On the statistical errors in the estimate of acoustical energy density by using two microphones in a one dimensional field

Jean-Claude Pascal[a] and Jean-Hugh Thomas[b]
Laboratoire d'Acoustique de l'Université du Maine (CNRS UMR 6613) and Ecole Nationale Supérieure d'Ingénieurs du Mans (ENSIM), Université du Maine, rue Aristote, 72085 Le Mans Cedex, France

Jing-Fang Li[c]
Visual VibroAcoustics, 51 rue d'Alger, 72000 Le Mans, France

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It was recently shown that the statistical errors of the measurement in the acoustic energy density by the two microphone method in waveguide have little variation when the losses of coherence between microphones increase. To explain these intervals of uncertainty, the variance of the measurement is expressed in this paper as a function of the various energy quantities of the acoustic fields—energy densities and sound intensities. The necessary conditions to reach the lower bound are clarified. The results obtained are illustrated by an example of a one-dimensional partially coherent field, which allows one to specify the relationship between the coherence functions of the pressure and particle velocity and those of the two microphone signals.
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I. INTRODUCTION

From the analysis of the statistical error of the acoustic energy density measured by two microphones, Ghan *et al.*[1] showed that the normalized standard deviation for a Gaussian broadband signal varies weekly in function of the kind of sound field, within an upper bound $1/\sqrt{BT}$ and a lower bound $1/\sqrt{2BT}$ (where B is the bandwidth of the analysis and T is the time length of acquisition). The upper bound is that of the normalized standard deviation of a potential energy density (or quadratic pressure) measured by a single microphone. The fact that in certain cases the standard deviation of the energy density (sum of potential and kinetic energy densities) can even be $\sqrt{2}$ times weaker than the upper bound is surprising at a first look. Indeed, studies of the statistical errors of the acoustic intensity[2–7] using a probe consisting of two microphones have shown that the normalized standard deviation always has $1/\sqrt{BT}$ for lower bound and can take much more significant values being dependent on the phase and the loss of coherence between the two microphone signals. To understand the significance of this result, in this paper, the statistical error of the energy density is analyzed for the case of two partially coherent waves in opposite directions. It is often thought that the use of a discretization scheme by a finite difference approximation is the cause of the increase of statistical errors. It is shown in this paper that the use of the finite difference approximation has a little influence on the statistical errors of the energy density which can entirely be expressed as a function of the quadratic quantities (intensities and energy densities) of acoustic fields.

[a]Electronic mail: jean-claude.pascal@univ-lemans.fr
[b]Electronic mail: jean-hugh.thomas@univ-lemans.fr
[c]Electronic mail: jinfang.li@visualvibroacoustics.com

II. EXPRESSION FOR THE STATISTICAL ERRORS OF THE ENERGY DENSITIES

Given a one-dimensional sound field with only the x-component of the particle velocity u_x, the potential V, kinetic T, and total E energy densities are written by

$$V = \frac{1}{2\rho_0 c^2} G_{pp}(\omega), \quad T = \frac{\rho_0}{2} G_{u_x u_x}(\omega), \quad E = V + T, \tag{1}$$

where ρ_0 is the mass density of the fluid and c is the speed of sound. $G_{pp}(\omega)$ and $G_{u_x u_x}(\omega)$ are, respectively, the autospectral power densities of pressure and x-particle velocity. The pressure measured by two microphones with spacing of Δ ($\Delta < \lambda/3$) allows one to give the finite-difference approximation expressions[8] for the particle velocity $u_x \approx (p_2 - p_1)/(-j\rho_0 ck\Delta)$ and the pressure estimation $p \approx (p_2 + p_1)/2$, with the wavenumber $k = \omega/c$. Using these expressions, the approximation of the energy densities can then be expressed in the following form:

$$D_a(\omega) = \alpha[G_{11}(\omega) + G_{22}(\omega)] + \beta[2C_{21}(\omega)]. \tag{2}$$

The subscript a indicates an approximate quantity. D_a corresponds to V_a, T_a, or E_a according to the values of the coefficients α and β in Table I. The one-sided autospectral $G_{11}(\omega)$, $G_{22}(\omega)$ and cross-spectral $G_{12}(\omega)$ densities are given by

$$G_{ij}(\omega) = \lim_{T_W \to \infty} \frac{2}{T_W} E\{p_j(\omega, T_W) p_i^*(\omega, T_W)\} \quad (i,j = 1,2), \tag{3}$$

$2C_{21}(\omega)$ is the shortened notation for $\mathrm{Re}\{G_{21}(\omega)\} + j\,\mathrm{Im}\{G_{21}(\omega)\}$. In Eq. (3), $p_1(\omega, T_W)$ and $p_2(\omega, T_W)$ are the finite Fourier transforms of length T_W.

TABLE I. Coefficients α and β used in the expressions of the energy densities by the finite-difference approximations (Eqs. (2) and (9)).

Energy quantity	α	β
potential energy V_a	$1/(8\rho_0 c^2)$	$1/(8\rho_0 c^2)$
kinetic energy T_a	$1/(2\rho_0 c^2 k^2 \Delta^2)$	$-1/(2\rho_0 c^2 k^2 \Delta^2)$
total energy E_a	$1/(8\rho_0 c^2)+1/(2\rho_0 c^2 k^2 \Delta^2)$	$1/(8\rho_0 c^2)-1/(2\rho_0 c^2 k^2 \Delta^2)$

$E\{p_1(\omega,T_W)p_2^*(\omega,T_W)\}$ denotes the expected value of $p_1(\omega,T_W)p_2^*(\omega,T_W)$.

By evaluating the cross-spectral power densities,

$$\hat{G}_{21}(\omega) = (2/n)\sum_{i=1}^{n} [p_1(\omega,T_W)p_2^*(\omega,T_W)]/T_W, \tag{4}$$

with a time window of length T_W and an average of n records, the statistical errors in the estimator appear if the pressures are represented by random signals. For bivariate random processes, the variance of the energy densities can be written in terms of the covariance of power densities by

$$\mathrm{var}\{\hat{D}_a(\xi_1,\xi_2,\ldots)\} \approx \sum_{i,j} \frac{\partial \hat{D}_a}{\partial \xi_i}\frac{\partial \hat{D}_a}{\partial \xi_j}\mathrm{cov}\{\xi_i,\xi_j\}$$

$$\text{with } \xi_i,\xi_j \in \{\hat{G}_{11}(\omega),\hat{G}_{22}(\omega),\hat{C}_{21}(\omega)\}. \tag{5}$$

Analytical expressions of covariances were given by Jenkins and Watts[9] for Gaussian processes as a function of the factor BT (B is the width of an elementary filter corresponding to a frequency bin of the fast Fourier transform FFT analysis, which leads to $BT_W=1$, $T=nT_W$, and $BT=n$)

$$\mathrm{cov}\{\hat{G}_{ii},\hat{G}_{jj}\} = |G_{ij}|^2/n,$$

$$\mathrm{cov}\{\hat{G}_{ii},\hat{C}_{ij}\} = G_{ii}C_{ij}/n \quad \text{with } i,j=\{1,2\}, \tag{6}$$

and

$$\mathrm{cov}\{\hat{G}_{ii},\hat{G}_{ii}\} = \mathrm{var}\{\hat{G}_{ii}\} = G_{ii}^2/n, \tag{7}$$

$$\mathrm{cov}\{\hat{C}_{21},\hat{C}_{21}\} = \mathrm{var}\{\hat{C}_{21}\} = (G_{11}G_{22} + C_{21}^2 - Q_{21}^2)/(2n). \tag{8}$$

When considering the coherence function $\gamma_{21}^2(\omega)$ between the two microphone signals and by using the relation $C_{21}^2 + Q_{21}^2 = |G_{21}|^2 = \gamma_{21}^2 G_{11}G_{22}$, the expression for the variance of the energy densities is written in the following form:

$$\begin{aligned}\mathrm{var}\{\hat{D}_a(\omega)\} &\approx \frac{1}{n}[D_a^2(\omega) - 2(\alpha^2-\beta^2)(G_{11}(\omega)G_{22}(\omega) \\ &\quad - |G_{21}(\omega)|^2)] \\ &\approx \frac{1}{n}[D_a^2(\omega) - 2(\alpha^2-\beta^2)G_{11}(\omega) \\ &\quad \times G_{22}(\omega)(1-\gamma_{21}^2(\omega))].\end{aligned} \tag{9}$$

To simplify the notations, we will consider, in the rest of this paper, the variance $\mathrm{var}\{\hat{D}_a(\omega)\}$ rather than the normalized standard deviation $\varepsilon\{\hat{D}_a(\omega)\}$ by knowing that $\varepsilon\{\hat{D}_a(\omega)\} = (\mathrm{var}\{\hat{D}_a(\omega)\})^{1/2}/\hat{D}_a(\omega)$. By considering the potential and kinetic energy densities for which $\alpha^2=\beta^2$ (Table I), the following equation can be obtained as

$$\mathrm{var}\{\hat{V}_a(\omega)\} \approx \frac{V_a^2(\omega)}{n} \quad \text{and } \mathrm{var}\{\hat{T}_a(\omega)\} \approx \frac{T_a^2(\omega)}{n}. \tag{10}$$

The variance of the statistical errors for a probe with two microphones is the same as that obtained from Eq. (1) for a single pressure sensor $\hat{V}(\omega)=\hat{G}_{pp}(\omega)/(2\rho_0 c^2)$ and for a single particle velocity sensor $\hat{T}(\omega)=\rho_0\hat{G}_{uu}(\omega)/2$, because the variance of an autospectrum[9] is always equal to $G_{ii}^2(\omega)/n$. This result had already been obtained by Elko.[5] The loss of coherence between the two microphones has no influence, whether it is caused by the nature of the sound fields or by the independent electronic noise of each channel. However, it is not the same for the total energy density since

$$\alpha^2 - \beta^2 = 1/(2\rho_0 c^2 k\Delta)^2, \tag{11}$$

and the second term on the right-hand side of Eq. (9) depending on the coherence function is not equal to zero. This term is always positive and will always result in subtraction from E_a^2. Since the probes are used in the range $k\Delta < 1$, this term may have significant values at low frequencies, in particular, when the coherence $\gamma_{21}^2(\omega)$ is appreciably lower than 1. Ghan *et al.*[1] studied this expression of the variance of the total energy density and defined its intervals of variation as follows:

$$\frac{E_a^2(\omega)}{2n} \le \mathrm{var}\{\hat{E}_a(\omega)\} \le \frac{E_a^2(\omega)}{n}. \tag{12}$$

The lower bound can seem paradoxical inasmuch as the variance could be smaller than (i) that of the measurement by using only one sensor or (ii) that of a measurement made in a sound field having a full spatial coherence $\gamma_{21}^2=1$ (which corresponds to the upper bound).

A different factorization from Eq. (9) allows the variance of the total energy density to be written in the following form:

$$\mathrm{var}\{\hat{E}_a(\omega)\} \approx \frac{1}{n}\left[V_a^2(\omega) + T_a^2(\omega) + \frac{4Q_{21}^2(\omega) + (G_{11}(\omega)-G_{22}(\omega))^2}{2(2\rho_0 c^2 k\Delta)^2}\right], \tag{13}$$

in which one can recognize the approximate expressions, respectively, for the active acoustic intensity $I_a(\omega) = Q_{21}(\omega)/(\rho_0 ck\Delta)$ and the reactive acoustic intensity $J_a(\omega) = (G_{11}(\omega)-G_{22}(\omega))/(2\rho_0 ck\Delta)$ for a probe with two microphones.[3,10] Thus Eq. (13) takes the following remarkable form:

$$\mathrm{var}\{\hat{E}_a(\omega)\} \approx \frac{1}{n}\left[V_a^2(\omega) + T_a^2(\omega) + \frac{I_a^2(\omega)}{2c^2} + \frac{J_a^2(\omega)}{2c^2}\right]. \tag{14}$$

III. INTERPRETATION

Without considering the finite difference approximations, the variance of the total energy density can be calculated by applying Eq. (5) to Eq. (1):

$$\mathrm{var}\{\hat{E}(\omega)\} \approx \left(\frac{1}{2\rho_0 c^2}\right)^2 \mathrm{var}\{G_{pp}(\omega)\} + \left(\frac{\rho_0}{2}\right)^2 \mathrm{var}\{G_{uu}(\omega)\} + \frac{1}{2c^2}\mathrm{cov}\{G_{pp}(\omega), G_{uu}(\omega)\}. \quad (15)$$

The coherence function between the pressure and the particle velocity in the sound field was defined as[11]

$$\gamma_{up}^2(\omega) = \frac{|G_{up}(\omega)|^2}{G_{pp}(\omega)G_{uu}(\omega)}, \quad (16)$$

and by using Eq. (6), the covariance term becomes $\mathrm{cov}\{G_{pp}(\omega), G_{uu}(\omega)\} \approx (1/n)\gamma_{up}^2(\omega)G_{pp}(\omega)G_{uu}(\omega)$. Using Eq. (7), the variance of the total energy density is finally expressed in the following two forms:

$$\mathrm{var}\{\hat{E}(\omega)\} \approx \frac{1}{n}[V^2(\omega) + T^2(\omega) + 2\gamma_{up}^2(\omega)V(\omega)T(\omega)] \quad (17a)$$

or

$$\mathrm{var}\{\hat{E}(\omega)\} \approx \frac{1}{n}[E^2(\omega) - 2(1-\gamma_{up}^2(\omega))V(\omega)T(\omega)]. \quad (17b)$$

Equation (17b) allows one to obtain the bounds of the variance according to whether the coherence function between pressure and particle velocity takes a value 0 or 1:

$$\frac{V^2(\omega)}{n} + \frac{T^2(\omega)}{n} \le \mathrm{var}\{\hat{E}(\omega)\} \le \frac{E^2(\omega)}{n}. \quad (18)$$

The lower bound is thus reached when the covariance term is zero. It is noted that this lower bound is two times smaller than the upper bound, when the potential energy density is equal to the kinetic energy density. One can also note that a zero value of the potential or kinetic energy density will cancel the cross term and will result in reaching the upper bound.

Now by considering the third term on right-hand side of Eq. (15) which can be written in the form $\mathrm{cov}\{G_{pp}(\omega), G_{uu}(\omega)\} \approx (1/n)|G_{up}(\omega)|^2$ and by expressing the complex intensity[3,10] as $G_{up}(\omega) = I(\omega) + jJ(\omega)$, the variance takes the same form as that in the expression (14) without, however, using the approximation values:

$$\mathrm{var}\{\hat{E}(\omega)\} \approx \frac{1}{n}\left[V^2(\omega) + T^2(\omega) + \frac{I^2(\omega)}{2c^2} + \frac{J^2(\omega)}{2c^2}\right]. \quad (19)$$

Equation (19) shows clearly that the lower bound $E^2(\omega)/2n$ is reached in a sound field where the active and reactive intensities are equal to zero, and where the energy densities are equal to $V(\omega) = T(\omega) = E(\omega)/2$. It now remains to specify this type of sound field.

The probe consisting of two microphones for measurement of the energy densities can be used only in one-dimensional fields because only one component of the particle velocity is measured for computing the kinetic energy density. A general model of a one-dimensional partially coherent field can be represented by two random plane waves traveling in opposite directions:

$$p(x,t) = a\left(t - \frac{x}{c}\right) + b\left(t + \frac{x}{c}\right), \quad (20)$$

where a and b are, respectively, the random amplitudes of two plane waves traveling in opposite directions. The energy quantities can thus be expressed as a function of the auto- and the cross-spectra between the amplitudes of the waves:

$$V(\omega) = \frac{G_{AA}(\omega) + G_{BB}(\omega) + 2G_{AB}(\omega)\cos 2kx}{2\rho_0 c^2},$$

$$T(\omega) = \frac{G_{AA}(\omega) + G_{BB}(\omega) - 2G_{AB}(\omega)\cos 2kx}{2\rho_0 c^2}, \quad (21)$$

$$I(\omega) = \frac{G_{AA}(\omega) - G_{BB}(\omega)}{\rho_0 c}, \quad J(\omega) = 2\frac{G_{AB}(\omega)}{\rho_0 c}\sin 2kx,$$

where $G_{AB}(\omega) = \gamma_{AB}(\omega)\sqrt{G_{AA}(\omega)G_{BB}(\omega)}e^{j\varphi_{AB}}$ (the phase φ_{AB} is set to 0 in what follows in order to simplify the expressions without loss of generality). The coherence $\gamma_{AB}^2(\omega)$ between the two components can vary from 0 (two independent progressive plane waves) to 1 (quasistanding wave). The approximate expression for the energy densities of Eq. (4) is obtained from the computations of the pressure at the two microphone positions $x_1 = x - \Delta/2$ and $x_2 = x + \Delta/2$ by the use of Eq. (20):

$$D_a(\omega) = \alpha[2(G_{AA}(\omega) + G_{BB}(\omega)) + 4G_{AB}(\omega)\cos 2kx \cos k\Delta] + \beta[2(G_{AA}(\omega) + G_{BB}(\omega))\cos k\Delta + 4G_{AB}(\omega)\cos 2kx]. \quad (22)$$

From Eqs. (21) and (22), one can derive the expressions for the potential and kinetic energy densities:

$$V_a(\omega) = V(\omega)\cos^2 k\Delta/2, \quad T_a(\omega) = T(\omega)\left(\frac{\sin k\Delta/2}{k\Delta/2}\right)^2, \quad (23)$$

and the expressions for the active and reactive acoustic intensities:[10]

$$I_a(\omega) = I(\omega)\frac{\sin k\Delta}{k\Delta}, \quad J_a(\omega) = J(\omega)\frac{\sin k\Delta}{k\Delta}. \quad (24)$$

IV. DISCUSSION

According to Eq. (19), the lower bound of the variance $E^2(\omega)/2n$ is reached when the acoustic intensities are equal to zero and when the potential and kinetic energies are equal. Equation (21) shows that the active intensity is canceled when the two opposite waves have identical amplitudes. To obtain, at the same time, a cancellation of the reactive intensity and an equality of the potential and kinetic energy densities, it is necessary that the fluctuations due to the interferences disappear, i.e., that the two waves are completely incoherent. The same reasoning is transposable with the

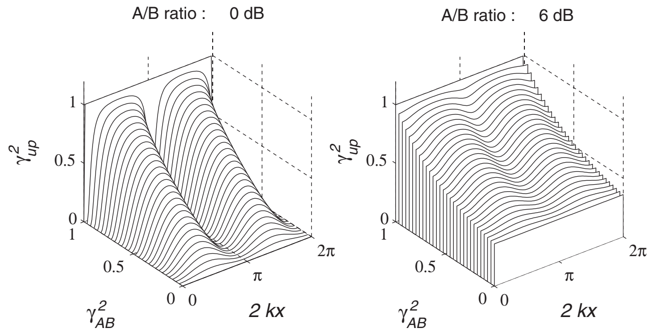

FIG. 1. Pressure-velocity coherence function γ^2_{up} of two waves, respectively, with amplitudes A and B traveling in opposite directions vs the coherence γ^2_{AB} of the two waves and $2kx$ for two A/B ratios indicated.

measurement by a probe consisting of two microphones by considering Eq. (14). Because of the systematic errors due to the finite-difference-approximation method used for evaluation of the potential and kinetic energy densities, the value of the variance of $\hat{E}(\omega)$ is always slightly higher than the theoretical value of the lower bound when $k\Delta$ increases.

According to the above results, the coherence function between the pressure and the particle velocity that appeared in Eqs. (17a) and (17b) and is given by Eq. (16) can also be written as follows:

$$\gamma^2_{up}(\omega) = \frac{I^2(\omega) + J^2(\omega)}{4c^2 V(\omega) T(\omega)}. \tag{25}$$

Uncorrelated two opposite waves ($\gamma^2_{AB}=0$) results in canceling the reactive intensity $J(\omega)$, but it is also necessary that these waves have identical amplitudes, so that the active intensity is canceled. Equation (25) shows that these two conditions lead to $\gamma^2_{up}(\omega)=0$. The null coherence between pressure and velocity is thus a sufficient condition to reach the lower bound of the variance, as it is predicted by Eqs. (17a) and (17b). The function of the coherence between pressure and velocity γ^2_{up} can be expressed in terms of the coherence γ^2_{AB} between the two opposite waves, the A/B ratio $r^2(\omega) = G_{BB}(\omega)/G_{AA}(\omega)$, and the position in the quasistanding wave

$$\gamma^2_{up}(\omega) = \frac{(1 - r^2(\omega))^2 + 4\gamma^2_{AB}(\omega) r^2(\omega)\sin^2 2kx}{(1 + r^2(\omega))^2 - 4\gamma^2_{AB}(\omega) r^2(\omega)\cos^2 2kx}. \tag{26}$$

Figure 1 shows the evolution of the coherence between the pressure and velocity γ^2_{up} versus γ^2_{AB} and kx for two amplitude ratios r^2. When two opposite waves have the same amplitudes ($r^2=0$ dB), the coherent part of the field creates quasistationary waves and γ^2_{up} depends also on the position of nodes ($kx=n\pi/2$, n is an integer number). When two opposite waves are incoherent, the coherence between pressure and velocity γ^2_{up} vanishes (neither active nor reactive intensity exists). When the amplitude of one wave is much greater than the others (r^2 is around 6 dB), the phenomenon is less evident and γ^2_{up} is never zero.

The variance of the energy density is often calculated from Eq. (9) in which the coherence between microphones appears. It is related to the pressure-velocity coherence function but evolves differently. For the two microphones positioned, respectively, at $x_1=x-\Delta/2$ and $x_2=x+\Delta/2$, using Eq. (20), the coherence between the two microphones can be obtained:

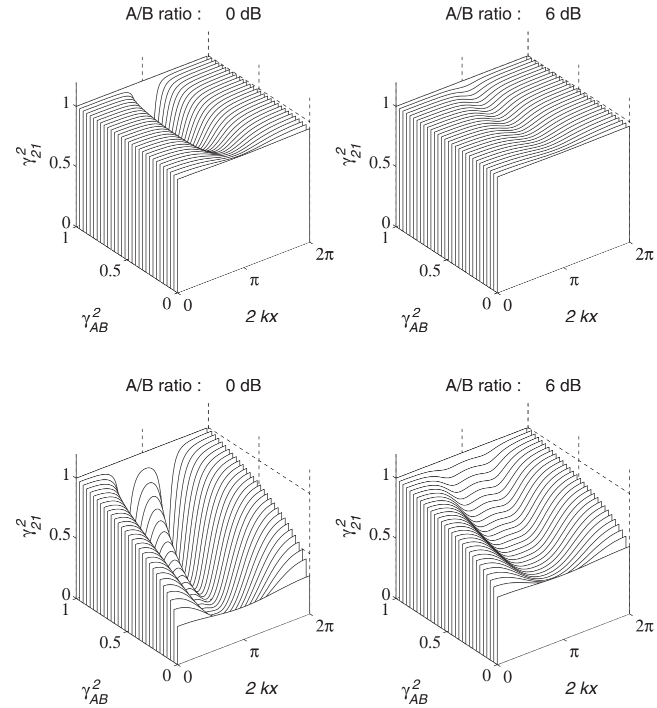

FIG. 2. Coherence function between two microphone signals γ^2_{12} vs the coherence of two waves traveling in opposite directions γ^2_{AB} and $2kx$ for two A/B ratios indicated and when $k\Delta=0.25$ (top), $k\Delta=1$ (bottom).

$$\gamma^2_{21}(\omega) = \frac{|G_{21}(\omega)|^2}{G_{11}(\omega)G_{22}(\omega)} = \frac{f(r(\omega),\gamma_{AB}(\omega),kx,k\Delta) - 4r^2(\omega)\sin^2 k\Delta}{f(r(\omega),\gamma_{AB}(\omega),kx,k\Delta) - 4\gamma^2_{AB}(\omega) r^2(\omega)\sin^2 k\Delta}, \tag{27}$$

where

$$f(r(\omega),\gamma_{AB}(\omega),kx,k\Delta) = (1 + r^2(\omega))^2 + 4\gamma_{AB}(\omega) r(\omega) \times (1 + r^2(\omega))\cos 2kx \cos k\Delta + 4\gamma^2_{AB}(\omega) r^2(\omega)\cos^2 2kx.$$

γ^2_{21} is equal to 1 when the two opposite waves are totally coherent. If this condition is not satisfied, it tends toward 1 when the microphone spacing becomes small in comparison to the wavelength (in the low frequency range, when $k\Delta \ll 1$). Figure 2 shows the coherence between the two microphones under the same conditions as in Fig. 1 for the values $k\Delta=0.25$ and $k\Delta=1$ (upper bound of the frequency range). The coherence between the two microphones γ^2_{21} is very different from that between the pressure and velocity γ^2_{up}. γ^2_{21} is never zero and is dependent on the microphone spacing $k\Delta$, where $k\Delta=2\pi\Delta/\lambda$, λ is the wavelength. Any loss of coherence between the signals of the two microphones will tend to approach the lower bound of the statistical error, just like the addition of independent noise to each of the two microphone channels.

V. CONCLUSION

The normalized variance of the total energy density has an upper bound equal to E^2/n $(n=BT)$, like any quadratic value of a physical quantity. It is shown that this variance can be expressed as a function of the various energy quantities of sound fields, such as potential and kinetic energy densities, and active and reactive acoustic intensities. It is the nature of the sound field that determines the statistical errors not the use of the discretization by finite difference. It appears that the value of the variance becomes two times lower when the active and the reactive acoustic intensities vanish simultaneously. However, this condition is not sufficient. It is also necessary that the potential and kinetic energy densities are equal at all points of the acoustic field. This can be obtained only when the coherence between the pressure and acoustic particle velocity is equal to zero. In a one-dimensional sound field, this condition is obtained when the two waves traveling in opposite directions are independent and of equal amplitudes. The relationships between the pressure-velocity coherence function and the coherence measured between the two microphone signals, by which the energy quantities can be experimentally determined, were specified.

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