A non-linear vibration spectroscopy model for structures with closed cracks

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Ensuring an uninterrupted service in critical complex installations requires parameter health monitoring of the vibrating structures. Tools for monitoring structural modifications through changes in the measured dynamic responses are necessary in order to detect the advent and evolution of cracks before the occurrence of catastrophic failures. It is shown, both theoretically and experimentally, that the equation for the modes of vibration of a structure with closed (breathing) cracks and whose surfaces enter into contact during vibration can be modeled using the Hertz contact theory. The damping chosen is a fractional order derivative to investigate the order matching the experimental data. A perturbation solution technique, combining the Multiple Time Scales and Lindsted–Poincaré methods, has been employed to construct analytical approximations to the resulting non-linear equation of vibration. A 3D finite element model of the structure has been employed to compute the eigenvalues of the sound structure, providing a means to validate the measured resonance frequencies and also allowing the visualization of the modal deformations thus giving greater insight into the physics of the problem.

1. Introduction

The objective of this study is to develop a model that can capture the characteristics of a vibrating structure and also follow the changes in its measured dynamic response which can lead to the identification of structural modification like the advent of micro- and macrocracks and their growth. The evolution of the response can eventually facilitate the follow-up of the degradation of the structural flaw allowing for a correction of the defect before the occurrence of a catastrophic failure. This topic touches the non-destructive evaluation (NDE) and parameter health monitoring of vibrating structures for ensuring an uninterrupted service in critical complex installations like aircraft (rotor dynamics), ships, steel bridges, and sea platforms.

In micro-inhomogeneous materials (microdamaged, with microcracks), the non-linear hysteretic stress/strain relationship behavior has been modeled using mesoscopic mechanical elements with stress (strain) dependent parameters. To this end, the phenomenological Preisach–Mayergoysz (PM) spaces have been employed to model the hysterical non-linearity of the mesoscopic elements composed of individual mechanical elements (internal contacts) that open and close at different stress/strain level thresholds [1–3]. A micro-contact based theory similar to the PM model for microdamaged materials has also been proposed in Refs. [4,5]. The material considered had a large number of isotropic oriented penny-shaped cracks with rough internal surfaces in contact. The Hertzian contact forces for surface roughness and adhesion were modeled using the Johnson–Kendal–Roberts adhesion theory. Similarly, the Preisach hysteretic formalism has been applied to the more complex but general case of the Hertz–Mindlin system [6] (model used for predicting dry effective elastic moduli in unconsolidated sediment) dealing with frictional and elastic interactions of two spheres in contact subjected to a varying oblique force [7].

When the samples are small in size and have well defined geometries, the non-linear resonance ultrasound spectroscopy (NRUS) characterization technique [8] can be employed. The NRUS technique with an interaction model based on non-linear constitutive equations, has been applied successfully to solve an inverse problem of defect imaging using experimental data [9].

Most engineering materials contain small crack-like defects (microcracks) which they can also spontaneously develop during service under fluctuating loads. Macrocracks result from the accumulation of damage due to fatigue loading. Failure can occur from repeated fluctuating stresses or strains, i.e., fatigue, causing catastrophic failure. Fatigue loading is also believed to cause accumulation of damage in osteoporotic bones in terms of microcracks [10] whose coalescence may lead to initiation of macrocracks that...
result in the catastrophic failure of bone [11]. This may consequently occasion fractures especially at common osteoporotic fracture sites (the wrist, femoral neck, the hip and the spine).

The presence of a breathing macrocrack in a structure has been known to introduce a local flexibility. This has been modeled as a continuous flexibility by using the displacement field in the vicinity of the crack [12].

In this study we consider a single mechanical, structural vibration mode approach in which the restoring force attached to the mass is made up of a spring in parallel with a deformation force due to the presence of the crack. The crack is modeled by supposing that the two crack surfaces enter into contact during vibration and that the contact obeys the basic Hertzian contact law [13–17] with the effect of adhesion due to the van der Waals forces [18] neglected. In parallel to the spring is a dashpot whose damping behavior obeys a fractional-order derivative whose order is sought by matching the model to vibration spectroscopy experimental data.

The resulting mechanical system equation drawn from the developed model is a non-linear ordinary differential equation (NLODE) with a fractional-order derivative. NLODE is complicated to solve exactly. Numerical methods, such as finite difference methods and multi-grid methods, can be employed, but they only provide approximate numerical solutions that do not easily provide direct insight as into the physics of the problem related to the structural parameters. NLODE of dynamical systems is often solved by searching approximate periodic solutions using different perturbation techniques. These approximate analytical solutions are close to the true ones. Their advantage over the numerical solutions is that they are analytical expressions (instead of just a long list of numbers) that enable one to gain some insight into the underlying physics of the problem. Some solutions are based on the combination of linearization of the governing NLODE using the popular method of harmonic balance [19], with some approaches, unlike the classical harmonic balance method, performing linearization prior to proceeding with harmonic balancing [20].

Most of the perturbation methods, such as the Krylov–Bogoliubov–Mitropolskii (KBM) [21], Variational iteration methods [22,23], Averaging methods [24], Method of Matched Asymptotic Expansions [25,26], Renormalization method [27–29], Method of Multiple Scales [30,31], are developed in the time domain. The NLODE with fractional-order derivative model equation is solved herein by combining the Multiple Time Scales and the Lindstedt Poincaré perturbation methods [32] in the frequency domain. The solutions are sought in this domain for the ease of observation and interpretation of the non-linear phenomenon.

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2. Theory

The elastic deformation of the impacting crack surfaces (Fig. 1) is accounted for using the Hertz contact theory for two connected Hookean bodies where the restoring contact force between them is given by

\[ f(\delta) = k\delta^3, \]  

where \( \delta \) is the local deformation (penetration) and \( k \) is a constant dependent on the elastic and geometric properties of the contact surfaces. Both \( k \) and the exponent \( \chi \) can be determined experimentally. In the literature the exponent for planar contacts of various materials and surface textures/state [33,34] is \( 1.6 < \chi < 3.3 \).

The differential equation governing the motion of the structure during contact, obtained after summing the forces on the mass \( M \) (Fig. 2) is

\[ M\ddot{x} + C \dot{x} + kx - \kappa \dot{x}^\omega = E \cos(\Omega t) \]  

where \( x \) is the absolute displacement of the mass relative to a fixed reference, \( k \) and \( c \) are the linear elastic stiffness and viscous damping coefficient respectively. \( E \) and \( \Omega \) are the excitation force amplitude and angular frequency respectively. \( \dot{x}^\omega \) is the fractional derivative in the Caputo sense, \( 0 < \eta \leq 1 \) is the order. The Caputo fractional derivative is defined by

\[ \frac{\ddot{x}^\eta}{\partial t^\eta} = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-\tau)^{n-\eta-1} \frac{\ddot{x}(\tau)}{\partial \tau^n} \, d\tau \]  

where \( \Gamma(\cdot) \) is the Gamma function and \( n-1 < \eta \leq n, n \in \mathbb{N} \).

By expanding the contact force non-linearity around a static point \( l_0 (\delta = x - l_0) \) using the Maclaurin power series expansion

\[ \delta^\omega \approx (-\delta \omega_0^3) (1-\delta \omega_0^3 x + 1/2(\omega_0^3 - 1)\delta \omega_0^3 x^2 - 1/6(\omega_0^3 - 2)(\omega_0^3 - 1)\delta \omega_0^3 x^3) + o(x^4). \]  

In the presence of a crack, a downward shift of the resonance angular frequency \( \omega_0 \) and a soft spring non-linear vibration behavior occurs [35]. This imposes that \( \chi = 3 \) in Eq. (4). The dynamic equation in Eq. (2) then has the form

\[ \frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \alpha \dot{x} \ddot{x} + \alpha x^3 + b\dot{x} = F \cos(\Omega t), \]  

where \( \lambda = c/2M \) is the damping ratio, \( \omega_0 = \sqrt{1 - 3\kappa_0^2/M} \), the non-linear multipliers \( a = 3\kappa_0^2/M \) and \( \beta = -\kappa/M \). The modified normalized (using the mass) excitation force \( F = E/M \). The constant term \( \kappa_0^2/M \) with no dependency on \( x(t) \) has been neglected because it appears in the solution of the response as an offset that can be removed by high-pass filtering.

It can be noticed from Eq. (5) that the vibration response of the cracked structure has become non-linear due to the presence of the quadratic and cubic terms in the equation of motion. The cubic term gives the frequency-response curve of a soft non-linear
spring behavior (as $b < 0$) [30]. This softening effect tends to bend the response curves towards lower frequencies.

If $\eta = 1$ then a damped non-linear single degree of freedom equation with quadratic and cubic non-linearities also known as the damped Helmholtz–Duffing (HD) equation [36,37], is obtained. A solution to this equation based on the use of Jacobi elliptic functions [37] has previously been proposed, but in the time domain. Similarly, approximate solutions to the weakly non-linear equation with damping of fractional order, using the multiple time scale perturbation method have been given in [38] in the time domain. It is important to note that variants of the HD have been solved in the time domain but often without damping [39]. The dynamic equation obtained for the behavior of the cracked structure is solved herein in the frequency domain using a combination of two perturbation methods.

2.1. Multiple Time Scales–Lindstedt Poincaré perturbation method of solution for the cracked structure dynamics

The purpose of employing the Multiple Time Scales–Lindstedt Poincaré perturbation method is to obtain solutions to the second order non-linear differential equation with a fractional-order damping derivative term (describing the cracked structure dynamics), valid for both weakly and strongly non-linear systems (with small or large physical parameters or perturbation quantities in governing equations).

The standard Lindstedt–Poincare method, that gives uniformly valid asymptotic expansions for the periodic solutions of weakly non-linear oscillations, introduces a new time scaled transformation [30] variable $\tau$ (distorted time) such that $\tau = \omega t$, and consequently

$$\frac{d}{dt} = \omega \frac{d}{d\tau}, \quad \frac{d^n}{dt^n} = \omega^n \frac{d^n}{d\tau^n}$$

where the harmonic frequency, $f = \omega / 2\pi$.

The fast and slow time scales transformation is given by

$$T_0 = \tau, \quad T_1 = \epsilon \tau, \quad T_2 = \epsilon^2 \tau,$$

where $\epsilon$ is a small parameter.

The integer first, second and fractional order derivatives with respect to the time scale variables are approximated by

$$\frac{d}{d\tau} = \sum_{m=0}^{N} \sum_{l=0}^{m} (D_l \chi_{m-l}) + O(\epsilon^{N+1}),$$

$$\frac{d^2}{d\tau^2} = \sum_{m=0}^{N} \sum_{l=0}^{m} \sum_{n=0}^{l} \left( \sum_{i=0}^{n} (D_i \chi_{m-l-n}) \right) + O(\epsilon^{N+1})$$

$$\left( \frac{d}{d\tau} \right)^q = D_0^q + qD_1^q - \frac{1}{2!} \left[ 2e^2 - 2(q-1)D_0^q + 2D_1^q \right] + \cdots,$$

where $D_{ij} = d^i/dT^j$.

Both the solution (supposed periodic) and the frequency are expanded approximately in powers of $\epsilon$. For the solution, the following perturbation scheme is employed

$$x(\tau; \epsilon, N) = \sum_{n=0}^{N} \epsilon^n x(T_0, T_1, \ldots, T_N) + O(\epsilon^{N+1})$$

The expansion of the fundamental non-linear frequency $\omega^2$ is given by [40],

$$\omega^2 = \omega_0^2 + \sum_{n=1}^{N} \epsilon^n \omega_n,$$

where $\omega_0$ are unknown constants.

The nearness to the excitation frequency is expressed as,

$$\Omega = \omega(1 + \epsilon^2 \sigma).$$

By rearranging the excitation term, the perturbed dynamic equation is finally written

$$\omega^2 \frac{d^2x}{d\tau^2} + \epsilon^2 2\omega \alpha \epsilon \frac{d^2x}{d\tau^2} + \epsilon^2 \omega x + \epsilon \alpha x^2 + \epsilon \beta x^3 = \epsilon^2 F \cos \left( \Omega \frac{T_0}{\omega} \right)$$

(14)

with the initial conditions $x(0) = 0$ and $dx/d\tau = 0$.

Substituting Eqs. (8)–(13) into Eq. (14), then collecting and equating coefficients of like powers of $\epsilon$ to zero leads to the following succession of differential equations

$$C(e^0) : D_0^3x_0(T_0, T_1, T_2) + x_0(T_0, T_1, T_2) = 0,$$

$$C(e^1) : \omega^2 (D_0^2x_1(T_0, T_1, T_2)) + x_1(T_0, T_1, T_2) = -2\alpha^2 x_0(T_0, T_1, T_2) x_1(T_0, T_1, T_2)$$

$$-2\alpha^2 x_0(T_0, T_1, T_2) x_1(T_0, T_1, T_2) - 2\alpha^2 x_0(T_0, T_1, T_2) x_1(T_0, T_1, T_2),$$

$$C(e^2) : \omega^2 (D_0^2x_2(T_0, T_1, T_2)) + \alpha^3 x_1(T_0, T_1, T_2) x_2(T_0, T_1, T_2)$$

$$= -3\beta x_0(T_0, T_1, T_2) x_1(T_0, T_1, T_2) - 2\alpha^2 x_0(T_0, T_1, T_2),$$

$$\omega^2 x_1(T_0, T_1, T_2) + 1 + 1/2 F \left( e^{i\Omega} e^{i\sigma} + e^{i\Omega} e^{-i\sigma} \right).$$

(15)

An exact analytical solution to the equation at order $C(e^0)$ in Eq. (15) is

$$x_0(T_0, T_1, T_2) = \Re(A(T_1, T_2) e^{i\Omega T_1} + A(T_1, T_2) e^{-i\Omega T_1})$$

(16)

where the line over $A(T_1, T_2)$ denotes the complex conjugate of the complex amplitude.

Substitution of this solution into the equation at order $C(e^1)$ in Eq. (15) gives

$$2\alpha^2 D_0^2x_1(T_0, T_1, T_2) - \alpha_1 x_1(T_0, T_1, T_2) + 3b(\Re(A(T_1, T_2)) e^{i\Omega T_1}$$

$$-2 \alpha_2 x_1(T_0, T_1, T_2) x_1(T_0, T_1, T_2) e^{2i\Omega T_1}$$

$$+ 2\alpha_2 x_1(T_0, T_1, T_2) x_2(T_0, T_1, T_2) e^{2i\Omega T_1} + b(A(T_1, T_2))^3 e^{3i\Omega T_1}$$

$$+ 2\alpha x_1(T_0, T_1, T_2) A(T_1, T_2) e^{i\Omega T_1}$$

$$+ 3\alpha x_1(T_0, T_1, T_2) A(T_1, T_2) e^{i\Omega T_1} = 0.$$}

(17)

Then eliminating the secular terms from Eq. (17), we get

$$3b(A(T_1, T_2))^2 A(\Re(T_1, T_2)) - \alpha_1 x_1(T_0, T_1, T_2) + 2i(D_0 x_1(T_0, T_1, T_2)) = 0.$$}

(18)

Since the frequency ($\omega$) is real valued, the only possible choice is to let $(D_0 x_1(T_0, T_1, T_2)) = 0$. Consequently, from Eq. (18),

$$\alpha_1 = 3bA(T_1, T_2) A(T_1, T_2).$$

(19)

Solving for $x_1(T_0, T_1, T_2)$ from the remaining equations of Eq. (17)

$$\alpha x_1(T_0, T_1, T_2) = -a(A(T_1, T_2))^3 e^{-3i\Omega T_1}$$

$$+ b(A(T_1, T_2))^3 e^{3i\Omega T_1} + 2\alpha A(T_1, T_2) A(T_1, T_2) e^{3i\Omega T_1}$$

$$+ 2\alpha x_1(T_0, T_1, T_2) A(T_1, T_2) = 2A(T_1, T_2) \Re(A(T_1, T_2)) e^{i\Omega T_1}$$

(20)

gives

$$x_1(T_0, T_1, T_2) = \frac{\Re(A(T_1, T_2))^2 e^{2i\Omega T_1} + (\Re(A(T_1, T_2)))^2 e^{-2i\Omega T_1}}{3\alpha}$$

(21)

Substituting $x_1(T_0, T_1, T_2)$ into the order $C(e^2)$ of Eq. (15) and removing the secular terms gives

$$4i\alpha^2 D_0^2 x_1(T_0, T_1, T_2) + 3b(\Re(A(T_1, T_2))^2 A(T_1, T_2))^3$$

$$+ \alpha x_1(T_0, T_1, T_2) + 2b\Re(A(T_1, T_2)) A(T_1, T_2) e^{i\Omega T_1}$$

$$- \alpha x_1(T_0, T_1, T_2) - 1/2 F e^{i\Omega T_1} = 0.$$}

(22)
The intact glass is then placed upside-down on a massive wooden table (contact with friction) to recover the eigenfrequencies and visualize the mode deformations. Then the response spectrum is computed using Linear perturbation analysis. This a direct-solution steady-state dynamic analysis that allows the computation of the steady-state dynamic linearized response of a system to harmonic excitation [44,35,45]. A unit force is applied at the same position as the exciter PZT while the acceleration response is obtained on an element at the position of the sensor PZT. The specimen material behavior was chosen as isotropic linear elastic. The numerical code associated with the 3D FEM elastic model [described by 14,484 nodes, 8461 tetrahedral elements of quadratic geometric order [46]] is constructed with the help of a commercial FEM software (Abaqus [44]).

The elastic moduli (Young’s modulus, 30 GPa, Poisson ratio, 0.22) were obtained by solving an inverse eigenvalue problem employing the FEM software and the measured resonance frequencies as detailed in [45]. The density (3700 kg/m³) of the glass was obtained from the volume of the glass measured using Archimedes’ principle. The value of the density was verified by assuring the conformance of the weight measured on the balance with that found through FEM computation. The 3D glass geometry is sketched using a computer-aided design program, SolidWorks (Dassault Systèmes, Vélizy, France).

The intact glass is then placed upside-down on a massive wooden pine table (a contact friction boundary condition (CFBC) prevails at their interface). Its vibration response is acquired in the same manner to visualize the behavior when the boundary condition is simple contact with friction.
4. Results

The amplitude of vibration is obtained from the frequency equation given by Eq. (26) which is solved using Maple computer algebra software employing the command function, solve. The numerical solutions for $0 < \eta < 1$ are sought using the function \texttt{fsolve}.

The measured and 3D FEM computed wide band vibrational responses spectra for the sound glass are shown in Fig. 4a. The first three computed modes agree well with the experiment but as the modal density and frequency become important not all the modes appear in the computed and measured responses. A mode of interest (MOI) is singled out from this spectrum. In order to fit the experimental curves (the response PZT amplifier gain was 80 dB i.e. $\times 10,000$), the first experimental response level curve measured in Volts (V, is directly proportional to the applied strain) is fitted by varying the oscillator parameters. Once the parameters found, the excitation voltage is increased by doubling the voltage input for each level.

The comparison between the amplitudes of the measured frequency response of a sound glass for the MOI with those computed using Multiple Time Scales –Lindsted Poincaré perturbation method solutions and the exact solution for the dynamic equation is shown in Fig. 4b.

The mass normalized excitation force, $F$ is then sought from the curves acquired around the MOI for each level. $F$ takes the following values $F = (1.0e5, 2.2e5, 4.8e5, 10.2e5) \text{ m s}^{-2}$ with $\lambda = 56$ and the resonance frequency at 3577 Hz.

For this case, the response curves do not bend to lower frequencies as the amplitude of the voltage applied to the exciter PZT is increased.

The slight accident on the right side of the resonance curve (near 3610 Hz) corresponds to the out-of-plane vibration mode.

![Fig. 4. Vibrational response amplitude versus excitation frequency of a sound glass (without crack). (a) Large band spectrum vibrational response, comparison between experiment and 3D FEM computation. (b) Comparison between the MTSLP solution for the non-linear parameters $(a,b) = (0,0)$ and $\eta = 1$ (solid line) against the acquired experimental response (dashed line) and the exact solution (dotted line) for the mode of interest (MOI).](image-url)
The in-plane and out-of-plane modes of a perfectly symmetric structure are ideally identical. The difference in this case can be due to the imperfection in the stiffness symmetry due to the geometry or inhomogeneities of the glass structure.

The corresponding 3D FEM computed out-of-plane (OP) and in-plane (IP) deformation mode shapes of the resonance frequency peaks and for the one around which the measurements were taken for the study are shown in Fig. 5. The difference of 2 Hz found between the IP and OP modes from numeric computation of the MOI is due to the dissymmetry introduced by the finite element meshing.

An identical glass but with a closed crack, ≈ 7 cm long starting from the lip of the glass towards its bottom, is tested. The glass’s wide band spectrum response and the splitting of modes due to the presence of the crack are shown in Fig. 6a. The split is well witnessed by the separation of the unique peak pertaining to the first mode that was originally at 1800 Hz. The third mode previously at 3839 Hz which was only slightly visible is now prominent, but shifted downwards. The other modes at frequencies >5000 Hz that were less apparent are now more conspicuous compared to the peaks on the response of the sound glass. This manifestation combined with mode-splitting makes the spectrum of the cracked glass more densely populated with resonance peaks as compared to the sound one consequently more rich.

Zooming on the MOI shows that the resonance frequencies have shifted downwards from 3577 Hz due to the presence of the crack in the glass. This mode shifting and splitting can be confirmed using FEM simulations of a glass with an open crack as was done in Ref. [35].

The frequency-response curves exhibit a soft non-linear spring behavior that tends to bend the response curves to lower frequencies as the excitation amplitude of the voltage applied to the PZT increases (Fig. 6b). In order to fit the experimental level curves for the MOI using the MTSLP perturbation method, the following parameters of the model were found \((a, b) = (1.0e5, 7.5e5)\) and \(\lambda = 125\), \(\eta = 1\), \(\omega_0 = 2\pi \times 3490\) rad/s. These excitation normalized force levels were found \(F = (3.2e5, 16.6e5, 80.5e5)\) m s\(^{-2}\). The results show that the resonance frequency has shifted 87 Hz downwards (the frequency sweep step in the experiment is 5 Hz).

The same experimental data acquisition procedure is repeated for the sound glass placed upside-down in contact with a massive wooden pine table. The response curves are depicted in Fig. 7. It is observed in the same manner, as for the cracked glass, that the frequency-response curves exhibit a soft non-linear spring behavior that tends to bend the response curves to lower frequencies as the excitation amplitude of the voltage applied to the PZT increases. The MTSLP perturbation method parameters are found to be \((a, b) = (1.0e4, 350e5)\) and \(\lambda = 220\), \(\eta = 1\), \(\omega_0 = 2\pi \times 3814\) rads/s. The following excitation force levels were found \(F = (4.5e5, 13.7e5, 22.5e5, 30.5e5)\) m s\(^{-2}\). In this case the damping coefficient \(\lambda\) and the non-linear coefficient \(b\) have considerably increased.

The influence as regards the order of the fractional derivative damping on the response is shown in Fig. 8. An implicit solution of Eq. (26b) is recovered in the first instance for \(\eta = 1\) employing the parameters recovered for the experimental curve of the lowest level of Fig. 6. It is denoted MTSLP1. The numerical solution
denoted MTSLP2) is also obtained with the same parameters but this curve presents jumps due to the instabilities in the numerical solution at \(3480\) Hz and \(3489\) Hz. For \(\eta = 1\), the MTSLP2 solution in the region without jumps is the same as for the implicit one. Then other numerical solutions are obtained for \(\eta = 19/20\) and \(\eta = 4/5\). In the two cases, the resonance frequency remains the same while the amplitude of the responses increases with the diminution of the fractional order. Consequently the fractional derivative order, \(\eta\), provides an extra fine tuning variable (in addition to \(\lambda\)) for the damping of the vibration amplitude.

![Figure 6](image6.png)

**Fig. 6.** Vibration response amplitude versus excitation frequency of varying amplitudes for a glass with a crack. (a) Large band spectrum vibrational response. (b) Comparison between the MTSLP solution for the non-linear parameters \((a, b) = (1.0e5, 80.5e5)\), and \(\omega_0 = 2\pi \times 3490\) rad/s, \(\eta = 1\) (solid line) against the acquired experimental non-linear response (dashed line).

![Figure 7](image7.png)

**Fig. 7.** Vibration response amplitude versus excitation frequency of varying amplitudes for a glass lying upside down on a pine table. Comparison between the MTSLP solution for the non-linear parameters \((a, b) = (1.0e4, 350e5)\), and \(\omega_0 = 2\pi \times 3814\) rad/s, \(\eta = 1\) (solid line) against the acquired experimental non-linear response (dashed line).
5. Discussion and concluding remarks

It has been demonstrated in this study that the dynamics of a cracked structure can be modeled by adding the Hertz contact force in parallel to that due to the spring force and fractional-order derivative dashpot appearing in the linear single degree of freedom mechanical vibration systems. The deformation in the Hertz contact model was expanded using a power series expansion. The value of the exponent that was found to best fit the experimental observation of the soft spring behavior and the downward shift of the resonance frequency is three (deformation raised to the third power). The resulting system of equations governing the motion of the structure was then found to be non-linear with quadratic and cubic terms and a fractional order damping close to unity. Consequently the equation of motion of a cracked structure is a Duf-\textit{fi}ng\textit{-}\textit{fi}ng equation, which is a dynamic mechanical system equation with quadratic and cubic non-linearities whose coefficients are related to the mechanical parameters of the structure. The two non-linear terms give the structure a non-linear response behavior that is observed through the downward drift of the amplitude of the response as the voltage applied to the PZT (excitation force) increases. The model also confirms that the resonance frequency shifts downwards due to the presence of the crack. The jump phenomenon observed when the excitation frequency sweeps upward and the hysterical aspect noticed when the frequency sweeps downward of the hard spring non-linear oscillator behavior [47] were avoided. This was done by ensuring that the peak amplitude of the response does not enter the unstable zone as the excitation force is increased.

The crack introduced into the glass through thermal shock reduced its effective stiffness asymmetrically in two orthogonal directions. This resulted in a separation (splitting) and shifting of its in-plane and out-of-plane resonance frequencies with the gap between them depending on the size of the crack. This aspect of mode splitting is pertinent to damage evaluation [35]. The phenomenon of mode splitting will be investigated in future work using two degrees of freedom system consisting of two coupled equations with one of the degrees of freedom being linear and the second non-linear [35].

The Multiple Time Scales–\textit{Lindstedt Poincaré perturbation method can equally be employed to recover the non-linear parameters when a contact friction boundary condition prevails at the interface of the glass with another structure e.g. the intact glass turned upside down on a table or placed on an elastic plate or surface. It behaves likewise in a non-linear manner, that is, the amplitude of the response drifts downwards with the increase in the amplitude of the exciting force. This drift is accentuated when a crack is introduced in the glass making the system even more non-linear [35]. The non-linear behavior of the sound glass in a contact friction boundary condition (CFBC) confirms indeed that the two crack surfaces vibrate in an identical manner i.e. as a structure with CFBC.

References


![Fig. 8. Vibration response amplitude versus excitation frequency to study the influence of the fractional derivative order \( \eta \) on the response. The experimental curve corresponds to the lowest level of Fig. 7.](image-url)


