoustic heat pumping in the stack.

Complete switch-off of the wave would be linked to different time scales (stabilization time of acoustic streaming seems to play an important role).
Appendix
other experimental results

Figure: \( x_s = 34.1 \, \text{cm}, Q_{onset} = 17 \, \text{W} \)

Figure: \( x_s = 26.6 \, \text{cm}, Q_{onset} = 16.9 \, \text{W} \)
Figure: Stability curve as function of the location $x_s$ of the stack.
Figure: Amplification rate $\epsilon_g$ and normalised frequency $\frac{\Omega}{\Omega_{res}}$ in function of temperature for 4 locations of the stack.
\[ \varphi_{ac} = \Im \{ g \} J - \Re \{ g \} \mathcal{I} - \lambda_{ac} \frac{\partial T}{\partial x}, \]

with

\[ \mathcal{I} = \frac{1}{2} \Re \{ \tilde{p}_1 \langle \tilde{v}_{1,x}^* \rangle \} \]
\[ J = \frac{1}{2} \Im \{ \tilde{p}_1 \langle \tilde{v}_{1,x}^* \rangle \} \]
\[ g = \frac{f_\kappa - f_\nu^*}{(\sigma + 1)(1 - f_\nu)} \]
\[ \lambda_{ac} = \rho_0 C_p \frac{\Im \{ \sigma f_\nu - f_\kappa \} |\langle \tilde{v}_{1,x} \rangle|^2}{(\sigma^2 - 1)|1 - f_\nu|^2} \frac{1}{2\omega} \]
Appendix
spatial distributions of acoustic streaming

![Graph 1: 3D plots of spatial distributions of acoustic streaming](image1)

![Graph 2: 2D plots of spatial distributions of acoustic streaming](image2)
Appendix

taking into account heat convection at interface
Appendix
other simulations of transient regimes

Matthieu Guédra et al.  Thermoacoustic instability in a standing-wave engine
Appendix
other simulations of transient regimes

Figure: At \( t = 0 \), \( Q = Q_{\text{onset}} \), \( \frac{\Delta Q}{Q} = 1\% \).
— : \( v^{(f)}_{\text{str}} = 0 \).
— : both non-linear effects are taken into account.