Weakly nonlinear acoustic oscillations in gas columns in the presence of temperature gradients

G. Penelet[(a)], T. Chareyre[(a)], J. Gilbert[(a)]

[(a)] Laboratoire d'Acoustique de l'Université du Maine, UMR CNRS 6613,
avenue Olivier Messiaen, 72085 Le Mans cedex 9, France

Acoustics 2012,Nantes, 26 April 2012
session « Thermoacoustics »

PLAN

1.- Introduction

- Nonlinear acoustics already has a **long history** and many applications

  [Rudenko and Soluyan, *Theoretical foundations of Nonlinear Acoustics*, Consultants Bureau, NY, 1977]
  [Hamilton and Blackstock, *Nonlinear Acoustics*, Acoustical Society of America, NY, 2008]

- Considering NL propagation of plane waves in ducts, many experimental and theoretical studies made in the past decades.

- In particular, when assuming a low mach number ($M=v_{ac}/c_0<<1$), it is well known that weakly NL propagation can be described by the Burgers equation, which is derived using the Multiple Scale Method.

However the effect of a temperature gradient on non linear propagation of plane guided waves has not been studied a lot

=> interest in the study of the operation of thermoacoustic engines

2.- The Burgers equation in a medium with temperature gradient.

2.1.- Establishment of the Burgers equation

• Governing equations

$$\begin{cases} \rho\left(\partial_t\vec{v}+\left(\vec{v}.\vec{\nabla}\right)\vec{v}\right) &= -\vec{\nabla}p+\eta\vec{\Delta}\vec{v}+\left(\xi+\frac{\eta}{3}\right)\vec{\nabla}\left(\vec{\nabla}.\vec{v}\right) \\ \partial_t\rho+\vec{\nabla}.\left(\rho.\vec{v}\right) &= 0 \\ p &= p(\rho,s) \\ \rho T\left(\partial_t s+\left(\vec{v}.\vec{\nabla}\right)s\right) &= \lambda\Delta T+F_d \end{cases}$$

(F_d : rate of dissipation of mechanical energy)

• Assumptions:

- inviscid fluid (μ=0,ξ=0), no heat conduction (λ=0),
- 1-D propagation along the x-axis
- weakly non linear propagation: $\frac{p-p_0}{p_0}=\frac{p'}{p_0}\sim\mu\,,\ \frac{\rho-\rho_0}{\rho_0}=\frac{\rho'}{\rho_0}\sim\mu\,,\ \frac{v}{c_0}\sim\mu\,,\ \mu<<1$
- adiabatic process: $p'\simeq c_0^2\rho'+\frac{\gamma-1}{2\rho_0}c_0^2\rho'^2$
- inhomogeneous temperature gradient $T=T_0(x)$: $\rho_0(x)=\frac{M_{mol}p_0}{RT_0(x)},\ c_0^2(x)=\frac{\gamma p_0}{\rho_0(x)}$

$$\Rightarrow \boxed{\begin{aligned} \left(\rho_0+\rho'\right)\partial_t v+\rho_0 v\partial_x v+c_0^2\partial_x\rho'+\frac{\gamma-1}{\rho_0}c_0^2\rho'\partial_x\rho' &= -c_0^2\frac{d_xT_0}{T_0}\left(\rho'+\frac{\gamma-1}{2\rho_0}\rho'^2\right)+o\left(\mu^2\right) \\ \partial_t\rho'+v\partial_x\rho'\left(\rho_0+\rho'\right)\partial_x v &= \rho_0 v\frac{d_xT_0}{T_0}+o\left(\mu^2\right) \end{aligned}}$$

2.- The Burgers equation in a medium with temperature gradient.

2.1.- Establishment of the Burgers equation

$$\begin{cases} (\rho_0+\rho')\,\partial_t v+\rho_0 v\partial_x v+c_0^2\partial_x\rho'+\dfrac{\gamma-1}{\rho_0}c_0^2\rho'\partial_x\rho' &= -c_0^2\dfrac{d_xT_0}{T_0}\left(\rho'+\dfrac{\gamma-1}{2\rho_0}\rho'^2\right)+o\left(\mu^2\right) \\ \partial_t\rho'+v\partial_x\rho'\,(\rho_0+\rho')\,\partial_x v &= \rho_0 v\dfrac{d_xT_0}{T_0}+o\left(\mu^2\right) \end{cases}$$

If $v/c_0 \ll 1$, non linear effects are essentially cumulative (local nonlinear effects neglected) => use of the Multiple Scale Method:

$x \leftarrow x,\ t \leftarrow \tau+\frac{x}{c_{ref}}$, with $c_{ref}=c_0(x_0)$ => $\dfrac{\partial}{\partial t}\leftarrow\dfrac{\partial}{\partial\tau}$, $\dfrac{\partial}{\partial x}\leftarrow\dfrac{\partial}{\partial x}-\dfrac{1}{c_{ref}}\dfrac{\partial}{\partial\tau}$

(simple wave propagating along x↑)

~ μ

and additional assumption: $\dfrac{d_xT_0}{T_0}\sim\mu$

=> Apply the above mentioned change of variables in Eqs. (1) and (2) (retain only variables of order $\leq \mu^2$, and eliminate ρ') leads after some calculations to:

$$\boxed{\partial_x v_+ - f_1(x)v_+\partial_\tau v_+ = f_2(x)\partial_\tau v_+ + f_3(x)v_+}$$

, with $\frac{d_xT_0}{T_0}\sim\mu,\ T_{ref}=T_0(x_0),\ c_{ref}=c_0(x_0)$

$$f_1(x) = \frac{1}{c_{ref}^2}\frac{\gamma\frac{T_{ref}}{T_0}+\sqrt{\frac{T_{ref}}{T_0}}\left(2-\frac{T_{ref}}{T_0}\right)}{1+\sqrt{\frac{T_{ref}}{T_0}}} \qquad f_2(x) = \frac{1}{c_{ref}}\left(1-\sqrt{\frac{T_{ref}}{T_0}}\right) \qquad f_3(x) = \frac{1+\frac{1}{2}\sqrt{\frac{T_{ref}}{T_0}}}{1+\sqrt{\frac{T_{ref}}{T_0}}}\frac{d_xT_0}{T_0}$$

2.- The Burgers equation in a medium with temperature gradient.

2.1.- Establishment of the Burgers equation

Summary: if $v/c_0 \ll 1$, $d_x T_0/T_0 \ll 1$, the resulting Burgers equation is

$$\partial_x v_+ - \frac{1}{c_{ref}^2}\frac{\gamma\frac{T_{ref}}{T_0}+\sqrt{\frac{T_{ref}}{T_0}}\left(2-\frac{T_{ref}}{T_0}\right)}{1+\sqrt{\frac{T_{ref}}{T_0}}}v_+\partial_\tau v_+ = \frac{1}{c_{ref}}\left(1-\sqrt{\frac{T_{ref}}{T_0}}\right)\partial_\tau v_+ + \frac{1+\frac{1}{2}\sqrt{\frac{T_{ref}}{T_0}}}{1+\sqrt{\frac{T_{ref}}{T_0}}}\frac{d_x T_0}{T_0}v_+$$

NB1: if $T_0=T_{ref}=c^{te}$, then $\partial_x v_+ - \frac{\epsilon}{c_{ref}^2}v_+\partial_\tau v_+ = 0, \quad \epsilon = \frac{\gamma+1}{2}$

NB2: if $(x-x_0)\frac{d_x T_0}{T_0}\sim\mu$, then $\partial_x v_+ - \frac{\epsilon}{c_{ref}^2}v_+\partial_\tau v_+ = \frac{1}{2c_{ref}}(x-x_0)\frac{d_x T_0}{T_0}\partial_\tau v_+ + \frac{3}{4}\frac{d_x T_0}{T_0}v_+$

NB3: if a simple wave propagating along x↓ is considered, then one gets

$$\partial_x v_- + f_1(x)v_-\partial_\tau v_- = -f_2(x)\partial_\tau v_- - f_3(x)v_-$$

2.- The Burgers equation in a medium with temperature gradient.

2.2.- Generalized Burgers equation

Additional effects can be easily included in the RHS of the Burgers equation:

Volumetric losses (Mendousse, J. ac. Soc. Am., 1953)	Boundary layer losses (Chester, J. Fluid Mech., 1964)	Varying diameter D(x) (Chester,Proc. Roy. Soc., 1994)
$+ \quad \frac{b}{2\rho_0 c_0^3}\frac{\partial^2 v_+}{\partial \tau^2}$	$- \quad \frac{B}{c_0}\frac{\partial^{1/2} v_+}{\partial \tau^{1/2}}$	$-\frac{d_x D}{D} v_+$
$b = \frac{4}{3}\eta + \xi + \lambda\left(\frac{1}{C_v} - \frac{1}{C_p}\right)$	$B = \sqrt{\frac{\eta}{\rho_0}}\left(1 + \frac{\gamma - 1}{\sqrt{pr}}\right)$	$d_x D \quad << \quad kD$

Introducing the dimensionless variables $\theta = \omega\tau,\ \sigma = \frac{\epsilon U \omega x}{c_0^2},\ q_+ = \frac{v_+}{U}$

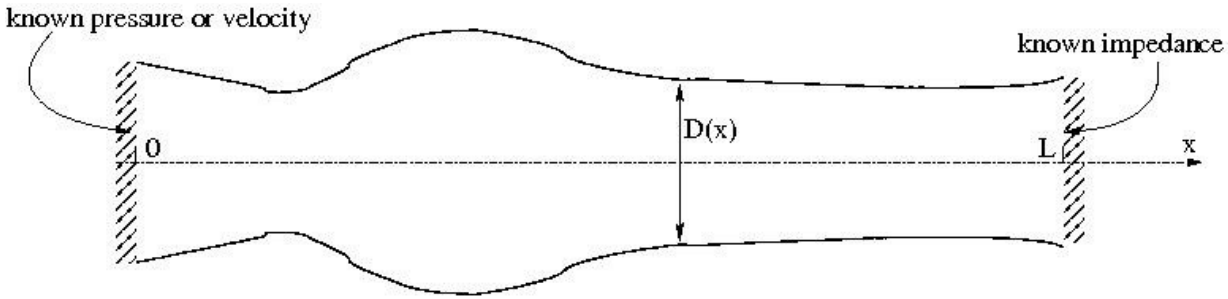
$$\partial_\sigma q_+ - f_1(\sigma)\, q_+ \partial_\theta q_+ = f_2(\sigma)\, \partial_\theta q_+ + f_3(\sigma) q_+ + f_4(\sigma)\partial^2_{\theta\theta} q_+ + f_5(\sigma)\frac{\partial^{1/2} q_+}{\partial \theta^{1/2}}$$

$$f_1(\sigma) = \frac{T_0}{\epsilon T_{ref}} \frac{\left[\gamma \frac{T_{ref}}{T_0} + \sqrt{\frac{T_{ref}}{T_0}}\left(2 - \frac{T_{ref}}{T_0}\right)\right]\left(1 - \sigma \frac{\partial_\sigma T_0}{T_0}\right)}{1 + \sqrt{\frac{T_{ref}}{T_0}}} \qquad f_2(\sigma) = \frac{T_0 c_{ref}}{\epsilon T_{ref} U}\left(1 - \sqrt{\frac{T_{ref}}{T_0}}\right)\left(1 - \sigma \frac{\partial_\sigma T_0}{T_0}\right)$$

$$f_3(\sigma) = \frac{1 + \frac{1}{2}\sqrt{\frac{T_{ref}}{T_0}}}{1 + \sqrt{\frac{T_{ref}}{T_0}}} \frac{\partial_\sigma T_0}{T_0} - \frac{\partial_\sigma D}{D} \qquad f_4(\sigma) = \frac{S c_{ref}}{\epsilon U}\sqrt{\frac{T_0}{T_{ref}}}\left(1 - \sigma \frac{\partial_\sigma T_0}{T_0}\right) \qquad f_5(\sigma) = -\frac{c_{ref}}{\Gamma U}\sqrt{\frac{T_0}{T_{ref}}}\left(1 - \sigma \frac{\partial_\sigma T_0}{T_0}\right)$$

3.- Applications

3.1.- Solving process



$$\partial_\sigma q_+ - f_1(\sigma)\, q_+ \partial_\theta q_+ = f_2(\sigma)\, \partial_\theta q_+ + f_3(\sigma) q_+ + f_4(\sigma) \partial^2_{\theta\theta} q_+ + f_5(\sigma) \frac{\partial^{1/2} q_+}{\partial \theta^{1/2}} \qquad \text{(Burg}^+\text{)}$$

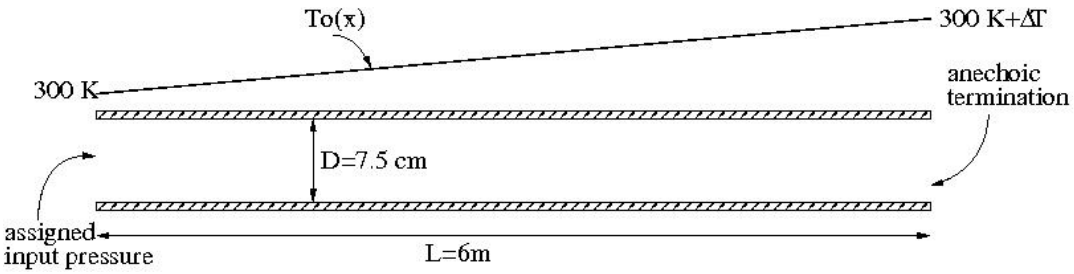
=> we seek a solution in the form $q_\pm = \sum_{n=1} \left(a_n^\pm(\sigma) \sin(n\theta) + b_n^\pm(\sigma) \cos(n\theta) \right)$

1. Choose $a_n^+(\sigma = 0)$ and $b_n^+(\sigma = 0)$ arbitrarily
2. Solve $(Burg^+)$ up to $\sigma(x = L)$ (Finite Difference scheme)
3. Assigned impedance at position $x = L$ => obtain $a_n^-(\sigma(x = L))$ and $b_n^-(\sigma(x = L))$, and solve $(Burg^-)$ up to $\sigma = 0$
4. Compare the resulting $q^+(0) + q^-(0)$ with the assigned one $q_{ass}(0)$
5. Choose a new $(a_n^+(0), b_n^+(0))$ and repeat steps $1 \rightarrow 4$ until $q^+(0) + q^-(0) = q_{ass}(0)$ (Newton-Raphson method)

NB: discarding nonlinear interaction of counterpropagating waves is a reasonable assumption in the frame of a weakly nonlinear theory [Menguy et al., Acta Acust 86:798, 2000]

3.- Applications

3.2.- Application 1: propagation of a simple wave

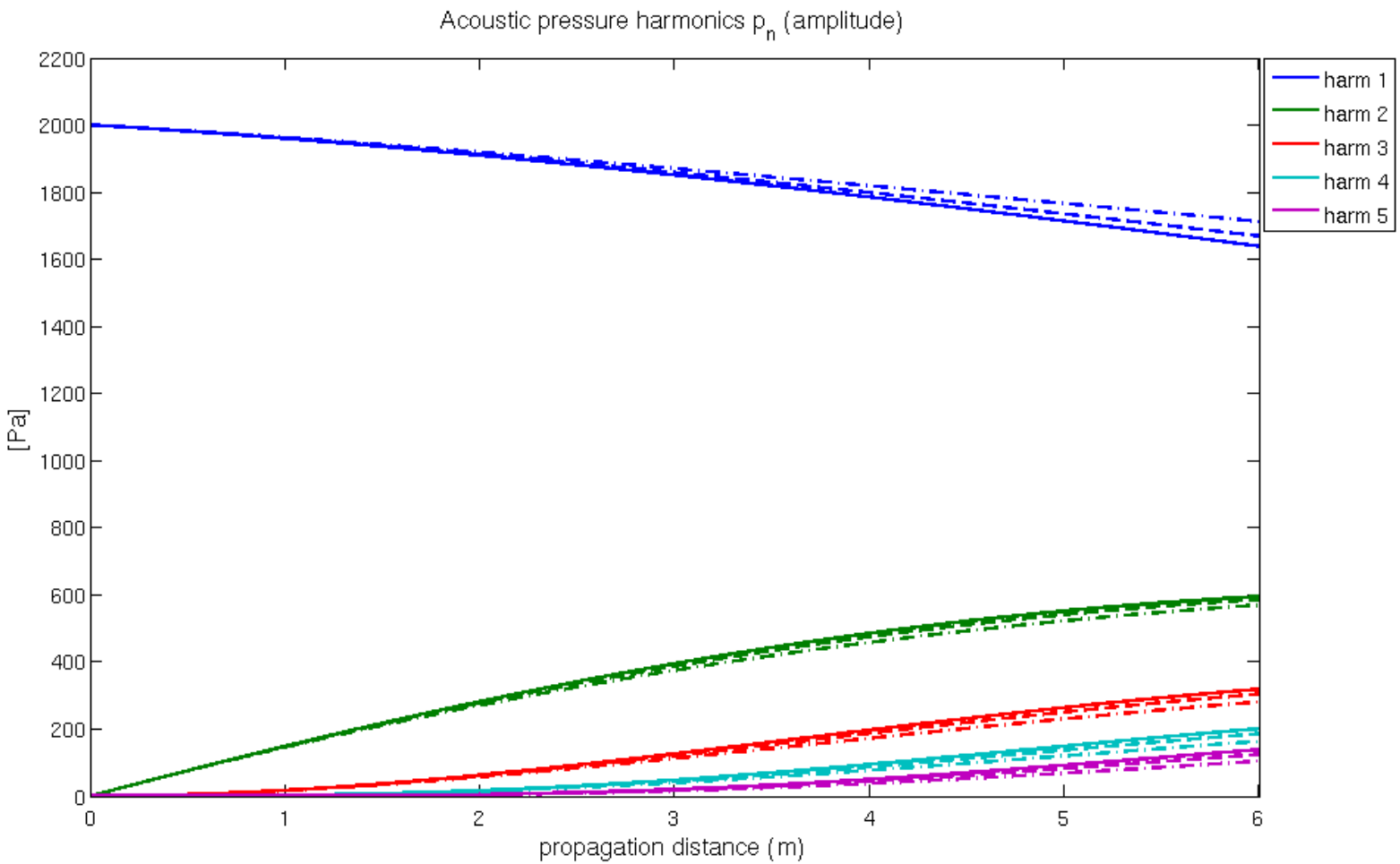


$p_{pk}(x=0)=2000$ Pa, f=500 Hz, $U/c_0=1.4$ %

solid line: $\Delta T=0$

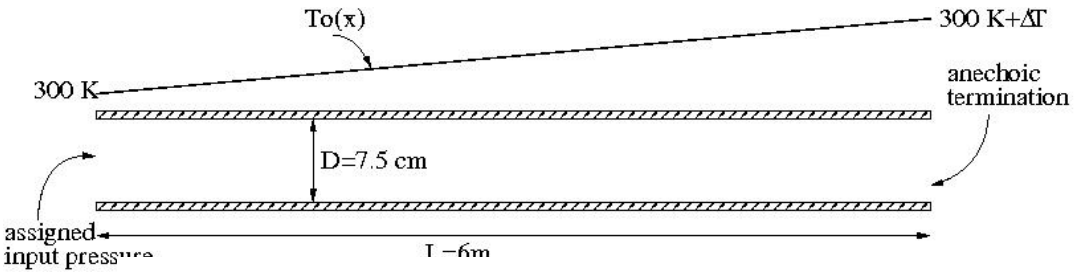
dashed line $\Delta T=30$ K ($d_xT_0/T_0=1.7\ 10^{-2}$ m^{-1})

dash-dotted line: $\Delta T=80$ K ($d_xT_0/T_0=4.4\ 10^{-2}$ m^{-1})



3.- Applications

3.2.- Application 1: propagation of a simple wave

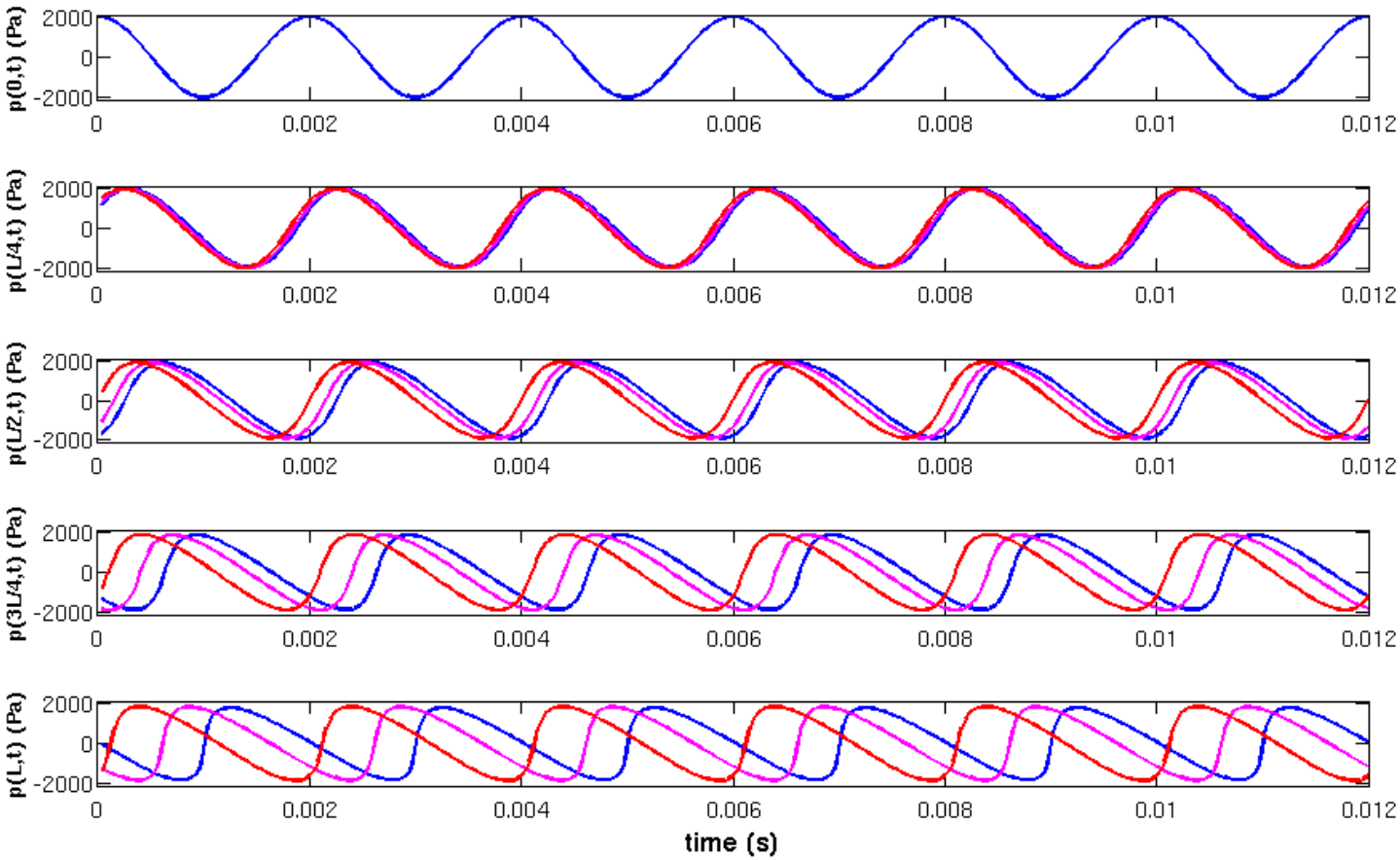


$p_{pk}(x=0)=2000$ Pa, f=500 Hz, $U/c_0=1.4$ %

blue line: $\Delta T=0$

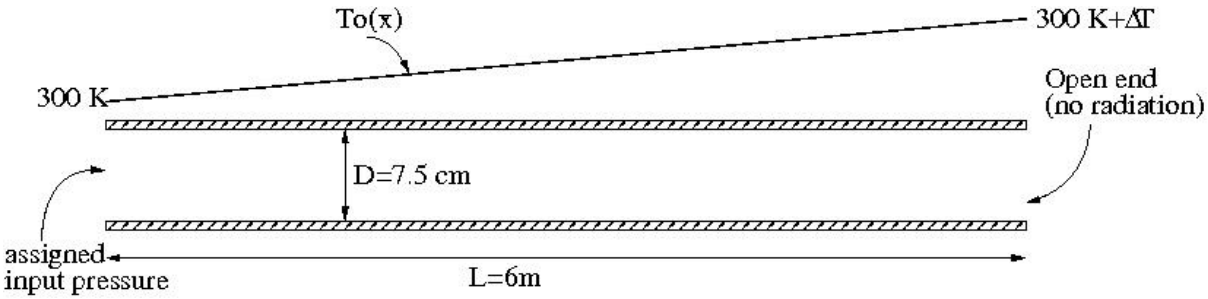
pink line $\Delta T=30$ K ($d_x T_0/T_0=1.7\ 10^{-2}$ m^{-1})

red line: $\Delta T=80$ K ($d_x T_0/T_0=4.4\ 10^{-2}$ m^{-1})



3.- Applications

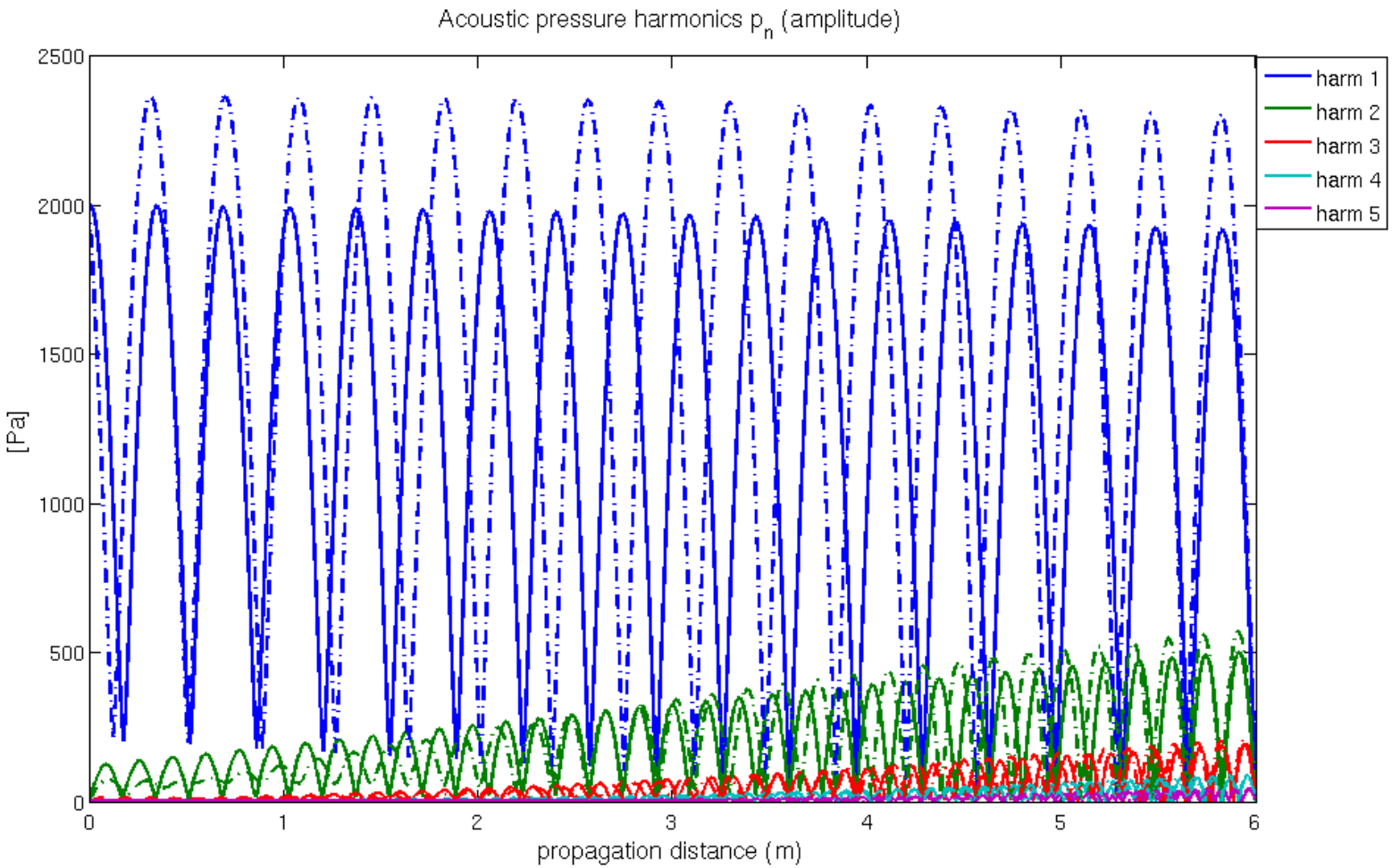
3.3.- Application 2: propagation into an open ended waveguide



$p_{pk}(x=0)=2000$ Pa, f=500 Hz, $U/c_0=1.4$ %

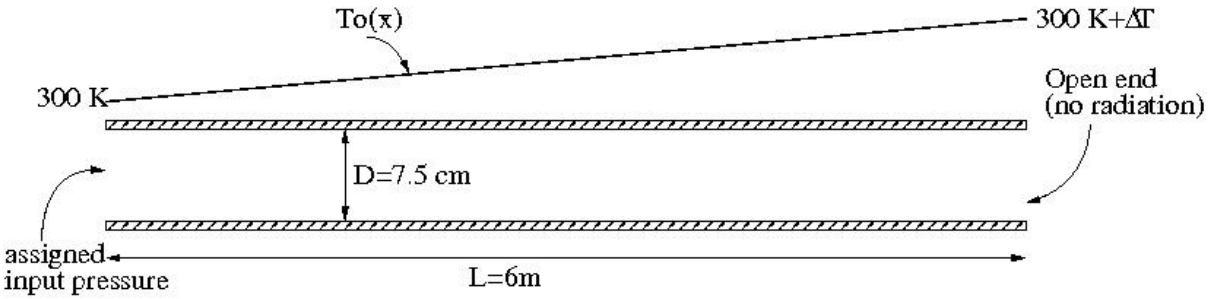
solid line: $\Delta T=0$

dash-dotted line: $\Delta T=80$ K ($d_x T_0/T_0=4.4\ 10^{-2}$ m^{-1})



3.- Applications

3.3.- Application 2: propagation into an open ended waveguide

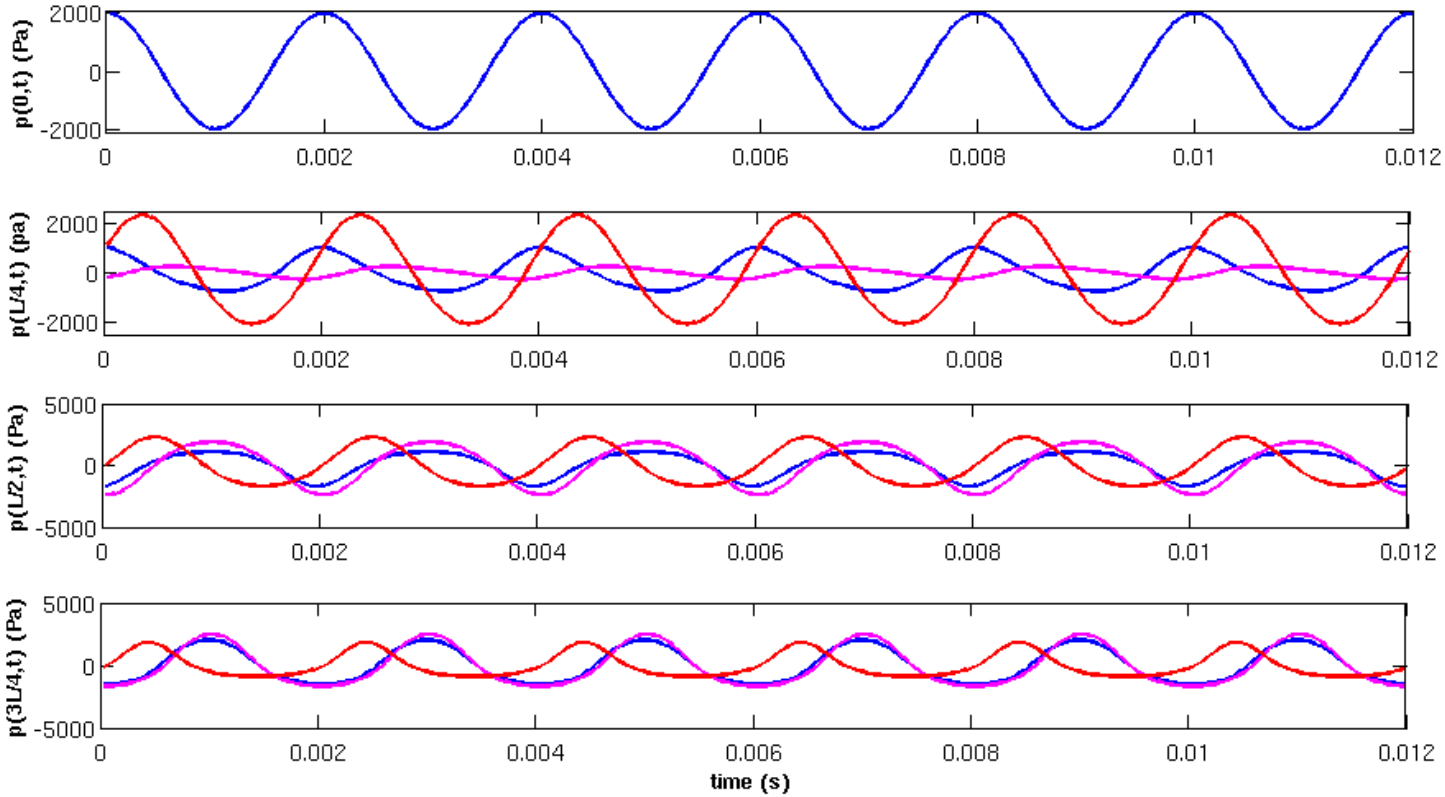


$p_{pk}(x=0)=2000$ Pa, f=500 Hz, $U/c_0=1.4$ %

blue line: $\Delta T=0$

pink line $\Delta T=30$ K ($d_x T_0/T_0=1.7\ 10^{-2}$ m^{-1})

red line: $\Delta T=80$ K ($d_x T_0/T_0=4.4\ 10^{-2}$ m^{-1})



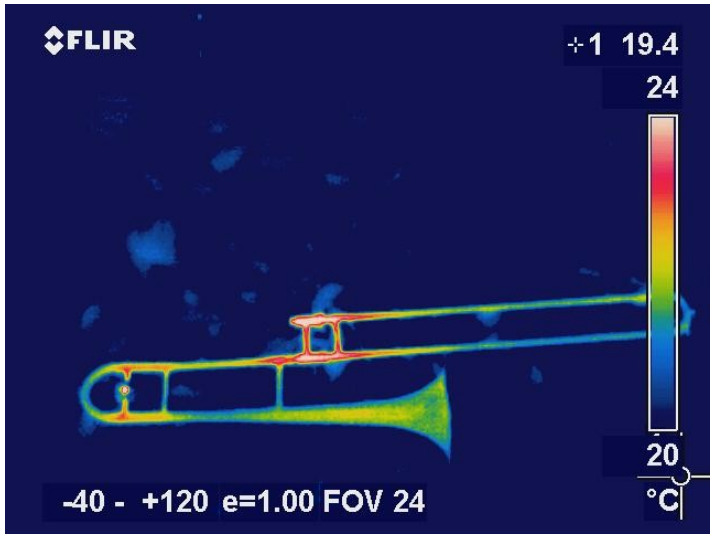
3.- Applications

3.4.- Application 3: On the influence of ΔT on the brassiness of trombones

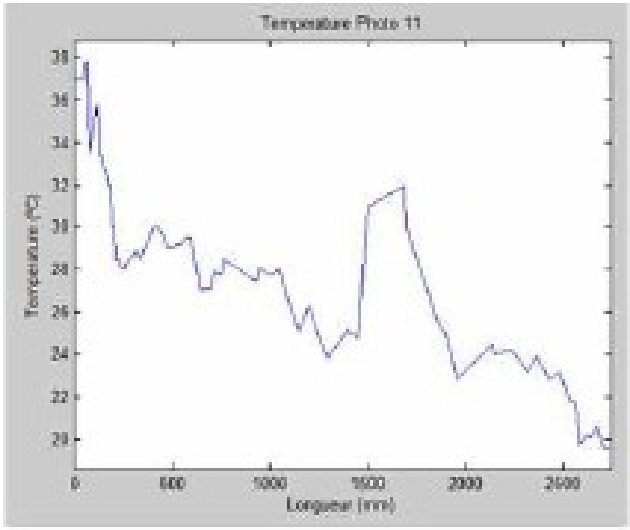
- Nonlinear acoustic propagation is worth considering when studying brass instruments

- At high dynamic levels, sounds generated by brass instruments have strong high frequency components, which are charcateristic of their « brassiness »

- In actual playing conditions, there exist temperature gradients along the waveguide:



IR thermogram of a valve trombone. From Gilbert et al. , Actes du 8ième Congrès Français d'Acoustique, Tours, April 2006

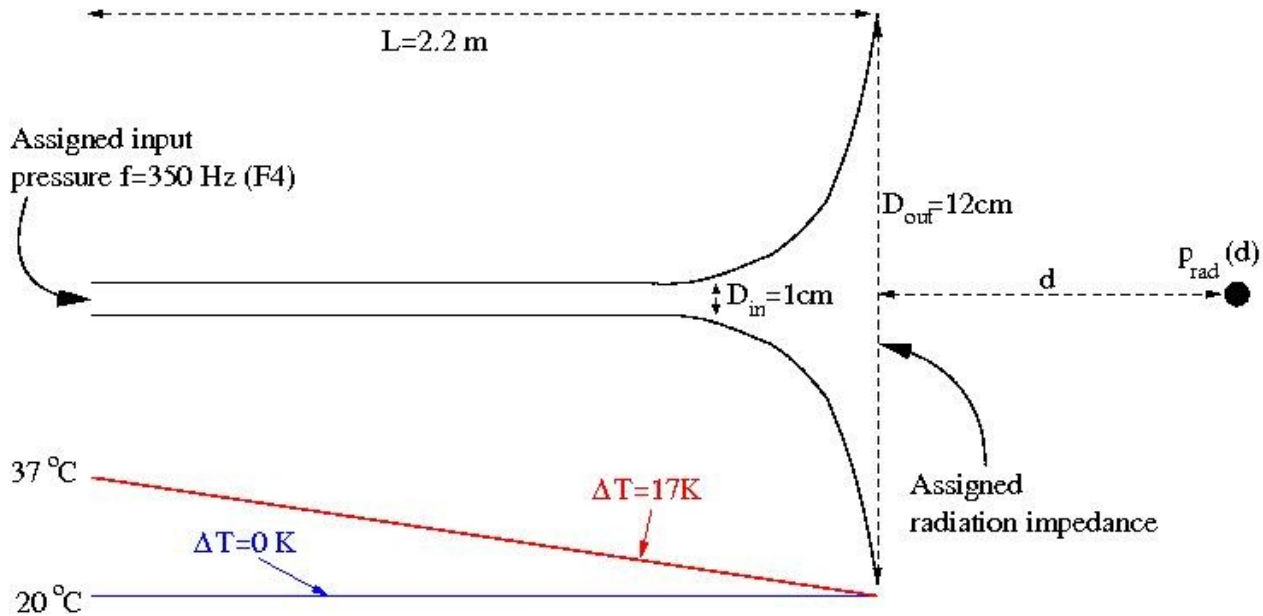


Spatial variation of the temperature along the unwrapped length of a valve trombone . From Gilbert et al. , Actes du 8ième Congrès Français d'Acoustique, Tours, April 2006

Question: does the presence of temperature gradients influences significantly the spectral enrichment of some brass instrument?

3.- Applications

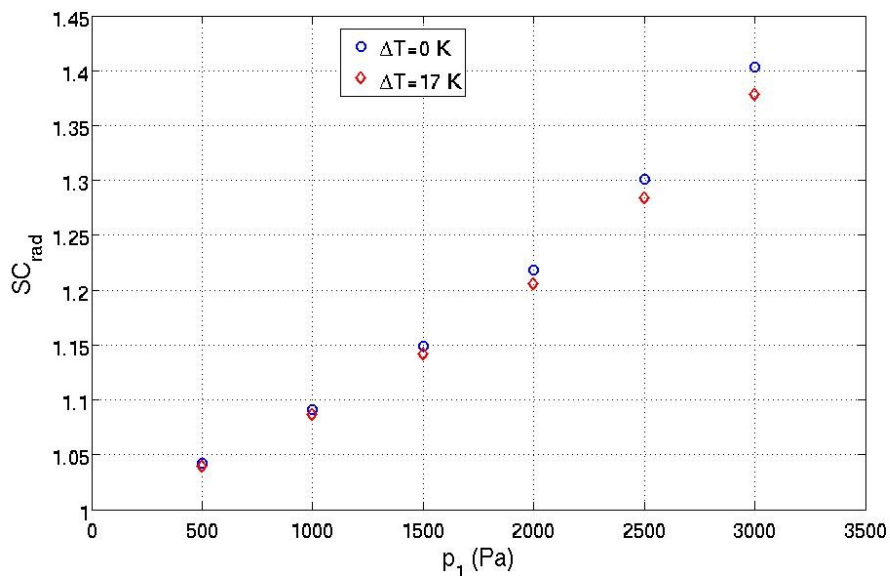
3.4.- Application 3: On the influence of ΔT on the brassiness of trombones



=> calculate NL propagation, and compute the spectral centroïd of the radiated acoustic pressure

$$SC_{rad} = \frac{\sum_n n p_n(d)}{\sum_n p_n}$$

which is indicative of the brassiness of the instrument (SC depends on loudness of excitation, fingering, bore geometry...)

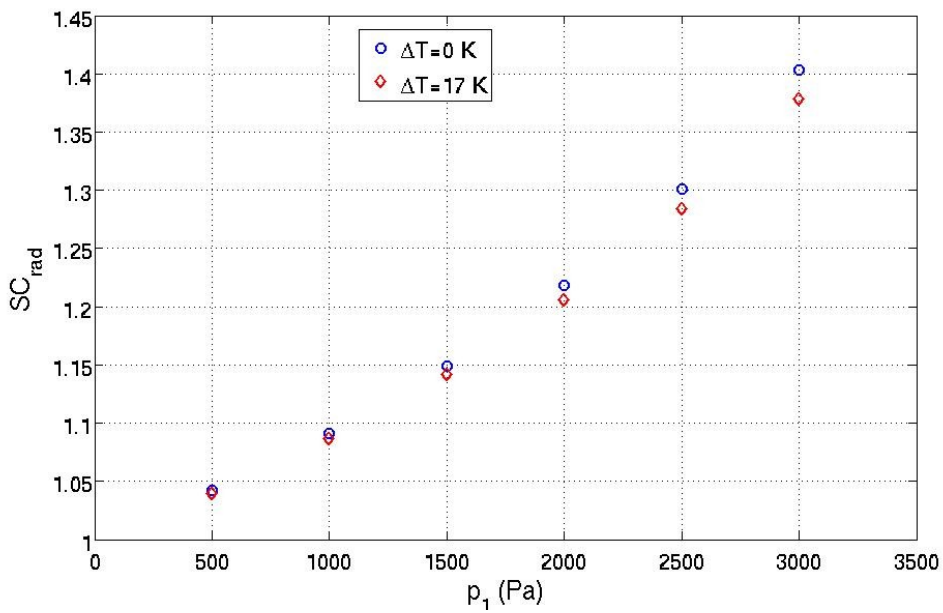


3.- Applications

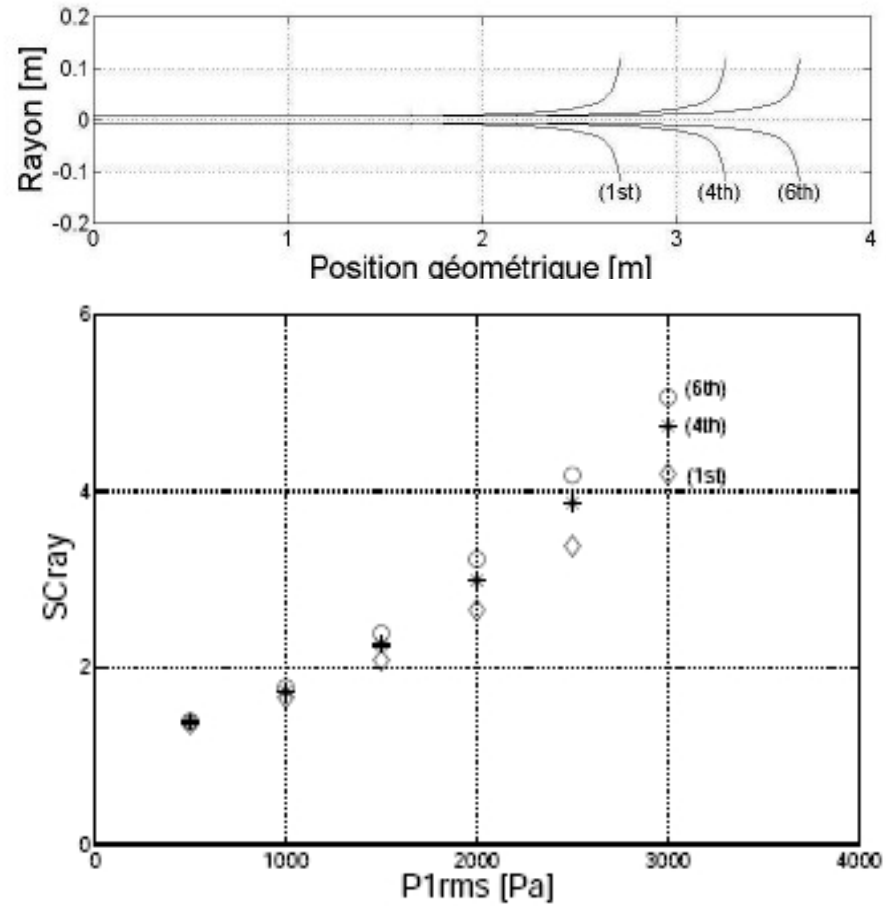
3.5.- Concluding remarks

- The presence of a ΔT impacts both linear and nonlinear propagation
- Considering NL propagation, an increasing ΔT tends to reduce wave steepening

But the effect is weak (e.g. SC of a trombone) ...



Spectral centroïd of radiated acoustic pressure for one particular fingering (1st position) with or without a temperature gradient



Spectral centroïd of radiated acoustic pressure for 3 different fingerings associated to 3 bore geometries. NB: the input pressure signal is experimental.

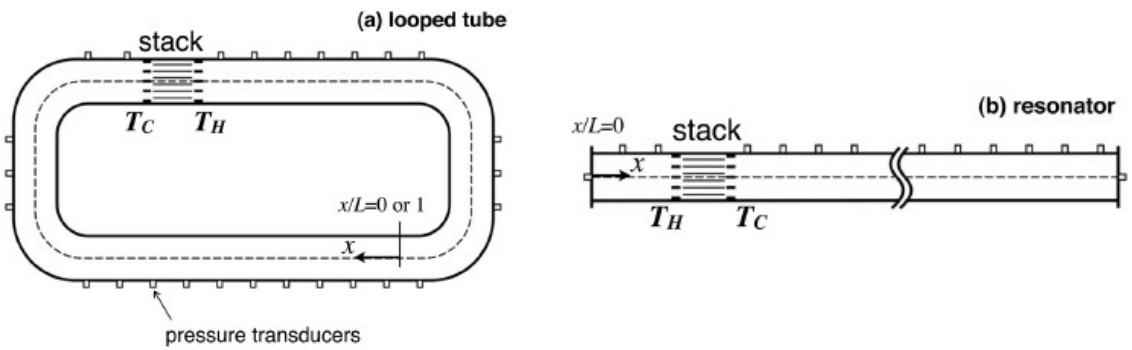
4.- Future prospects

1.- Experimental validation

2.- Extend the theory to $d_x T_0/T_0 \sim 1$?

=> interest for the study of thermoacoustic engines
... but there exist complications because
- separating counterpropagating waves is impossible even in linear regime when $d_x T_0/T_0 \sim 1$
- one should also account for the variations of $\eta, \xi, \gamma, \lambda$ with temperature
- ..

3.- Try to reproduce recent experiments on thermoacoutic engines by Biwa et al.



T. Biwa, T. Takahashi, T Yazaki, « observation of traveling thermoacoustic shock waves », J.Acoust. Soc. Am. 130:3558, 2011

- ΔT = 250 K, fixed
- SW engine => **no shock waves**
- Annular engine => T**raveling shock wave**

=> Adapt the present simulation tool to model thermoacoustic engines
- frequency dependent boundary condition at the interfaces of the thermoacoustic core
- NL propagation in the remaining of the waveguide (complication in the TBT in which $d_x T_0/T_0 \sim 1$)