Weakly nonlinear acoustic oscillations in gas columns in the presence of temperature gradients

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2.- The Burgers equation in a medium with a temperature gradient
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1.- Introduction

• Nonlinear acoustics already has a long history and many applications
  [Rudenko and Soluyan, Theoretical foundations of Nonlinear Acoustics, Consultants Bureau, NY, 1977]
  [Hamilton and Blackstock, Nonlinear Acoustics, Acoustical Society of America, NY, 2008]

• Considering NL propagation of plane waves in ducts, many experimental and theoretical studies made in the past decades.

• In particular, when assuming a low mach number \((M=\frac{v_{ac}}{c_0}<<1)\), it is well known that weakly NL propagation can be described by the Burgers equation, which is derived using the Multiple Scale Method.

However the effect of a temperature gradient on non linear propagation of plane guided waves has not been studied a lot

=> interest in the study of the operation of thermoacoustic engines
2.- The Burgers equation in a medium with temperature gradient.

2.1.- Establishment of the Burgers equation

**Assumptions:**
- inviscid fluid ($\mu=0, \zeta=0$), no heat conduction ($\lambda=0$),
- 1-D propagation along the x-axis
- weakly non linear propagation: $\frac{p-p_0}{p_0} = \frac{\rho}{p_0} \sim \mu, \frac{p-p_0}{\rho_0} = \frac{\rho'}{\rho_0} \sim \mu, \frac{v}{c_0} \sim \mu, \mu << 1$
- adiabatic process: $p' \sim c_0^2 \rho' + \frac{\gamma-1}{2\rho_0} c_0^2 \rho'^2$
- inhomogeneous temperature gradient $T=T_0(x)$: $\rho_0(x) = \frac{M_{mot} p_0}{R T_0(x)}, c_0^2(x) = \frac{\gamma p_0}{\rho_0(x)}$

\[
\begin{align*}
\begin{array}{l}
\rho \left( \partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p + \eta \Delta \vec{v} + \left( \xi + \frac{\eta}{3} \right) \nabla \left( \nabla \cdot \vec{v} \right) \\
\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0 \\
\rho T \left( \partial_t s + (\vec{v} \cdot \nabla) s \right) = \lambda \Delta T + F_d
\end{array}
\end{align*}
\] (F_d : rate of dissipation of mechanical energy)
2.- The Burgers equation in a medium with temperature gradient.

2.1.- Establishment of the Burgers equation

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( \rho_0 + \rho' \right) \partial_t v + \rho_0 v \partial_x v + c_0^2 \partial_x \rho' + \frac{\gamma - 1}{\rho_0} c_0^2 \rho' \partial_x \rho' &= -c_0^2 \frac{d_x T_0}{T_0} \left( \rho' + \frac{\gamma - 1}{2 \rho_0} \rho'^2 \right) + o(\mu^2) \\
\partial_t \rho' + v \partial_x \rho' (\rho_0 + \rho') \partial_x v &= \rho_0 v \frac{d_x T_0}{T_0} + o(\mu^2)
\end{array} \right.
\end{align*}
\]

If \( \nu/c_0 \ll 1 \), non linear effects are essentially cumulative (local nonlinear effects neglected)

\[ \Rightarrow \] use of the Multiple Scale Method:

\[ x \leftarrow x, \ t \leftarrow t + \frac{x}{c_{\text{ref}}}, \text{ with } c_{\text{ref}} = c_0(x_0) \]

(simple wave propagating along \( x \uparrow \))

and additional assumption:

\[ \frac{d_x T_0}{T_0} \sim \mu \]

\[ \Rightarrow \] Apply the above mentioned change of variables in Eqs. (1) and (2) (retain only variables of order \( \leq \mu^2 \), and eliminate \( \rho' \)) leads after some calculations to:

\[
\partial_x v_+ - f_1(x) v_+ \partial_\tau v_+ = f_2(x) \partial_\tau v_+ + f_3(x) v_+
\]

\[ \text{, with } \frac{d_x T_0}{T_0} \sim \mu, \ T_{\text{ref}} = T_0(x_0), \ c_{\text{ref}} = c_0(x_0) \]

\[ f_1(x) = \frac{1}{c_{\text{ref}}} \left( \frac{1}{1 + \sqrt{\frac{T_{\text{ref}}}{T_0}}} \right) \]

\[ f_2(x) = \frac{1}{c_{\text{ref}}} \left( 1 - \sqrt{\frac{T_{\text{ref}}}{T_0}} \right) \]

\[ f_3(x) = \frac{1 + \frac{1}{2} \sqrt{\frac{T_{\text{ref}}}{T_0}}}{1 + \sqrt{\frac{T_{\text{ref}}}{T_0}}} \frac{d_x T_0}{T_0} \]
2. The Burgers equation in a medium with temperature gradient.

2.1. Establishment of the Burgers equation

**Summary:** if \( v/c_0 \ll 1 \), \( d_x T_0 / T_0 \ll 1 \), the resulting Burgers equation is

\[
\partial_x v_+ - \frac{1}{c_{\text{ref}}^2} \frac{\gamma T_{\text{ref}}}{T_0} + \frac{\sqrt{T_{\text{ref}}/T_0}}{1 + \sqrt{T_{\text{ref}}/T_0}} \left( 2 - \frac{T_{\text{ref}}}{T_0} \right) v_+ \partial_\tau v_+ = \frac{1}{c_{\text{ref}}} \left( 1 - \sqrt{T_{\text{ref}}/T_0} \right) \partial_\tau v_+ + \frac{1 + \frac{1}{2} \sqrt{T_{\text{ref}}/T_0}}{1 + \sqrt{T_{\text{ref}}/T_0}} d_x T_0 T_0 \partial_\tau v_+
\]

**NB1:** if \( T_0 = T_{\text{ref}} = \text{cte} \), then

\[
\partial_x v_+ - \frac{\epsilon}{c_{\text{ref}}^2} v_+ \partial_\tau v_+ = 0, \quad \epsilon = \frac{\gamma + 1}{2}
\]

**NB2:** if \( (x - x_0) d_x T_0 / T_0 \sim \mu \), then

\[
\partial_x v_+ - \frac{\epsilon}{c_{\text{ref}}^2} v_+ \partial_\tau v_+ = \frac{1}{2c_{\text{ref}}} (x - x_0) \frac{d_x T_0}{T_0} \partial_\tau v_+ + \frac{3}{4} \frac{d_x T_0}{T_0} v_+
\]

**NB3:** if a simple wave propagating along \( x \) is considered, then one gets

\[
\partial_x v_- + f_1(x) v_- \partial_\tau v_- = -f_2(x) \partial_\tau v_- - f_3(x) v_-
\]
2.- The Burgers equation in a medium with temperature gradient.
2.2.- Generalized Burgers equation

Additional effects can be easily included in the RHS of the Burgers equation:

Volumetric losses
(Mendousse, J. ac. Soc. Am., 1953)

\[ b = \frac{4}{3} \eta + \xi + \lambda \left( \frac{1}{C_v} - \frac{1}{C_p} \right) \]

Boundary layer losses
(Chester, J. Fluid Mech., 1964)

\[ B = \sqrt{\frac{\eta}{\rho_0}} \left( 1 + \frac{\gamma - 1}{\sqrt{\gamma r}} \right) \]

Varying diameter \(D(x)\)

\[ d_x D << kD \]

Introducing the dimensionless variables
\[ \theta = \omega \tau, \quad \sigma = \frac{\epsilon U \omega x}{c_0^2}, \quad q_+ = \frac{v_+}{U} \]

\[ \partial_{\sigma} q_+ - f_1(\sigma) q_+ \partial_{\theta} q_+ = f_2(\sigma) \partial_{\theta} q_+ + f_3(\sigma) q_+ + f_4(\sigma) \partial_{\theta \theta} q_+ + f_5(\sigma) \frac{\partial^{1/2} q_+}{\partial \theta^{1/2}} \]

\[ f_1(\sigma) = \frac{T_0}{\epsilon T_{ref}} \left[ \frac{T_{ref}}{T_0} + \sqrt{\frac{T_{ref}}{T_0} \left( 2 - \frac{T_{ref}}{T_0} \right)} \right] \left( 1 - \sigma \frac{\partial_{\sigma} T_0}{T_0} \right) \]

\[ f_2(\sigma) = \frac{T_0 c_{ref}}{\epsilon T_{ref} U} \left( 1 - \sqrt{\frac{T_{ref}}{T_0}} \right) \left( 1 - \sigma \frac{\partial_{\sigma} T_0}{T_0} \right) \]

\[ f_3(\sigma) = \frac{1 + \frac{1}{2} \sqrt{\frac{T_{ref}}{T_0}} \frac{\partial_{\sigma} T_0}{T_0} - \frac{\partial_{\sigma} D}{D}}{1 + \sqrt{\frac{T_{ref}}{T_0}}} \]

\[ f_4(\sigma) = \frac{S c_{ref}}{\epsilon U} \sqrt{\frac{T_0}{T_{ref}}} \left( 1 - \sigma \frac{\partial_{\sigma} T_0}{T_0} \right) \]

\[ f_5(\sigma) = -\frac{c_{ref}}{\epsilon U} \sqrt{\frac{T_0}{T_{ref}}} \left( 1 - \sigma \frac{\partial_{\sigma} T_0}{T_0} \right) \]
3.- Applications
3.1.- Solving process

\[ \partial_\sigma q_+ - f_1(\sigma) q_+ \partial_\theta q_+ = f_2(\sigma) \partial_\theta q_+ + f_3(\sigma) q_+ + f_4(\sigma) \partial^2_\theta q_+ + f_5(\sigma) \frac{\partial^{1/2} q_+}{\partial \theta^{1/2}} \]  

(Burg\(^+\))

=> we seek a solution in the form

\[ q_\pm = \sum_{n=1} \left( a_n^\pm(\sigma) \sin(n\theta) + b_n^\pm(\sigma) \cos(n\theta) \right) \]

1. Choose \( a_n^+(\sigma = 0) \) and \( b_n^+(\sigma = 0) \) arbitrarily
2. Solve \((Burg^+)\) up to \( \sigma(x = L) \) (Finite Difference scheme)
3. Assigned impedance at position \( x = L \) => obtain \( a_n^-(\sigma(x = L)) \) and \( b_n^-(\sigma(x = L)) \), and solve \((Burg^-)\) up to \( \sigma = 0 \)
4. Compare the resulting \( q^+(0) + q^-(0) \) with the assigned one \( q_{ass}(0) \)
5. Choose a new \((a_n^+(0), b_n^+(0))\) and repeat steps 1 \(\rightarrow\) 4 until \(q^+(0) + q^-(0) = q_{ass}(0)\) (Newton-Raphson method)

**NB:** discarding nonlinear interaction of counterpropagating waves is a reasonable assumption in the frame of a weakly nonlinear theory [Menguy et al., Acta Acust 86:798, 2000]
3.- Applications
3.2.- Application 1: propagation of a simple wave

\[ p\text{pk}(x=0)=2000 \text{ Pa}, \quad f=500 \text{ Hz}, \quad U/c_0=1.4\% \]

solid line: \( \Delta T=0 \)
dashed line \( \Delta T=30 \text{ K} \) \( (d_x T_0/T_0=1.7 \times 10^{-2} \text{ m}^{-1}) \)
dash-dotted line: \( \Delta T=80 \text{ K} \) \( (d_x T_0/T_0=4.4 \times 10^{-2} \text{ m}^{-1}) \)
3.- Applications

3.2.- Application 1: propagation of a simple wave

\[ p_{pk}(x=0)=2000 \text{ Pa}, \ f=500 \text{ Hz}, \ U/c_0 = 1.4 \% \]

- blue line: \( \Delta T=0 \)
- pink line: \( \Delta T=30 \text{ K} \) (\( d_x T_0/T_0 = 1.7 \times 10^{-2} \text{ m}^{-1} \))
- red line: \( \Delta T=80 \text{ K} \) (\( d_x T_0/T_0 = 4.4 \times 10^{-2} \text{ m}^{-1} \))
3.- Applications

3.3.- Application 2: propagation into an open ended waveguide

\[ p_{pk}(x=0)=2000 \text{ Pa}, f=500 \text{ Hz}, U/c_0=1.4 \% \]

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3.- Applications

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3. Applications

3.4.- Application 3: On the influence of $\Delta T$ on the brassiness of trombones

- Nonlinear acoustic propagation is worth considering when studying brass instruments
- At high dynamic levels, sounds generated by brass instruments have strong high frequency components, which are characteristic of their « brassiness »
- In actual playing conditions, there exist temperature gradients along the waveguide:

Question: does the presence of temperature gradients influence significantly the spectral enrichment of some brass instrument?
3.- Applications

3.4.- Application 3: On the influence of $\Delta T$ on the brassiness of trombones

$\Rightarrow$ calculate NL propagation, and compute the spectral centroïd of the radiated acoustic pressure

$$SC_{rad} = \frac{\sum_n n p_n (d)}{\sum_n p_n}$$

which is indicative of the brassiness of the instrument ($SC$ depends on loudness of excitation, fingering, bore geometry...)
3.- Applications
3.5.- Concluding remarks

- The presence of a $\Delta T$ impacts both linear and nonlinear propagation
- Considering NL propagation, an increasing $\Delta T$ tends to reduce wave steepening
  But the effect is weak (e.g. SC of a trombone) ...

Spectral centroid of radiated acoustic pressure for one particular fingering (1st position) with or without a temperature gradient

Spectral centroid of radiated acoustic pressure for 3 different fingerings associated to 3 bore geometries. NB: the input pressure signal is experimental.
4.- Future prospects

1.- Experimental validation

2.- Extend the theory to $d_x T_0/T_0 \sim 1$ ?

   => interest for the study of thermoacoustic engines
   ... but there exist complications because
   - separating counterpropagating waves is impossible even in linear regime when $d_x T_0/T_0 \sim 1$
   - one should also account for the variations of $\eta, \xi, \gamma, \lambda$ with temperature
   - ..

3.- Try to reproduce recent experiments on thermoacoustic engines by Biwa et al.

- $\Delta T = 250$ K, fixed
- SW engine => no shock waves
- Annular engine => Traveling shock wave

=> Adapt the present simulation tool to model thermoacoustic engines
   - frequency dependent boundary condition at the interfaces of the thermoacoustic core
   - NL propagation in the remaining of the waveguide (complication in the TBT in which $d_x T_0/T_0 \sim 1$)