Using simple shape three-dimensional rigid inclusions to enhance porous layer absorption

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The absorption properties of a metaporous material made of non-resonant simple shape three-dimensional rigid inclusions (cube, cylinder, sphere, cone, and ring torus) embedded in a rigidly backed rigid-frame porous material are studied. A nearly total absorption can be obtained for a frequency lower than the quarter-wavelength resonance frequency due to the excitation of a trapped mode. To be correctly excited, this mode requires a filling fraction larger in three-dimensions than in two-dimensions for purely convex (cube, cylinder, sphere, and cone) shapes. At long wavelengths compared to the spatial period, a cube is found to be the best purely convex inclusion shape to embed in a cubic unit cell, while the embedment of a sphere or a cone cannot lead to an optimal absorption for some porous material properties and dimensions of the unit cell. At a fixed position of purely convex shape inclusion barycenter, the absorption coefficient only depends on the filling fraction and does not depend on the shape below the Bragg frequency arising from the interaction between the inclusion and its image with respect to the rigid backing. The influence of the incidence angle and of the material properties, namely, the flow resistivity is also shown. The results of the modeling are validated experimentally in the case of cubic and cylindrical inclusions.

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I. INTRODUCTION

Acoustic porous materials are widely used in noise control applications for their good sound absorbing properties in the middle and high frequency range due to their ability to dissipate sound through thermal and viscous losses. This results in limitations: To effectively absorb sound, the porous layer thickness should be on the order of the quarter of the wavelength. Thus, to absorb low frequency sound, bulky and heavy porous material treatments are required.

These last decades, several ways to enhance low frequency absorption of the porous materials have been proposed, mainly by combining resonant phenomena with the traditional viscous and thermal losses. Generally, the idea is to excite modes of the structure that will trap the energy and enhance the absorption, either with air inclusion (double porosity) or with various shape rigid inclusions. The absorbing properties of periodic rigid inclusions of various shapes embedded in low and medium flow resistivity porous layers were studied in two-dimensions when the porous layer was either backed by a rigid backing or possibly incorporating cavities, or radiated in a semi-infinite half-space. The increased absorption was explained by the excitation of (a) a local mode of the inclusion or of the cavities of the rigid backing, (b) a trapped mode (TM) that traps the energy between the inclusion and the rigid backing, and (c) a modified mode of the layer by coupling the layer mode with the Bloch waves originated by the added periodic heterogeneities. The embedment of rigid inclusion is particularly efficient for enhancing the absorption properties of low and medium flow resistivity foam layers, whose thicknesses do not lead to a unit absorption at the so-called quarter wavelength frequency. In this last case, the embedment of rigid inclusions can lead to a unit absorption coefficient at the same frequency but for thinner structure. For large flow resistivity foam layers (typically $\sigma \geq 50 \text{kN m}^{-2}$), when the absorption is not the unity because the penetration depth is much smaller than the thickness of the layer, embedment of air or low resistivity foam inclusions should be preferred. Indeed, the main parameter is the large contrast between the layer and the inclusion materials. The effects of the inplane periodicity, which mainly acts on the modified mode of backed layer (MMBL) were discussed in detail in Ref. 9 in the case of parallelepipedic irregularities of the rigid backing. These effects will not be investigated in details here. Only square lattice are considered, i.e., the periodicities are identical in both directions of the plane.

In this article, the influence of the periodic embedment of three-dimensional rigid elementary shape inclusions in low and medium resistivity porous layers rigidly backed is studied by use of an in-house Finite Element (FE) code. It also extends previous works performed in 2D configurations to 3D ones which are more realistic and more complex. These inclusions cover a wide range of topological characteristics. For instance, cube, cylinder, sphere, and cone have...
purely convex geometry, while the ring torus is not convex and presents some concave faces. Furthermore, the investigated shapes present different degrees of symmetry and volume repartition with respect to their barycenter.

The present paper is organized as follows. The problem is described in Sec. II. The FE method is then validated numerically and experimentally in Sec. III. In Sec. IV, various numerical examples with different inclusion shapes are discussed.

II. DESCRIPTION OF THE CONFIGURATION

A parallelepipedic unit cell of the 3D scattering problem is shown in Fig. 1. Before the addition of the inclusions, the layer is a rigid frame porous material saturated by air (e.g., a foam) which is modeled as a macroscopically homogeneous equivalent fluid $M^p$ using the Johnson-Champoux-Allard model.10,11 The parameters of the considered foams are the porosity $\phi$, tortuosity $x_\infty$, flow resistivity $\sigma$, and thermal $\Lambda$ and viscous $\Lambda'$ characteristic lengths.

The upper and lower flat and mutually parallel boundaries of the layer, whose $x_3$ coordinates are $L$ and 0, are designated by $\Gamma_L$ and $\Gamma_0$, respectively. The upper semi-infinite material $M^a$, i.e., the ambient fluid that occupies $\Omega^a$, and $M^p$ are in a firm contact at the boundary $\Gamma_L$, i.e., the pressure and normal velocity are continuous across $\Gamma_L$. A Neumann type boundary condition is applied on $\Gamma_0$, i.e., the normal velocity vanishes on $\Gamma_0$.

Inclusions, with a common spatial periodicity $d = (d_1, d_2, 0)$, are embedded in the porous layer and create a two-dimensional diffraction grating in the plane $x_1 - x_2$. The periodicities $d_1$ and $d_2$ along the $x_1$ and $x_2$ directions are identical, $d_1 = d_2 = d$. In the following, five different infinitely rigid simple shape inclusions, depicted in Fig. 2, are considered: A cubic inclusion of edge $a$, a cylindrical inclusion of radius $r$ and length $h$, a spherical inclusion of radius $r$, a conic inclusion of radius $r$ and height $h$, and a toric ring inclusions of neutral axis radius $r$ and tore radius $r'$. A Cartesian coordinate system, with the three unit vectors $\mathbf{i}_j$, $j = 1, 2, 3$, is attached to each inclusion barycenter. The position and orientation of the inclusion in the unit cell coordinate system are referred to by its barycenter $x^{inc}$, its azimuth $\psi^{inc}$, and its elevation $\theta^{inc}$.

The incident wave propagates in $\Omega^a$ and is expressed by $p^i(x) = A^i e^{i(k_1 x_1 + k_2 x_2 - k_3 x_3 - \omega t)}$, wherein $k_1 = -k \sin \theta \cos \psi$, $k_2 = -k \sin \theta \sin \psi$, $k_3 = k \cos \theta$ and $A^i = A^i(\omega)$ is the signal spectrum. The azimuth of the incident wave vector is $\psi^i$ and its elevation $\theta^i$.

In each domain $\Omega^a$ ($\alpha = a, p$), the pressure field fulfills the Helmholtz equation

$$\nabla \cdot \left( \frac{1}{\rho^x} \nabla \rho^x \right) + \frac{(k^x)^2}{\rho^x} \rho^x = 0,$$

with the density $\rho^x$ and the wave number $k^x = \omega/c^x$, defined as the ratio between the angular frequency $\omega$ and the acoustic wave phase velocity $c^x$.

As the problem is periodic in space and the excitation is due to a plane wave, each field ($X$) satisfies the Floquet-Bloch relation

$$X(x + d) = X(x) e^{ik^p \cdot d},$$

FIG. 1. Example of a $d$-periodic fluid-like porous sheet backed by a rigid wall with a periodic inclusion embedded in.

FIG. 2. (Color online) Sketch of the different simple shape inclusions considered (of barycenter $x^{inc}$): (a) cubic inclusions, (b) cylindrical inclusions, (c) spherical inclusions, (d) conic inclusion, and (e) toric ring inclusion.
where $\mathbf{k}_i^l = (k_{1i}^l, k_{2i}^l, 0)$ is the in-plane component of the incident wave number. Consequently, it suffices to examine the field in the elementary cell of the material to get the fields via the Floquet relation in the other cells.

The periodic wave equation is solved with a FE method. Only the most important steps are presented below, for all fields, via the Floquet relation, in the other cells. The computations are carried out using periodic Lagrange finite elements in the fluid and porous domains. The nonreflecting boundary conditions are implemented using the Floquet mode decomposition on surface $\Gamma_{\infty}$. On this boundary, the total pressure reads $p^\infty = p' + p^\infty$ and the scattered pressure can be expanded as

$$p^\infty (\mathbf{x}, \omega) |_{\Gamma_{\infty}} = \sum_{m,n \in \mathbb{Z}^2} A_{mn} \phi_{m,n},$$

with

$$\phi_{m,n} = \frac{1}{\sqrt{S}} e^{i(k_{1mn}x_1 + k_{2mn}x_2)},$$

where $A_{mn}$ are the amplitudes of the Floquet mode $(m,n)$, $k_{1m} = k_{11}^l + m(2\pi/d)$, $k_{2n} = k_{22}^l + n(2\pi/d)$, $k_{3mn}^q = \sqrt{(k_{1mn}^q)^2 - k_{11}^2 - k_{22}^2}$, and $S = d^2$ is the surface of the elementary cell. To satisfy the radiation condition, i.e., the field remains bounded when $x_1 \to \infty$, the values of $k_{3mn}^q$ are chosen to account for both propagative and evanescent waves in $\mathbb{Q}^3$.

Once the wave amplitudes have been evaluated, the integration of the acoustic intensity leading to the energy balance is performed. This integration runs over the unit cell using the orthogonality relation of the Floquet modes. In practice, the number of propagating modes in $\mathbb{Q}^3$ is very small (since $| \mathbf{k}_1^l \cdot \mathbf{d} | \ll 1$) and often reduced to the fundamental mode $(m,n) = (0,0)$ (specular reflection) and the first evanescent modes.

Thanks to the conservation of the energy, the absorbed power is given by $P_{abs} = P_i - P_r$, where the reflected power in the $x_3$ direction is

$$P_r = \sum_{m,n \in \mathbb{Z}^2} \text{Re}(k_{3mn}^q) |A_{mn}|^2 / (\rho^i \omega),$$

and the incident power is

$$P_i = S|A|^2 k_{11}^i / (\rho^i \omega).$$

The absorption coefficient is then defined as the ratio of the absorbed power to the incident power

$$A = \frac{P_{abs}}{P_i} = \frac{P_i - P_r}{P_i}.$$ 

### III. NUMERICAL AND EXPERIMENTAL VALIDATIONS

A large tortuosity ($x_\infty = 1.42$) 20-mm-thick foam (Fireflex) sheet $S1$ and a medium resistivity ($\sigma = 11500 \text{ Ns m}^{-1}$) 22-mm-thick foam (Melamine) sheet $S2$ are mainly used throughout the article. Two other porous materials, $S3$ with a very low resistivity and a wool $S4$ ($32 \text{ nu, ISOVER}$) with larger resistivity, are used to show the influence of the matrix material flow resistivity on the absorption when 3D inclusions are embedded in. The parameters of these porous materials are reported in Table I. These parameters have been evaluated using the traditional methods (Flowmeter for the resistivity and ultrasonic methods for the four other parameters, together with a cross-validation by impedance tube measurement) described in Ref. 12. The first three low and medium flow resistivity foams are examples of porous layers whose absorption properties can be enhanced by embedment of rigid inclusions, while the fourth one is a limit example for which the absorption properties are not that much enhanced for fixed thickness.

Extruded 2D configuration has first been used to validate the proposed 3D FE method by comparison with 2D results. This 2D configuration has been extensively validated with the multipole method, with a modal approach, and with FE method. This last 2D FE method was based on a slightly different approach than the present 3D one. The configuration consists in a 20-mm-thick foam $S1$ with centered rigid infinitely long circular cross section cylinders of radius $r = 7.5 \text{ mm}$ embedded in with a spatial periodicity $d = 20 \text{ mm}$. The comparison of the absorption coefficient calculated with the present FE method and the multipole method is presented in Fig. 3, showing a good agreement for both oblique and normal incidences. Around 10 linear elements per edges (20 mm) on the elementary cell leads to less than 1% of error on the absorption coefficient below 10 kHz and around 5% above. With the same mesh, quadratic elements yield to less than 1% of error on the absorption curves up to 20 kHz. Quadratic elements are used in this paper when calculations are run through 20 kHz, while linear

![Absorption coefficient](image-url)

**FIG. 3.** (Color online) Absorption coefficient for an infinitely long cylinder inclusion, when the layer is occupied by the foam $S1$ [see Table I], excited at normal incidence (black) or with $\theta = \pi/3$ (gray): reference results from Ref. 1, i.e., 2D results (solid line), linear elements (○) and quadratic elements (□). The inset shows the snapshot of the pressure field magnitude at $2674 \text{ Hz}$.
mental and calculated with the present FE method. Both absorption coefficients of these samples measured experi-
mentally and without inclusions, were also performed and shows a good agreement with the model in Fig. 5. The experiments highlight an increase of the absorption coefficient (almost total) at relatively low frequency (2000–4000 Hz) due to the excitation of the trapped mode. This increase is almost the same for both inclusion shapes.

IV. NUMERICAL RESULTS

Numerical simulations have been performed for various geometric parameters, various shape inclusions and within the frequency range of audible sound, particularly at low frequencies. One of the main constraints in designing acoustically absorbing materials is the size and weight of the configuration. In this sense, a low frequency improvement implies good absorption for wavelength larger than the thickness of the structure.

The dimensions of the main studied configurations are listed in Table II.

A. Rigid cubic inclusions

First, a rigid $a$-edge cubic inclusion is considered in a cubic unit cell $(d, d, L) = (20 \text{ mm}, 20 \text{ mm}, 20 \text{ mm})$. The cube is centered in the unit cell, i.e., $x^{inc} = (d/2, d/2, L/2)$. The inclusion is oriented such that $(\theta^{inc}, \psi^{inc}) = (0, 0)$, i.e., the faces of the cube are parallel to those of the unit cell. Figure 6 depicts the evolution of the absorption coefficient at normal incidence for various edge lengths $a$ from 0 mm to 17.5 mm leading to different filling fractions $\frac{ff}{L}$ from 0 to $\approx 0.67$.

Similarly to the analysis carried out in the two-dimensional case,$^{1,3}$ a trapped mode (TM) is excited by the presence of the inclusion. The TM excitation frequency $\nu'$ becomes lower when the filling fraction increases for fixed position of the barycenter. This frequency is always lower than the quarter wavelength resonance one ($a = 0$), when the absorption coefficients are in good agreement. The small differences can be attributed to imperfection in the sample manufacturing incorporating glue layers not accounted for in the model and possible thin air layers inside the sample at the interfaces of the different elements that are gathered back. Measurements of the initial 22-mm-thick foam absorption coefficient, i.e., without inclusions, were also performed and shows a good agreement with the model in Fig. 5.
barycenter is higher or equal to half of the layer thickness $L$. The absorption coefficient possesses a maximum as a function of the edge of the cube. The absorption is nearly total for $a = 16$ mm edge cubic inclusion (configuration C1), which corresponds to a filling fraction of $\alpha_f \approx 0.51$. When compared to the results obtained in the 2D case\footnote{For a $r = 7.5$ mm infinitely long cylinder centered in the same matrix material with identical geometry (Fig. 3), the required filling fraction for the absorption peak to be total is larger in the 3D case, i.e., $\alpha_f \approx 0.51$, than in the 2D case, $\alpha_f \approx 0.44$. In the same way, when the nearly total absorption peak is reached, $\nu'$ is higher in the 3D case ($\nu' = 2860$ Hz) than in the 2D case ($\nu' = 2680$ Hz).

An important issue is the balance between the flow resistivity, the thickness of the initial layer and the size of the inclusions to be embedded in, to enhanced the absorption at frequencies lower than the so-called quarter-wavelength resonance. The same numerical experiments were repeated when the material layer is a lower resistivity foam S3 and a wool S4 with a larger resistivity. Figures 7(a) and 7(b) depict the evolution of the absorption coefficient at normal incidence for various edge lengths $a$ from 0 to 17.5 mm leading to different filling fractions $\alpha_f$ from 0 to $\approx 0.67$ for these two materials, respectively. When the resistivity is low, the periodic embedment of cubic inclusions leads to enhanced absorption peak at frequencies $\nu'$ lower and lower as $\alpha_f$ increases. A total absorption peak is not reached because the required $\alpha_f$ cannot be obtained. When the resistivity is large, the periodic embedment of cubic inclusions leads to a lowering of the maximum absorption frequency $\nu'$ when $\alpha_f$ increases. Nevertheless, its amplitude rapidly decreases with $\alpha_f$, because a total absorption peak is already reached at the quarter-wavelength frequency of the initial plate. This results can be summarized as follows: (a) periodically embedding inclusions leads to a lowering of the first maximum absorption peak frequency and (b) the amplitude of this absorption peak increases with $\alpha_f$ till unity and decreases for larger $\alpha_f$. This last assertion strongly depends on the thickness and flow resistivity of the material layer. When the amplitude of the absorption peak is already unity at the quarter-wavelength resonance, the embedment of rigid inclusions will decrease this amplitude. In this case, it is better to consider air inclusions. When the amplitude of the absorption peak is not unity at the quarter wavelength resonance, the embedment of rigid inclusions will increase this amplitude till the feasible $\alpha_f$, which cannot be necessarily unity.

We now focus on numerical results when the S2 foam around the foil. Figure 8 depicts a cross-sectional view ($x_1-x_3$ plane) of the pressure field magnitude inside the unit cell at $x_2 = d/2$ of the configuration C1 at $\nu' = 2860$ Hz, showing that the wave is trapped between the inclusion and the rigid backing. Similarly to the 2D case, at fixed edge $a$, $\nu'$ becomes smaller when the distance between the inclusion and the rigid backing is larger, i.e., when $x_3^{inc}$ increases. Therefore, this increased absorption could be explained by the first Fabry-Perot interference between the inclusions and its image with respect to the rigid backing, as can be shown in the transmission case. Nevertheless, the Fabry-Perot interference appears when the vertical distance between two adjacent inclusions is equal to the quarter of the projection on the vertical axis of the wave vector, which is impossible here for two reasons: (1) it is not possible that a quarter-wavelength stands between the inclusions and their image for symmetry reasons; (2) $\nu'$ would be identical for each cube edges and equal to the quarter-wavelength resonance frequency for a centered inclusion. Nevertheless, this frequency provides a good approximation of $\nu'$.

Once the optimal edge size is determined to have a nearly total absorption peak at $\nu'$, the absorption coefficient.
is calculated for the total frequency range of audible sound in Fig. 9. The first Bragg interference, which corresponds to the maximum of reflected energy leads to a minimum of absorption around 6000 Hz. This corresponds to constructive interferences between the scattered waves by the inclusion and its image with respect to the rigid backing. This minimum appears when $2x_{3}^{inc}$ is equal to half of the wavelength in the case of normal incidence.

The modified mode of the backed layer (MMBL), which traps the energy inside the porous plate and corresponds to an evanescent wave in the upper half plate and a propagative wave inside the porous plate is excited at $\nu_{MMBL} \approx 17000$ Hz. This corresponds to the intersection of the longitudinal mode of the porous plate, which cannot be excited by a plane incident wave without heterogeneity, with the first Bloch wave, as explained in Refs. 3, 9, and 13. This mode is excited at relatively high frequency because the periodicity is here relatively small ($d = 20$ mm). For a larger periodicity, this mode would have a larger influence on the absorption coefficient at a low frequency. The enhanced absorption due to MMBL was extensively explained in the case of parallelepiped irregularities of the rigid backing in Ref. 9 and used in Ref. 14.

The absorption coefficients calculated in the case of a centered $a = 12.5$ mm ($\eta \approx 0.24$) edge cube with upper and lower interfaces parallel to $\Gamma_{L}$ and $\Gamma_{0}$ rotated around $i_{3}$ are almost identical below the first MMBL. In particular, the absorption coefficients calculated for $\psi_{inc} = 0$ and $\psi_{inc} = \pi/4$ ($\theta_{inc} = 0$) are identical, while it is well known,15,16 that a 2D sonic crystal composed of square cross-section scatterers possesses full bandgap when $\psi_{inc} = \pi/4$ and only a bandgap at normal incidence ($\Gamma X$), when $\psi_{inc} = 0$. Some differences between these configurations were noticed near grazing incidence but are not significant in the case of an acoustic excitation by an airborne plane wave impinging the structure from the upper half-space. The bandgaps were shown and were clearly of interest because the excitation was performed in between the parallelepipedic scatterers in Ref. 17.

Parallelepiped scatterers were tested, exhibiting similar influence on the absorption. The advantage is that $\nu'$ can be smaller because the parallelepipeded can be placed further from the hard backing $\Gamma_{0}$, letting $x_{3}^{inc}$ being larger, but the

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**FIG. 7.** Absorption coefficient (linear elements) of a cubic inclusion of edge length $a$ centered in a cubic unit cell $(d_{1}, d_{2}, L) = (20$ mm, 20 mm, 20 mm) occupied by the foam $S3$ (a) and $S4$ (b) when excited at normal incidence: from the thinnest to the thickest curves $a = 0$, $a = 5$ mm, $a = 7.5$ mm, $a = 10$ mm, $a = 12.5$ mm, $a = 15$ mm, and $a = 17.5$ mm.

**FIG. 8.** (Color online) Snapshot of the pressure field magnitude (linear elements) along a cross-sectional ($x_{1} - x_{3}$ plan) plan view at $x_{2} = d/2$ in the configuration C1, when the layer is occupied by the foam $S1$, excited at normal incidence at $\nu = 2860$ Hz.

**FIG. 9.** (Color online) Absorption coefficient (quadratic elements) of the configuration C1, when the layer is occupied by the foam $S1$, excited at normal incidence. The absorption coefficient of the homogeneous layer is depicted by the dashed line.
failing fraction is then lower and a nearly total absorption peak is then difficult to reach at this frequency.

B. Other rigid simple shape inclusions at normal incidence

We first focus on the three purely convex simple inclusion shapes that possess geometric symmetry with respect to their barycenter.

The absorption coefficients for different simple shape inclusions embedded in the same porous material with the same periodicity and at identical filling fraction are calculated and compared. For an identical material layer and dimensions of the unit cell, the large filling fraction, \( ff \approx 0.51 \), required to reach an almost total absorption coefficient at \( \nu' \) in the case of a rigid cube would impose \( a r \approx 9.9 \) mm radius sphere. Such a filling fraction is also impossible to realize in practice when spheres are embedded with a periodicity \( d = 20 \) mm. For this comparison, the cube edge is \( a = 15 \) mm in order for the filling fraction to be \( ff \approx 0.42 \). This means that in a cubic unit cell, a cubic inclusion seems to be the best choice in the sense that this inclusion shape enables a large filling fraction when compared with other simple shape inclusions, like a sphere or cylinder. While the embedding of spheres in a porous materials probably does not lead to an optimal absorption, i.e., a unit amplitude absorption, at a low frequency, this solution can be preferred.\(^{18}\)

A comparison between the absorption coefficients calculated for centered inclusions in the unit cell of a \( a = 15 \) mm cube (configuration C2), \( h = 15 \) mm and \( r = 8.5 \) mm cylinder with different orientations (configuration C3: \( \theta_{inc} = 0 \) vertical cylinder and \( \theta_{inc} = \pi/2 \) horizontal cylinder), and a \( r = 9.3 \) mm sphere (configuration C4) is shown in Fig. 10. Several observations can be made. First, for the fixed properties of a material layer, the inclusion barycenter position, periodicity, and the filling fraction, the absorption coefficient is identical for the different inclusion shapes below the first Bragg frequency. This means that the absorption coefficient is mainly driven by the filling fraction below this frequency. Second, whatever the inclusion shape, the MMBL is excited at the same frequency \( \nu_{MMBL} \approx 17 \) kHz, because it only depends on the layer material properties and periodicity. Third, absorption coefficients are almost identical between a vertical cylinder (configuration C3, \( \theta_{inc} = 0 \)) and a cube (configuration C2). This means that at a higher frequency, flat interfaces parallel with the boundaries \( \Gamma_0 \) and \( \Gamma_L \) have more influence on the absorption coefficient than the lateral shape. Fourthly, absorption coefficients for a flat interface inclusions parallel with \( \Gamma_L \) and \( \Gamma_0 \) are completely different from those for non-flat interface inclusions. Fifthly, while cubic inclusions enable a larger filling fraction, spherical and horizontal cylindrical inclusions lead to larger absorption at a higher frequency. The higher order Bragg interferences seem to be more excited for these two last inclusion shapes than in the case of a cube and a vertical cylinder. This is in accordance with the conclusion of Ref. 19 in which it is shown in 2D that square cross-section scatterer in square lattice provides a larger bandgap than other scatterer shapes. The absorption is almost always larger than the one of the homogeneous layer between the Bragg frequency and the first MMBL frequency for spheres and horizontal cylinders. In average, the best absorption coefficient is obtained with an horizontal cylindrical inclusion.

Calculations performed for each inclusion shape show that \( \nu' \) decreases and that the absorption amplitude increases when the filling fraction increases at fixed barycenter position. Geometric dimensions required to obtain a nearly total absorption peak were realistic for cylinders but were not for spheres without contact. Once the optimum is reached, if the filling fraction \( ff \) is still increased, both frequency \( \nu' \) and amplitude of the associated absorption peak decrease.

The last two shapes studied here present particularities: The cone does not present geometric symmetry with respect to its barycenter and the ring torus is not of convex shape. Concerning conic inclusions, besides an apparent dependence of \( \nu' \) on the orientation of the cone, all the phenomena are in accordance with the previously studied shapes. Figure 11(a) depicts the absorption coefficient of a \( h = 15 \) mm, \( r = 8.5 \) mm cone, configuration C5, embedded in a porous sheet S1, and geometrically centered in the unit cell for different orientations: Cone up, i.e., the cone vertex is oriented toward the air medium, \( x_{inc} = (10 \) mm, 10 mm, 6.25 mm), \( \theta_{inc} = \pi/2 \); cone horizontal, \( x_{inc} = (6.25 \) mm, 10 mm, 10 mm), \( \theta_{inc} = 0 \); and cone down, i.e., the cone vertex is oriented toward the rigid backing, \( x_{inc} = (10 \) mm, 10 mm, 13.25 mm), \( \theta_{inc} = -\pi/2 \). The corresponding filling fraction is \( ff \approx 0.15 \). It is impossible to reach a sufficiently large filling fraction to obtain a nearly total absorption peak with the conic inclusion in this case. When the cone is horizontal, the \( x_3 \)-coordinate of the barycenter is located in the middle height unit cell, i.e., \( x^c_3 = L/2 \). The configuration being periodic, it is possible to find a unit cell such that the barycenter is the center of the unit cell (in the \( x_1-x_2 \) plane). In this case, the absorption peak associated with the excitation of the trapped mode is very close to the one of a \( a = 10 \) mm cube, \( ff = 0.125 \) (rigorously the edge of the cube should be \( a = 10.5 \) mm for \( ff \approx 0.15 \)). When the cone vertex is oriented toward the air medium, i.e., cone up, the barycenter is lower than the middle of the unit cell, while when the cone vertex is oriented toward the rigid backing, i.e., cone down, the

![FIG. 10. (Color online) Absorption coefficients (quadratic elements) of the configuration C2 (□), C3 (\( \theta_{inc} = 0 \) (○) and \( \theta_{inc} = \pi/2 \) (△), and C4 (○), when the layer is made of the foam S1, excited at normal incidence. The absorption coefficient of the homogeneous layer is depicted by the dashed line.](image-url)
obtained at 11(b). For this shape, a nearly total absorption peak can be obtained at a very low frequency (h/C23) when \( x_{inc}^3 \) is increased at \( x_{inc}^3 = 3L/4 \), Fig. 11(b). Nevertheless, this last shape should be considered as a particular case because it is not composed of only convex part and is becoming a complex 3D inclusion. In particular, ring torus is obviously resonant at high frequency because it possesses concave shape in its inner part in which the acoustic energy can be trapped, while the other shapes cannot be.

C. Numerical results at oblique incidence

Figure 12 depicts the absorption coefficient of the configuration C1 when the layer material is the foam S1 for \( \theta' \in [0, \pi/3] \) \( \psi' \) and for \( \psi' \in [0, \pi/4] \) with \( \theta' = \pi/4 \). For symmtery reason, performing calculation for larger \( \psi' \) is useless. The absorption coefficients for \( \psi' = \pi/3 \) and \( \psi' = \pi/6 \) when \( \theta' = \pi/4 \), were found to be identical, which provides another verification of the method. The frequency of excitation of the trapped mode is slightly modified when \( \theta' \) increases: \( \nu' \) increases when \( \theta' \) increases. The absorption is nearly total up to \( \theta' \approx \pi/3 \). At fixed \( \theta' \), \( \psi' \) only influences the results for frequencies higher than the first Bragg frequency. Similar results were found for the other inclusion shapes.

V. CONCLUSION

The influence of the periodic embedment of non-resonant three-dimensional rigid simple shape inclusions (cube, cylinder, sphere, cone, and ring torus) in a rigidly backed porous layer modeled in the rigid frame...
approximation is studied numerically and experimentally. Similarly to the 2D case, the absorption coefficient of these structures is enhanced, because of the excitation of different types of modes, in particular at a low frequency because of the excitation of a trapped mode that traps the energy between the inclusions and the rigid backing. This entrapment leads to a nearly total absorption peak for a specific filling fraction which is larger in 3-D than in 2-D and the frequency of excitation of this trapped mode is higher in 3-D than in 2-D. This frequency shifts down when the filling fraction increases. The absorption enhancement depends on the filling fraction, but also on the material properties, mainly the flow resistivity, and thickness of the porous layer. The FE results are validated experimentally in the case of cubic and cylindric inclusions. Focusing on the absorption enhancement at a low frequency, a cube is better than a sphere in a cubic unit cell because it allows a larger filling fraction. It is shown that in some cases, it is impossible to obtain a nearly total absorption peak by embedding spheres because the required filling fraction cannot be reached. At a fixed filling fraction and the position of the barycenter, the absorption coefficients are identical below the first Bragg frequency for the various non-resonant inclusions. In other words, for only convex inclusions, the absorption coefficient only depends on the filling fraction and the position of the barycenter and not on the shape of the inclusions below the first Bragg frequency. The embedment of the ring torus shape inclusion requires a lower filling fraction in order to reach a nearly total absorption peak at a frequency which is smaller than for the other shapes. Differences in terms of the absorption coefficient are particularly noticeable at frequencies higher than the first Bragg frequency, and allow for classification of the inclusion shape, either possessing faces parallel to the interface porous/air or the rigid backing, or not. In particular, horizontal cylinders and ring torus lead to larger absorption coefficients than the other shape inclusions. The so-designed structures are obviously anisotropic. The trapped mode is not strongly affected by the angle of incidence, when compared to the effect of the angle of incidence on the modified mode of the backed layer. At a fixed elevation angle of incidence, the trapped mode is not affected by the azimuthal angle. These results pave the way to the development of the porous layers with enhanced absorption properties which can be achieved through optimization procedures and embedment of 3D resonant inclusions to combine resonant feature of the inclusions together with trap mode excitation.

**APPENDIX A: FINITE ELEMENT FORMULATION**

The weak form associated to the Helmholtz equation (1), required for FE resolution, is

\[ -\frac{1}{\rho^2} \nabla \bar{q}^\beta \cdot \nabla \bar{p}^\alpha \mathrm{d} \Omega + \int_{\Omega} \frac{(k\gamma)^2}{\rho^2} \bar{q}^\beta \bar{p}^\alpha \mathrm{d} \Omega = -\int_{\partial \Omega} \frac{1}{\rho^2} \bar{q}^\beta \bar{p}^\alpha \cdot \bar{n} \mathrm{d} \Gamma \]  \quad (A1)

for all the test functions \( q^\alpha, \alpha = \alpha, \beta \). The bar \( \bar{q}^\beta \) denotes the complex conjugate of \( q^\beta \).

The solution being periodic, the pressure fields \( \bar{p}^\alpha \) and the test function \( q^\beta \) are demodulated so as to use the periodic part of the pressure denoted \( \bar{p}^\alpha \) such that \( \bar{p}^\alpha = e^{-ik\gamma \xi} \bar{p}^\alpha_{\beta} \), and of the test function \( \bar{q}^\beta = e^{ik\gamma \xi} \bar{q}^\beta_{\alpha} \). Introducing these expressions in the weak form (A1) leads to

\[ -\int_{\Omega} \frac{1}{\rho^2} \nabla \bar{q}^\beta \cdot \nabla \bar{p}^\alpha \mathrm{d} \Omega + \int_{\Omega} \frac{(k\gamma)^2}{\rho^2} \bar{q}^\beta \bar{p}^\alpha \mathrm{d} \Omega = -\int_{\partial \Omega} \frac{1}{\rho^2} \bar{q}^\beta \bar{p}^\alpha \cdot \bar{n} \mathrm{d} \Gamma, \]  \quad (A2)

where the shifted gradient operator reads as \( \nabla^{\prime} = \nabla - ik_\xi \), \( \nabla = \nabla + ik_\xi \).

The normal derivative of the pressure vanishes on the bottom surface \( \Gamma_0 \) and on \( \Gamma_{inc} \) because the scatterer is infinitely rigid. The boundary term pairs on the lateral boundaries \( \Gamma_r \) (right), \( \Gamma_l \) (left) and \( \Gamma_b \) (back), \( \Gamma_f \) (front) vanish due to the periodicity of \( \bar{p}^\alpha \) and \( \bar{q}^\beta \). The normal velocity continuity on \( \Gamma_L \) is automatically accounted for by removing the boundary integral, while the pressure continuity or more precisely \( \bar{p}^\alpha = \bar{p}^\alpha_{\beta} \) on \( \Gamma_L \) is ensured by use of a Lagrange multiplier \( \lambda \) and its associated test function \( \zeta \). To do that, the two following integrals are evaluated on the boundary \( \Gamma_L \)

\[ \int_{\Gamma_L} \bar{q}^\beta (\bar{p}^\alpha - \bar{p}^\alpha_{\beta}) \mathrm{d} \Gamma + \int_{\Gamma_L} \bar{q}^\beta (\bar{q}^\beta - \bar{q}^\beta_{\alpha}) \lambda \mathrm{d} \Gamma. \]  \quad (A3)

Note that using a Lagrange multiplier is not mandatory here and an algebraic condition or penalization can be used. The main interests are to deal with Hermitian matrix and to ease the implementation.

For the sake of computation, the radiating boundary \( \Gamma_\infty \) of height \( L_\infty \) that truncate \( \Omega^2 \) is introduced, Fig. 1. The radiation of the elementary cell can be handled in the FE method with (a) Dirichlet to Neumann (DtN) map, (b) Perfectly Matched Layer (PML), or (c) modal expansion. Note PML is not suitable for the long wave limit (\( \lambda > d \)) and is not safe for this application. In this paper, the last solution is preferred for its robustness and because the modal coefficients are required to compute the absorption coefficient. In practice, only a few modes are propagative. The FE degree of freedom on \( \Gamma_\infty \) is removed from the FEM matrix in favor of modal amplitude. The boundary term on \( \Gamma_\infty \) is easily computed thanks to the Floquet mode orthogonality and the modal expression of the scattered pressure \( \bar{p}^\alpha \) \((x, \omega) \) given in (3). The modal expansion for the periodic part of pressure field reads as

\[ \bar{p}^\alpha |_{\Gamma_\infty} = \sum_{m, n \in \mathbb{Z}^2} A_{mn} \hat{\phi}_{mn}^\alpha + A^\ast e^{ik^2 L_\infty}, \]  \quad (A4)

with \( \hat{\phi}_{mn}^\alpha = (1/\sqrt{S})e^{i\int_{0}^{L^2} m \mathrm{d}x_1 + n \mathrm{d}x_2} \), and the modal expansion coefficient \( A_{mn} \), used to compute the absorption of the material [see Eq. (7)] can be cast in the vector \( \mathbf{A} \). The modal profile is changed, but the value of \( k^2_{mn} \) remains the same.

It is worth noting that the FE discretization of \( \Omega^2 \) encapsuled by \( \Gamma_\infty \) is not mandatory if the interface \( \Gamma_L \) is a plane surface. In this case, the radiation condition can be applied directly on \( \Gamma_L \) instead of \( \Gamma_\infty \). The general formulation proposed here can tackle with corrugated porous material surface.


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The weak formulation arising from (A2) yields after FE discretization (the boundary integral on $\Gamma_{\infty}$ will be stated later)

$$V K U = VF. \quad (A5)$$

The unknown vector $U$ can be cast into a vector $U_\infty$ containing the FE degree of freedom (dof) on the radiation boundary $\Gamma_{\infty}$ and a vector $U'$ containing the other dof. The unknown vector can be expressed with (A4)

$$\begin{pmatrix} U' \\ U_\infty \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & p^T \end{pmatrix} \begin{pmatrix} U' \\ A \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & p' \end{pmatrix} \begin{pmatrix} 0 \\ A' \end{pmatrix}. \quad (A6)$$

Here, $P$ stands for the modal projection matrix. The Floquet modes are stored in columns and the raw contains the nodal value. The same form is chosen for the test function

$$V = T^T \begin{pmatrix} V' \\ B \end{pmatrix}, \quad (A7)$$

where $V'$ and $B$ are associated to the FE and to the modal dof, respectively.

Introducing (A6) and (A7) in (A5) leads to the modified system

$$T^T K T \begin{pmatrix} U' \\ A \end{pmatrix} = T^T F - T^T K G \begin{pmatrix} 0 \\ A' \end{pmatrix}, \quad (A8)$$

where $^T$ is for the Hermitian transpose. The boundary term on $\Gamma_{\infty}$, using the modal expansion from (A4) and for the associated test function from (A7), yields

$$\int_{\Gamma_{\infty}} \frac{1}{\rho^2} \frac{\partial}{\partial r} \left( \frac{p'}{\rho} \right) \nabla \rho \cdot \mathbf{n} d\Gamma = \sum_{m,n \epsilon \Lambda^2} B_{mn} \varepsilon^{\frac{k_{mn}^2}{\rho^2}}{\Lambda_{mn}} - B_{00} \varepsilon^{\frac{k_{00}^2}{\rho^2}}{L_{\infty}} A' \quad (A9)$$

These terms can be easily added at the end of $F$ and on the diagonal of $K$. The last step is to solve the modified FE matrix $K U = F$ with a sparse solver.19

The implementation of the proposed method has been performed with the open source software. The finite element library FreeFEM++20 is used (version 3.20) with linear (P1) or quadratic (P2) Lagrangian tetrahedral finite element, periodic boundary conditions and parallel computing facilities. The meshes were realized with Gmsh22 (version 2.7) with coincident mesh constraint on each opposite lateral sides of the elementary cell. The inclusions have been designed with FreeCAD23 (version 0.13).

23FreeCAD is a general purpose feature-based, parametric 3D modeler (version 0.13),” http://www.freecadweb.org/ (last viewed).