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Optimized reactive silencers composed of closely-spaced elongated side-branchn resonators

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This paper reports a theoretical study of the sound propagation in a rectangular waveguide loaded by closely-spaced elongated side-branchn resonators forming a simple low-frequency broadband reactive silencer. Semi-analytical calculations account for the evanescent modes both in the main waveguide and side-branchn resonators and for the viscothermal losses in the silencer elements. Reasonable accuracy is maintained in the evaluation of transmission, reflection, and absorption coefficients, while the calculation time is reduced by a few hundred times in comparison with the finite element method. Therefore, the proposed method is particularly suitable for optimization procedure. The lengths of the individual equally spaced side-branchn resonators are optimized by a heuristic evolutionary algorithm that maximizes the minimum transmission loss (TL) over a pre-defined frequency range. Numerical results indicate that the minimum TL of the optimized silencers is reduced due to the destructive effect of the evanescent coupling from the resonators of the nearest side-branches. In the opposite, the TL increases linearly with the number of the side-branchn resonators. © 2019 Acoustical Society of America. https://doi.org/10.1121/1.5097167

I. INTRODUCTION

Reduction of low-frequency noise propagating in heating, ventilation, or air-conditioning ductwork is now a major issue, which is traditionally dealt with by means of silencers (mufflers).1

Dissipative silencers employ porous absorbing materials to convert the acoustic energy into heat. These silencers are usually not efficient at low frequencies and their use is problematic under strict hygienic conditions or when the air is humid, dirty, or greasy.

Reactive silencers make use of the impedance mismatch between the upstream and the treated elements of a duct to reflect the acoustic energy. Therefore, the dissipation of the acoustic energy plays a minor role for this latter type of silencer. Typical examples of reactive silencers are expansion chamber1 or side-branchn resonators. These resonators are often Helmholtz resonators (HRs) or quarter-wavelength resonators (QWRs), but each independent resonator operates efficiently only in a relatively narrow frequency range. Detuned HRs are thus arranged in series, parallel, or both2–6 to broaden the silencer efficiency frequency range. These detuned HR arrays can then be optimized.2,7 Arrays of side-branchn QWRs of varying lengths according to a simple relation were also examined.8,9 The opening of a broad stop band resulting from the hybridization of QWR resonances with Bragg interference was further studied in Ref. 10. The lengths and separating distances of side-branchn QWRs flush-mounted to a waveguide were optimized to maximize the minimum value of the transmission loss (TL) over a given frequency range in a previous work.11 The separating distances between the adjacent side-branchn QWRs were found to strongly affect the performance of the silencers and to be comparable to the QWR lengths in optimal configurations, thus making the length of the silencers rather large. Nevertheless, these large separating distances allow us to use a fairly simple one-dimensional mathematical model, e.g., the transmission matrix method (TMM), during the optimization process.

In this work, we optimize silencers consisting of narrow side-branchn QWRs when their separating distances are deeply sub-wavelength, thus making the silencers compact. The evanescent coupling between the side-branchn QWRs must also be accounted for, see, e.g., Refs. 12 and 13, because of the close proximity of each individual QWR. As a result, the physical behaviour of the system cannot be described anymore by TMM. Moreover, the use of a purely numerical approach like the finite element method (FEM), see, e.g., Refs. 8 and 9, even if possible at least in principle, would require rather substantial computational effort preventing it from being used effectively in combination with multi-parametric optimization algorithms. For these reasons, a semi-analytical model accounting for both evanescent modes in the main waveguide and transverse modes in the side-branchn QWRs is proposed in this work. This model accelerates the typical calculation duration by a few hundred times while maintaining reasonable accuracy, and is then employed to optimize a compact silencer.

II. MATHEMATICAL MODEL

A. Model geometry

The geometry of the studied problem is depicted in Fig. 1. The main waveguide is assumed to have a rectangular cross section of height $a$ and width $b$ sufficiently large for the walls to be considered as rigid ones. This duct is supposed to be of an infinite extent or anechoic at both ends, i.e., the reflections from its both ends are null. The analysis is performed in the linear harmonic regime at the circular frequency $\omega$ with the implicit time dependence $e^{i\omega t}$. The excitation takes the form of a plane incident wave $\tilde{p}_i = \tilde{p}_0 e^{-i\omega s}$ of amplitude $\tilde{p}_0$, coming from the left (upstream) side of the duct. The analysis is conducted at low frequency, therefore, the frequency of excitation $\omega$ is much below the cut-on frequency of the main duct, i.e., only the fundamental mode is propagative, the higher order ones being evanescent. The evanescent coupling is thus operated through these higher modes. The silencer consists of a series of $M$ rigid-wall narrow side-branch QWRs flush-mounted to the upper side of the main duct. The QWRs have also a rectangular cross section of widths $d$ and $b$, which means that they cover the entire width of the main waveguide. The lengths of the individual QWRs are $l_0$, $l_1$, $l_2$, ..., $l_M$, labelled from the upstream to the downstream parts of the main duct, while the distance between the centres of the adjacent side-branch QWRs is constant and equal to $L$ with $L > d$. Only two modes are accounted for in each QWR, one propagative and another one evanescent.

B. Acoustic field radiated by a piston in a rectangular duct

The loading side-branch QWRs are modelled as virtual air-pistons located on the upper side of the main waveguide following Ref. 8. The acoustic field radiated by a piston into an infinitely long rectangular rigid-wall duct is therefore briefly introduced in this subsection. The analysis is based on the theory developed in Refs. 14 and 15.

Consider an infinitely long rigid-wall duct with rectangular cross section of height $a$ and width $b$ (see Fig. 2) and a rectangular piston of length $d$, located on the waveguide upper side ($y = 0$) and covering the entire duct width $b$. The centre of the piston is at the abscissa $x = 0$. The complex amplitude of the acoustic pressure $\tilde{p}(x,y,z)$ radiated by the piston into the waveguide takes the form $^{14,15}$

$$\tilde{p}(x,y,z) = \sum_{m,n} \rho_0 c_{mn} \psi_{mn}(y,z) \left( \frac{1}{2ab} \right) \int_{S} \tilde{V}(x',y',z') \times \psi_{mn}^*(y',z') \left[ H(x-x') e^{-i\omega(x-x')/c_0} \right. + H(x-x') e^{i\omega(x-x')/c_0} \right] dS' ,$$  

(1)

where $\rho_0$ is the density of the ambient fluid, $c_{mn}$ is the (complex) modal wave speed, defined as

$$\left( \frac{c_0}{c_{mn}} \right)^2 = 1 - \left( \frac{\omega_{mn}}{\omega} \right)^2 ,$$

where

$$\left( \frac{\omega_{mn}}{c_0} \right)^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 ,$$

(2)

where $c_0$ is the isentropic sound speed, $\psi_{mn}$ is the normalized modal function

$$\psi_{mn} = \sqrt{2(2-\delta_{0n})(2-\delta_{0m})} \cos \left( \frac{m\pi y}{a} \right) \cos \left( \frac{n\pi z}{b} \right) ,$$

(3)

with $\delta_{ij}$ being the Kronecker delta, the asterisk stands for the complex conjugate ($\psi_{mn}^* = \psi_{mn}$, in this particular case), $\tilde{V}(x',y'=0,z')$ is the source strength, i.e., the piston complex velocity amplitude distribution, and $H(x)$ is the Heaviside function. The surface integral runs over the piston surface, $dS' = dx'dz'$, with $x' \in [-d/2,d/2]$, $y' = 0$, $z' \in [0,b]$.

We further assume that the piston complex velocity is given as

$$\tilde{V}(x',y'=0,z') = \tilde{V}(x') = \tilde{V}_0^{(0)} + \tilde{V}_0^{(1)} \sin \left( \frac{\pi x'}{d} \right),$$

$$x' \in \left[ -\frac{d}{2} , \frac{d}{2} \right] , \text{ and } z' \in [0,b] ,$$

(4)

where $\tilde{V}_0^{(0)}$ represents the monopolar oscillation velocity and $\tilde{V}_0^{(1)}$ represents the dipolar oscillation velocity, i.e., only two modes are considered along the $x$-direction of the QWR. Effectively, no lateral mode is driven along the $z$-axis ($n = 0$) because of the configuration symmetry—due to the fact that the piston width equals the waveguide width. Therefore, the problem reduces to a two-dimensional (2D) one because $\int_0^b \cos(n\pi z/b) dz = b \delta_{0n}$.

The total acoustic pressure radiated by the piston then reads as

$$\tilde{p}(x,y,z) = \tilde{p}(x,y) = \tilde{p}^{(0)}(x,y) + \tilde{p}^{(1)}(x,y) ,$$

where $\tilde{p}^{(0)}(x,y)$ and $\tilde{p}^{(1)}(x,y)$ are the pressure components radiated by the monopolar and dipolar modes, respectively.
The result of the integration Eq. (1) depends on whether $|x| < d/2$, or $|x| > d/2$ because of the presence of the Heaviside functions. The $\hat{p}^{(0)}(x, y)$ component reads as15

$$\hat{p}^{(0)}(x, y) = \begin{cases} \sum_{m=0}^{\infty} \rho_0 c_m V_0^{(0)}(2 - \delta_m) \cos \left( \frac{m \pi y}{a} \right) \times c_m \frac{\sin \left( \frac{\pi x}{d} \right)}{\sinh \left( \frac{k_m d}{2} \right)} e^{-ik_m |x|} & |x| \leq d/2, y \geq 0, \\ \sum_{m=0}^{\infty} \rho_0 c_m V_0^{(0)}(2 - \delta_m) \cos \left( \frac{m \pi y}{a} \right) \times c_m \frac{\sin \left( \frac{\pi x}{d} \right)}{\sinh \left( \frac{k_m d}{2} \right)} e^{-ik_m |x|} & |x| > d/2, y < 0 \end{cases}$$

(5)

and

$$\hat{p}^{(0)}(x, y) = \begin{cases} \sum_{m=0}^{\infty} \rho_0 c_m V_0^{(0)}(2 - \delta_m) \cos \left( \frac{m \pi y}{a} \right) \times c_m \frac{\sin \left( \frac{\pi x}{d} \right)}{\sinh \left( \frac{k_m d}{2} \right)} e^{-ik_m |x|} & |x| \leq d/2, y \geq 0, \\ \sum_{m=0}^{\infty} \rho_0 c_m V_0^{(0)}(2 - \delta_m) \cos \left( \frac{m \pi y}{a} \right) \times c_m \frac{\sin \left( \frac{\pi x}{d} \right)}{\sinh \left( \frac{k_m d}{2} \right)} e^{-ik_m |x|} & |x| > d/2, y < 0 \end{cases}$$

(6)

where $c_m^{-1} = c_0^{-1} \sqrt{1 - \left( \frac{m \pi c_0}{a} \right)^2}$ from Eq. (2). Assuming the excitation frequency is below the cut-off frequency, i.e., $\omega < \omega_{\text{cut}} = \frac{m \pi c_0}{a}$, $c_m$ is purely imaginary for $m \geq 1$. Thus, $c_m = i |c_m| \text{ for } m \geq 1$, and Eqs. (5) and (6) reduce to

$$\hat{p}^{(0)}(x, y) = \frac{1}{ik} \left[ 1 - e^{-ikd/2} \cos(kx) \right]$$

$$\hat{p}^{(0)}(x, y) = \frac{1}{ik} \left[ 1 - e^{-ikd/2} \cos(kx) \right] + \frac{2ik}{a} \sum_{m=1}^{\infty} \frac{1}{|k_m|^2} \cos \left( \frac{m \pi y}{a} \right) \times \left[ 1 - e^{-ik_m |x|} \right]$$

(7)

and

$$\hat{p}^{(0)}(x, y) = \frac{2ik}{a} \sum_{m=1}^{\infty} \frac{1}{|k_m|^2} \cos \left( \frac{m \pi y}{a} \right) \times \left[ 1 - e^{-ik_m |x|} \right]$$

(8)

where $k = \omega/c_0$, and $|k_m| = \omega/|c_m|$. In both Eqs. (7) and (8), the terms under the sum sign represent the evanescent modes, the amplitude of which exponentially decays away from the piston. The $\hat{p}^{(1)}$ component (for $\omega < \omega_{\text{cut}}$) is evaluated in a similar way and takes the forms

$$\hat{p}^{(1)}(x, y) = \frac{1}{ik} \left[ 1 - e^{-ikd/2} \cos(kx) \right]$$

$$\hat{p}^{(1)}(x, y) = \frac{2ik}{a} \sum_{m=1}^{\infty} \frac{1}{|k_m|^2} \cos \left( \frac{m \pi y}{a} \right) \times \left[ 1 - e^{-ik_m |x|} \right]$$

(9)

C. Waveguide loaded by $M$ side-branch QWRs

Within this subsection, the formulas introduced in Sec. II B are employed to describe the scattering of an incoming plane wave with $\omega < \omega_{\text{cut}}$ by $M$ flush-mounted QWRs. We remind that due to the symmetry of the configuration, the acoustic field becomes independent on the $z$-coordinate and the problem reduces to a 2D one; see Fig. 3. The loading QWRs are first replaced by equidistant air-pistons of identical width $d$, the central positions of which being $x_m = mL$, where $m = 0, 1, 2, \ldots, M - 1$.

The total pressure field on the surface of the $L$th piston is then given as the sum of the pressure of the incoming wave, the pressure radiated by the $n$th piston itself (see Appendix A), and the pressure radiated by all the other pistons

$$\hat{p}_{\text{tot}}(x - nL) = \hat{p}_n + \hat{p}_n^{(0)} + \sum_{m=0}^{M-1} (\hat{p}_m^{(0)} + \hat{p}_m^{(1)})$$

(11)

The latter expression is then projected on both Fourier components

$$\hat{p}_{\text{tot}}^{(0)} = \frac{1}{a^2} \int_{nL-d/2}^{nL+d/2} \hat{p}_{\text{tot}}^{(0)} \, dx$$

$$= \hat{J}_n^{(0)} + \hat{J}_n^{(1)} + \sum_{m=0}^{M-1} \left( \hat{V}_m^{(0)} G_{nm}^{(0)} + \hat{V}_m^{(1)} G_{nm}^{(1)} \right)$$

(12a)

and

$$\hat{p}_{\text{tot}}^{(1)} = \frac{2}{a^2} \int_{nL-d/2}^{nL+d/2} \hat{p}_{\text{tot}}^{(1)} \cos \left( \frac{\pi(x - nL)}{d} \right) \, dx$$

$$= \hat{J}_n^{(1)} + \hat{J}_n^{(1)} + \sum_{m=0}^{M-1} \left( \hat{V}_m^{(0)} G_{nm}^{(10)} + \hat{V}_m^{(1)} G_{nm}^{(11)} \right)$$

(12b)

FIG. 3. (Color online) 2D geometry of a waveguide with flush-mounted side-branch QWRs.
where \( J^{(0)}_n \) and \( J^{(1)}_n \) are the driving terms, \( F^{(0)} \) and \( F^{(1)} \) represent the loading of the air-pistons by the self-radiated pressure, and \( G^{(0)}_{nm}, G^{(1)}_{nm}, G^{(10)}_{nm} \), and \( G^{(11)}_{nm} \) account for the coupling between the monopolar and the dipolar oscillation velocity of the \( n \)th piston on the \( n \)th piston. The expression of every element is provided in Appendix B.

Central to the method is the expression of the Fourier components at the opening of each QWR. The latter are, respectively, related to \( \tilde{V}^{(q)}_n \) and \( \tilde{V}_n \) via impedance relations

\[
\tilde{p}'_n = -Z^{(q)}_n \tilde{V}^{(q)}_n = -i \rho_0 c_0 k \tilde{z}^{(q)} \coth(z^{(q)} l_n), \quad q = 0, 1, \tag{13}
\]

where \( z^{(q)} = \sqrt{\eta^2 - d^2 - k^2} \) (see Appendix C for demonstration). Please note that the zeroth order directly provides the usual impedance of locally reacting materials \( Z^{(0)}_n = -i \rho_0 c_0 \cot(k l_n) \). The minus sign in front of the impedance symbol \( Z^{(q)}_n \) in Eq. (13) represents the fact that the “positive direction” of the velocity in Eq. (1) is the inward direction (into the main waveguide).

Equations (12) and (13) lead to a set of 2\( M \) linear equations for the calculation of monopolar and dipolar oscillation velocities of the \( M \) QWRs,

\[
\begin{align*}
(F^{(0)} + Z^{(0)}_n) \tilde{V}^{(0)}_n &+ \sum_{m=0}^{M-1} (\tilde{V}^{(0)}_m G^{(00)}_{nm} + \tilde{V}^{(1)}_m G^{(10)}_{nm}) = J^{(0)}_n, \tag{14a} \\
(F^{(1)} + Z^{(1)}_n) \tilde{V}^{(1)}_n &+ \sum_{m=0}^{M-1} (\tilde{V}^{(0)}_m G^{(10)}_{nm} + \tilde{V}^{(1)}_m G^{(11)}_{nm}) = J^{(1)}_n. \tag{14b}
\end{align*}
\]

These equations can be recast into a matrix form

\[
\begin{bmatrix}
\Gamma^{(0)}_0 & 0 & G^{(00)}_{01} & G^{(00)}_{02} & \cdots \\
0 & \Gamma^{(1)}_0 & G^{(10)}_{01} & G^{(10)}_{02} & \cdots \\
G^{(00)}_{10} & G^{(00)}_{11} & \tilde{V}^{(0)}_0 & 0 & \cdots \\
G^{(10)}_{10} & G^{(10)}_{11} & 0 & \tilde{V}^{(1)}_0 & \cdots \\
G^{(00)}_{20} & G^{(00)}_{21} & G^{(00)}_{22} & \Gamma^{(0)}_1 & \cdots \\
G^{(10)}_{20} & G^{(10)}_{21} & G^{(10)}_{22} & 0 & \Gamma^{(1)}_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots 
\end{bmatrix}
\begin{bmatrix}
\tilde{V}^{(0)}_0 \\
\tilde{V}^{(1)}_0 \\
\tilde{V}^{(0)}_1 \\
\tilde{V}^{(1)}_1 \\
\tilde{V}^{(0)}_2 \\
\tilde{V}^{(1)}_2 \\
\vdots 
\end{bmatrix}
= 
\begin{bmatrix}
J^{(0)}_0 \\
J^{(1)}_0 \\
J^{(0)}_1 \\
J^{(1)}_1 \\
J^{(0)}_2 \\
J^{(1)}_2 \\
\vdots
\end{bmatrix}, \tag{15}
\]

where \( \Gamma^{(0)}_n = F^{(0)} + Z^{(0)}_n \), and \( \Gamma^{(1)}_n = F^{(1)} + Z^{(1)}_n \). Once the velocities of the individual air-pistons have been calculated, the total transmitted (downstream) and reflected (upstream) pressures can be evaluated from Eqs. (A1) and (A2) (see Appendix A); far away from the silencer, i.e., when only the propagative waves are accounted for, the formulas take the form

\[
\tilde{p}_r(x \to \infty) = \left[ \tilde{p}_0 + \rho_0 c_0 \sin \left( \frac{k d}{2} \right) \sum_{m=0}^{M-1} \tilde{V}^{(0)}_m e^{ikmL} \right] e^{-iks},
\]

\[
\tilde{p}_r(x \to -\infty) = \rho_0 c_0 \left[ \frac{1}{k d} \sin \left( \frac{k d}{2} \right) \sum_{m=0}^{M-1} \tilde{V}^{(0)}_m e^{-ikmL} \right] - \frac{ikd^2}{a(\pi^2 - k^2 d^2)} \cos \left( \frac{k d}{2} \right) \sum_{m=0}^{M-1} \tilde{V}^{(1)}_m e^{-ikmL} e^{iks}.
\]

From the latter expressions, the transmission, reflection, and absorption coefficients can be calculated as

\[
|T| = \left| \frac{\tilde{p}_r}{\tilde{p}_i} \right|, \quad |R| = \left| \frac{\tilde{p}_r}{\tilde{p}_i} \right|, \quad A = \sqrt{1 - |T|^2 - |R|^2}.
\]

In particular, the TL is calculated as

\[
\text{TL} = -20 \log |T|.
\]

**D. Accounting for viscothermal losses**

The thermoviscous losses due to the thermal and viscous boundary layers are accounted for only in the side-branch QWRs because the resonance takes place in these side-branch QWRs, and these side-branch QWRs are close to each other. The equivalent fluid model of the properties of plane waves propagating in slits\(^{16}\) of width \( d \) with complex ambient density \( \rho_c \) and sound speed \( c_c \) is thus used

\[
\rho_c = \frac{\rho_0}{\Psi_v},
\]

\[
c_c = c_0 \sqrt{\frac{\Psi_v}{\gamma - (\gamma - 1)\Psi_h}},
\]

where \( \gamma \) is the adiabatic exponent, and

\[
\Psi_j = 1 - \tanh \left( \frac{\lambda_j d}{2} \right), \quad j = v, h,
\]

with

\[
\lambda_v = \sqrt{\frac{i\eta \rho_0 c_v}{\mu}}, \quad \lambda_h = \sqrt{\frac{i\eta \rho_0 c_p}{\kappa}},
\]

where \( \mu \) is the dynamic viscosity, \( c_p \) is the specific heat at constant pressure, and \( \kappa \) is the coefficient of thermal conduction.
E. FEM

The transmission properties evaluated numerically via the proposed semi-analytical model are validated against FEM results. The calculations are performed using the commercial software COMSOL Multiphysics (Acoustics Module, Pressure Acoustics Interface, frequency-domain, 2D Cartesian coordinate system). The waveguide and QWR boundaries (black lines in Fig. 3) are considered rigid. An incident plane wave is imposed on the upstream side, while plane-wave radiation conditions are imposed on both ends of the computational domain; see Fig. 3. The distances between the first and last side-branch QWRs to the radiation-condition boundaries are long enough for the evanescent modes of the main duct to vanish. The Helmholtz equation is solved numerically with the COMSOL’s Pressure Acoustics Interface. The thermoviscous losses due to the thermal and viscous boundary layers in the side-branch QWRs are modelled via the Narrow Region Acoustics fluid model17 of the Pressure Acoustics Interface. Within this model, complex equivalent fluid density $\rho_c$ and speed of sound $c_c$ are introduced mimicking the sound attenuation in slits.16

F. Optimization procedure

The optimization procedure adopted here is essentially the same as that used in our previous work.11 The objective function $Q(q)$ to be maximized is the minimum value of the TL in a pre-defined frequency range. The optimized parameters (represented by the parameter vector $q$) are the lengths $l_j$ of each individual side-branch QWR. The parameter space is thus of dimension $M$, and

$$Q(q) = Q(l_0, l_1, \ldots, l_{M-1}).$$

To study the influence of the close distances between the side-branch QWRs on the silencer performance, the parameter $L$ is not subjected to the optimization procedure. The latter procedure is conducted for fixed sub-wavelength values of $L$.

The frequency bandwidth $\Delta f$ of central frequency $f_c$ over which the TL is maximized is discretized as follows:

$$f_j = f_{\text{min}} + (f_{\text{max}} - f_{\text{min}}) \frac{j}{N-1}, \quad j = 0, 1, 2, \ldots, N - 1,$$

where $f_{\text{min}} = f_c - \Delta f/2$, and $f_{\text{max}} = f_c + \Delta f/2$. The objective function to be maximized is then introduced as

$$Q(q) = \min\{\text{TL}(f_j), \quad j = 0, 1, 2, \ldots, N - 1\}.$$  \hspace{1cm} (19)

Maximization of the objective function Eq. (19) relies on a search in a multidimensional parameter space. Conventional deterministic search algorithms18 are prone to get stuck in a local extrema. Therefore, a self-adaptation variant $(\mu, \lambda)$—ES of evolution strategies19—already successfully used in similar optimization problems,11,20 is employed here.

The optimization algorithm, briefly described in Ref. 11, was implemented in MATLAB. By way of illustration, six maximizations of the objective function (six evolutions with different random initial guesses), for $M = 6$ side-branch QWRs and $N = 100$ frequencies, last 36 min of parallel running time on a regular personal computer (PC) with (six-core) central processing unit (CPU) Intel Core i7–5820K, 3.3 GHz (Intel, Santa Clara, CA).

III. NUMERICAL RESULTS

The waveguide as well as the side-branch QWRs are supposed to be filled with air with sound speed $c_0 = 340$ m s$^{-1}$, and the ambient density $\rho_0 = 1.2$ kg m$^{-3}$. The corresponding material parameters for the calculation of the thermoviscous losses are the adiabatic exponent $\gamma = 1.4$, the dynamic viscosity $\mu = 1.83 \times 10^{-5}$ Pa s, the specific heat at constant pressure $c_p = 1004$ J kg$^{-1}$ K$^{-1}$, and the coefficient of heat conduction $\kappa = 25.9 \times 10^{-3}$ W m$^{-1}$ K$^{-1}$. The height of the main waveguide was set to $a = 15$ cm.

A. Validation of the model

The modal decomposition proposed in this work results in infinite sums in Eqs. (7)–(10). These sums must be truncated in practice, the number of terms accounted for balancing the accuracy of the results with the speed of the calculations.

Figure 4 depicts the acoustic pressure amplitude radiated by a piston of width $d = 4$ cm and monopolar velocity amplitude $V_0^{(0)} = 1$ m s$^{-1}$ calculated with FEM, the current model [Eqs. (7) and (8)] for different numbers of modes accounted for, and the TMM at $f = 400$ Hz. To evaluate the acoustic pressure on the face of the piston (its position is delimited by the vertical dotted lines in Fig. 4) accurately enough, $N \sim 200$ modes have to be accounted for. On the other hand, it is enough to account for $N \sim 10$ modes to evaluate acoustic pressure outside the piston. It is also demonstrated that the TMM provides relatively accurate results only far away from the piston. The corresponding pressure amplitude was calculated as

$$|\tilde{p}_{\text{TMM}}| = \rho_0 c_0 |V_0^{(0)}| \times \frac{d}{2a}.$$  

As a result, the number of modes accounted for in the evaluation of the $F$-terms [Eqs. (B2a) and (B2b)] is fixed to

![FIG. 4. (Color online) Acoustic pressure amplitude for $y=0$, calculated employing FEM, the current model (Eqs. (7) and (8)), and TMM; $f=400$ Hz, $V_0^{(0)} = 1$ m s$^{-1}$](image-url)
$N = 200$ and the number of the modes accounted for in the $G$-terms [Eqs. (B3a)–(B4b)] is fixed to $N = 10$ in all the following calculations.

Figure 5 presents a comparison of the transmission coefficient calculated with the FEM (black line), the current model with (red line) and without the transverse mode in the side-branch QWRs (blue lines), and the TMM (with end corrections) for two different separating distances $L = 5$ cm and $L = 10$ cm. The thermoviscous losses are not accounted for in the $M = 6$ side-branch QWRs of the width $d = 4$ cm and lengths

$$l_m = (21.25, 24.08, 26.92, 29.75, 32.58, 35.42) \text{ cm}. \quad (20)$$

The results predicted by the current model (red line) and FEM (black line) are in excellent agreement Fig. 5(a) ($L = 5$ cm) only when the transverse modes in the side-branch QWRs are accounted for (red line). FEM results are in strong disagreement with the TMM (green line), because the latter one-dimensional model does not capture the evanescent coupling between the side-branch QWRs. In the opposite, the evanescent coupling of the side-branch QWRs is weaker when the distance between the centres of the side-branch QWRs is larger, i.e., $L = 10$ cm; see Fig. 5(b). The simplified semi-analytical model (without the transverse modes in the side-branch QWRs, blue line) provides quite good approximation to the FEM results, but the difference between the TMM and FEM can still be noticed. Therefore, accounting of the transverse modes in the side-branch QWRs is a key aspect here, especially for the optimization procedure, as will be shown in Sec. III B.

In spite of the fact that the current semi-analytical model provides numerical results with essentially the same accuracy as FEM, its computational demands are considerably lower. For example, the evaluation of 401 frequency samples of the transmission coefficient spectrum in Fig. 5(a) took 51 s by FEM (15 386 degrees of freedom), whereas the presented algorithm needed only 0.15 s for the same calculation (340 × speedup). The further speedup follows from the fact that during the optimization the transmission properties for individual parameter vectors are evaluated in parallel.

B. Numerical experiments

Various numerical experiments are now discussed. In all the cases, the current semi-analytical model is employed, and the thermoviscous losses in the side-branch QWRs are accounted for.

The TL of an optimized silencer with $M = 6$ side-branch QWRs, $d = 4$ cm, $L = 5$ cm, $f_c = 300$ Hz, and $\Delta f = 100$ Hz are depicted in Fig. 6. The lengths of the individual QWRs are

$$l_m = (25.14, 26.13, 28.37, 32.51, 30.46, 22.77) \text{ cm}. \quad (21)$$

The geometry of the silencer is provided in Fig. 7. It can be observed that the two shortest QWRs (which resonate at the highest frequencies) are located on the both end-sides of the silencer.

The optimization results, when the transverse mode is accounted for [Fig. 6(a)] or not [Fig. 6(b)], are compared to the corresponding FEM results. In the absence of the
transverse mode, a difference between the semi-analytical model and the FEM is noticed, e.g., FEM predicts the TL by 3.9 dB smaller than the analytical model around 255 Hz. On the other hand, the results of both the models are very similar when the transverse mode is accounted for; see Fig. 6(b). A small frequency shift can be observed between the results, though, which is for the highest frequency peak, 0.9 Hz (=0.26%). This latter frequency shift could be corrected by accounting for higher transverse modes in the side-branch QWRs.

We now focus on Fig. 6(b). To maintain the same minimum value between the TL dips, the frequencies of the TL peaks are irregularly spaced. These frequencies are also listed in Table I together with the TL-peak frequencies of the individual QWRs calculated in the absence of the other QWRs. The difference between these frequencies is quite noticeable, e.g., the difference reaches 6.2 Hz for the lowest frequencies. These shifts can be attributed to the evanescent coupling between the neighbouring QWRs. This behaviour was effectively not observed in the previous work\(^1\) where the distances between the adjacent QWRs were much larger. The locations of the side-branch QWRs within a silencer are found to greatly influence its efficiency. When re-ordered according to their lengths, the TL-peak frequencies are completely modified (see the third column in Table I), and some of the QWRs are fully coupled leading to coupled modes and therefore a reduction of the number of TL peaks. The silencer efficiency (minimum TL) also decreases; see Fig. 8. In particular, re-ordering the QWRs makes one of the TL peaks disappear and prominent dips appear 7.9 dB lower than the minimum TL encountered in optimal case, i.e., \(\text{TL}_{\text{min}} = 19.2\,\text{dB}\).

To evaluate the effect of the inherent viscothermal losses, the transmission, reflection, and absorption coefficients of the optimized silencer are plotted in Fig. 9(a). Because the structure is different from the structure of a rainbow trapping absorber,\(^2\) the absorption coefficient cannot be close to one over the entire optimization frequency range. Three absorption peaks are effectively noticed; the one at 306 Hz reaches the value \(A_{\text{max}} = 0.94\). The effects of losses are therefore much more visible on the reflection coefficient, which is not unity at the frequencies of the TL peaks. In the absence of losses, the maximum transmission coefficient increases to \(|T|_{\text{max}} = 0.117\), which corresponds to \(\text{TL}_{\text{min}} = 18.6\,\text{dB}\); see Fig. 9(b). In other words, the inherent losses only slightly improve the silencer performance.

The dotted lines in Fig. 9(a) show the frequency characteristics of the transmission, reflection, and absorption coefficients in the case where the sequence of the side-branch QWRs given by Eq. (21) is reversed, modelling the propagation of the sound waves through the silencer in the opposite direction. It can be seen that, whereas the transmission

![FIG. 7. (Color online) Geometry of the optimized silencer, \(M = 6\).](image)

![FIG. 8. (Color online) TL of a silencer with the side-branch QWRs given by Eq. (21), dashed blue lines; with the side-branch QWRs ordered according to their lengths, red line, and TL of the individual QWRs, dotted black lines.](image)

![FIG. 9. (Color online) Transmission, reflection, and absorption coefficient of an optimized silencer with \(M = 6, d = 4\,\text{cm}, L = 5\,\text{cm}\); (a) dissipation taken into account, (b) dissipation neglected.](image)

<table>
<thead>
<tr>
<th>Optimized silencer QWRs</th>
<th>Individual QWRs</th>
<th>Optimized silencer ordered QWRs</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_{\text{peak}},\text{Hz})</td>
<td>(f_{\text{peak}},\text{Hz})</td>
<td>(f_{\text{peak}},\text{Hz})</td>
</tr>
<tr>
<td>250.9</td>
<td>244.7</td>
<td>248.6</td>
</tr>
<tr>
<td>259.2</td>
<td>260.0</td>
<td>261.1</td>
</tr>
<tr>
<td>279.2</td>
<td>277.8</td>
<td>279.1</td>
</tr>
<tr>
<td>298.1</td>
<td>299.7</td>
<td>303.9</td>
</tr>
<tr>
<td>308.4</td>
<td>310.7</td>
<td>338.0</td>
</tr>
<tr>
<td>338.8</td>
<td>340.0</td>
<td>—</td>
</tr>
</tbody>
</table>
coefficients moduli $|T|$ and $|T_r|$ (blue lines) are the same—the silencer represents a reciprocal system—the reflection and absorption coefficients spectra depend on the wave-propagation direction very much—the silencer represents a non-symmetric system.

C. Role of the distance between the side-branch QWRs

In this subsection, the role of the (sub-wavelength) distances $L$ separating the centres of the adjacent side-branch QWRs is investigated. Figure 10 depicts, for individual frequency-ranges $\Delta f$, the minimum TL of the optimized silencers for $M = 6$ QWRs [Fig. 10(a)] and $M = 12$ QWRs [Fig. 10(b)] as a function of $L$. In all the cases, $f_c = 300$ Hz. The circular markers represent the calculated data, while the dashed lines are the third-order polynomial fits.

For a given number $M$ of QWRs, the wider the frequency range $\Delta f$ is, the lower the minimum TL is, and for a given $\Delta f$, the bigger the number $M$ of the side-branch QWRs is, the higher the minimum TL is. These two facts are in accordance with the results obtained earlier. In addition, decreasing $L$ leads to the decrease of the minimum TL. For example, for $M = 6$ and $\Delta f = 100$ Hz, $T_{L_{\text{min}}} = 25.9$ dB for $L = 10$ cm, which reduces to the value of $T_{L_{\text{min}}} = 17.8$ dB for $L = 4.2$ cm (reduction by 31.2%). Similarly, for $M = 12$ and $\Delta f = 100$ Hz, $T_{L_{\text{min}}} = 56.2$ dB for $L = 10$ cm, which reduces to the value of $T_{L_{\text{min}}} = 41.0$ dB for $L = 4.2$ cm (reduction by 27.1%). Analysis of the numerical data shows that the percentage decrease of $T_{L_{\text{min}}}$ is stronger for a smaller number of the side-branch QWRs. The decrease of $T_{L_{\text{min}}}$ with decreasing $L$ could be explained by the destructive effect of the evanescent coupling of the resonators with similar frequencies. The optimization algorithm tries to minimize this decrease by placing the shortest side-branch QWRs (with the highest resonant frequencies) to the opposite sides of the silencer; see the example in Fig. 7.

Figure 11 shows the dependence of the minimum TL of the optimized silencers on the number of the side-branch QWRs for different values of $\Delta f$ and $L$. In a similar way as it is shown in Fig. 10, for given $M$, the minimum TL increases with decreasing $\Delta f$ and increasing $L$. Moreover, it is also apparent that for given values of $\Delta f$ and $L$, the minimum TL increases linearly with the number of the side-branch QWRs.

A particular example of the TL characteristics for different values of $L$ can be seen in Fig. 12. In all the cases, $M = 8$, $d = 4$ cm, and $\Delta f = 100$ Hz. Similarly as in Fig. 6, the individual peaks of TL are distributed irregularly in frequency to maintain the same value of the $T_{L_{\text{min}}}$, but their frequency locations strongly depend on the value of $L$. This is a consequence of complex interactions taking place among the individual side-branch QWRs.

D. Maximization of the absorption

Repeated numerical experiments have shown that if the minimum value of TL is maximized and the side-branch QWRs width $d$ is also a subject of the optimization, then the objective function is $(M + 1)$-dimensional,

$$Q'(q) = Q'(l_0, l_1, \ldots, l_{M-1}, d),$$

the algorithm always converges to the maximum allowed value of $d$ because the bigger the ratio of $d/a$, the wider the resonance peaks of the individual QWRs.

It has been shown in Fig. 9(a) that the absorption plays a minor role in the performance of the studied optimized silencer, which mostly reflects the acoustic energy. To determine if it is better to maximize the minimum value of either
TL or the absorption coefficient, the objective function was modified; for Eq. (19) was replaced by

$$Q'(q) = \min\{A(f_j), j = 0, 1, 2, \ldots, N - 1\}. \quad (23)$$

The latter objective function optimizes both the lengths of the QWRs and their width $d$. Figure 13 shows what happens if the minimum value of the absorption coefficient $A$ is maximized instead of TL for $f_c = 300$ Hz and $\Delta f = 100$ Hz. Here, the parameters $M = 6$ and $L = 5$ cm are fixed. In order to maximize the absorption in the silencer, the algorithm decreased the width of the side-branch QWRs to $d_{\text{opt}} = 8.2$ mm, resulting in $A_{\text{min}} = 0.747$. However, the increased absorption coefficient also results in a reduced reflection, and thus to an increased transmission through the silencer; the minimum TL decreases to a mere 3.79 dB.

As a result, even if this type of a silencer benefits from the inherent losses, reflection of acoustic energy plays the most important role in the reduction of the transmission through the silencer. Nevertheless, this result should be tempered and it requires further comments in view of the efficiency of the rainbow trapping absorbers, where perfect absorption is achieved in transmission problem. First, the side-branch resonators considered in the present case are QWRs, the quality factor of which is higher than the quality factor of HRs. Therefore, the absorption peaks associated with these types of resonators are very narrow. This emphasizes the fact that broadband large absorption is conditioned by the choice of the side-branch resonators. Second, to create a stop band efficiently, the flux pulsed by the lowest-frequency QWR should be important, thus, imposing wide QWRs. Therefore, the width of the treatment should be much larger than the width imposed in the present study to achieve perfect absorption with QWRs. Third, the main duct section being relatively large, the flux pulsed by a QWR should be larger than the one for a smaller-dimension main duct, thus, imposing a collective behaviour of QWRs along the treatment. To sum up, this result emphasizes the fact that reaching perfect absorption in a duct employing side-branch resonators is conditioned by a balance among the type of side-branch resonators, the targeted frequency range of perfect absorption, and the section of the main duct.

**IV. CONCLUSIONS**

A 2D semi-analytical model for the calculation of acoustic field in a rectangular waveguide loaded by narrow flush-mounted side-branch QWRs has been developed. The model accounts for evanescent modes in the waveguide as well as the first transverse mode in the side-branch QWRs. Thermoviscous losses in the side-branch QWRs have been introduced employing the equivalent fluid model of wave-propagation in slits. Comparison with the numerical results obtained using FEM has shown that the presented semi-analytical model can be used for the prediction of the transmission properties of a compact silencer comprising closely packed side-branch QWRs. The accuracy of the results is reasonable, and the calculations are a few hundred times faster compared to the FEM. Therefore, a simple heuristic optimization algorithm has been implemented for the maximization of the minimum TL over a pre-defined frequency range. It has been found that the $T_{\text{min}}$ of an optimized silencer decreases with the decreasing value of the distances between the adjacent side-branch QWRs. It has also been demonstrated that for a given distance between the QWRs and targeted frequency range, the $T_{\text{min}}$ of an optimized silencer increases linearly with the number of the side-branch QWRs. The numerical experiments have also revealed that the inherent absorption in the side-branch QWRs slightly improves the performance of an optimized silencer. However, the maximization of the absorption coefficient does not improve the TL of this type of a silencer. This emphasizes the fact that the absorption of acoustic energy is a more complex process than its reflection and it involves more constraints to be achieved.

The methods adopted within this work directly allow the design of optimized silencers with QWRs placed symmetrically on the opposite main-waveguide walls, or, the model can be extended in a straightforward way to cover QWRs placed non-symmetrically. Also, the switchover from the rectangular geometry to the cylindrical geometry represents a direct process.

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**APPENDIX A: PRESSURE RADIATED BY THE nth PISTON**

The pressure radiated by the nth piston for \(|x - mL| > d/2\) and \(y = 0\) can be calculated as [see Eqs. (8) and (10)]

\[
\tilde{p}^{(0)}_m(x) = \frac{\rho_0 c_0 V_m^{(0)}}{ka} \sin \left( \frac{kld}{2} \right) e^{-ik|x - mL|}
+ \frac{2i\rho_0 c_0 a V_m^{(0)}}{a} \sum_{s=1}^{\infty} \frac{1}{|k_s|^2} \sinh \left( \frac{|k_s|d}{2} \right) e^{-|k_s||x - mL|},
\]

(A1)

and

\[
\tilde{p}^{(1)}_m(x) = \frac{i\rho_0 c_0 kd^2 V_m^{(1)}}{a(n^2 - k^2d^2)} \cos \left( \frac{kld}{2} \right) \frac{x - mL}{|x - mL|} e^{-ik|x - mL|}
+ \frac{2i\rho_0 c_0 kd^2 V_m^{(1)}}{a} \sum_{s=1}^{\infty} \frac{1}{n^2 + |k_s|^2d^2} \cosh \left( \frac{|k_s|d}{2} \right) \frac{x - mL}{|x - mL|} e^{-|k_s||x - mL|},
\]

(A2)

**APPENDIX B: COEFFICIENTS INVOLVED IN EQ. (12)**

The coefficients involved in Eq. (12) have the form

\[
f^{(0)}_n = \tilde{p}_0 \frac{\sin \left( \frac{kld}{2} \right)}{kd/2} e^{-ikdL},
\]

(B1a)

\[
f^{(1)}_n = -\frac{4ikd^2 \tilde{p}_0}{\pi^2 - k^2d^2} \cos \left( \frac{kld}{2} \right) e^{-ikdL},
\]

(B1b)

\[
F^{(0)} = \frac{\rho_0 c_0}{ika} \left[ 1 - e^{-ikd/2} \sin \left( \frac{kd}{2} \right) \frac{kld}{kd/2} \right]
+ \frac{2i\rho_0 c_0 k}{a} \sum_{m=1}^{\infty} \frac{1}{|k_m|^2} \left[ 1 - e^{-|k_m|d/2} \sinh \left( \frac{|k_m|d}{2} \right) \frac{|k_m|d}{|k_m|d/2} \right],
\]

(B2a)

\[
F^{(1)} = \frac{\rho_0 c_0 kd^2}{a(n^2 - k^2d^2)} \left[ 1 + \frac{4kld}{\pi^2 - k^2d^2} \sin \left( \frac{kld}{2} \right) e^{-ikd/2} \right]
+ \frac{2i\rho_0 c_0 kd^2}{a} \sum_{m=1}^{\infty} \frac{1}{n^2 + |k_m|^2d^2} \left[ 1 - \frac{4|k_m|d}{\pi^2 + |k_m|^2d^2} \right]
\times \cosh \left( \frac{|k_m|d}{2} \right) e^{-|k_m|d/2},
\]

(B2b)

\[
G^{(00)}_{nm} = \frac{\rho_0 c_0}{ka} \sin \left( \frac{kd}{2} \right) \sin \left( \frac{kld}{2} \right) e^{-ik|m-n|L} + \frac{2i\rho_0 c_0 k}{a} \sum_{s=1}^{\infty} \frac{1}{|k_s|^2} \sinh \left( \frac{|k_s|d}{2} \right) e^{-|k_s||m-n|L},
\]

(B3a)

\[
G^{(01)}_{nm} = \frac{i\rho_0 c_0 d}{a(n^2 - k^2d^2)} \sin(kd) \frac{n - m}{|n - m|} e^{-ik|m-n|L}
+ \frac{2i\rho_0 c_0 k}{a} \sum_{s=1}^{\infty} \frac{1}{|k_s|^2 + |k_s|^2d^2} \sinh(|k_s|d) \frac{n - m}{|n - m|} e^{-|k_s||m-n|L}.
\]

(B3b)

Obviously, \(G^{(00)}_{nm} = G^{(00)}_{nm}\) and \(G^{(01)}_{nm} = -G^{(01)}_{nm}\).

**APPENDIX C: TRANSVERSE MODE IN A SIDE-BRANCH QWR**

Let us assume a rectangular channel with a length \(L\) and a width \(d\); see Fig. 14. The walls at \(x = \pm d/2\) and \(y = l\) are supposed to be rigid, and pressure at \(y = 0\) is supposed to be given as

\[
\tilde{p}(x, y = 0) = \tilde{p}^{(1)}_0 \sin \left( \frac{\pi x}{d} \right),
\]

(C1)

Let us assume that the acoustic pressure in the channel has the form

\[
\tilde{p}(x, y) = \tilde{p}_y(y) \sin \left( \frac{\pi x}{d} \right),
\]

(C2)

which satisfies the rigid-wall boundary conditions at \(x = \pm d/2\).

Substitution of Eq. (C2) into the Helmholtz equation results in

\[
\frac{d^2 \tilde{p}_y}{dy^2} - \alpha^2 \tilde{p}_y = 0, \quad \text{where} \quad \alpha = \sqrt{\frac{\pi^2}{d^2} - k^2}
\]

(C3)

is supposed to be real. The solution of Eq. (C3) reads

\[
\tilde{p}^{(1)}_0 \sin \left( \frac{\pi x}{d} \right)
\]

\(y\)

\(\frac{d}{2}\)

\(x\)

FIG. 14. (Color online) A side-branch QWR.
\[ \tilde{p}_y = Ae^{\gamma y} + Be^{-\gamma y}, \]

the rigid-wall boundary condition at \( y = l \) is satisfied if \( B = \exp(2\pi l) \), so that (normalized) Eq. (C2) can be written as

\[ \tilde{p}(x, y) = \tilde{p}_0 \frac{\exp(z \gamma y) + \exp(z(2\gamma - y))}{1 + \exp(2\pi l)} \sin \left( \frac{\pi x}{d} \right), \]  \hspace{1cm} (C4)

which satisfies the boundary condition given by Eq. (C1). Employing the linearised Euler equation, the \( y \)-component of the acoustic velocity vector can be calculated as

\[ \tilde{v}(x, y) = -\frac{1}{\rho_0 c_0} \frac{\partial \tilde{p}}{\partial y}, \]

\[ \tilde{v}(x, y) = \frac{izp_0^{(1)} e^{\gamma y} - e^{z(2\gamma - y)}}{\rho_0 c_0} \sin \left( \frac{\pi x}{d} \right). \]  \hspace{1cm} (C5)

Substitution of \( y = 0 \) into Eq. (C5) results in

\[ \tilde{v}(x, y = 0) = \tilde{v}_0^{(1)} \sin \left( \frac{\pi x}{d} \right), \]

\[ \text{where} \quad \tilde{v}_0^{(1)} = -\frac{izp_0^{(1)}}{\rho_0 c_0} \tanh(\alpha l). \]  \hspace{1cm} (C6)

From here, we can write

\[ \tilde{p}_0^{(1)} = Z_v^{(1)} \tilde{v}_0^{(1)}, \quad \text{where} \quad Z_v^{(1)} = i\rho_0 c_0 \frac{k}{\alpha} \coth(\alpha l). \]  \hspace{1cm} (C7)

In a limit, we get

\[ \lim_{d \to \infty} Z_v^{(1)} = i\rho_0 c_0 k \lim_{d \to \infty} \frac{\coth \left( \sqrt{\gamma^2 + k^2 l} \right)}{\sqrt{\gamma^2 + k^2 l}} \]

\[ = -i\rho_0 c_0 \cot(kl) = Z_v^{(0)}. \]  \hspace{1cm} (C8)

---


