Waves in Random and Complex Media

Acoustic response of a periodic distribution of macroscopic inclusions within a rigid frame porous plate

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Acoustic response of a periodic distribution of macroscopic inclusions within a rigid frame porous plate

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The acoustic response (in particular, the transmission) of a periodic distribution of macroscopic inclusions within a rigid frame porous plate (similar to a sonic crystal) is studied by the multipole method. Numerical results show that the addition of grating stacks leads to bandgaps within the audible frequency range for a small number of stacks, this being associated with a large decrease of the transmission coefficient of the initial plate. The first bandgap is of practical interest for noise shielding, i.e. very low transmission. The second bandgap enables total acoustic absorption within a narrow frequency range due to the fact that a modified mode of the plate lies within this bandgap.

1. Introduction

This work was initially motivated by the study of multiscattering of acoustic waves in porous materials. This subject was given much attention in homogeneous or layered porous materials after the first work of Biot \cite{1,2} and later contributions \cite{3–5}. Recently \cite{6} the equation that describes acoustic wave propagation in a macroscopically inhomogeneous (whose spatial properties vary in a discontinuous as well as continuous manner) rigid frame porous medium was derived from the alternative formulation of Biot’s theory \cite{2}.

Multiscattering in rigid frame porous materials was initiated in \cite{7} whose authors considered the transmission of an acoustic wave through a porous medium in which randomly-arranged small-sized metallic rods are imbedded transversally. This medium was converted, by a procedure called ISA\textbeta, into an equivalent homogeneous medium, and shown to exhibit decreased transmission. This finding, together with evidence of increased optical absorption in granular media (first studied by Wood \cite{8}, and partially explained in Wirgin and Lopez-Rios \cite{9} by appealing to the ideas of Cutler \cite{10}), led Groby et al. \cite{11} to investigate, by the so-called multipole method, the acoustic properties of a porous plate in which is imbedded a periodic set of scatterers (grating), each of whose size is not small compared with the wavelength. The system exhibits decreased transmission and increased absorption for high contrast scatterers (Neumann-type boundary condition) at frequencies near and beyond the natural frequency of the first modified mode of the plate.

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Fibre-reinforced porous matrix composites are increasingly used in situations involving dynamic loading, where the evaluation of wave propagation will be a main concern. The influence of porosity on the dynamic behaviour of composite materials has been the focus of much research in recent years mainly in connection with the non-destructive evaluation (NDE) of porous fibre-reinforced composite materials [12, 13]. The analytical computation of the far-field radial and shearing stress amplitudes by a pair of rigid or elastic fibres embedded in a porous matrix insonified by a plane compressional or shear wave has been recently been carried out [14]. The effect of visco-thermal losses in a 2D regular square array of rigid cylinders embedded in air was investigated in [15] where it was shown that an accurate prediction of the transmission coefficient outside the bandgaps requires accounting for the visco-thermal losses, whereas transmission within the bandgaps is mainly due to multidiffusion effects.

Sonic crystals should have a full bandgap, a complete bandgap, or an acoustic bandgap, wherein acoustic plane waves cannot propagate in any direction at all. Since the first theoretical investigation [16] of the possibility of a full bandgap in periodic elastic composites [17], sonic and phononic crystals have been widely investigated experimentally and theoretically particularly as concerns the bandgap calculation for various geometries and host/scatterers media couples [18–22], for guiding acoustic waves [23, 24] or for obtaining negative refraction of acoustic waves [25].

Herein, we study, in an acoustical setting, the influence on the hemispherical transmission, hemispherical reflection and absorption of the introduction of periodic sets of fluid- or solid-like circular cylinders (grating stacks) into a plate filled with a macroscopically-homogeneous porous medium which is homogenized by means of the equivalent fluid model. The media, including the cylinders, filling such plates constitute sonic-like crystals. A drastic decrease of the transmission coefficient is to be expected within the frequency range of the bandgaps. Rigorously, we would not expect to find any ‘real’ full bandgap in such structures because porous media are strongly dissipative, and also because the medium is infinite in only one direction, the other direction being finite (the thickness of the plate) and relatively small for practical purposes.

Since the properties of the rigid frame porous medium are close to those of air, we shall use the sonic crystal parameters available in the literature [20,24]. The finite thickness of the plate in which the sonic crystal is imbedded is accounted for in this study. In fact, the plate possesses its own modes which are modified by the presence of the sonic crystal. This results in the creation of so-called modified plate modes, which result from a complex interaction between the plate and the sonic-crystal and which influence the transmission within some bandgaps.

2. Formulation of the problem

Both the incident plane acoustic wave and the plate are assumed to be invariant with respect to the Cartesian coordinate $x_3$. A sagittal $x_1-x_2$ plane view of the 2D scattering problem is given in Figure 1.

Before the introduction of the cylindrical inclusions, the plate is made of a porous material (e.g., a foam) which is modelled (by homogenization) as a (macroscopically-homogeneous) equivalent fluid $M^{[1]}$. Another equivalent fluid material or an elastic solid $M^{[2]}$ occupies each cylindrical inclusion. In the sagittal plane, the unit cell is composed of $N$ cylinders, such that the $j$th cylinder is the circular disk $\Omega^{[2]}_{[2j]}$. The set of indices by which the cylinders within the unit cell are identified is denoted by $\mathcal{N} \in \mathbb{N}$. Two subspaces $\Omega^{[1]}_{[\pm]} \in \Omega^{[1]}$ are also created, respectively, corresponding to the upper and lower part of the plate not containing inclusions. The host media $M^{[0]}$ and $M^{[3]}$ occupy the two half-spaces $\Omega^{[0]}$ and $\Omega^{[3]}$, respectively. Thus, we are dealing with a macroscopically-inhomogeneous plate, the heterogeneity being periodic in the $x_1$ direction with period $d$. The general basis vector of the lattice is $e^1 = (d, 0)$. The vector
Figure 1. Sagittal plane representation of the configuration of plane wave solicitation of a $d$-periodic fluid-like porous plate containing inclusions (of radius $R$).

$$e^2 = (e_1^2, e_2^2) = \|e^2\|(\cos(\beta), \sin(\beta))$$

Together with the vector $e^1$ would describe the general lattice vector in the case of sonic crystals of infinite spatial extent.

The upper and lower flat, mutually-parallel, boundaries of the plate are designated by $\Gamma_a$ and $\Gamma_b$, respectively. The $x_2$ coordinates of these lines are $a$ and $b$, the thickness $L$ of the plate being $L = a - b$. The circular boundary of $\Omega_{1[2]}$ is $\Gamma_{(j)}$. The centre of the $j = 0$ disk is at the origin $O$ of the laboratory system $Ox_1x_2x_3$.

The wavevector $k^j$ of the incident plane wave lies in the sagittal plane and the angle of incidence is $\theta^i$ measured counterclockwise from the positive $x_1$ axis. Usually, such problems are solved by the transfer matrix method, with the multipole method in the background [26–28]. Here, a slightly different method, derived from [11], is followed, in the sense that the multipole method is used as the main tool together with the transfer matrix method in the background. The reflected and transmitted fields clearly split into the usual reflected and transmitted fields by the plate in the absence of the grating stack and the diffracted reflected and diffracted transmitted fields linked to the presence of the grating stacks.

3. Wave equations

We designate the total pressure, wavenumber and wave speed by the generic symbols $p$, $k$ and $c$, respectively, with $p = p^{[0]}$, $k = k^{[0]} = \omega/c^{[0]}$ in $\Omega_{[0]}$, $p = p^{[1]}$, $k = k^{[1]} = \omega/c^{[1]}$ in $\Omega_{[1]}$, $p = p^{[2]}$, $k = k^{[2]} = \omega/c^{[2]}$ in $\Omega_{[2]}$ and $p = p^{[3]}$, $k = k^{[3]} = \omega/c^{[3]}$ in $\Omega_{[3]}$.

Rather than solve directly for the pressure $\overline{p}(x, t)$ (with $x = (x_1, x_2)$), we prefer to deal with $p(x, \omega)$, related to $\overline{p}(x, t)$ by the Fourier transform:

$$\overline{p}(x, t) = \int_{-\infty}^{\infty} p(x, \omega)e^{-i\omega t} d\omega. \quad (1)$$
Henceforth, we drop the $\omega$ in $p(x, \omega)$ so as to designate the latter by $p(x)$. This function satisfies the Helmholtz equations

$$\left[ \triangle + (k_{m1}^2) \right] p(x) = 0; \quad x \in \Omega_{[m]}, \quad m = 0, 1, 2, 3. \quad (2)$$

In medium $M^{[1]}$, the formulation derived for the macroscopically inhomogeneous equivalent media in [6] is preferred to the one dedicated to macroscopically homogeneous media [3] because it accounts for the continuity conditions at the interface between the air medium and the porous plate without the help of a jump condition.

The compressibility and density, linked to the sound speed through $c^{[1]} = (1/(K^{[1]} \rho^{[1]}))^{1/2}$, are

$$\frac{1}{K^{[1]}} = \frac{\gamma P_0}{\phi (\gamma - (\gamma - 1)(1 + i \frac{\omega _G}{Pr} G(Pr\omega))^{-1})},$$

$$\rho^{[1]} = \frac{\rho_f \alpha_\infty}{\phi} \left( 1 + i \frac{\omega _G}{\omega} F(\omega) \right)$$

wherein $\omega_c = \sigma \phi/((\rho_f \alpha_\infty)$ is the Biot characteristic frequency, $\gamma$ the specific heat ratio, $P_0$ the atmospheric pressure, $Pr$ the Prandtl number, $\rho_f$ the density of the fluid in the (interconnected) pores, $\phi$ the porosity, $\alpha_\infty$ the tortuosity, and $\sigma$ the flow resistivity. The correction functions $G(Pr\omega)$ [29], $F(\omega)$ [5] are given by

$$G(Pr\omega) = \left( 1 - 4i \frac{\eta \rho_f \alpha_\infty^2}{\sigma^2 \phi^2 \Lambda^2} Pr\omega \right)^{1/2},$$

$$F(\omega) = \left( 1 - 4i \frac{\eta \rho_f \alpha_\infty^2}{\sigma^2 \phi^2 \Lambda^2} \omega \right)^{1/2} \quad (4)$$

where $\eta$ is the viscosity of the fluid, $\Lambda'$ the thermal characteristic length, and $\Lambda$ the viscous characteristic length.

The incident wave propagates in $\Omega_{[0]}$ and is expressed by

$$p^i(x) = A^i e^{i(k_1(x_1-k_2^{[0]})(x_2-a))} \quad (5)$$

wherein $k_1 = -k^{[0]} \cos \theta^i$, $k_2^{[0]} = k^{[0]} \sin \theta^i$ and $A^i = A^i(\omega)$ is the signal spectrum.

The new feature, with respect to the canonical case considered in [11], is the fact that the unit cell contains more than one inclusion.

The plane wave nature of the incident wave, and the periodic nature of $\bigcup_{j \in \mathcal{N}} \Omega_{[2^j]}$ imply the Floquet relation

$$p((x_1 + nd, x_2)) = p((x_1, x_2)) e^{i k_{n1} d}; \quad \forall x \in \mathbb{R}^2; \quad \forall n \in \mathbb{Z}. \quad (6)$$

Consequently, it suffices to examine the field in the central cell of the plate which includes the disks $\Omega_{[2^j]}$, $j \in \mathcal{N}$ in order to obtain the fields, via the Floquet relation, in the other cells.
4. Boundary and radiation conditions

Since $M^{[0]}$ and $M^{[1]}$ are fluid-like, the pressure and normal velocity are continuous across the interfaces $\Gamma_a$ and $\Gamma_b$:

\begin{align}
\rho^{[0]}(x) - \rho^{[1]}(x) &= 0, \quad \forall x \in \Gamma_a \\
\rho^{[1]}(x) - \rho^{[1]}(x) &= 0, \quad \forall x \in \Gamma_b,
\end{align}

wherein $\mathbf{n}$ denotes the generic unit vector normal to a boundary and $\partial_n$ designates the operator $\partial_n = \mathbf{n} \cdot \nabla$.

The boundary conditions across the interfaces $\Gamma_{(j)}$, $j \in \mathcal{N}$ depend on the type of inclusions, i.e. either elastic or fluid-like. If $M^{[2]}$ is fluid-like, the pressure and normal velocity are continuous across the interfaces $\Gamma_{(j)}$:

\begin{align}
\rho^{[2]}(x) - \rho^{[1]}(x) &= 0, \quad \forall x \in \Gamma_{(j)}, \quad j \in \mathcal{N},
\end{align}

whereas if $M^{[2]}$ is elastic, the traction and normal component of the displacement are continuous across the interfaces $\Gamma_{(j)}$:

\begin{align}
\sigma^{[2]}(x) \cdot \mathbf{n} + p^{[1]}(x) \mathbf{n} &= 0, \quad \forall x \in \Gamma_{(j)}, \quad j \in \mathcal{N},
\end{align}

When the contrast between the media $M^{[1]}$ and $M^{[2]}$ is high, i.e. high contrast inclusions, the latter can be approximated as infinitely rigid. In this case, the boundary conditions across the interface $\Gamma_{(j)}$ no longer depend on the fields inside the inclusions, i.e. on the internal geometry of $\Omega_{(j)}$ and on the material characteristics of $M^{[2]}$ (except its rigid behaviour), and reduce to Neumann type boundary conditions:

\begin{align}
\partial_n p^{[1]}(x) = 0, \quad \forall x \in \Gamma_{(j)}, \quad j \in \mathcal{N}.
\end{align}

The uniqueness of the solution to the forward-scattering problem is assured by the radiation conditions:

\begin{align}
p^{[0]}(x) - p'(x) &\sim \text{outgoing waves}; \quad |x| \to \infty, \quad x_2 > a, \\
p^{[3]}(x) &\sim \text{outgoing waves}; \quad \forall |x| \to \infty, \quad x_2 < b.
\end{align}

5. Field representations in $\Omega_0$, $\Omega_3$ and $\Omega_{1\pm}$

We first consider the equations of continuity across the interfaces $\Gamma_a$ and $\Gamma_b$, so that the field representation in $\Omega_{[0]}$, $\Omega_{[3]}$ and $\Omega_{1\pm}$ are needed as the first step. The continuity conditions across
\[ p^{[0]}(x) = \sum_{p \in \mathbb{Z}} \left[ A_{p} e^{-i k_{1p}^2 (x-x_0)} + R_{p} e^{i k_{2p}^2 (x-x_0)} \right] e^{i k_{1p} x_1}, \quad \forall x \in \Omega_{[0]}, \quad (18) \]

\[ p^{[3]}(x) = \sum_{p \in \mathbb{Z}} T_{p} e^{i k_{3p} (x-x_3)}, \quad \forall x \in \Omega_{[3]} \quad (19) \]

where \( \delta_p \) is the Kronecker symbol, \( k_{1p} = k_1^2 + \frac{2 \pi p}{a} \), \( k_{2p}^2 = ((k_{1p})^2 - (k_{1p})^2)^{1/2} \), with \( \text{Re}(k_{2p}^2) \geq 0 \) and \( \text{Im}(k_{2p}^2) \geq 0 \). 

It is convenient to use Cartesian coordinates \((x_1, x_2)\) to write the field representations in \( \Omega_{[1]}. \) Both fields are composed of the diffracted field in the plate \( p_{s}^{[1]}(x) \) and the fields diffracted by the other inclusions \( p_{d}^{[1\pm]}(x) \), whose forms depend on the position of \( x \), either below or above the inclusions [30].

Because of the periodic nature of the configuration, the diffracted field in the plate can be written in Cartesian coordinates as:

\[ p_{s}^{[1]}(x) = \sum_{p \in \mathbb{Z}} \left( f_{p}^{[1]} e^{-i k_{2p} x_2} + f_{p}^{[1]} e^{i k_{2p} x_2} \right) e^{i k_{1p} x_1}, \quad \forall x \in \Omega_{[1]}, \quad (20) \]

where \( k_{2p}^2 = ((k_{1p})^2 - (k_{1p})^2)^{1/2} \), with \( \text{Re}(k_{2p}^2) \geq 0 \) and \( \text{Im}(k_{2p}^2) \geq 0 \). 

The field diffracted by the inclusion is expressed, in the Cartesian coordinate system \( x_j = (x_{1(j)}, x_{2(j)}) \) linked to each cylinder, by

\[ p_{d}^{[1]}(x) = \sum_{j \in \mathbb{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} K_{pl}^{\pm(j)} B_{l}^{(j)} e^{i k_{j} x_j} e^{i k_{2p} x_2}, \quad \forall x_j \in \Omega_{[1]}, \quad (21) \]

wherein the signs + and - correspond to \( x_{2(j)} > R \) and \( x_{2(j)} < -R \), respectively, \( B_{l}^{(j)} \) are the coefficients of field scattered by the \( j \)-th cylinder of the unit cell, and \( K_{pl}^{\pm(j)} = \frac{2(1-i)^{m}}{d k_{2p}^{\pm(j)}} e^{i m \theta_p} \) with \( \theta_p \) such that \( k_{1p}^2 e^{i \theta_p} = k_{1p} + i k_{2p}^2 \) [30, 31]. In particular, for \( x_j \in \Omega_{[1+]} \) (respectively, \( x_j \in \Omega_{[1-]} \)), \( x_{2(j)} \) is greater than \( R \) \( \forall j \in \mathbb{N} \) (respectively, \( x_{2(j)} \) is smaller than \( -R \) \( \forall j \in \mathbb{N} \)). Dropping the \((j)\) in \( K_{pl}^{\pm(j)} \), and transforming the local Cartesian coordinate to the global Cartesian coordinates by means of \( x_{1(j)} = x_{1} - j e_1 \) and \( x_{2(j)} = x_{2} - j e_2 \), leads to:

\[ p_{d}^{[1\pm]}(x) = \sum_{j \in \mathbb{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} K_{pl}^{\pm} B_{l}^{(j)} e^{i k_{j} (x_{1} - j e_1)} e^{i k_{2p} x_2}, \quad \forall x_j \in \Omega_{[1\pm]}, \quad (22) \]

The field representation in \( \Omega_{[1\pm]} \) finally takes the form:

\[ p^{[1\pm]}(x) = \sum_{p \in \mathbb{Z}} \left( f_{p}^{[1]} e^{-i k_{2p} x_2} + f_{p}^{[1]} e^{i k_{2p} x_2} \right) e^{i k_{1p} x_1} + \sum_{j \in \mathbb{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} K_{pl}^{\pm} B_{l}^{(j)} e^{i k_{j} (x_{1} - j e_1)} e^{i k_{2p} x_2}, \quad \forall x \in \Omega_{[1\pm]}, \quad (23) \]
6. Determination of the unknowns

6.1. Application of the continuity conditions across $\Gamma_a$ and $\Gamma_b$

Applying successively $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(k_{1n}-k_{1b})x_1} dx_1$ to the continuity of the pressure field and of the normal component of the velocity across $\Gamma_a$ and $\Gamma_b$, Equations (7)–(10), introducing the appropriate field representation therein, Equations (18), (19) and (23), and making use of the orthogonality relation

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(k_{1n}-k_{1b})x_1} dx_1 = d\delta_{nl}, \quad \forall (l, n) \in \mathbb{Z}^2, \quad (24) \]

give rise to the system of linear equations

\[ A^i \delta_p + R_p - f_p^{[1]} e^{-i k_p^{[1]} a} - f_p^{[1]} e^{i k_p^{[1]} a} - \sum_{j \in \mathcal{N}} \sum_{l \in \mathbb{Z}} k_{pl} B_l^{(j)} e^{-i k_{pl} e_1 + i k_{pr} (a - j e_2)} = 0, \quad (25a) \]

\[ -\alpha_p^{[0]} A^i \delta_p + \alpha_p^{[0]} R_p + \alpha_p^{[1]} f_p^{[1]} e^{-i k_p^{[1]} a} - \alpha_p^{[1]} f_p^{[1]} e^{i k_p^{[1]} a} - \alpha_p^{[1]} \sum_{j \in \mathcal{N}} \sum_{l \in \mathbb{Z}} k_{pl} B_l^{(j)} e^{-i k_{pl} e_1 + i k_{pr} (a - j e_2)} = 0, \quad (25b) \]

\[ T_p - f_p^{[1]} e^{-i k_p^{[1]} a} - f_p^{[1]} e^{i k_p^{[1]} a} - \sum_{j \in \mathcal{N}} \sum_{l \in \mathbb{Z}} k_{pl} B_l^{(j)} e^{-i k_{pl} e_1 - i k_{pr} (b - j e_2)} = 0, \quad (25c) \]

\[ -\alpha_p^{[1]} T_p + \alpha_p^{[1]} f_p^{[1]} e^{-i k_p^{[1]} a} - \alpha_p^{[1]} f_p^{[1]} e^{i k_p^{[1]} a} - \alpha_p^{[1]} \sum_{j \in \mathcal{N}} \sum_{l \in \mathbb{Z}} k_{pl} B_l^{(j)} e^{-i k_{pl} e_1 - i k_{pr} (b - j e_2)} = 0, \quad (25d) \]

for the resolution of $R_p$, $T_p$, $f_p^{[1]} \pm$, in terms of $B_m^{(j)}$, $\forall m \in \mathbb{Z}$ and $\forall j \in \mathcal{N}$ and wherein $\alpha_p^{[i]} = k_{2_p}^{[i]} / \rho_{[i]}$, $i = 0, 1, 2, 3$.

6.2. Application of the multipole method

Introducing the expressions of $f_p^{[1]} \pm$ and of $f_p^{[1]} -$ in terms of $B_l^{(j)}$ obtained from (6.1) into (23) leads to

\[ p_{[1]}^{[\pm]}(x) = \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} k_{pl}^{[\pm]} B_l^{(j)} e^{i(k_{pl}(x_1 - j e_1) \pm i k_{pl}^{[1]}(x_2 - j e_2))} \]
\[ + \sum_{p \in \mathbb{Z}} \left( F_p^{[1]} e^{-i k_p^{[1]} x_2} + F_p^{[1]} e^{i k_p^{[1]} x_2} \right) e^{i k_{pl} x_1} \]
\[ - \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} B_l^{(j)} e^{i k_{pl}(x_1 - j e_1) + i k_{pr}^{[1]} L} \left( \alpha_p^{[1]} - \alpha_p^{[3]} \right) \left( \alpha_p^{[0]} - \alpha_p^{[1]} \right) \]
\[ \times \left( K_{pl} e^{-i k_{pl}^{[1]}(x_2 - j e_2)} + K_{pl} e^{-i k_{pl}^{[1]}(x_2 - j e_2)} \right) \]
\[
+ \sum_{j \in J} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} B_l^{(j)} e^{ik_{lp}(x_l-x_{j,l})} \frac{D_p}{(rJ(x_l-x_{j,l}))^m} \left[ (\alpha_p^{(0)} + \alpha_p^{(1)}) (\alpha_p^{(3)} - \alpha_p^{(1)}) K_{n,l}^{-1} e^{-ik_{lp}(a+b)} e^{ik_{lp}(x_l-x_{j,l})} \right]
- (\alpha_p^{(3)} + \alpha_p^{(1)}) (\alpha_p^{(0)} - \alpha_p^{(1)}) K_{n,l}^{+1} e^{ik_{lp}(a+b)} e^{-ik_{lp}(x_l-x_{j,l})} \right]. \] (26)

wherein

\[
D_p = (\alpha_p^{(1)} - \alpha_p^{(3)})(\alpha_p^{(0)} - \alpha_p^{(1)}) e^{ik_{lp} L} + (\alpha_p^{(0)} + \alpha_p^{(1)})(\alpha_p^{(3)} + \alpha_p^{(1)}) e^{-ik_{lp} L} \]

and

\[
F_{1p}^{(1)-} = \frac{2\alpha_p^{(0)} (\alpha_p^{(3)} + \alpha_p^{(1)})}{D_p} e^{ik_{lp} b} A_i \delta_p \]

\[
F_{1p}^{(1)+} = \frac{2\alpha_p^{(0)} (\alpha_p^{(1)} - \alpha_p^{(3)})}{D_p} e^{-ik_{lp} b} A_i \delta_p. \] (27)

Central to the multipole method are the local field expansions or multipole expansions around each inclusion [30–32]. Because \( p(r_j), \forall j \in J \) satisfy a Helmholtz equation inside and outside each cylinder of the unit cell, in the vicinity of the \( J \)-th cylinder we can write

\[
p^{(1)}(r_j) = \sum_{m \in \mathbb{Z}} B_m^{(J)} H_m^{(1)} (k^{(1)} r_j) e^{im \theta_j} + \sum_{m \in \mathbb{Z}} A_m^{(J)} H_m (k^{(1)} r_j) e^{im \theta_j}, \forall J \in J, \] (28)

wherein \( A_m^{(J)} \) are the coefficients of the locally-incident field to the \( J \)-th cylinder, \( H_m^{(1)} \) the \( m \)-th order Hankel function of first kind and \( J_m \) the \( m \)-th order Bessel function.

To proceed further, we first convert the Cartesian form to the cylindrical harmonic form in the coordinate system linked to a generic cylinder \( J \in J \) by means of \( x_{1,l} = x_1 - Je_1 \) and \( x_{2,l} = x_2 - Je_2 \), and then transform the Cartesian coordinates to polar coordinates by means of \( x_{1,l} = r_j \cos(\theta_j) \) and \( x_{2,l} = r_j \sin(\theta_j) \) together with the identity \( e^{ikr \cos(\theta)} = \sum_{m \in \mathbb{Z}} (i)^m J_m(kr) e^{im \theta} \):

\[
p^{(1)}(r_j) = \sum_{m \in \mathbb{Z}} B_m^{(J)} H_m^{(1)} (k^{(1)} r_j) e^{im \theta_j} + \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} S_{m-l} B_l^{(J)} J_m (k^{(1)} r_j) e^{im \theta_j} + \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} J_{m,l}^{(1)-} e^{-ik_{lp}(x_l-x_{j,l})} + J_{m,l}^{(1)+} e^{ik_{lp}(x_l-x_{j,l})} \] \[
+ \sum_{j \in J} \sum_{m \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} J_{m,l}^{(j),+} e^{ik_{lp}(x_l-x_{j,l})} J_m (k^{(1)} r_j) e^{im \theta_j}, \forall J \in J, \] (29)

with \( J_{m,l}^{(1)} = i^m e^{(1)m \theta} \) and

\[
S_{m-l} = \sum_{i=1}^\infty H_{m-j}^{(1)}(k^{(1)} i d) \left[ e^{ik_{lp}(x_l-x_{j,l})} + (-1)^{m-l} e^{-ik_{lp}(x_l-x_{j,l})} \right].
\]

\[
Q_{m,l}^{(j)} = \frac{2(-i)^{j-m} e^{ik_{lp}(J-j)e_1}}{dk_{2p} D_p} \times \left[ (\alpha_p^{(0)} + \alpha_p^{(1)}) (\alpha_p^{(3)} - \alpha_p^{(1)}) e^{-ik_{lp}(a+b-(J-j)e_2-\hat{l}(l+m)\theta_p)} \right].
\]
Appendix A.

By identifying (30) with (28), we find

$$A_J^l = \sum_{p \in \mathbb{Z}} \left( J_{mp} F_p^{[1]} e^{-i k_{2p} I e_1} + J_{lp} F_p^{[1]} e^{i k_{2p} I e_1} \right) e^{i k_{1p} J e_1} + \sum_{l \in \mathbb{Z}} S_{m-l} B_l^{(j)} + \sum_{j \in \mathcal{N}} \sum_{l \in \mathbb{Z}} \left( Q_{mlp}^{(j,j)} + G_{mlp}^{(j,j)} - B_{mlp}^{(j,j)} \right) B_l^{(j)}, \quad \forall J \in \mathcal{N}. \quad (31)$$

At this point, we account for Equations (11) and (12), (13) and (14), or (15). It is well-known that the coefficients of the scattered field and those of the locally-incident field are linked by a matrix relation depending on the parameters of the cylinder only, i.e.,

$$B_m = V_m A_m, \quad \forall m \in \mathbb{Z} \quad (32)$$

wherein $V_m$ are the cylindrical harmonic reflection coefficients whose expressions are derived in Appendix A.

Denoting by $B$ the infinite column matrix of components $B_m^{(j)}$, (31) together with (32) may be written in the matrix form

$$(I - VS - V(Q + G - P))B = VF, \quad (33)$$

with $F$ the column matrix of element $\sum_{p \in \mathbb{Z}} J_{mp} F_p^{1} e^{-i k_{2p} J e_2} + J_{lp} F_p^{1} e^{i k_{2p} J e_2}$, $I$ the identity matrix, $V$ the diagonal matrix of components $V_m$ and $S$, $G$, $Q$ and $P$ four square matrices of elements, $S_{m-l} \delta_{jj}$, $G_{mlp}^{(j,j)}$, $Q_{mlp}^{(j,j)}$ and $B_{mlp}^{(j,j)}$, respectively.

The elements of the matrix $S$, called the Schlömilch series and often referred to as lattice sums, account for the field scattered by the cylinders of the $J$-th array excepting the $J$-th cylinder. The elements of the matrix $G$ account for the field scattered by the other arrays, and the elements of the vector $F$ account for the field scattered by the plate. These three terms deal with fields that are directly incoming to the $J$-th cylinder whereas the elements of the matrices $Q$ and $P$ account for the field scattered by all the inclusions which are then reflected by the plate and finally incoming to the $J$-th cylinder.
7. Evaluation of the transmitted and reflected fields

From the linear system (25) we obtain expressions for \( R_p \) and \( T_p \) in terms of \( B_{l}^{(j)} \):

\[
R_p = \frac{A}{D_0} \left[ (\alpha^{[1][j]} + \alpha^{[0][j]})(\alpha^{[1][j]} - \alpha^{[3][j]}) e^{i k_z^{[1][j]} L} \right.
\]
\[
- (\alpha^{[1][j]} - \alpha^{[0][j]})(\alpha^{[3][j]} + \alpha^{[1][j]}) e^{- i k_z^{[1][j]} L} \]
\[
+ \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{8 \alpha^{[1][j]}(-i m) B_{l}^{(j)} e^{-i k_{1p} j e_1}}{d k_{2p} D_p} \times \left( i \alpha^{[1][j]} \sin \left( m \theta_p + k_{2p}(b - j e_2) \right) + \alpha^{[1][j]} \cos \left( m \theta_p + k_{2p}(b - j e_2) \right) \right),
\]
\[
T_p = \frac{4A^i \alpha^{[1][j]} \alpha^{[0][j]} L_0^2}{D_0} + \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \frac{8 \alpha^{[1][j]}(-i m) B_{l}^{(j)} e^{-i k_{1p} j e_1}}{d k_{2p} D_p} \times \left( -i \alpha^{[1][j]} \sin \left( m \theta_p + k_{2p}(a - j e_2) \right) + \alpha^{[1][j]} \cos \left( m \theta_p + k_{2p}(a - j e_2) \right) \right). \quad (34)
\]

After introduction into (18), the expression of the pressure field in \( \Omega_{[0]} \) becomes:

\[
p_{[0]}(x) = A^i e^{i k_{z_1} x_1 - i k_{z_2}^{[0]}(x_2-a)}
\]
\[
+ \frac{A^i e^{i k_{z_1} x_1 + i k_{z_2}^{[0]}(x_2-a)}}{D_0} \left[ (\alpha^{[1][j]} + \alpha^{[0][j]})(\alpha^{[1][j]} - \alpha^{[3][j]}) e^{i k_z^{[1][j]} L} \right.
\]
\[
- (\alpha^{[1][j]} - \alpha^{[0][j]})(\alpha^{[3][j]} + \alpha^{[1][j]}) e^{- i k_z^{[1][j]} L} \]
\[
+ \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} \frac{8 \alpha^{[1][j]}(-i m) B_{l}^{(j)} e^{i k_{1p} j e_1 + i k_{z_2}^{[0]}(x_2-a)}}{d k_{2p} D_p} \times \left( i \alpha^{[1][j]} \sin \left( m \theta_p + k_{2p}(b - j e_2) \right) + \alpha^{[1][j]} \cos \left( m \theta_p + k_{2p}(b - j e_2) \right) \right), \quad (35)
\]

and, after introduction into (19), the expression of the pressure field in \( \Omega_{[3]} \) takes the form

\[
p_{[3]}(x) = \frac{4A^i \alpha^{[1][j]} \alpha^{[0][j]} e^{i k_{z_1} x_1 - i k_{z_2}^{[0]}(x_2-b)}}{D_0}
\]
\[
+ \sum_{j \in \mathcal{N}} \sum_{p \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \frac{8 \alpha^{[1][j]}(-i m) B_{l}^{(j)} e^{i k_{1p} j e_1 - i k_{z_2}^{[0]}(x_2-b)}}{d k_{2p} D_p} \times \left( -i \alpha^{[1][j]} \sin \left( m \theta_p + k_{2p}(a - j e_2) \right) + \alpha^{[1][j]} \cos \left( m \theta_p + k_{2p}(a - j e_2) \right) \right). \quad (36)
\]

The reflected and transmitted fields are expressed as a sum of (i) the field in the absence of the inclusions (whose expressions are the same as those in [33]) with (ii) the field due to the presence of the inclusions (whose expressions reduce to those in [11] when \( N = 1 \)).

The procedure described in Section 6 and the expression of both the transmitted and reflected fields have been validated with the help of our finite-element code [34] in the case of a dissipative...
plate in which is imbedded a number of dissipative inclusions large enough for the situation in the central part of the configuration to be very close to that described by the multipole method as applied to a configuration with an infinite number of inclusions.

8. Evaluation of the reflection, transmission and absorption coefficient

The conservation of energy relation, in case of \( N \) arrays of cylinders, takes the form

\[
1 = A^N + R^N + T^N
\]  

with \( A^N, R^N \) and \( T^N \) the absorption, hemispherical reflection and transmission coefficients, respectively defined by

\[
R^N = \sum_{p \in \mathbb{Z}} \frac{\text{Re} \left( \frac{k_{2p}^{[0]}}{k_2^{[0]}} \right)}{k_{2p}^{[0]}} \| R_p \|^2 = \sum_{p = -\infty}^{\infty} \frac{k_{2p}^{[0]}}{k_2^{[0]}} \| R_p \|^2
\]

\[
T^N = \frac{\rho_0^{[0]}}{\rho_0^{[3]}} \sum_{p \in \mathbb{Z}} \frac{\text{Re} \left( \frac{k_{2p}^{[3]}}{k_2^{[0]}} \right)}{k_{2p}^{[0]}} \| T_p \|^2 = \frac{\rho_0^{[0]}}{\rho_0^{[3]}} \sum_{p = -\infty}^{\infty} \frac{k_{2p}^{[3]}}{k_2^{[0]}} \| T_p \|^2
\]

\[
A^N = \frac{1}{d k_2^{[10]}} \text{Im} \int_{\Omega_{[1]}} \frac{\rho_0^{[0]}}{\rho_0^{[1]}} \left( k_{11}^{[1]} \right)^2 \| p^{[1]}(x, \omega) \| d\tilde{\omega}
\]

\[
+ \sum_{j \in N} \frac{1}{d k_2^{[10]}} \text{Im} \int_{\Omega_{[2]}} \frac{\rho_0^{[0]}}{\rho_0^{[2]}} \left( k_{12}^{[2]} \right)^2 \| p^{[2]}(x, \omega) \| d\tilde{\omega}
\]  

wherein \( d\tilde{\omega} \) is the differential element of surface in the sagittal plane, \( \tilde{p}^{(j)} \) is such that \( (k_1 + 2\pi \tilde{p}^{(j)} + 1)^2 > (k_1 + 2\pi \tilde{p}^{(j)})^2 \) for \( j = 0, 3 \), and the expression of \( R_p, T_p \) are given by Equation (34). Because of the complicated shape of \( \Omega_{[1]} \), \( A^N \) will not be calculated by the expression given in (38), but rather by \( A^N = 1 - R^N - T^N \).

In addition, as the plate is filled with a porous medium, dissipation effects occur even in the absence of the inclusions. For the sake of comparison and to quantify the influence of the sonic crystal, we also introduce the relative absorption, hemispherical reflection and transmission coefficients \( A^{(N, 0)} = A^N / A^0, R^{(N, 0)} = R^N / R^0 \) and \( T^{(N, 0)} = T^N / T^0 \), respectively.

9. Numerical results and discussion

The ambient and saturating fluid is air \( (\rho_0^{[0]} = \rho_0^{[3]} = \rho_f = 1.213 \text{ kg m}^{-3}, c_0^{[0]} = c_0^{[3]} = (\gamma P_0/\rho_0)^{1/2}) \), with \( P_0 = 1.01325 \times 10^5 \text{ Pa} \) and \( \gamma = 1.4, \eta = 1.839 \times 10^{-5} \text{ kg m}^{-1} \text{ s}^{-1} \). The layer (of infinite lateral extent) is filled with a polymer foam \( M^{[1]} \), characterized by \( \phi = 0.96, \alpha_{\infty} = 1.07, \Lambda = 273 \times 10^{-6} \text{ m}, \Lambda' = 672 \times 10^{-6} \text{ m}, \sigma = 2843 \text{ N s m}^{-4} \). In what follows, and for all the configurations considered, the plate is symmetric with respect to the abscissa \( x_2 = (a + b)/2 \).
9.1. Numerical recipes

The infinite sum $\sum_{m \in \mathbb{Z}}$ over the indices of the modal representation of the field diffracted by a cylinder is truncated as $\sum_{m=-M}^{M}$ such that

$$M = \text{int} \left( \text{Re} \left( 4.05 \times (k^1 R)^{1/3} + k^1 R \right) \right) + 10, \quad (39)$$

with 10 a security term [35]. The infinite sum $\sum_{p \in \mathbb{Z}}$ over the indices of the $k_{1p}$ is found to depend on the frequency and on the period of the grating. "We approximate the infinite sum by the finite $\sum_{P=-P}^{P}$, with $P$ defined by":

$$P = \text{int} \left( \frac{d}{2\pi} \left( 3\text{Re}(k^{[1]}) - k^{[1]} \right) \right) + 5. \quad (40)$$

In the latter equations, $\text{int}(a)$ represents the integer part of $a$. Considering the foam plate without dissipation, $k_{2p}^{[1]}$ is the last vertical wavenumber to become purely imaginary when $p$ increases. The previous numerical rule also ensures that $k_{2p}^{[1]} = i2^{1/2} \|k^{[1]}\|$ (non-dissipative case) with an added security term equal 5.

The infinite sum in Equation (30), $\sum_{i=1}^{\infty}$ (lattice sum), is found to be slowly convergent, particularly in the absence of dissipation, and to be strongly dependent on the indices $m-l$. A large literature exists on this problem [36,37]. Here, the fact that the medium $M^{[1]}$ is dissipative, greatly simplifies the evaluation of the Schlömilch series. Let us denote by the superscript $I$ in $S_{m-l}^{I}$ the upper limit of the sum $\sum_{i=1}^{I}$. The latter sum is computed until the condition

$$\left| \text{Re} \left( \frac{S_{m-l}^{I+1} - S_{m-l}^{I}}{S_{m-l}^{I}} \right) \right| \leq 10^{-5} \quad \text{and} \quad \left| \text{Im} \left( \frac{S_{m-l}^{I+1} - S_{m-l}^{I}}{S_{m-l}^{I}} \right) \right| \leq 10^{-5} \quad (41)$$

is reached.

9.2. Aluminium and acrylic inclusions

The existence of a pronounced sound attenuation band [38] is strongly connected with a large acoustic impedance ratio between the scatterers and the matrix material. The bandwidth and depth vary with the density of the scatterers inside the sonic crystal. The centre frequencies are always given by the Bragg condition $\bar{\nu}(n) = nc/(2d), n \in \mathbb{N}$, where $c$ is the longitudinal sound velocity of the matrix material and $d$ the lattice constant. Rather than calculating the structure of the bandgaps (for example by use of the so-called plane wave method [22]), we employ the sonic crystal parameters available in the literature for air matrix materials, because the mechanical properties of a rigid-frame porous medium are close to those of the air medium.

From [24], it is well known (experimentally) that a square lattice of aluminium ($c^{[2]}_L = 6374$ m s$^{-1}$ and $\rho^{[2]}_L = 2698$ kg m$^{-3}$) rods in air possesses a full bandgap between 15.1 kHz and 18.8 kHz for $d = 12$ mm and $R = 5$ mm (i.e. $R/d \approx 0.417$), while from [20], it is known (experimentally as well) that a square lattice of acrylic ($c^{[2]}_L = 2211$ m s$^{-1}$ and $\rho^{[2]}_L = 1103$ kg m$^{-3}$) rods in air possesses a full bandgap between 7 kHz and 9.5 kHz for $d = 24$ mm and $R = 10.2$ mm (i.e. $R/d \approx 0.425$). We note that the centre frequencies of the first bandgap experimentally noticed in both [20, 24] are higher than the Bragg condition. In both cases, the first bandgap stand for frequencies that occur, respectively, in the middle and in the high frequency limit of the audible frequency range (i.e. $\nu \approx [20$ Hz, $20$ kHz]). The bandgap also occurs at frequencies of interest...
for noise suppression. Nevertheless, this assertion has to be moderated by the fact that the major difficulties in the noise problem appear for frequencies that are situated near the lower limit of the audible frequency range at which the rigid frame approximation of the porous medium is no longer valid (i.e. the full Biot model [2] has to be employed) because of the possible vibrations of the frame. Hereafter, we will concentrate on noise transmission and attenuation at the middle and high frequencies of the audible frequency range.

Aluminium and acrylic inclusions are treated herein as fluid inclusions, because the impedance contrast is very high when they are embedded in a rigid frame porous medium. This approximation is validated in Appendix B. The addition of a square lattice of aluminium inclusions to a porous plate defines the so-called type 1 configuration and the addition of a square lattice of acrylic inclusions to a porous plate defines the so-called type 2 configuration.

The design of acoustic materials is of course connected to the acoustic efficiency of the latter over some frequency range, and, in addition, to the thermal efficiency of the system, to constraints in terms of weight, in terms of thickness, depending on its practical use. Avoiding considerations of the thermal efficiency, it is clear that the use of a type 1 or a type 2 configuration is first related to the bandgap. For a constant \( N \), the type 1 configuration \((L = 56 \text{ mm for } N = 4)\) is thinner than the type 2 configuration, this being mainly due to the period of the sonic crystal \((d = 12 \text{ mm for type 1 and } d = 24 \text{ mm for type 2})\), while the density per unit cell (square of edge \(d\)) is larger for type 1 \((\rho^1_\text{eff} = 1472 \text{ kg m}^{-3})\) than for type 2 \((\rho^2_\text{eff} = 626.5 \text{ kg m}^{-3})\). Furthermore, the domains \(\Omega^j_2\) do not have to be completely filled with an elastic (at least a metallic) material for the calculated acoustic response to be accurate, because in this case the boundary condition across \(\Gamma_2\) is dominant; tubes are an example of such inclusions [39]. Effectively, as shown in Appendix B, both acrylic and aluminium inclusions are high contrast inclusions and therefore can be modelled as being infinitely rigid. In practice, and for the frequency range considered in this paper, inclusions involved in both type 1 and type 2 configurations can be made of tubes, whose thickness has to be carefully checked (if it is too thin, the tube can vibrate) and filled with acrylic or sufficiently rigid material for a Neumann type boundary condition to be employed. The result is also a drastic decrease of the weight of the configuration.

Figure 2 depicts the hemispherical transmission, hemispherical reflection and absorption coefficients and their associated relative coefficients for a constant plate thickness \(L = 56 \text{ mm}\) with aluminium inclusions, creating a type 1 configuration, such that \( N \) varies from 0 (plate without inclusions) to 4. The system possesses a bandgap in a frequency range (transmission in case of \( N = 4 \) is less than \(10^{-3}\) over \([8.3 \text{ kHz}, 16 \text{ kHz}]\) that is lower and wider than the one obtained in [24] for an air matrix. This is partly due to the normal incidence of the plane wave that impinges on the plate and partly due to the material of the matrix. The minimum of the hemispherical transmission coefficient varies from \(6.6 \times 10^{-5}\) for \( N = 4\) to \(7 \times 10^{-2}\) for \( N = 1\) within the frequency range of the first bandgap. In particular for \( N = 4\), \(T^4\) is \(2.3 \times 10^{-4}\) times less than the transmission coefficient \(T^0\) at \(12.15 \text{ kHz}\) and is always lower than \(10^{-1}\) for \(\nu \geq 5.5 \text{ kHz}\).

The sound transmission decreases in proportion to the number \( N \) of consecutive scatterers encountered by the incoming acoustic wave, as is usually noticed for sonic crystals. The hemispherical transmission coefficient is approximatively divided by 10 each time a grating is added to the configuration. For \( N \) larger than 2, the minimum of the hemispherical transmission, within the first bandgap, occurs approximately at the same frequency for all \( N \), and this frequency satisfies the Bragg condition. In our case, the longitudinal sound velocity of the matrix material is frequency-dependent and imaginary. \(\nu_{(n)}\) corresponds to the intersection of \(2d\nu/n\) with \(\text{Re}(c^{(1)})\).

Figure 3 depicts \(\nu_{(n)}\) for both rigid-frame porous and air matrix materials: \(\nu_{(1)}^{\text{rf}} \approx 13.2 \text{ kHz}\) (in accordance with the numerical results of Figure 2) for the porous matrix material and \(\nu_{(1)}^{\text{a}} \approx 13.2 \text{ kHz}\) for the air matrix material. The differential in the two curves is due to the dissipation of the acoustic wave that impinges on the plate and partly due to the material of the matrix. The minimum of the hemispherical transmission coefficient varies from \(6.6 \times 10^{-5}\) for \( N = 4\) to \(7 \times 10^{-2}\) for \( N = 1\) within the frequency range of the first bandgap. In particular for \( N = 4\), \(T^4\) is \(2.3 \times 10^{-4}\) times less than the transmission coefficient \(T^0\) at \(12.15 \text{ kHz}\) and is always lower than \(10^{-1}\) for \(\nu \geq 5.5 \text{ kHz}\).
Figure 2. The top panel depicts the transmission (a) and the relative transmission (b) coefficients, the middle panel the reflection (c) and the relative reflection (d) coefficients and the bottom panel the absorption (e) and the relative absorption (f) coefficients over the frequency range \( \nu \in [3 \, \text{kHz}, 22 \, \text{kHz}] \) for a constant plate thickness \( L = 56 \, \text{mm} \) with aluminium inclusions such that \( N \) varies from 0 (plate without inclusions) to 4 creating a type 1 configuration.

14.25 kHz for the air matrix material. As the sound velocity in a rigid-frame porous medium is lower than in the air medium, the centre frequency of the first bandgap is lower for the porous matrix material than for the air matrix material. For a fixed period \( d \), the relative frequency shift is

\[
\Delta \tilde{\nu}_{n,arf} = (\tilde{\nu}_{n,a} - \tilde{\nu}_{n,rf})/\nu_{n,a} = 1 - \text{Re}(c^{[1]}(\nu))/c^{[0]} < 1,
\]

and is all the larger the smaller is \( \text{Re}(c^{[1]}(\nu)) \). This also means that for a fixed centre frequency \( \tilde{\nu}_n \), the size of the configuration can be less for a porous material matrix than for an air material matrix. The relative spatial period shift is also of the form

\[
\Delta d_{arf} = (d_{a} - d_{rf})/d_{a} = 1 - \text{Re}(c^{[1]}(\nu))/c0 < 1
\]

and is all the larger the smaller is \( \text{Re}(c^{[1]}(\nu)) \). The number \( N \) of gratings required for the central frequency of the first bandgap to match the Bragg condition can be associated with the number \( N \) at which a bandgap
is initially created. This number is $N = 3$ and matches the one hypothesized in the study of the approximate dispersion relation of Appendix C.2.

On the other hand, the addition of inclusions induces a large increase of the hemispherical reflection coefficient associated with a large decrease of the absorption coefficient at low frequencies in the neighbourhood of the first bandgap, in particular for $N = 4$. The hemispherical reflection coefficient increases within the bandgap with the number of gratings of the configuration. The large oscillations of $R^{(N,0)}$ for $N \geq 1$ are due to the division of $R^N$ by $R^0$, the latter possessing a large number of minima. The hemispherical reflection coefficient also possess fewer minima in the presence of inclusions than in their absence. In practice, this means that the possibilities of large decreases of the absorption coefficient (partly linked to minima of the hemispherical reflection coefficient) can only occur at a small number of frequencies and for a small frequency range. Outside the frequency range of the first bandgap, the depth of the latter is close to zero around 20 kHz, so as to make the absorption more than 0.9 between 18.3 kHz and 21.3 kHz. This peak cannot be explained by the excitation of a modified mode of the plate, because none of these modes are excited in the range [3 kHz, 23 kHz], Figure 4. The modified modes correspond to zeros of the approximate dispersion relation given in Appendix C.2, which were largely studied.
in [11] for the case of $N = 1$. Such a mode $c^{*}_{n,(p)} = \omega / k^{*}_{1,(n,p)}$ corresponds to the intersection of $c^{*}_{n} = \omega / k^{*}_{1(n)}$ (the $n$-th mode of the plate without inclusions as depicted in Appendix C.1 with $c_{p}(\omega) = \omega / (k_{1(p)})$. For modified modes to be excited in the range $[3 \text{ kHz}, 23 \text{ kHz}]$ we have to employ a type 2 configuration, Figure 4.

Figure 5 depicts the absorption, hemispherical transmission and reflection coefficients, and their associated relative coefficients, for a plate of thickness $L = 102$ mm with acrylic inclusions, creating a type 2 configuration, such that $N$ varies from 0 (plate without inclusions) to 4. The system possesses two bandgaps in the audible frequency range (the transmission for $N = 4$ is less than $10^{-3}$ over $[3.8 \text{ kHz}, 8.1 \text{ kHz}] \cup [11.1 \text{ kHz}, 15.7 \text{ kHz}]$) that are once again lower and wider than those of the corresponding bandgap observed for an air matrix. The minimum of the hemispherical transmission coefficient varies from $3.5 \times 10^{-5}$ for $N = 4$ to $4 \times 10^{-2}$ for $N = 1$. 

Figure 5. The top panel depicts the transmission (a) and the relative transmission (b) coefficients, the middle panel the reflection (c) and the relative reflection (d) coefficients and the bottom panel the absorption (e) and the relative absorption (f) coefficients for $\nu \in [3 \text{ kHz}, 22 \text{ kHz}]$ and a constant plate thickness $L = 102$ mm with acrylic inclusions such that $N$ varies between 0 (plate without inclusions) and 4 creating a type 2 configuration.
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within the frequency range of the first bandgap. In particular, for \( N = 4 \), \( T^4 \) is \( 1.8 \times 10^{-4} \) times less than the transmission coefficient \( T^0 \) at 6 kHz and is always lower than \( 10^{-1} \). The same remarks concerning the depth and central frequency of the bandgaps can be made for this configuration as those for the type 1 configuration. The addition of inclusions again induces a large increase of the reflection coefficient associated with a large decrease of the absorption coefficient at low frequencies in the neighbourhood of the first bandgap, in particular for \( N = 4 \). The hemispherical reflection coefficient is close to zero around 9.5 kHz and 13 kHz, so as to make the absorption coefficient more than 0.9 between 9 kHz and 10.5 kHz and between 12.5 kHz and 13.5 kHz. The absorption coefficient is nearly equal to 1 for frequencies near 13 kHz. While the first minimum of the hemispherical reflection coefficient is not related to a modified plate mode, the second one is.

Effectively, the particular feature of the type 2 configuration is that the first modified plate mode is excited at 13 kHz, Figure 4. When a modified plate mode is excited, the structure of the associated waves (propagative waves in the plate and evanescent waves in the ambient fluid) allow the entrapment of a part of the energy inside the plate and thus increased absorption. \( c_{(n,p)} \) lies between \( \text{Re}(c^{[1]}) \) and \( c^{[0]} \) so that \( n\text{Re}(c^{[1]})/d \leq \nu^*_{(n,p)} \leq nc^{[0]}/d = \bar{\nu}_{(2n)} \). The modified plate mode natural frequencies always lie between the central frequency of the odd bandgap in the case of rigid-frame porous and air material matrices. In our particular case, the first modified plate mode is excited around the central frequency of the second bandgap. The velocity of the wave that locally (in the slab) impinges on the sonic crystal is \( c^*_{(n,p)} \) at \( \nu \approx \nu^*_{(n,p)} \) and \( \text{Re}(c^{[1]}) \), thus modifying the Bragg condition as well as the bandgap. While the modified plate modes are associated with a decrease of the hemispherical transmission coefficient for \( N = 1 \) (and \( N = 2 \)), they are associated with an increase of the hemispherical transmission coefficient within the odd bandgap (at least the second one) taking the form of a peak when a well-developed sonic crystal is created inside the slab (i.e. \( N \geq 3 \)). This peak is a particular feature of defects in sonic/phononic crystals. The excitation of modified plate modes, in the sonic crystal sense, is also associated with a defect behaviour. However, the hemispherical transmission coefficient is still very low and so, together with the minimum value of the hemispherical reflection coefficient, the absorption is close to 1 at \( \nu^*_{(n,p)} \).

While, the first bandgap is of practical importance for the attenuation of the sound transmission, the second bandgap is of practical importance for sound attenuation. To our knowledge, none of the proposed procedures to increase the absorption of the porous material [11, 40] lead to an absorption coefficient as close to 1 as obtained herein within the second bandgap.

10. Conclusion

The purpose of this paper was: (i) to investigate the existence of bandgaps in a lattice of scatterers embedded in a rigid-frame porous plate and (ii) to consider some of the design issues for the attenuation and/or the absorption of sound by means of discontinuous porous materials. We have shown, on two examples of a square lattice of acrylic and aluminium cylinders embedded in a foam plate, that such structures possess bandgaps within the audible frequency range whose depth can be sufficient for practical use with a relatively small numbers of arrays and thus a relatively small thickness of the plate. The central frequencies of the bandgaps of a rigid frame porous material matrix follow the Bragg condition and are lower than those of an air matrix. The transmission coefficient in the presence of the inclusions can be \( 10^{-4} \) times the transmission coefficient in the absence of the inclusions for only four arrays of inclusions. While the first bandgap is of practical use for acoustic shielding (very low transmission, but relatively small absorption), the second bandgap appears to be of practical use for sound absorption, with an
absorption coefficient very close to 1 for a particular frequency. The latter corresponds to the frequency of excitation of the first modified plate mode whose structure enables the entrapment of the energy inside the plate. The excitation of this mode occurs within the second bandgap frequency range as a defect.

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Appendix

A. Expressions of the cylindrical harmonic reflection coefficients

To derive the expression of the cylindrical harmonic reflection coefficients, the field representations in the vicinity of the J-th cylinder (28) and in the J-th inclusion are needed. The latter depends on the type of inclusion and takes either the form

\[ p^{[2](j)}(r_j) = \sum_{l \in \mathbb{Z}} C_j l (k^{[2]} r_j) e^{i l \theta_j}, \quad \forall (r_j, \theta_j) \in \Omega_{[2];j}, \quad \forall j \in \mathcal{N}, \]  

(42)
in the case of fluid-like inclusions, or
\[
\begin{align*}
\phi^{(2,j)}(r_j) &= \sum_{l \in \mathbb{Z}} C_l J_l \left( k_l^{[2]} r_j \right) e^{i\theta_j}, \\
\psi^{(2,j)}(r_j) &= \sum_{l \in \mathbb{Z}} C_l J_l \left( k_l^{[2]} r_j \right) e^{i\theta_j}, \quad \forall (r_j, \theta_j) \in \Omega_{[2]}, \quad \forall j \in \mathcal{N},
\end{align*}
\]

in the case of solid-like inclusions, wherein \( \phi^{(2,j)}(r_j) \) and \( \psi^{(2,j)}(r_j) = \psi^{(2,j)}(r_j)i_3 \) are the scalar and vector potentials, respectively, such that \( \mathbf{u}^{(2,j)} = \nabla \phi^{(2,j)} + \nabla \times \psi^{(2,j)} \), and \( k_L \) and \( k_S \) are the wavenumbers associated with the bulk and shear waves, respectively. In the fluid-like case we apply (11) and (12), together with \( \int_0^{2\pi} e^{im-\beta \phi} \, d\theta = 2\pi \delta_{ml} \), to find:
\[
V_m = \frac{B_m}{A_m} = \frac{\gamma^{[1]} J_m(\chi^{[1]}) J_l(\chi^{[2]}) - \gamma^{[2]} J_m(\chi^{[2]}) J_l(\chi^{[2]})}{\gamma^{[2]} J_l(\chi^{[2]}) H^{[1]}(\chi^{[1]}) - \gamma^{[1]} H_l^{[1]}(\chi^{[1]}) J_l(\chi^{[2]})}
\]

wherein \( \dot{Z}_m(x) = dZ_m(x)/dx \), with \( Z_m = H_m^{[1]} \) or \( Z_m = J_m \), \( \forall m \in \mathbb{N} \), \( \gamma^{[1]} = k^{[1]}/\rho^{[1]} \) and \( \chi^{[1]} = k^{[1]} R \), \( j = 1, 2 \). In the solid-like case we apply (13) and (14) as previously to obtain
\[
V_m = \frac{B_m}{A_m} = \frac{K_m \chi^{[1]} J_m(\chi^{[1]}) - W_m R^2 J_m(\chi^{[1]})}{W_m R^2 H_m^{[1]}(\chi^{[1]}) - K_m \chi^{[1]} H_m^{[1]}(\chi^{[1]})}
\]

wherein
\[
W_m = \left( K_c/(k^{[1]^2}) \right) \left( \chi_L^{[2]} J_m(\chi_L^{[1]}) - \chi_S^{[2]} J_m(\chi_S^{[1]}) - m^2 J_m(\chi_S^{[2]}) \right) \\
-2m^2 J_m(\chi_S^{[2]}) \left( J_m(\chi_L^{[2]}) - \chi_L^{[2]} J_m(\chi_L^{[2]}) \right)
\]
\[
K_m = \left( (\chi_S^{[2]} J_m(\chi_S^{[1]}) - \chi_S^{[2]} J_m(\chi_S^{[1]}) + m^2 J_m(\chi_S^{[2]}) \right) \\
\times \left( \chi_L^{[2]} J_m(\chi_L^{[1]}) + \chi_L^{[2]} J_m(\chi_L^{[1]}) - m^2 J_m(\chi_L^{[2]}) \right) + 2\mu^{[2]}(\chi_S^{[2]} J_m(\chi_S^{[2]})) \\
-4\mu^{[2]} m^2 \left( J_m(\chi_S^{[2]}) - \chi_S^{[2]} J_m(\chi_S^{[1]}) \right) \left( J_m(\chi_L^{[2]}) - \chi_L^{[2]} J_m(\chi_L^{[2]}) \right)
\]

with \( \chi^{[2]} \) and \( \mu^{[2]} \) the Lamé coefficients of \( M^{[2]} \) and wherein \( \chi^{[1]} = k^{[1]} R \), \( j = 1, 2 \), \( i = L, S \). When the contrast between \( M^{[1]} \) and \( M^{[2]} \) is large, the boundary conditions reduce to Neumann type ones. We apply (15) as previously to obtain
\[
V_m = \frac{B_m}{A_m} = \frac{-J_l(\chi^{[1]})}{H_l^{[1]}(\chi^{[1]})},
\]

### B. Validation of the treatment of aluminium or acrylic inclusions as fluids

Solid scatterers embedded in a light fluid-like air medium or rigid frame porous medium are usually treated as fluid scatterers. We validate the latter assumption on both type 1 systems \( (c_i^{[2]} = 3120 \text{ m s}^{-1}) \) and \( 2 \) \( (c_i^{[2]} = 1300 \text{ m s}^{-1}) \) for \( N = 4 \). Moreover, the contrast between \( M^{[1]} \) and \( M^{[2]} \) being high, both type (aluminium or acrylic) of inclusions can be modelled as infinitely rigid inclusions. We also validate the latter assumption for \( N = 4 \). The results are shown in Figure 6 and match well.
Figure 6. Transmission coefficient over [3 kHz, 22 kHz]: (a) for a constant plate thickness $L = 102$ mm with acrylic inclusions creating a type 2 configuration such that $N = 4$ and (b) for a constant plate thickness $L = 56$ mm with aluminium inclusions creating a type 2 configuration such that $N = 4$: (---) hemispherical transmission coefficient with inclusions treated as a solid medium, (—) hemispherical transmission coefficient with inclusions treated as a fluid medium, and (···) hemispherical transmission coefficient with infinitely rigid (Neumann type boundary conditions) inclusions.

C. Modal analysis

In this section, we consider the media $M^{[0]}$ and $M^{[3]}$ to be identical. Therefore, all the equations that are presented in Sections 5–7 are simplified.

C.1 Modal analysis in the absence of the inclusions

In the absence of inclusions, the coefficients of the field scattered by the inclusions vanish ($B_{l}^{(j)} = 0, \forall l \in \mathbb{N}$ and $\forall j \in \mathcal{N}$) and all the unknowns that are indexed by $p$ reduce to their value for $p = 0$. The problem reduces to the determination of $R = R_{p} \delta_{R0}$, $T = T_{p} \delta_{R0}$, $f^{[1]+} = f^{[1]+}_{p} \delta_{R0}$ and $f^{[1]-} = f^{[1]-}_{p} \delta_{R0}$ from the linear system associated with the corresponding system of Equations (25). After some algebra to eliminate $f^{[1]+}$ and $f^{[1]-}$, the latter system reduces to:

$$
\begin{bmatrix}
(\alpha^{[1]} - \alpha^{[0]}) - e^{-i \frac{\omega}{2} L} (\alpha^{[1]} + \alpha^{[0]}) \\
(\alpha^{[1]} + \alpha^{[0]}) - e^{i \frac{\omega}{2} L} (\alpha^{[1]} - \alpha^{[0]})
\end{bmatrix}
\begin{bmatrix}
T \\
R
\end{bmatrix}
= \begin{bmatrix}
A' e^{-i \frac{\omega}{2} L} (\alpha^{[1]} - \alpha^{[0]}) \\
A' e^{i \frac{\omega}{2} L} (\alpha^{[1]} + \alpha^{[0]})
\end{bmatrix}.

(49)
$$

The natural frequencies of the modes of the configuration are obtained by turning off the excitation (i.e. $A' = 0$). The resulting matrix equation possesses a non-trivial solution only if the determinant of the matrix vanishes, i.e.

$$
i \left((\alpha^{[0]})^{2} + (\alpha^{[1]})^{2}\right) \sin \left(\frac{1}{2} \kappa_{1}^{[1]} L\right) - 2 \alpha^{[1]} \alpha^{[0]} \cos \left(\frac{1}{2} \kappa_{2}^{[1]} L\right) = 0,

(50)
$$

whose roots, take the form of a couple $(k_{1}', \omega)$, defining the wavenumbers and natural frequencies of the modes of the configuration. Figure 4 depicts the real and the imaginary parts of the solution $c^{*}_{\omega}(\omega) = \omega/(k_{1}')$ of Equation (50) with the mechanical characteristics used in Section 9 for plates $L = 56$ mm and $L = 102$ mm thick. The solution is normalized by $c^{[0]}$ defining $c^{*}_{\omega}(\omega) = c^{[0]}(\omega)/c^{[0]}$. To solve the latter, we proceed as in [11]. An important remark is that the dispersion relation cannot vanish when the incident wave takes the form of a plane bulk wave, because we must have $(\Re(k_{1}'))^{2} \geq (k^{0})^{2}$, whereas $k_{1}' \in [-k_{0}, k_{0}]$ in this case. In the absence of the inclusions, no modes of the configuration can be excited by an incident plane (bulk) wave.
C.2 Approximate modal analysis in the presence of inclusions: the emergence of the modified plate modes and the issue of how many gratings are required for a sonic crystal to be created

For the plate with periodic inclusions, the problem reduces to the resolution of the linear system (33). As previously, the natural modes of the configuration are obtained by turning off the excitation, embodied in the vector $VF$. The resulting equations possess non-trivial solutions only if the determinant of the matrix vanishes:

$$\det (I - VS - V(Q + G - P)) = 0.$$  

(51)

The roots of the latter equation that correspond to propagative waves in $\Omega_{[0]}$ and $\Omega_{[3]}$ (propagative mode) define the band structure of the finite configuration. To be convinced of this, it is simpler to consider an infinite number $N$ of gratings and $M^{[i]} = M^{[0]}. The quasi-periodicity in the $x_2$ direction is then characterized by $B_m^{[i]} = B_m e^{j/k^{[0]}\xi_2}. Introduced into (33), the corresponding determinant (51) enables the calculation of the band structure as in [41]. In what follows, we will not look for bulk wave (propagative) modes but rather for surface wave modes, which are associated with evanescent waves in $\Omega_{[0]}$ and $\Omega_{[3]}$ enabling the entrapment of the energy in the plate.

A procedure, called the partition method, for solving Equation (51), is not easy to apply because the off-diagonal elements of the matrix are not small compared to the diagonal elements. Proceeding as in [11], an iterative scheme can be employed to solve (33) so as to obtain an approximate dispersion relation. The latter focuses on the sonic crystal and not on the whole configuration. We rewrite this equation in the form:

$$(I - V_m (S_0 + G_{mn} + Q_{mn} - P_{mn})) B_m = V_m \sum_{l \in \mathbb{Z}} (S_{ml} + G_{ml} + Q_{ml} - P_{ml}) B_l (I - I\delta_{lm}) + V_m F_m, \quad (52)$$

wherein $B_m$ is the $m$-th component of the infinite column matrix $B$, $F_m$ the $m$-th component of the infinite column matrix $F$, $S_{ml}$, $G_{ml}$, $Q_{ml}$ and $P_{mn}$ the $ml$-th components of the four square matrices $S$, $G$, $Q$ and $P$, respectively.

The iterative procedure for solving this linear set of equations is:

$$\begin{cases} 
B_m^{[0]} = (I - V_m (S_0 + G_{mn} + Q_{mn} - P_{mn}))^{-1} \cdot V_m F_m \\
B_m^{[n]} = (I - V_m (S_0 + G_{mn} + Q_{mn} - P_{mn}))^{-1} \\
\quad \cdot V_m \left(F_m + \sum_{l \in \mathbb{Z}} (S_{ml} + G_{ml} + Q_{ml} - P_{ml}) B_l^{[n-1]} (I - I\delta_{lm})\right). 
\end{cases} \quad (53)$$

Recalling that $A^{-1} = \frac{\text{com}A^T}{\det(A)}$, wherein $\text{com}A^T$ is the transpose of the matrix whose elements are the cofactors of $A$, it becomes apparent from (53) that the solution $B_m^{[n]}$, to any order of approximation $n \in \mathbb{N}$, is expressed as a fraction, the denominator of which $D_m$ (not depending on the order of approximation), can become small for certain couples $(k_1, \omega)$ so as to make $B_m^{[n]}$, and (possibly) the field large.

When this happens, a (surface wave) mode of the configuration, comprising the grating stack and the plate, is excited, this taking the form of a resonance with respect to $B_m^{[n]}$, i.e. with respect to a plane wave component of the field in the plate relative to the grating stack.

The approximate dispersion relation

$$D_m = \det (I - V_m (S_0 + G_{mn} + Q_{mn} - P_{mn})) = 0, \quad (54)$$

is a combination of a term linked to the stack of gratings, i.e. the finite (in the $x_2$ direction) size sonic crystal, embodied in $I - V_m (S_0 + G_{mn})$, and a term linked to the plate embodied in $-V_m ((Q_{nm} - P_{mn})$. From (44) in the case of fluid-like inclusions, or from (45) in the case of solid-like inclusions, it is clear that if media $M^{[1]}$ and $M^{[2]}$ have properties that are close, then the approximate dispersion relation $D_m = 0$ is never satisfied because $V_m \approx 0$ (and so $D_m \approx 1$). The contrast between the medium $M^{[1]}$ and $M^{[2]}$ has to...
be large for the approximate dispersion relation to be satisfied. This is often encountered in sonic crystals, for which, the existence of a pronounced loss of the sound transmission is strongly connected with a large acoustic impedance ratio between the scatterers and the matrix material.

The zeroth-order Schlömilch series $S_0$ can be rewritten \[37\] in the form $2/d \sum_{p \in \mathbb{Z}} 1/k_{2p}^{[1]}$ (additional constants are neglected). From the latter transformation it becomes clear that the elements $V_m \left( S_0 + Q_{mm}^{(j,j)} - P_{mm}^{(j,j)} \right)$ and $V_m \left( G_{mm}^{(j,j)} + Q_{mm}^{(j,j)} - P_{mm}^{(j,j)} \right)$ have the same form

$$
\left( V_m \left( S_0 + Q_{mm} - P_{mm} \right) V_m \left( G_{mm} + Q_{mm} - P_{mm} \right) \right)^{(j,j)} = -V_m \sum_{p \in \mathbb{Z}} \frac{e^{i k_{2p}^{[1]} D_p}}{d k_{2p}^{[1]} D_p} N_p^{(j,j)},
$$

wherein

$$
N_p^{(j,j)} = -4 \left( (\alpha_p^{[0]} )^2 - (\alpha_p^{[1]} )^2 \right) \cos \left( k_{2p}^{[1]} (a + b - (J - j)e_2) + 2m\theta_p \right) + 4 \left( (\alpha_p^{[0]} )^2 + (\alpha_p^{[1]} )^2 \right) \cos \left( k_{2p}^{[1]} (L \pm (J - j)e_2) \right)
- 4i \alpha_p^{[0]} \alpha_p^{[1]} \sin \left( k_{2p}^{[1]} (L \pm (J - j)e_2) \right).
$$

The partition method for solving Equation (54) consists in first considering the matrix to have one row and one column:

$$
1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} = 0
$$

which is the approximate dispersion relation obtained in [11] in the case of a single grating imbedded in a porous plate. From [11], the latter equation is satisfied for $D_p = 0$, corresponding to a modified plate mode.

Next, consider the matrix to have two rows and two columns:

$$
det \left( 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,2)}}{d k_{2p}^{[1]} D_p} \right)
\left( -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,1)}}{d k_{2p}^{[1]} D_p} - 1 + V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,2)}}{d k_{2p}^{[1]} D_p} \right)
= 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} + \frac{N_p^{(2,2)}}{d k_{2p}^{[1]} D_p}
- V^2 \left[ \left( \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} \right) \left( \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,1)}}{d k_{2p}^{[1]} D_p} \right) \right]
- \left( \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} \right) \left( \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,2)}}{d k_{2p}^{[1]} D_p} \right) = 0.
$$

From this relation, together with Equation (56), it appears that the off-diagonal elements cannot be neglected. While the diagonal elements account for the interaction of each grating with the plate (as previously studied in [11]), the off-diagonal elements account for the complex interaction between the gratings (and perhaps between the latter and the plate), leading to the dispersion effects associated with those of a sonic crystal. For $N = 2$ further simplifications can be made. $N_p^{(j,j)}$ being of the same order $\forall (J, j) \in (1, 2)$, the second
order term become negligible so as to reduce (58) to

\[
\det \left( \begin{array}{ccc}
1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,2)}}{d k_{2p}^{[1]} D_p} \\
-V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,1)}}{d k_{2p}^{[1]} D_p} & 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,2)}}{d k_{2p}^{[1]} D_p} \\
-V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(3,1)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(3,2)}}{d k_{2p}^{[1]} D_p} \\
\end{array} \right) \\
\approx 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)} + N_p^{(2,2)}}{d k_{2p}^{[1]} D_p} = 0. \quad (59)
\]

The latter equation is of the same form as Equation (57). Once again, by referring to [11], Equation (59) is satisfied for \( D_p = 0 \), corresponding to a modified plate mode. This shows that the case \( N = 2 \) corresponds to a transition configuration between a case in which the phenomena are related to those of a grating to a configuration in which the phenomena are related to those of a sonic crystal. Effectively, when the matrix with three (or more) rows and three (or more) columns is considered:

\[
\det \left( \begin{array}{ccc}
1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,1)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,2)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(1,3)}}{d k_{2p}^{[1]} D_p} \\
-V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,1)}}{d k_{2p}^{[1]} D_p} & 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,2)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(2,3)}}{d k_{2p}^{[1]} D_p} \\
-V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(3,1)}}{d k_{2p}^{[1]} D_p} & -V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(3,2)}}{d k_{2p}^{[1]} D_p} & 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(3,3)}}{d k_{2p}^{[1]} D_p} \\
\end{array} \right) = 0 \quad (60)
\]

no simplifications can be made any more, so that all the terms have to be accounted for. For \( N \geq 3 \), strong interaction between the gratings occurs, this leading to the creation of the bandgap. The resolution of (60) is not easy. Nevertheless, from the expression of the reflected and transmitted fields (Equations (35) and (36)) and from the cases \( N = 2 \), it is clear that something happens when \( D_p = 0 \).

To get a grip on this, another iterative scheme can be employed to solve (33) and obtain another approximate dispersion relation. The latter focuses on each grating of the sonic crystal rather than on the whole sonic crystal. We rewrite this equation in the form

\[
(1 - V_m \left( S_0 + Q_{mm}^{(J)} - P_{mm}^{(J)} \right)) B_m^{(J)} = V_m \sum_{l \in \mathbb{Z}, j \in \mathbb{N}} \left( S_{ml} \delta_{lj} + G_{ml}^{(J)} + Q_{ml}^{(J)} - P_{ml}^{(J)} \right) B_l^{(j)} (1 - \delta_{lm}) + V_m F_m^{(J)}. \quad (61)
\]

The iterative procedure for solving this linear set of equations is:

\[
\begin{align*}
(B_m^{(J)})^{[0]} &= \frac{V_m F_m^{(J)}}{1 - V_m \left( S_0 + Q_{mm}^{(J)} - P_{mm}^{(J)} \right)}, \\
(B_m^{(J)})^{[n]} &= V_m \sum_{l \in \mathbb{Z}, j \in \mathbb{N}} \left( S_{ml} \delta_{lj} + G_{ml}^{(J)} + Q_{ml}^{(J)} - P_{ml}^{(J)} \right) B_l^{(j)} (1 - \delta_{lm}) \quad \text{for } \left( B_m^{(J)} \right)^{[n]} \\
&+ \frac{V_m F_m^{(J)}}{1 - V_m \left( S_0 + Q_{mm}^{(J)} - P_{mm}^{(J)} \right)}, \quad (62)
\end{align*}
\]

from which it becomes apparent that the solution \((B_m^{(J)})^{[n]}\), to any order of approximation, is expressed as a fraction, the denominator of which \( D_m^{(J)} \) (not depending on the order of approximation), can become small for certain couples \((k_1, \omega)\) so as to make \((B_m^{(J)})^{[n]}\), and (possibly) the field, large.
When this happens, a natural mode of the configuration, comprising the inclusions and the plate, is excited, this taking the form of a resonance with respect to \((B_m^{(j)})^{[n]}\), i.e. with respect to a plane wave component of the field in the plate relative to the inclusions. As \((B_m^{(j)})^{[n]}\) is related to \(f_p^{(1)\pm}, T_p\) and \(R_p\), the structural resonance manifests itself for the same \((k_{1p}, \omega)\) in the fields within the plate and in the air.

The approximate dispersion relation

\[
\mathcal{D}_m^{(j)} = 1 - V_m \left( S_0 + \sum_{p \in \mathbb{Z}} \left( Q_{mmp}^{(jj)} - P_{mmp}^{(jj)} \right) \right) = 0, \quad \forall J \in \mathcal{N} \tag{63}
\]

is the sum of a term linked to the \(J\)-th grating embodied in \(1 - V_m S_0\) and a term linked to the plate embodied in \(-V_l \sum_{p \in \mathbb{Z}} (Q_{mmp}^{(jj)} - B_{mmp}^{(jj)})\). The latter becomes

\[
\mathcal{D}_m^{(j)} = 1 - V_m \sum_{p \in \mathbb{Z}} \frac{N_p^{(jj)}}{d_{2p} k_{1p}^2 D_p} = 0, \quad \forall J \in \mathcal{N} \tag{64}
\]

whose roots correspond to those of the modified modes of the plate.