Malliavin Calculus for Markov Chains using Perturbations of Time

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Abstract

In this article, we develop a Malliavin calculus associated to a time-continuous Markov chain with finite state space. We apply it to get a criterion of density for solutions of SDE involving the Markov chain and also to compute greeks.

keywords: Dirichlet form; Integration by parts formula; Malliavin calculus; Markov chain; computation of greeks.
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1 Introduction

The aim of this article is to construct a Dirichlet structure associated to a time-homogeneous Markov chain with finite state space in the spirit of Carlen and

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Pardoux ([8]) who obtained criteria of density for stochastic differential equations (SDEs) driven by a Poisson process. More precisely, we develop a Malliavin calculus on the canonical space by "derivating" with respect to the jump times of the Markov chain, the main difficulty is that the times of jumps of Markov chain we consider are no more distributed according to an homogeneous Poisson process.
Extensions of Malliavin calculus to the case of SDEs with jumps have been soon proposed and gave rise to an extensive literature. The approach is either dealing with local operators acting on the size of the jumps (cf [2] [9] [15] etc.) or acting on the instants of the jumps (cf [8] [10]) or based on the Fock space representation of the Poisson space and finite difference operators (cf [17] [18] [12] etc.). Let us mention that developping a Malliavin Calculus or a Dirichlet structure on the Poisson space is not the only way to prove the absolutely continuity of the law of the solution of SDE's driven by a Lévy process, see for example [16] or the recent works of Bally and Clément [1] who consider a very general case.
In this article we consider a Markov chain and we construct "explicitly" a *gradient operator* and a local Dirichlet form. With respect to the Malliavin analysis or the integration by part formula approach , what brings the Dirichlet forms approach is threefold: a) The arguments hold under only Lipschitz hypotheses, e.g. for density of solutions of stochastic differential equations cf [7], this is due to the celebrated property that contractions operate on Dirichlet forms and the Émile Picard iterated scheme may be performed under the Dirichlet norm. b) A general criterion exists, the energy image density property (EID), proved on the Wiener space for the Ornstein-Uhlenbeck form, and in several other cases (but still a conjecture in general since 1986 cf [6]), which provides an efficient tool for obtaining existence of densities in stochastic calculus. c) Dirichlet forms are easy to construct in the infinite dimensional frameworks encountered in probability theory (cf [7] Chap.V) and this yields a theory of errors propagation, especially for finance and physics cf [3].
Moreover, since the gradient operator may be calculated easily, this permits to make numerical simulations, for example in order to compute greeks of an asset in a market with jumps.
The plan of the article is as follows. In Section 2, we describe the probabilistic framework and introduce the Markov chain. Then in Section 3, we introduce the directional derivative w.r.t. an element of the Cameron-Martin and give some basic properties. The next section is devoted to the construction of the local Dirichlet structure and the associated operators namely the gradient and divergence, and also to the establishment of an integration by parts formula. In Section 5, we prove that this Dirichlet form satisfies the *Energy Image Density (EID)* property that we apply in Section 6 to get a criterion of density for solution of SDEs involving the Markov chain. In the last section, we show that this

Malliavin calculus may be applied to compute some greeks in finance.

2 Probability Space

Let $C = (C_t)_{t\geq 0}$ be a time-homogeneous Markov chain, with finite state space $\mathcal{K} = \{k_1, k_2, \cdots, k_l\}$. The one-step transition probability matrix is $\mathcal{P} = (p_{ij})_{1\leq i,j\leq l}$, and the infinitesimal generator matrix $\Lambda = (\lambda_{ij})_{1\leq i,j\leq l}$. We assume that $(C_t)_{t\geq 0}$ has a finite number of jumps over a bounded horizon of time so that we consider that it is defined on the canonical probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where Ω is the set of $\mathcal{K}$-valued right continuous maps $w : [0, \infty) \to \mathcal{K}$ starting at $w(0)= k_{i_0} \in \mathcal{K}$ with $i_0 \in \{1, \cdots, l\}$, and there exists sequences $i_1, i_2, \cdots \in \{1, \cdots, l\}$ and $0 = t_0 < t_1 < t_2 < \cdots$ such that

$$w(t) = \sum_{n=0}^{\infty} k_{i_n} \mathbf{1}_{[t_n, t_{n+1}[}(t). \tag{2.1}$$

The filtration $(\mathcal{F}_t)_{t\geq 0}$ we consider is the natural filtration of the Markov chain $(C_t)_{t\geq 0}$ satisfying the usual hypotheses.

Let $(T_n)_{n\in\mathbb{N}}$ denote the sequence of successive jump times of C and Z_n denote the position of C at time T_n. More explicitly, for any $n \in \mathbb{N}$, the random variables T_n and Z_n are defined as follow

$$\begin{cases} T_0 = 0, Z_0 = k_{i_0}, \\ T_n = inf\{t > T_{n-1} : C_t \neq Z_{n-1}\}, \\ Z_n = C_{T_n}. \end{cases}$$

We have $0 = T_0 < T_1 < T_2 < \cdots$, and $T_n \to \infty$ when $n \to \infty$ almost surely. These are two well-known properties

1. $\mathbb{P}(T_n - T_{n-1} > t | Z_{n-1} = k_i) = e^{\lambda_{ii} t}$.
2. $\mathbb{P}(Z_n = j | Z_{n-1} = k_i) = p_{ij} = -\dfrac{\lambda_{ij}}{\lambda_{ii}}$.

Let $u_n = (u_n^i)_{1\leq i\leq l}$ be the distribution of Z_n, i.e., $u_n^i = \mathbb{P}(Z_n = k_i), i = 1, \cdots, l$. From the second property and the law of total probability we have

$$u_n^j = \sum_{i=1}^{l} \mathbb{P}(Z_n = k_j | Z_{n-1} = k_i)\mathbb{P}(Z_{n-1} = k_i) = \sum_{i=1}^{l} p_{ij} u_{n-1}^i.$$

Hence, $u_n = \mathcal{P}u_{n-1} = \mathcal{P}^2 u_{n-2} = \cdots = \mathcal{P}^n u_0$. The first property means that conditionally on the position $Z_{n-1} = k_i$ at the jump time T_{n-1}, the random time that elapses until the next jump has an exponential probability law with parameter $-\lambda_{ii} > 0$. Moreover, we have the following proposition whose proof is obvious:

Proposition 2.0. *Conditionally on the positions $Z_1 = k_{i_1}, \cdots, Z_{n-1} = k_{i_{n-1}}$,*

1. *the increments $\tau_1 = T_1, \tau_2 = T_2 - T_1, \cdots, \tau_n = T_n - T_{n-1}$ are independent and exponentially distributed with parameters $\lambda_1 = -\lambda_{i_0 i_0}, \cdots, \lambda_n = -\lambda_{i_{n-1} i_{n-1}}$;*

2. *the joint law $(T_1, T_2, \cdots, T_n)$ has the following density function on $\mathbb{R}^n$*

$$\prod_{i=1}^{n} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} \mathbf{1}_{\{0 < t_1 < \cdots < t_n\}}(t_1, \cdots, t_n).$$

We can also compute the law of $\tau_n = T_n - T_{n-1}$ in the same way as Z_n:

$$\mathbb{P}(\tau_n > t) = \sum_{i=1}^{l} \mathbb{P}(T_n - T_{n-1} > t | Z_{n-1} = k_i) \mathbb{P}(Z_{n-1} = k_i) = \sum_{i=1}^{l} e^{\lambda_{ii} t} u_{n-1}^{i}. \tag{2.2}$$

Let $(N_t)_{t \geq 0}$ be the process that counts the number of jumps of the Markov chain $(C_t)_{t \geq 0}$ up to t

$$N_t = \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\}}. \tag{2.3}$$

We now define the process λ by

$$\lambda(t) = \sum_{n \geq 0} \sum_{i=1}^{l} -\lambda_{ii} \mathbf{1}_{\{Z_n = k_i\}} \mathbf{1}_{\{T_n \leq t < T_{n+1}\}},$$

which is nothing but the intensity of N. The three processes $(C_t)_{t \geq 0}, (N_t)_{t \geq 0}$ and $(\lambda(t))_{t \geq 0}$ have the same jump times. More importantly, the compensated process defined by

$$\tilde{N}_t = N_t - \int_0^t \lambda(u) du \tag{2.4}$$

is an $\mathcal{F}_t$-martingale. Indeed, conditionally on the positions $Z_1, Z_2, \cdots, Z_{n-1}$, $\int_0^{t \wedge T_n} \lambda(u) du$ is the compensator of the process $N_{t \wedge T_n}$. Hence for every $0 \leq s \leq t$ and $i_1, \cdots, i_{n-1}$ in $\{1, \cdots, l\}$

$$\mathbb{E}[N_{t \wedge T_n} - N_{s \wedge T_n} - \int_{s \wedge T_n}^{t \wedge T_n} \lambda(u) du | \mathcal{F}_s, Z_1 = k_{i_1}, \cdots, Z_{n-1} = k_{i_{n-1}}] = 0,$$

which implies

$$\begin{aligned} & \mathbb{E}[N_{t \wedge T_n} - N_{s \wedge T_n} - \int_{s \wedge T_n}^{t \wedge T_n} \lambda(u) du | \mathcal{F}_s] \\ = & \sum_{1 \leq i_1, \cdots, i_{n-1} \leq l} \mathbb{E}[N_{t \wedge T_n} - N_{s \wedge T_n} - \int_{s \wedge T_n}^{t \wedge T_n} \lambda(u) du | \mathcal{F}_s, Z_1 = k_{i_1}, \cdots, Z_{n-1} = k_{i_{n-1}}] \\ & \times \mathbb{P}(Z_1 = k_{i_1}, \cdots, Z_{n-1} = k_{i_{n-1}}) \\ = & \ 0. \end{aligned}$$

By making n tend to $+\infty$, we get $\mathbb{E}[N_t - N_s - \int_s^t \lambda(u)du|\mathcal{F}_s] = 0$, so the process $(\tilde{N}_t)_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

3 Directional derivation

In this section, we will consider the Markov chain (C_t) defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0\leq t\leq T})$ where $\mathcal{F} = \mathcal{F}_T$ for a fixed time horizon $0 < T < \infty$. We apply the same approach as in [8] to define the directional derivative using the reparametrization of time with respect to a function in a Cameron-Martin space.

Let $\mathcal{H}$ be the closed subspace of $L^2([0,T])$ orthogonal to the constant functions:

$$\mathcal{H} = \{m \in L^2([0,T]) : \int_0^T m(t)dt = 0\}.$$

In a natural way , $\mathcal{H}$ inherits the Hilbert structure of $L^2([0,T])$ and we denote by $\| \|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the norm and the scalar product on it.

From now on in this section, we fix a function $m \in \mathcal{H}$. The condition $\int_0^T m(t)dt = 0$ ensures that the change of intensity that we are about to define simply shifts the jump times without affecting the total number of jumps. We define

$$\tilde{m}_\epsilon(t) = \begin{cases} \frac{-1}{3\epsilon} & \text{if } m(t) \leq \frac{-1}{3\epsilon}; \\ m(t) & \text{if } \frac{-1}{3\epsilon} \leq m(t) \leq \frac{1}{3\epsilon}; \\ \frac{1}{3\epsilon} & \text{if } m(t) \geq \frac{1}{3\epsilon}. \end{cases}$$

and $m_\epsilon(t) = \tilde{m}_\epsilon(t) - \frac{1}{T}\int_0^T \tilde{m}_\epsilon(t)dt$. Then we have again $\int_0^T m_\epsilon(t)dt = 0$, and $\frac{1}{3} \leq 1 + \epsilon m_\epsilon(t) \leq \frac{5}{3}$ (since $-\frac{1}{3\epsilon} \leq \tilde{m}_\epsilon(t) \leq \frac{1}{3\epsilon}$). Moreover, $||m - m_\epsilon||_{\mathcal{H}} \to 0$ as $\epsilon \to 0$. We define the reparametrization of time with respect to m_ϵ as follow

$$\tau_\epsilon(t) = t + \epsilon \int_0^t m_\epsilon(s)ds, \; t \geq 0.$$

Notice that $\tau_\epsilon(0) = 0, \tau_\epsilon(T) = T$, and $\tau'_\epsilon(t) = 1 + \epsilon m_\epsilon(t) > 0$, so the number and the order of jump times between 0 and T remain unchanged. Let $\mathcal{T}_\epsilon : \Omega \to \Omega$ be the map defined by

$$(\mathcal{T}_\epsilon(w))(t) = w(\tau_\epsilon(t)) \text{ for all } w \in \Omega,$$

$\mathcal{T}_\epsilon F = F \circ \mathcal{T}_\epsilon$ for every $F \in L^2(\Omega)$, and $\mathbb{P}^\epsilon$ be the probability measure $\mathbb{P}\mathcal{T}_\epsilon^{-1}$. We denote

$$\mathbb{D}_m^0 = \{F \in L^2(\Omega) : \frac{\partial \mathcal{T}_\epsilon F}{\partial \epsilon}|_{\epsilon=0} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}(\mathcal{T}_\epsilon F - F) \, in \, L^2(\Omega) \, exists\}.$$

For $F \in \mathbb{D}_m^0$, $D_m F$ is defined as the above limit.

Example 3.0. *Now we give some examples of random variables whose directional derivatives can be computed directly from the definition.*

1. *The random variables* Z_n *do not change under* $\mathcal{T}_\epsilon$*, so* $Z_n \in \mathbb{D}_m^0$ *and* $D_m Z_n = 0$.

2. *Let* $w \in \Omega$ *have the form* (2.1)*. Then*

$$(\mathcal{T}_\epsilon(w))(t) = w(\tau_\epsilon(t)) = \sum_{n=0}^{\infty} k_{i_n} 1_{[t_n, t_{n+1}[}(\tau_\epsilon(t)) = \sum_{n=0}^{\infty} k_{i_n} 1_{[\tau_\epsilon^{-1}(t_n), \tau_\epsilon^{-1}(t_{n+1})[}(t), \tag{3.5}$$

which implies $\mathcal{T}_\epsilon T_n(w) = T_n(\mathcal{T}_\epsilon(w)) = \tau_\epsilon^{-1}(t_n) = \tau_\epsilon^{-1}(T_n(w))$*. We put* $\bar{T}_n = T_n \wedge T$ *and clearly we also have* $\mathcal{T}_\epsilon \bar{T}_n = \tau_\epsilon^{-1} \circ \bar{T}_n$*. By the mean value theorem, we have*

$$\begin{aligned}
\frac{1}{\epsilon}(\mathcal{T}_\epsilon \bar{T}_n(w) - \bar{T}_n(w)) &= \frac{1}{\epsilon}\left(\tau_\epsilon^{-1}(\bar{T}_n(w)) - \tau_\epsilon^{-1}(\tau_\epsilon(\bar{T}_n(w)))\right) \\
&= \frac{1}{\epsilon}(\tau_\epsilon^{-1})'(S(w))\left(\bar{T}_n(w) - \tau_\epsilon(\bar{T}_n(w))\right) \\
&= -(\tau_\epsilon^{-1})'(S(w)) \int_0^{\bar{T}_n(w)} m_\epsilon(s)ds,
\end{aligned}$$

where $S(w)$ *is some value between* $\bar{T}_n(w)$ *and* $\tau_\epsilon(\bar{T}_n(w))$*. It is clear that for any* $s \in [0, T]$, $(\tau_\epsilon^{-1})'(s)$ *is bounded by a deterministic constant which does not depend on* ϵ *and that it converges to* 1 *as* ϵ *tends to* 0*, hence we easily conclude that* $\bar{T}_n \in \mathbb{D}_m^0$*, and* $D_m \bar{T}_n = -\int_0^{\bar{T}_n} m(t)dt$.

3. *Since the number of jumps of* $(C_t)_{t\in[0,T]}$ *does not change after the reparametrization of time, we have* $\mathcal{T}_\epsilon N_T = N_T$*. Hence* $N_T \in \mathbb{D}_m^0$*, and* $D_m N_T = 0$.

Remark .1. *If the assumption* $\int_0^T m(t)dt = 0$ *was relaxed, the number of jumps in* $[0, T]$ *would change, which does not ensure that* $N_T \in \mathbb{D}_m^0$*. Without loss of generality, we can assume that* $\int_0^T m(t)dt > 0$*, which deduces* $\tau_\epsilon(T) > T$ *for* $\epsilon > 0$*. Indeed,*

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{\epsilon}(\mathcal{T}_\epsilon N_T - N_T)\right]^2 &= \frac{1}{\epsilon^2}\mathbb{E}\left[(\mathcal{T}_\epsilon N_T - N_T)^2\right] \\
&\geq \frac{1}{\epsilon^2}\mathbb{E}\left[(\mathcal{T}_\epsilon N_T - N_T)^2 | \mathcal{T}_\epsilon N_T - N_T \geq 1\right] \mathbb{P}(\mathcal{T}_\epsilon N_T - N_T \geq 1) \\
&\geq \frac{1}{\epsilon^2}\mathbb{P}(\mathcal{T}_\epsilon N_T - N_T \geq 1).
\end{aligned}$$

From (3.5)*, we have* $\mathcal{T}_\epsilon N_T(w) = N_T(\mathcal{T}_\epsilon(w)) = \sum_{n\geq 1} \mathbf{1}_{\{\tau_\epsilon^{-1}(t_n)\leq T\}} = \sum_{n\geq 1} \mathbf{1}_{\{t_n \leq \tau_\epsilon(T)\}}$*. In other words,* $\mathcal{T}_\epsilon N_T$ *is the number of jumps up to* $\tau_\epsilon(T)$*, so the inequality* $\mathcal{T}_\epsilon N_T - N_T \geq 1$ *is equivalent to* $T_{N_T+1} < \tau_\epsilon(T)$*, which can be ensured if*

$T_{N_T+1} - T_{N_T} < \tau_\epsilon(T) - T$, *since* $T_{N_T} < T$. *Put* $\bar{\lambda} = max_{1\le i\le l}\lambda_{ii}$, *then from* (2.2) *and notice that* $\bar{\lambda} < 0$, *we have* $\mathbb{P}(T_n - T_{n-1} \le t) = 1 - \sum_{i=1}^{l} e^{\lambda_{ii}t}u_{n-1}^i \ge 1 - e^{\bar{\lambda}t} > 0$ *for every* $n \ge 1, t > 0$. *In summary, we have*

$$\begin{aligned}\mathbb{E}\left[\frac{1}{\epsilon}(\mathcal{T}_\epsilon N_T - N_T)\right]^2 &\ge \frac{1}{\epsilon^2}\mathbb{P}(T_{N_T+1} - T_{N_T} \le \tau_\epsilon(T) - T)\\ &\ge \frac{1}{\epsilon^2}\left(1 - e^{\bar{\lambda}(\tau_\epsilon(T)-T)}\right) \to \infty \text{ as } \epsilon \downarrow 0.\end{aligned}$$

Thus, in this case we would have $N_T \notin \mathbb{D}_m^0$. *This explains why we need the assumption* $\int_0^T m(t)dt = 0$ *in our construction.*

Let us define the set $\mathcal{S}$ of "smooth" functions. A map $F : \Omega \to \mathbb{R}$ belongs to $\mathcal{S}$ if and only if there exists $a \in \mathbb{R}, d \in \mathbb{N}^*$ and for any $n \in \{1, \cdots, d\}$, $i_1, \cdots, i_n \in \{1, \cdots, l\}$, there exists a function $f_n^{i_1,\cdots,i_n} : \mathbb{R}^n \to \mathbb{R}$ such that

1. $F = a\mathbf{1}_{A_0} + \sum_{n=1}^{d}\sum_{i_1,\cdots,i_n\in\{1,\cdots,l\}} f_n^{i_1,\cdots,i_n}(T_1, \cdots, T_n)\mathbf{1}_{A_n^{i_1,\cdots,i_n}}$, where $A_0 = \{N_T = 0\}, A_n^{i_1,\cdots,i_n} = \{N_T = n, Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}$;
2. for any $n \in \{1, \cdots, d\}$, any $i_1, \cdots, i_n \in \{1, \cdots, l\}$, $f_n^{i_1,\cdots,i_n}$ is smooth with bounded derivatives of any order.

It is known that $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
We recall that for all i, $\bar{T}_i = T_i \wedge T$.

Proposition 3.0. *Let* $n \in \mathbb{N}^*$, *for every smooth function with bounded derivatives of any order* $f : \mathbb{R}^n \to \mathbb{R}$, *we have* $f(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n) \in \mathbb{D}_m^0$, *and*

$$D_m f(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n) = -\sum_{j=1}^{n}\frac{\partial f}{\partial t_j}(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n)\int_0^{\bar{T}_j} m(t)dt.$$

As a consequence, $\mathcal{S}$ *is a subset of* $\mathbb{D}_m^0$.

Proof. By the definition of D_m we have

$$\begin{aligned}D_m f(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n) &= \frac{\partial \mathcal{T}_\epsilon f(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n)}{\partial \epsilon}|_{\epsilon=0}\\ = \frac{\partial}{\partial\epsilon}f(\mathcal{T}_\epsilon\bar{T}_1, \mathcal{T}_\epsilon\bar{T}_2, \cdots, \mathcal{T}_\epsilon\bar{T}_n)|_{\epsilon=0} &= \sum_{j=1}^{n}\frac{\partial f}{\partial t_j}\frac{\partial}{\partial\epsilon}\mathcal{T}_\epsilon\bar{T}_j|_{\epsilon=0}\\ = \sum_{j=1}^{n}\frac{\partial f}{\partial t_j}D_m\bar{T}_j &= -\sum_{j=1}^{n}\frac{\partial f}{\partial t_j}(\bar{T}_1, \bar{T}_2, \cdots, \bar{T}_n)\int_0^{\bar{T}_j} m(t)dt.\end{aligned}$$

□

Proposition 3.0. *(Functional calculus properties)*

1. If $F, G \in \mathcal{S}$ then $FG \in \mathcal{S}$ and $D_m(FG) = (D_mF)G + F(D_mG)$.

2. If $F_1, F_2, \cdots, F_n \in \mathcal{S}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function then $\Phi(F_1, F_2, \cdots, F_n) \in \mathcal{S}$ and

$$D_m\Phi(F_1, F_2, \cdots, F_n) = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j}(F_1, F_2, \cdots, F_n) D_m F_j.$$

Now we study the absolute continuity of $\mathbb{P}^\epsilon$ with respect to $\mathbb{P}$. Let $\mathbb{E}^\epsilon$ be the expectation taken under the probability $\mathbb{P}^\epsilon$. For every $n \in \mathbb{N}, i_1, \cdots, i_n \in \{1, \cdots, l\}$, $\lambda_1, \cdots, \lambda_n$ as in Proposition 2.0, and every measurable function $f : \mathbb{R}^n \to \mathbb{R}$ we have

$$\begin{aligned}
& \mathbb{E}^\epsilon[f(T_1, \cdots, T_n)\mathbf{1}_{\{N_T=n\}}|Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}] \\
= \ & \mathbb{E}[f(\mathcal{T}_\epsilon^{-1}T_1, \cdots, \mathcal{T}_\epsilon^{-1}T_n)\mathbf{1}_{\{N_T=n\}}|Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}] \\
= \ & \mathbb{E}[(f \circ \Phi^{-1})(T_1, \cdots, T_n)\mathbf{1}_{\{T_n \le T < T_{n+1}\}}|Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}] \\
= \ & \iint_{0<t_1<\cdots<t_n \le T < t_{n+1}} (f \circ \Phi^{-1})(t_1, \cdots, t_n) \prod_{i=1}^{n+1} \lambda_i e^{-\lambda_i(t_i - t_{i-1})} dt_1 \cdots dt_{n+1} \\
= \ & \iint_{0<t_1<\cdots<t_n \le T} (f \circ \Phi^{-1}) \prod_{i=1}^{n} \lambda_i e^{-\lambda_i(t_i - t_{i-1})} e^{\lambda_{n+1} t_n} dt_1 \cdots dt_n \int_T^\infty \lambda_{n+1} e^{-\lambda_{n+1} t_{n+1}} dt_{n+1} \\
= \ & \iint_{0<t_1<\cdots<t_n \le T} (f \circ \Phi^{-1})(t_1, \cdots, t_n)\varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n \\
= \ & \iint_{0<u_1<\cdots<u_n \le T} f(u_1, \cdots, u_n)(\varphi \circ \Phi)(u_1, \cdots, u_n)|det J_\Phi| du_1 \cdots du_n \\
= \ & \iint_{0<u_1<\cdots<u_n \le T} f\varphi e^{\epsilon(\lambda_{n+1}\int_0^{u_n} m_\epsilon dt - \sum_{i=1}^n \lambda_i \int_{u_{i-1}}^{u_i} m_\epsilon dt)} \prod_{i=1}^{n}(1 + \epsilon m_\epsilon(u_i)) du_1 \cdots du_n \\
= \ & \mathbb{E}\left[f(T_1, \cdots, T_n)\mathbf{1}_{\{N_T=n\}} p_n | Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\right]
\end{aligned}$$

where

$$\begin{aligned}
\Phi(u_1, \cdots, u_n) &= (u_1 + \epsilon \int_0^{u_1} m_\epsilon(t)dt, \cdots, u_n + \epsilon \int_0^{u_n} m_\epsilon(t)dt) \\
\varphi(t_1, \cdots, t_n) &= e^{-\lambda_{n+1}(T - t_n)} \prod_{i=1}^{n} \lambda_i e^{-\lambda_i(t_i - t_{i-1})} \\
p_n &= e^{\epsilon(\lambda_{n+1}\int_0^{T_n} m_\epsilon dt - \sum_{i=1}^n \lambda_i \int_{T_{i-1}}^{T_i} m_\epsilon dt)} \prod_{i=1}^{n}(1 + \epsilon m_\epsilon(T_i)).
\end{aligned}$$

Notice that $\int_0^{T_n} m_\epsilon dt = -\int_{T_n}^T m_\epsilon dt$, and conditionally to $N_T = n$,

$$\sum_{i=1}^{n} \lambda_i \int_{T_{i-1}}^{T_i} m_\epsilon dt + \lambda_{n+1} \int_{T_n}^{T} m_\epsilon dt = \int_0^T \lambda(t) m_\epsilon(t) dt,$$

we can rewrite $p_n = e^{-\epsilon \int_0^T \lambda(t) m_\epsilon(t)dt} \prod_{i=1}^n (1+\epsilon m_\epsilon(T_i))$ and so we have proved:

Proposition 3.0. $\mathbb{P}^\epsilon$ *is absolutely continuous with respect to* $\mathbb{P}$ *with density*

$$\frac{d\mathbb{P}^\epsilon}{d\mathbb{P}} = e^{-\epsilon \int_0^T \lambda(t) m_\epsilon(t)dt} \prod_{i=1}^{N_T} (1+\epsilon m_\epsilon(T_i)) := G^\epsilon.$$

In case of Poisson process $\lambda_i = 1$, so $\int_0^T \lambda(t) m_\epsilon(t)dt = 0$. Hence we have again the result obtained in [8] for standard Poisson processes

$$\frac{d\mathbb{P}^\epsilon}{d\mathbb{P}} = \prod_{i=1}^{N_T} (1+\epsilon m_\epsilon(T_i)).$$

4 Gradient and divergence

For any $h \in \mathcal{H}$, set

$$\hat{\delta}(h) = \sum_{i=1}^{N_T} h(T_i) - \lambda(T_{N_T}) \int_{T_{N_T}}^T h(t)dt - \sum_{i=1}^{N_T} \lambda(T_{i-1}) \int_{T_{i-1}}^{T_i} h(t)dt \tag{4.6}$$

with convention that $\hat{\delta}(h) = 0$ if $N_T = 0$. We can rewrite (4.6) as follows

$$\hat{\delta}(h) = \int_0^T h(t)dN_t - \int_0^T \lambda(t)h(t)dt = \int_0^T h(t)d\tilde{N}_t.$$

where the compensated process $\tilde{N}$ is defined in (2.4).

Moreover, if we assume that $h \in \mathcal{H} \cap \mathcal{C}^1([0,T])$, by taking the directional derivative of (4.6), we have

$$\begin{aligned}
D_m\hat{\delta}(h) &= \sum_{i=1}^{N_T} h'(T_i)D_mT_i + \lambda(T_{N_T})h(T_{N_T})D_mT_{N_T} - \sum_{i=1}^{N_T} \lambda(T_{i-1})(h(T_i)D_mT_i - h(T_{i-1})D_mT_{i-1}) \\
&= -\sum_{i=1}^{N_T} h'(T_i) \int_0^{T_i} m(s)ds - \sum_{i=0}^{N_T} \lambda(T_i)h(T_i) \int_0^{T_i} m(s)ds + \sum_{i=1}^{N_T} \lambda(T_{i-1})h(T_i) \int_0^{T_i} m(s)ds \\
&= -\int_0^T h'(t) \int_0^t m(s)dsdN_t - \int_0^T \lambda(t)h(t) \int_0^t m(s)dsdN_t + \int_0^T \lambda(t-)h(t) \int_0^t m(s)dsdN_t \\
&= -\int_0^T h'(t) \int_0^t m(s)dsdN_t - \int_0^T (\lambda(t) - \lambda(t-))h(t) \int_0^t m(s)dsdN_t. \qquad (4.7)
\end{aligned}$$

By applying Itô formula for the process $f(t,\lambda(t)) = \lambda(t)h(t)\int_0^t m(s)ds$ with

$f(0,\lambda(0)) = f(T,\lambda(T)) = 0$, we have

$$\begin{aligned} 0 &= \int_0^T \lambda(t)[h'(t)\int_0^t m(s)ds + h(t)m(t)]dt + \sum_{n\geq 1, T_n \leq 1} (\lambda(T_n) - \lambda(T_n-))h(T_n)\int_0^{T_n} m(s)ds \\ &= \int_0^T \lambda(t)h'(t)\int_0^t m(s)dsdt + \int_0^T \lambda(t)h(t)m(t)dt + \int_0^T (\lambda(t)-\lambda(t-))h(t)\int_0^t m(s)dsdN_t. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we obtain

$$\begin{aligned} D_m\hat{\delta}(h) &= -\int_0^T h'(t)\int_0^t m(s)dsdN_t + \int_0^T \lambda(t)h'(t)\int_0^t m(s)dsdt + \int_0^T \lambda(t)h(t)m(t)dt \\ &= -\int_0^T h'(t)\int_0^t m(s)dsd\tilde{N}_t + \int_0^T \lambda(t)h(t)m(t)dt. \end{aligned}$$

By taking expectation, we get

$$\mathbb{E}[D_m\hat{\delta}(h)] = \mathbb{E}\left[\int_0^T \lambda(t)h(t)m(t)dt\right]. \tag{4.9}$$

4.1 Integration by parts formula in Bismut's way

In this section, we will establish an integration by parts formula directly from the perturbation of measure as Bismut's way (c.f. [2]).

Proposition 4.0. *For all $F \in \mathbb{D}_m^0$ and m in $\mathcal{H}$,*

$$\mathbb{E}[D_mF] = \mathbb{E}[\hat{\delta}(m)F].$$

Proof. By the definitions of $\mathcal{T}_\epsilon$ and $\mathbb{P}^\epsilon$, we have

$$\mathbb{E}[\mathcal{T}_\epsilon F] = \mathbb{E}^\epsilon[F] = \mathbb{E}\left[G^\epsilon F\right]. \tag{4.10}$$

By derivating in both sides of (4.10) with respect to ϵ at $\epsilon = 0$, we obtain

$$\mathbb{E}[D_mF] = \mathbb{E}\left[\frac{\partial G^\epsilon}{\partial \epsilon}|_{\epsilon=0}F\right].$$

Since we have

$$\frac{\partial G^\epsilon}{\partial \epsilon}|_{\epsilon=0} = -\int_0^T \lambda(t)m(t)dt + \sum_{i=1}^{N_T} m(T_i) = \hat{\delta}(m),$$

the result follows. □

Remark .2. *We can also retrieve the equality (4.9) by using the proposition 4.0 for $F = \hat{\delta}(h)$ as follow*

$$\mathbb{E}[D_m\hat{\delta}(h)] = \mathbb{E}[\hat{\delta}(m)\hat{\delta}(h)] = \mathbb{E}\left[\int_0^T m(t)d\tilde{N}_t \int_0^T h(t)d\tilde{N}_t\right] = \mathbb{E}\left[\int_0^T \lambda(t)h(t)m(t)dt\right].$$

Corollary 0. *For all $F, G \in \mathbb{D}_m^0$ such that $FG \in \mathbb{D}_m^0$ and m in $\mathcal{H}$,*

$$\mathbb{E}[GD_mF] = \mathbb{E}[F(\hat{\delta}(m)G - D_mG)].$$

Proof. The result is directly obtained by applying Proposition 4.0 for FG. $\square$

Corollary 0. *Let $\psi \in \mathbb{D}_m^0$, $\Phi = (\phi_1, \cdots, \phi_d)$ with $\phi_i \in \mathbb{D}_m^0, i = 1, \cdots, d$, and $f \in C^1(\mathbb{R}^d)$. Then*

$$\mathbb{E}[\psi D_m(f \circ \Phi)] = \mathbb{E}[\psi \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\Phi)D_m\phi_i] = \mathbb{E}[f(\Phi)(\hat{\delta}(m)\psi - D_m\psi)].$$

Proof. By applying Proposition 4.0 for $F = \psi f(\Phi)$, we obtain

$$\mathbb{E}[D_m(\psi f(\Phi))] = \mathbb{E}[\hat{\delta}(m)\psi f(\Phi)].$$

But $D_m(\psi f(\Phi)) = D_m\psi f(\Phi) + \psi D_m f(\Phi) = D_m\psi f(\Phi) + \psi \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\Phi)D_m\phi_i$. Hence

$$\begin{aligned}\mathbb{E}[\psi \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\Phi)D_m\phi_i] &= \mathbb{E}[\hat{\delta}(m)\psi f(\Phi)] - \mathbb{E}[D_m\psi f(\Phi)] \\ &= \mathbb{E}[f(\Phi)(\hat{\delta}(m)\psi - D_m\psi)].\end{aligned}$$

$\square$

Proposition 4.0. *(Closability of D_m) For any $m \in \mathcal{H}$, D_m is closable.*

Proof. To prove that D_m is closable, we have to prove that if $(F_i)_{i\in\mathbb{N}} \subset \mathcal{S}$ satisfying $F_i \to 0$ and $D_mF_i \to u$ in $L^2(\Omega)$ then $u = 0$ $\mathbb{P}$ a.s. From Corollary 0 we have

$$\mathbb{E}[GD_mF_i] = \mathbb{E}[F_i(\hat{\delta}(m)G - D_mG)].$$

Let i tend to infinite we get $\mathbb{E}[Gu] = 0$ for every $G \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $L^2(\Omega)$, we deduce $u = 0$ $\mathbb{P}$ a.s. $\square$

We shall identify D_m with its closed extension and denote by $\mathbb{D}_m^{1,2}$ its domain, i.e.,

$$\mathbb{D}_m^{1,2} = \{F \in L^2(\Omega) : \exists (F_i)_{i\in\mathbb{N}} \subset \mathcal{S} \text{ s.t } (F_i)_{i\in\mathbb{N}} \to F, (D_mF_i)_{i\in\mathbb{N}} \text{ converges in } L^2(\Omega)\}.$$

Obviously, $\mathcal{S} \subset \mathbb{D}_m^{1,2}$. For every $F \in \mathbb{D}_m^{1,2}$, we define $D_m F = \lim_{i\to\infty} D_m F_i$. Thanks to Proposition 4.0, this limit does not depend on the choice of the sequence $(F_i)_{i\in\mathbb{N}}$, so $D_m F$ is well-defined and coincides with the already defined $D_m F$ in case $F \in \mathcal{S}$. For every $F \in \mathbb{D}_m^{1,2}$, we have

$$||F||^2_{\mathbb{D}_m^{1,2}} = ||F||^2_{L^2(\Omega)} + ||D_m F||^2_{L^2(\Omega)} < +\infty.$$

4.2 The local Dirichlet form

Now we would like to define an operator D from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$ which plays the role of stochastic derivative in the sense that for every F in its domain and $m \in \mathcal{H}$,

$$D_m F =< DF, m >_{\mathcal{H}}= \int_0^T D_t F m(t) dt.$$

Let $(m_i)_{i\in\mathbb{N}}$ be an orthonormal basis of the space $\mathcal{H}$. Then every function $m \in \mathcal{H}$ can be expressed as $m = \sum_{i=1}^{+\infty} < m, m_i >_{\mathcal{H}} m_i$.

We now follow the construction of Bouleau and Hirsch [7]. We set

$$\mathbb{D}^{1,2} = \{X \in \bigcap_{i=1}^{+\infty} \mathbb{D}^{1,2}_{m_i} : \sum_{i=1}^{+\infty} \| D_{m_i} X \|^2_{L^2(\Omega)} < +\infty\}, \text{ and}$$
$$\forall X, Y \in \mathbb{D}^{1,2},\ \mathcal{E}(X,Y) = \sum_{i=1}^{+\infty} \mathbb{E}[D_{m_i} X D_{m_i} Y].$$

We denote $\mathcal{E}(X) := \mathcal{E}(X,X)$ for convenience. The next proposition defines the *local Dirichlet form*. Concerning the definition of local Dirichlet forms, our reference is [7].

Proposition 4.0. *The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a local Dirichlet admitting a gradient, D, and a carré du champ, Γ, given respectively by the following formulas:*

$$\begin{aligned}
\forall X \in \mathbb{D}^{1,2},\ DX &= \sum_{i=1}^{+\infty} D_{m_i} X\, m_i \in L^2(\Omega;\mathcal{H}),\\
\forall X, Y \in \mathbb{D}^{1,2},\ \Gamma[X,Y] &= \langle DX, DY \rangle_{\mathcal{H}}.
\end{aligned}$$

As a consequence $\mathbb{D}^{1,2}$ is a Hilbert equipped with the norm:

$$\forall X \in \mathbb{D}^{1,2},\ \| X \|^2_{\mathbb{D}^{1,2}} = ||X||^2_{L^2(\Omega)} + \mathcal{E}(X).$$

Proof. The proof is obvious and uses the same arguments as the proof of Proposition 4.2.1 in [7], Chapter II. Let us remark that the locality property is a direct consequence of the functional calculus, see Proposition 3.0. $\square$

Example 4.0. *By using the result in Example 3.0, for any $n \in \mathbb{N}$ we have*

1. $DZ_n = \sum_{i=1}^{+\infty} < DZ_n, m_i >_{\mathcal{H}} m_i = \sum_{i=1}^{+\infty} (D_{m_i} Z_n)\, m_i = 0$. *Similarly,* $DN_T = 0$.

2. *In the same way,*

$$\begin{aligned} D\bar{T}_n &= \sum_i < D\bar{T}_n, m_i >_{\mathcal{H}} m_i = \sum_i (D_{m_i}\bar{T}_n)\, m_i \\ &= -\sum_i m_i \int_0^{\bar{T}_n} m_i(s) ds = -\sum_i m_i \int_0^T m_i(s) \mathbf{1}_{[0,\bar{T}_n]}(s) ds \\ &= \sum_i < m_i, \frac{\bar{T}_n}{T} - \mathbf{1}_{[0,\bar{T}_n]} >_{\mathcal{H}} m_i = \frac{\bar{T}_n}{T} - \mathbf{1}_{[0,\bar{T}_n]}. \end{aligned}$$

Moreover, as a consequence of the functional calculus specific to local Dirichlet forms (see [7], Section I.6) the set of smooth function $\mathcal{S}$ is dense in $\mathbb{D}^{1,2}$ and if $F \in \mathcal{S}$, $F = a\mathbf{1}_{A_0} + \sum_{n=1}^{d} \sum_{i_1,\cdots,i_n \in \{1,\cdots,l\}} f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)\mathbf{1}_{A_n^{i_1,\cdots,i_n}}$, then

$$D_t F = -\sum_{n=1}^{d} \sum_{i_1,\cdots,i_n \in \{1,\cdots,l\}} \sum_{i=1}^{n} \frac{\partial}{\partial t_i} f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n) \times \left(\mathbf{1}_{[0,T_i]}(t) - \frac{T_i}{T} \right) \mathbf{1}_{A_n^{i_1,\cdots,i_n}}.$$

Remark .3. *The expression of the gradient operator D above prove that nor the Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ nor the gradient D depend on the choice of the orthonormal basis $(m_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$.*

4.3 Divergence operator

Let $\delta : L^2(\Omega;\mathcal{H}) \to L^2(\Omega)$ be the adjoint operator of D. Its domain, $Dom\delta$, is the set of $u \in L^2(\Omega;\mathcal{H})$ such that there exists $c > 0$ satisfying

$$\left| \mathbb{E}\left[\int_0^T D_t F u_t dt \right] \right| \leq c||F||_{\mathbb{D}^{1,2}}, \ \forall F \in \mathbb{D}^{1,2}.$$

It follows from the properties of D that δ is also a closed densely defined operator. We have the integration by parts formula by the duality:

$$\mathbb{E}[\delta(u)F] = \mathbb{E}\left[< u, DF >_{\mathcal{H}}\right] = \mathbb{E}\left[\int_0^T u_t D_t F dt \right], \ \forall F \in \mathbb{D}^{1,2}, u \in Dom\delta.$$

Proposition 4.0. *For every $u \in \mathbb{D}^{1,2} \otimes \mathcal{H}$, we have $u \in Dom\delta$ and*

$$\delta(u) = \int_0^T u_t d\tilde{N}_t - \int_0^T D_t u_t dt.$$

Proof. First of all, if $u = mG$ with $m \in \mathcal{H}, G \in \mathbb{D}^{1,2}$ then

$$\mathbb{E}\left[\int_0^T u_t D_t F\right] dt = \mathbb{E}\left[G\int_0^T m(t) D_t F dt\right] = \mathbb{E}[G D_m F] = \mathbb{E}[F(\hat{\delta}(m)G - D_m G)]$$

for every $F \in \mathbb{D}^{1,2}$. From the uniqueness of δ, we have

$$\delta(u) = \hat{\delta}(m)G - D_m G = G\int_0^T m(t)d\tilde{N}_t - \int_0^T D_t G m(t)dt = \int_0^T u_t d\tilde{N}_t - \int_0^T D_t u_t dt.$$

By linearity, this is true for every function in

$$\{u \in \mathbb{D}^{1,2} \otimes \mathcal{H} : u = \sum_{i=1}^n m_i G_i, m_i \in \mathcal{H}, G_i \in \mathbb{D}^{1,2}\}.$$

The result of the proposition follows since this set is dense in $\mathbb{D}^{1,2} \otimes \mathcal{H}$. □

Remark .4. *If $u \in \mathcal{H}$, then $D_s u_s = 0$ for every $s \in [0,T]$, so the divergence operator δ coincides with the integral w.r.t. the compensated Poisson process $\hat{\delta}$. From the proof of Proposition 4.0, we can retain that*

1. *$\delta(mG) = \delta(m)G - D_m G$ for every $m \in \mathcal{H}, G \in \mathbb{D}^{1,2}$.*
2. *$\mathbb{E}[G D_m F] = \mathbb{E}[F\delta(mG)]$ for every $m \in \mathcal{H}, F, G \in \mathbb{D}^{1,2}$.*

Corollary 0. *If $u \in L^2(\Omega;\mathcal{H})$ is an adapted process then*

$$\delta(u) = \int_0^T u_t d\tilde{N}_t.$$

Proof. We establish first the result for an elementary process u in $\mathbb{D}^{1,2} \otimes \mathcal{H}$ given by

$$u_t = \sum_{j=1}^q f_j(\bar{T}_1, \cdots, \bar{T}_n, Z_1, \cdots, Z_n)\mathbf{1}_{]t_{j-1},t_j]}(t),$$

where $q, n \in \mathbb{N}^*$, $0 \leq t_1 \leq \cdots \leq t_n \leq T$ and for each j, $f_j : \mathbb{R}^n \times \mathcal{K}^n \to \mathbb{R}$ is infinitely differentiable with bounded derivatives of any order with respect to variables in $\mathbb{R}^n$ and such that $f_j(\bar{T}_1, \cdots, \bar{T}_n, Z_1, \cdots, Z_n)$ is $\mathcal{F}_{t_{j-1}}$-measurable. As a consequence, on the set $\Delta_i^j = \{0 \leq s_1 \leq \cdots \leq s_i \leq t_{j-1} \leq s_{i+1} \leq \cdots \leq s_n \leq T\}$,

$$f_j(s_1, \cdots, s_n, c_1, \cdots, c_n) = f_j(s_1, \cdots, s_i, t_{j-1}, \cdots, t_{j-1}, c_1, \cdots, c_i, k_1, \cdots, k_1)$$

for every $(c_1, \cdots, c_n) \in \mathcal{K}^n$. Hence for each $k > i$,

$$\forall (s_1, \cdots, s_n) \in \Delta_i^j, \ \frac{\partial f_j}{\partial s_k}(s_1, \cdots, s_n, c_1, \cdots c_n) = 0.$$

As u belongs to $\mathbb{D}^{1,2} \otimes \mathcal{H}$, one has to keep in mind that $\int_0^T u_t dt = 0$. We note also that the set of such elementary adapted processes is dense in the (closed) subspace of adapted processes in $\mathbb{D}^{1,2} \otimes \mathcal{H}$.
We have

$$D_t u_t = \sum_{j=1}^{q} \sum_{i=1}^{n} \frac{\partial f_j}{\partial s_i}(\bar{T}_1, \cdots, \bar{T}_n, Z_1, \cdots, Z_n) \left(\frac{\bar{T}_i}{T} - \mathbf{1}_{[0,\bar{T}_i]}(t) \right) \mathbf{1}_{]t_{j-1},t_j]}(t).$$

Since u is adapted we have for each (i, j),

$$\frac{\partial f_j}{\partial s_i}(\bar{T}_1, \cdots, \bar{T}_n, Z_1, \cdots, Z_n) \mathbf{1}_{[0,\bar{T}_i]}(t) \mathbf{1}_{]t_{j-1},t_j]}(t) = 0.$$

Then, adopting obvious notations, we have for each i

$$\int_0^T \sum_{j=1}^{q} \frac{\partial f_j}{\partial s_i}(T_1, \cdots, T_n, Z_1, \cdots, Z_n) \frac{T_i}{T} \mathbf{1}_{]t_{j-1},t_j]}(t) dt = \frac{T_i}{T} \frac{\partial}{\partial s_i} \int_0^T u_s ds = 0,$$

hence $\int_0^T D_t u_t dt = 0$ which proves the result in this case. We conclude using a density argument. $\square$

The next proposition is an important property of the operator δ, which follows from the fact that D is a derivative.

Proposition 4.0. *Let $F \in \mathbb{D}^{1,2}$ and $X \in Dom\delta$ such that $F\delta(X) - \int_0^T D_t F X_t dt \in L^2(\Omega)$, then $FX \in Dom\delta$ and*

$$\delta(FX) = F\delta(X) - \int_0^T D_t F X_t dt.$$

Proof. For every $G \in \mathcal{S}$, we have

$$\begin{aligned} \mathbb{E}[\delta(FX)G] &= \mathbb{E} \int_0^T F X_t D_t G dt = \mathbb{E} \int_0^T X_t (D_t(GF) - G D_t F) dt \\ &= \mathbb{E} \left[G(F\delta(X) - \int_0^T D_t F X_t dt) \right]. \end{aligned}$$

$\square$

In particular, if $X = m \in \mathcal{H} \subset Dom\delta$ then $\delta(m) = \int_0^T m_t d\tilde{N}_t$ and $\int_0^T D_t F m(t) dt = D_m F$. Hence we have a simpler formula

$$\delta(mF) = F \int_0^T m(t) d\tilde{N}_t - D_m F.$$

Remark .5. *In this approach, the Clark-Ocone formula does not hold. For example, we have already shown that $D_m N_T = 0, \forall m \in \mathcal{H}$, so $D_t N_T = 0, \forall t \in [0, T]$, but we do not have $N_T = \mathbb{E} N_T$.*

5 An absolute continuity criterion

Recall that $A_0 = \{N_T = 0\}$, and $A_n^{i_1,\cdots,i_n} = \{N_T = n, Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}$ for all $n \in \mathbb{N}^*$, $i_1, \cdots, i_n \in \{1, \cdots, l\}$.

Lemma 5.0. *Let $\Delta_n = \{(t_1, \cdots, t_n) : 0 < t_1 < \cdots < t_n < T\}$ and $\lambda_1, \cdots, \lambda_n$ as in Proposition 2.0. The joint distribution of $(T_1, \cdots, T_n)$ conditionally to $A_n^{i_1,\cdots,i_n}$ has a density*

$$\begin{aligned} k_n^{i_1,\cdots,i_n} : \mathbb{R}^n \quad &\to \quad \mathbb{R}^+ \\ t = (t_1, \cdots, t_n) \quad &\mapsto \quad k_n^{i_1,\cdots,i_n}(t) = n!\, \mathbf{1}_{\Delta_n}(t) \prod_{i=1, \lambda_{i+1} \neq \lambda_i}^{n} \frac{(\lambda_{i+1} - \lambda_i) e^{t_i(\lambda_{i+1} - \lambda_i)}}{e^{(\lambda_{i+1} - \lambda_i)T} - 1} \end{aligned}$$

with respect to the Lebesgue measure ν_n on $\mathbb{R}^n$.

Proof. Let f be a measureable function on $\mathbb{R}^n$. Then

$$\begin{aligned} &\mathbb{E}[f(T_1, \cdots, T_n) | A_n^{i_1,\cdots,i_n}] = \mathbb{E}[\mathbf{1}_{A_n^{i_1,\cdots,i_n}} f(T_1, \cdots, T_n)] / \mathbb{P}(A_n^{i_1,\cdots,i_n}) \\ &= \frac{\mathbb{E}[f(T_1, \cdots, T_n) \mathbf{1}_{\{N_T = n\}} \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]}{\mathbb{E}[\mathbf{1}_{\{N_T = n\}} \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]} \\ &= \frac{\mathbb{E}[f(T_1, \cdots, T_n) \mathbf{1}_{\{N_T = n\}} | \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]\, \mathbb{E}[\mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]}{\mathbb{E}[\mathbf{1}_{\{N_T = n\}} | \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]\, \mathbb{E}[\mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]} \\ &= \frac{\mathbb{E}[f(T_1, \cdots, T_n) \mathbf{1}_{\{T_n \leq T < T_{n+1}\}} | \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]}{\mathbb{E}[\mathbf{1}_{\{T_n \leq T < T_{n+1}\}} | \mathbf{1}_{\{Z_1 = k_{i_1}, \cdots, Z_n = k_{i_n}\}}]} \\ &= \frac{\iint_{0 < t_1 < \cdots < t_n \leq T < t_{n+1}} f(t_1, \cdots, t_n) \prod_{i=1}^{n+1} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} dt_1 \cdots dt_{n+1}}{\iint_{0 < t_1 < \cdots < t_n \leq T < t_{n+1}} \prod_{i=1}^{n+1} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} dt_1 \cdots dt_{n+1}} \\ &= \frac{\iint_{\Delta_n} f \prod_{i=1}^{n} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} e^{\lambda_{n+1} t_n} dt_1 \cdots dt \int_T^\infty \lambda_{n+1} e^{-\lambda_{n+1} t_{n+1}} dt_{n+1}}{\iint_{\Delta_n} \prod_{i=1}^{n} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} e^{\lambda_{n+1} t_n} dt_1 \cdots dt \int_T^\infty \lambda_{n+1} e^{-\lambda_{n+1} t_{n+1}} dt_{n+1}} \\ &= \frac{\iint_{\Delta_n} f \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n}{\iint_{\Delta_n} \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n}, \end{aligned}$$

where

$$\varphi(t_1, \cdots, t_n) = e^{-\lambda_{n+1}(T - t_n)} \prod_{i=1}^{n} \lambda_i e^{-\lambda_i (t_i - t_{i-1})} = e^{-\lambda_{n+1} T} \prod_{i=1}^{n} \lambda_i e^{t_i (\lambda_{i+1} - \lambda_i)},$$

and

$$\begin{aligned} &\iint_{\Delta_n} \varphi(t_1, \cdots, t_n) dt_1 \cdots dt_n = \frac{1}{n!} \iint_{[0,T]^n} e^{-\lambda_{n+1} T} \prod_{i=1}^{n} \lambda_i e^{t_i (\lambda_{i+1} - \lambda_i)} dt_1 \cdots dt_n \\ &= \frac{e^{-\lambda_{n+1}} T}{n!} \prod_{i=1}^{n} \lambda_i \prod_{i=1, \lambda_{i+1} \neq \lambda_i}^{n} \frac{e^{(\lambda_{i+1} - \lambda_i)T} - 1}{\lambda_{i+1} - \lambda_i}. \end{aligned}$$

$\square$

Remark .6. *The density* $k_n^{i_1,\cdots,i_n}$ *is a positive function of class* $\mathcal{C}^\infty$ *on* Δ_n. *Particularly, in case* C *is a homogeneous Poisson process, we have* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+1}$, *so* $k_n^{i_1,\cdots,i_n}(t_1,\cdots,t_n) = n!\,\mathbf{1}_{\Delta_n}(t)$, *which means* $(T_1,\cdots,T_n)$ *has uniform distribution on* Δ_n *conditionnally to* $A_n^{i_1,\cdots,i_n}$.

Now fix $n \in \mathbb{N}^*$, $i_1,\cdots,i_n \in \{1,\cdots,l\}$, we consider $\mathbf{d}_n^{i_1,\cdots,i_n}$ the set of $\mathcal{B}(\mathbb{R}^n)$-measurable functions u in $L^2(k_n^{i_1,\cdots,i_n}dt)$ such that for any $i \in \{1,\cdots,n\}$, and ν_{n-1}-almost all $\bar{t} = (t_1,\cdots,t_{i-1},t_{i+1},\cdots,t_n) \in \mathbb{R}^{n-1}$, $u_{\bar{t}}^{(i)} = u(t_1,\cdots,t_{i-1},.,t_{i+1},\cdots,t_n)$ has an absolutely continuous version $\tilde{u}_{\bar{t}}^{(i)}$ on $\{t_i : (t_1,\cdots,t_n) \in \Delta_n\}$ such that

$$\sum_{i,j=1}^n \frac{\partial u}{\partial t_i}\frac{\partial u}{\partial t_j}\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) \in L^1(k_n^{i_1,\cdots,i_n}dt),$$

where $\dfrac{\partial u}{\partial t_i} = \dfrac{\partial \tilde{u}_{\bar{t}}^{(i)}}{\partial s}$. We consider the following quadratic form on $\mathbf{d}_n^{i_1,\cdots,i_n}$

$$e_n^{i_1,\cdots,i_n}(u,v) = \frac{1}{2}\int_{\mathbb{R}^n}\sum_{i,j=1}^n \frac{\partial u}{\partial t_i}\frac{\partial v}{\partial t_j}\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) k_n^{i_1,\cdots,i_n}dt, \forall u,v \in \mathbf{d}_n^{i_1,\cdots,i_n}.$$

As usual, we denote $e(u,u)$ by $e(u)$.

Proposition 5.0. 1. *The bilinear form* $(\mathbf{d}_n^{i_1,\cdots,i_n}, e_n^{i_1,\cdots,i_n})$ *defines a local Dirichlet form on* $L^2(k_n^{i_1,\cdots,i_n}dt)$ *which admits a "carré du champ" operator* γ_n *and a gradient operator* $\tilde{D}^n$ *given by*

$$\gamma_n[u,v](t) = \sum_{i,j=1}^n \frac{\partial u}{\partial t_i}\frac{\partial v}{\partial t_j}\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right),\ \tilde{D}_s^n u(t) = \sum_{i=1}^n \frac{\partial u}{\partial t_i}\left(\frac{t_i}{T} - \mathbf{1}_{[0,t_i]}(s)\right)$$

for all $u,v \in \mathbf{d}_n^{i_1,\cdots,i_n}$, $t = (t_1,\cdots,t_n) \in \mathbb{R}^n$, $s \in [0,T]$.

2. *The structure* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n^{i_1,\cdots,i_n}dt, \mathbf{d}_n^{i_1,\cdots,i_n}, \gamma_n)$ *satisfies the (EID) property, i.e., for every* $d \in \mathbb{N}^*, u = (u_1,\cdots,u_d) \in (\mathbf{d}_n^{i_1,\cdots,i_n})^d$,

$$u_*[(det\gamma_n[u]).k_n^{i_1,\cdots,i_n}\nu_n] \ll \nu_d,$$

where $\gamma_n[u]$ *denotes the matrix* $(\gamma_n(u_i,u_j))_{1\le i,j\le d}$.

Proof. The results are obtained by applying Proposition 1 and Theorem 2 in [4] for $\mathbf{d} = \mathbf{d}_n^{i_1,\cdots,i_n}, k = k_n^{i_1,\cdots,i_n}$ and $\xi_{ij}(t) = t_i \wedge t_j - \frac{t_i t_j}{T}, \forall 1 \le i,j \le n, t = (t_1,\cdots,t_n)$. We only have to prove that ξ is locally elliptic on Δ_n. Indeed, for every $\alpha = (\alpha_1,\cdots,\alpha_n) \in \mathbb{R}^n$, $t = (t_1,\cdots,t_n) \in \Delta_n$, we have

$$\alpha^*\xi(t)\alpha = \sum_{i,j=1}^n \alpha_i\alpha_j\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) = \int_0^T\left(\sum_{i=1}^n \alpha_i\left(\mathbf{1}_{[0,t_i]}(s) - \frac{t_i}{T}\right)\right)^2 ds \ge 0.$$

The equality occurs if and only if $\sum_{i=1}^n \alpha_i(\mathbf{1}_{[0,t_i]}(s) - \frac{t_i}{T}) = 0, \forall s \in [0,T]$. By taking $t_n < s < T$, we obtain $\sum_{i=1}^n \alpha_i t_i = 0$, so $\sum_{i=1}^n \alpha_i \mathbf{1}_{[0,t_i]}(s) = 0, \forall s \in [0,T]$. By taking $t_{n-1} < s < t_n$, we obtain $\alpha_n = 0$, so $\sum_{i=1}^{n-1} \alpha_i \mathbf{1}_{[0,t_i]}(s) = 0, \forall s \in [0,T]$. Continue this process and finally we obtain $\alpha_i = 0, \forall i = 1, \cdots, n$. $\square$

Remark .7. *Since the density function $k_n^{i_1,\cdots,i_n}$ is bounded in Δ_n both below and above by positive constants, we have $L^p(k_n^{i_1,\cdots,i_n}dt) = L^p(dt)$, so the domain $d_n^{i_1,\cdots,i_n}$ does not depend on $i_1,\cdots,i_n$.*

For every measurable random variable $F : \Omega \to \mathbb{R}$, there exists a constant $a \in \mathbb{R}$ and measurable functions $f_n^{i_1,\cdots,i_n} : \mathbb{R}^n \to \mathbb{R}, \forall n \in \mathbb{N}^*, i_1,\cdots,i_n \in \{1,\cdots,l\}$ such that

$$F = a\mathbf{1}_{A_0} + \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_n \in\{1,\cdots,l\}} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n) \quad \mathbb{P}-a.s.. \tag{5.11}$$

From now on, we will write $\sum_{n,i}$ instead of $\sum_{n=1}^{\infty}\sum_{i_1,\cdots,i_n\in\{1,\cdots,l\}}$ for simplicity.

Proposition 5.0. *For every measurable random variable F of the form (5.11), we have $F \in \mathbb{D}^{1,2}$ if and only if $f_n^{i_1,\cdots,i_n} \in \mathbf{d}_n^{i_1,\cdots,i_n}, \forall n \in \mathbb{N}^*, \forall i_1,\cdots,i_n \in \{1,\cdots,l\}$ and*

$$\sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{\mathbf{d}_n^{i_1,\cdots,i_n}} < \infty.$$

Moreover, if $F \in \mathbb{D}^{1,2}$ then

$$||F||^2_{\mathbb{D}^{1,2}} = a^2\mathbb{P}(A_0) + \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{\mathbf{d}_n^{i_1,\cdots,i_n}}. \tag{5.12}$$

Proof. For every F of the form (5.11), we have

$$\begin{aligned}
||F||^2_{L^2(\Omega)} &= \mathbb{E}[F^2] = a^2\mathbb{P}(A_0) + \sum_{n,i}\mathbb{E}\left[\mathbf{1}_{A_n^{i_1,\cdots,i_n}}\left(f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)\right)^2\right] \\
&= a^2\mathbb{P}(A_0) + \sum_{n,i}\mathbb{P}(A_n^{i_1,\cdots,i_n})\mathbb{E}\left[\left(f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)\right)^2|A_n^{i_1,\cdots,i_n}\right] \\
&= a^2\mathbb{P}(A_0) + \sum_{n,i}\mathbb{P}(A_n^{i_1,\cdots,i_n})\int_{\mathbb{R}^n}\left(f_n^{i_1,\cdots,i_n}(t)\right)^2 k_n^{i_1,\cdots,i_n}(t)dt \\
&= a^2\mathbb{P}(A_0) + \sum_{n,i}\mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{L^2(k_n^{i_1,\cdots,i_n}dt)}.
\end{aligned}$$

And the Malliavin derivative

$$\begin{aligned}
D_sF &= \sum_{n,i}\mathbf{1}_{A_n^{i_1,\cdots,i_n}}\left(\sum_{i=1}^n \frac{\partial f_n^{i_1,\cdots,i_n}}{\partial t_i}(T_1,\cdots,T_n)\left(\frac{T_i}{T} - \mathbf{1}_{[0,T_i]}(s)\right)\right) \\
&= \sum_{n,i}\mathbf{1}_{A_n^{i_1,\cdots,i_n}}\tilde{D}_s^n f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n),
\end{aligned}$$

which deduces

$$
\begin{aligned}
||DF||^2_{L^2(\Omega;\mathcal{H})} &= \int_0^T \mathbb{E}[(D_sF)^2]ds \\
&= \sum_{n,i} \int_0^T \mathbb{E}\left[\mathbf{1}_{A_n^{i_1,\cdots,i_n}} \left(\tilde{D}^n_s f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)\right)^2\right] ds \\
&= \sum_{n,i} \int_0^T \mathbb{P}(A_n^{i_1,\cdots,i_n})\mathbb{E}\left[\left(\tilde{D}^n_s f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)\right)^2 |A_n^{i_1,\cdots,i_n}\right] ds \\
&= \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n}) \int_0^T \int_{\mathbb{R}^n} \left(\tilde{D}^n_s f_n^{i_1,\cdots,i_n}(t)\right)^2 k_n^{i_1,\cdots,i_n}(t)dtds \\
&= \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||\tilde{D}^n f_n^{i_1,\cdots,i_n}||^2_{L^2((\mathbb{R}^n,k_n^{i_1,\cdots,i_n}dt);\mathcal{H})} \\
&= \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n}) e_n^{i_1,\cdots,i_n}[f_n^{i_1,\cdots,i_n}].
\end{aligned}
$$

Therefore,

$$
||F||^2_{L^2(\Omega)} + ||DF||^2_{L^2(\Omega;\mathcal{H})} = \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{\mathbf{d}_n^{i_1,\cdots,i_n}}.
$$

From here we obtain the condition for $F \in \mathbb{D}^{1,2}$. The equation (5.12) is obvious since the fact that $||F||^2_{\mathbb{D}^{1,2}} = ||F||^2_{L^2(\Omega)} + \mathcal{E}(F) = ||F||^2_{L^2(\Omega)} + ||DF||^2_{L^2(\Omega;\mathcal{H})}$. □

Remark .8. *In summary, we have the following relations between the Dirichlet structure $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{D}^{1,2}, \Gamma)$ and the Dirichlet structures $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n^{i_1,\cdots,i_n}dt, \mathbf{d}_n^{i_1,\cdots,i_n}, \gamma_n)$: for every $F \in \mathbb{D}^{1,2}$ having the form (5.11),*

1. $||F||^2_{L^2(\Omega)} = a^2\mathbb{P}(A_0) + \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{L^2(k_n^{i_1,\cdots,i_n}dt)}$.
2. $D_sF = \sum_{n,i} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} \tilde{D}^n_s f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)$, $\forall s \in [0,T]$.
3. $\Gamma[F] = \sum_{n,i} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} \gamma_n[f_n^{i_1,\cdots,i_n}](T_1,\cdots,T_n)$
4. $\mathcal{E}(F) = \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n}) e_n^{i_1,\cdots,i_n}[f_n^{i_1,\cdots,i_n}]$.
5. $||F||^2_{\mathbb{D}^{1,2}} = a^2\mathbb{P}(A_0) + \sum_{n,i} \mathbb{P}(A_n^{i_1,\cdots,i_n})||f_n^{i_1,\cdots,i_n}||^2_{\mathbf{d}_n^{i_1,\cdots,i_n}}$.

Now let $d \in \mathbb{N}^*$ and $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$, we denote by $\Gamma[F]$ the $d \times d$ symmetric matrix $(\Gamma[F_i, F_j])_{1 \le i,j \le d}$.

Theorem 5.1. *(Energy image density property)*
If $F \in (\mathbb{D}^{1,2})^d$, then $F_*[(det\Gamma[F]).\mathbb{P}]$ is absolutely continuous with respect to the Lebesgue measure ν_d on $\mathbb{R}^d$.

Proof. Let $B \subset \mathbb{R}^d$ such that $\nu_d(B) = 0$. We have to prove that

$$F_*[(det\Gamma[F]).\mathbb{P}](B) = 0 \text{ or equivalently, } \mathbb{E}[\mathbf{1}_B(F)det\Gamma[F]] = 0.$$

Thanks to the previous proposition, we write

$$F = a\mathbf{1}_{A_0} + \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_n \in \{1,\cdots,l\}} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n),$$

with $f_n^{i_1,\cdots,i_n} \in (\mathbf{d}_n^{i_1,\cdots,i_n})^d, \forall n \in \mathbb{N}^*, \forall i_1,\cdots,i_n \in \{1,\cdots,l\}$. Then

$$\begin{aligned}
\Gamma[F,F] &= \sum_{n,i} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} \gamma_n[f_n^{i_1,\cdots,i_n}]\,(T_1,\cdots,T_n)\,, \\
\mathbb{E}[\mathbf{1}_B(F)det\Gamma[F]] &= \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_n \in \{1,\cdots,l\}} \mathbb{P}(A_n^{i_1,\cdots,i_n}) \int_{\mathbb{R}^n} \mathbf{1}_B det\gamma_n[f_n^{i_1,\cdots,i_n}] k_n^{i_1,\cdots,i_n} dt.
\end{aligned}$$

By applying Part 2 of Proposition 5.0 for $u = f_n^{i_1,\cdots,i_n}$, we obtain

$$\int_{\mathbb{R}^n} \mathbf{1}_B det\gamma_n[f_n^{i_1,\cdots,i_n}] k_n^{i_1,\cdots,i_n} dt = 0, \forall n \in \mathbb{N}^*.$$

Hence $\mathbb{E}[\mathbf{1}_B(F)det\Gamma[F]] = 0$. □

6 Application to SDEs involving the Markov chain

Reminding that the original works on Malliavin calculus aimed to study the existence and the smoothness of densities of solutions to stochastic differential equations, this section is contributed to infer the existence in some sense of a density from the non-degeneracy of the Malliavin covariance matrix and deduce a simple result for Markov chain driven stochastic differential equations.

First notice that every random variable F in Ω can not have a density nor the Malliavin variance $\int_0^T |D_tF|^2 dt$ be a.s. strictly positive. Indeed, from the decomposition

$$F = a\mathbf{1}_{A_0} + \sum_{n=1}^{\infty} \sum_{i_1,\cdots,i_n \in \{1,\cdots,l\}} \mathbf{1}_{A_n^{i_1,\cdots,i_n}} f_n^{i_1,\cdots,i_n}(T_1,\cdots,T_n)$$

we deduce that the law of F has a point mass at a (since $\mathbb{P}(F = a) \geq \mathbb{P}(N_T = 0) = \mathbb{P}(T_1 > T) = e^{-\lambda_1 T} > 0$), and $\int_0^T |D_tF|^2 dt = 0$ on $\{N_T = 0\}$. Therefore, we shall rather give conditions under which $(\mathbf{1}_{\{N_T \geq 1\}}\mathbb{P})F^{-1}$ has a density.

Now we will study the regularity of the solution of a stochastic differential equation driven by the Markov chain C. Let $d \in \mathbb{N}^*$ and consider the SDE

$$X_t = x_0 + \int_0^t f(s, X_s, C_s)ds + \int_0^t g(s, X_{s-}, C_s)dN_s, \tag{6.13}$$

or in the differential form

$$dX_t = f(t, X_t, C_t)dt + g(t, X_{t-}, C_t)dN_t, \quad X_0 = x_0.$$

where $x_0 \in \mathbb{R}^d$ fixed, $(N_t)_{t\geq 0}$ is the process of cumulative of jumps of $(C_t)_{t\geq 0}$ defined by (2.3), the functions $f, g : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}^d$ are measurable and satisfy

1. $\forall t \in \mathbb{R}^+, k \in \mathcal{K}$, the maps $x \mapsto f(t, x, k)$, $x \mapsto g(t, x, k)$ are $\mathcal{C}^1$;
2. $\sup_{(t,x,k)} |\nabla_x f(t, x, k)| + |\nabla_x g(t, x, k)| < +\infty$.

Remark .9. *Here, the term $g(s, X_{s^-}, C_s)$ is not predictable but it is not a real problem since N is of finite variation so that that this equation may be solved pathwise. By adapting the proofs in [2], [13] or using the explicit expression of the solution given below, it is clear that Equation* (6.13) *admits a unique solution, X, such that*

$$\forall T > 0, \sup_{t\in[0,T]} |X_t| \in \bigcap_{p\geq 1} L^p(\Omega).$$

We would like to apply Theorem 5.1 to $F = X_T$. For each $k \in \mathcal{K}$, let $\{\Phi_{s,t}(x, k), t \geq s\}$ denote the deterministic flow defined by

$$\Phi_{s,t}(x, k) = x + \int_s^t f(u, \Phi_{s,u}(x, k), k)du,$$

or in the differential form

$$\partial_t \Phi_{s,t}(x, k) = f(t, \Phi_{s,t}(x, k), k), \quad \Phi_{s,s}(x, k) = x. \tag{6.14}$$

We can see that $\Phi_{s,t}(x, k)$ exists and is unique for every $x \in \mathbb{R}^d, k \in \mathcal{K}$. On the set $\{N_t = 0\}$, $(X_s)_{0\leq s\leq t}$ is the solution of

$$dX_s = f(s, X_s, Z_0)ds, \quad X_0 = x_0.$$

Hence we have $X_s = \Phi_{0,s}(x_0, Z_0), 0 \leq s \leq t$, and particularly $X_t = \Phi_{0,t}(x_0, Z_0)$. In case $N_t \geq 1$, we have the following result:

Proposition 6.1. *Let Ψ is the map: $(t, x, k) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{K} \mapsto \Psi(t, x, k) = x + g(t, x, k)$. Then for any $t \geq 0$ and $i \in \mathbb{N}^*$,*

$$X_t = \Phi_{T_i,t}(., Z_i) \circ \Psi(T_i, ., Z_i) \circ \cdots \circ \Phi_{T_1,T_2}(., Z_1) \circ \Psi(T_1, ., Z_1) \circ \Phi_{0,T_1}(x_0, Z_0) \tag{6.15}$$

$\mathbb{P}$ a.e. on the set $\{N_t = i\}$.

Proof. We will prove (6.15) by induction. On the set $\{N_t = 1\}$, the process (N_t) has a jump at T_1. The equation remain unchanged for $0 \leq s < T_1$, so $X_s = \Phi_{0,s}(x_0, Z_0), 0 \leq s < T_1$. For $T_1 \leq s \leq t$, X_s satisfies the following equation

$$dX_s = f(s, X_s, Z_1)ds, \quad X_{T_1} = \Psi(T_1, X_{T_1-}, Z_1).$$

We obtain $X_s = \Phi_{T_1,s}(X_{T_1}, Z_1) = \Phi_{T_1,s}(\Psi(T_1, \Phi_{0,T_1}(x_0, Z_0), Z_1), Z_1), T_1 \leq s \leq t$. Particularly,

$$X_t = \Phi_{T_1,t}(X_{T_1}, Z_1) = \Phi_{T_1,t}(., Z_1) \circ \Psi(T_1, ., Z_1) \circ \Phi_{0,T_1}(x_0, Z_0).$$

Now suppose that (6.15) holds until i. Let consider the set $\{N_t = i+1\}$, i.e., the set of trajectories of the Markov chain having $i+1$ jumps up to t. For $T_{i+1} \leq s \leq t$, X_s satisfies the following equation

$$dX_s = f(s, X_s, Z_{i+1})ds, \quad X_{T_{i+1}} = \Psi(T_{i+1}, X_{T_{i+1}-}, Z_{i+1}).$$

Hence $X_s = \Phi_{T_{i+1},s}(\Psi(T_{i+1}, X_{T_{i+1}-}, Z_{i+1}), Z_{i+1}))$. We obtain the formula for $\{N_t = i+1\}$ by using the induction assumption

$$X_{T_{i+1}-} = \Phi_{T_i,T_{i+1}}(., Z_i) \circ \Psi(T_i, ., Z_i) \circ \cdots \circ \Phi_{T_1,T_2}(., Z_1) \circ \Psi(T_1, ., Z_1) \circ \Phi_{0,T_1}(x_0, Z_0).$$

□

From Equation (6.14), the process $\partial_t \Phi_{s,t}(x,k)$ satisfies

$$\partial_t \left(\partial_t \Phi_{s,t}(x,k)\right) = \nabla_x f(t, \Phi_{s,t}(x,k), k) \partial_t \Phi_{s,t}(x,k), \quad \partial_t \Phi_{s,s}(x,k) = f(s,x,k),$$

which deduces

$$\partial_t \Phi_{s,t}(x,k) = f(s,x,k) exp\left(\int_s^t \nabla_x f(u, \Phi_{s,u}(x,k), k) du\right).$$

Similarly, the processes $\nabla_x \Phi_{s,t}(x,k)$ satisfies

$$\partial_t(\nabla_x \Phi_{s,t}(x,k)) = \nabla_x f(t, \Phi_{s,t}(x,k), k) \nabla_x \Phi_{s,t}(x,k), \quad \nabla_x \Phi_{s,s}(x,k) = I_d.$$

Hence,

$$\nabla_x \Phi_{s,t}(x,k) = exp\left(\int_s^t \nabla_x f(u, \Phi_{s,u}(x,k), k) du\right).$$

We define

$$K_t(x) = \nabla_x X_t(x), \forall t \geq 0, \tag{6.16}$$

$$K_t^s(x) = \left(\nabla_x X_t^s\right)(X_s(x)), \forall t \geq s \geq 0, \tag{6.17}$$

where $(X_t^s(x))_{t \geq s}$ is the solution of

$$X_t^s = x + \int_s^t f(u, X_u^s, C_u) du + \int_s^t g(u, X_{u-}^s, C_u) dN_u. \tag{6.18}$$

The process (K_t) defined by (6.16) satisfies the SDE

$$K_t = I_d + \int_0^t \nabla_x f(s, X_s, C_s) K_s ds + \int_0^t \nabla_x g(s, X_{s-}, C_s) K_{s-} dN_s. \qquad (6.19)$$

Suppose that $det[I_d + \nabla g(., x, .)] \neq 0$ and let $(\bar{K}_t)$ be the solution of the following SDE

$$\bar{K}_t = I_d - \int_0^t \nabla_x f(s, X_s, C_s) \bar{K}_s ds - \int_0^t (I_d + \nabla_x g(s, X_{s-}, C_s))^{-1} \nabla_x g(s, X_{s-}, C_s) \bar{K}_{s-} dN_s. \qquad (6.20)$$

Proposition 6.1. *If $det[I_d + \nabla g(., x, .)] \neq 0$, the processes (K_t) and $(\bar{K}_t)$ defined by* (6.16) *and* (6.20) *satisfy $K_t \bar{K}_t = 1, \forall t \geq 0$.*

Proof. Indeed, the process $Y_t = K_t \bar{K}_t$ satisfies $Y_0 = I_d$, and

$$\begin{aligned}
dY_t &= \bar{K}_{t-} dK_t + K_{t-} d\bar{K}_t + d[K, \bar{K}]_t \\
&= Y_{t-}(\nabla_x f(t, X_t, C_t) dt + \nabla_x g(t, X_{t-}, C_t) dN_t) \\
&\quad + Y_{t-}(-\nabla_x f(t, X_t, C_t) dt - (I_d + \nabla_x g(t, X_{t-}, C_t))^{-1} \nabla_x g(t, X_{t-}, C_t) dN_t) \\
&\quad - Y_{t-}(\nabla_x g(t, X_{t-}, C_t))^2 (I_d + \nabla_x g(t, X_{t-}, C_t))^{-1} dN_t \\
&= 0.
\end{aligned}$$

□

From Equations (6.19) and (6.20), we obtain the recurrence property of K_t and $\bar{K}_t$:

$$K_{T_i} = (I_d + \nabla_x g(T_i, X_{T_i -}, Z_{T_i})) K_{T_i -}, \quad \bar{K}_{T_i} = (I_d + \nabla_x g(T_i, X_{T_i -}, Z_{T_i}))^{-1} \bar{K}_{T_i -}.$$

Proposition 6.1. *If $det[I_d + \nabla g(., x, .)] \neq 0$, the processes (K_t), $(\bar{K}_t)$, and (K_t^s) defined by* (6.16), (6.20), *and* (6.17) *satisfy $K_t^s(x) = K_t(x) \bar{K}_s(x), \forall t \geq s \geq 0$.*

Proof. We fix $s \geq 0$. From (6.18), we deduce the S.D.E. satisfied by K_t^s

$$K_t^s = I_d + \int_s^t \nabla_x f(u, X_u^s(X_s(x)), C_u) K_u^s du + \int_s^t \nabla_x g(u, X_{u-}^s(X_s(x)), C_u) K_{u-}^s dN_u,$$

or

$$K_t^s = I_d + \int_s^t \nabla_x f(u, X_u, C_u) K_u^s du + \int_s^t \nabla_x g(u, X_{u-}, C_u) K_{u-}^s dN_u, \forall t \geq s.$$

In addition, K_t satisfies

$$K_t = K_s + \int_s^t \nabla_x f(u, X_u, C_u) K_u du + \int_s^t \nabla_x g(u, X_{u-}, C_u) K_{u-} dN_u, \forall t \geq s. \qquad (6.21)$$

Therefore, $K_t = K_t^s K_s$. □

Now we study the relationship between the Malliavin derivative and the derivative of flow of the solution. In the Brownian case, we have (cf. [5])

$$D_s X_t = f(s, X_s, C_s) K_t^s, \; \forall 0 \leq s \leq t.$$

In our case, the result is as follows:

Proposition 6.1. *Let $\varphi : \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{K} \to \mathbb{R}^d$ defined by: $\forall (t, x, k) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathcal{K}$,*

$$\varphi(t, x, k) = f(t, x + g(t, x, k), k) - (I_d + \nabla_x g(t, x, k)) f(t, x, k) - \partial_t g(t, x, k).$$

Then we have

1. $D_s X_T = -\int_0^T K_T^t \varphi(t, X_{t-}, C_t) \left(\frac{t}{T} - \mathbf{1}_{[0,t]}(s) \right) dN_t$.
2. $\Gamma[X_T] = \int_0^T \int_0^T K_T^t \varphi(t, X_{t-}, C_t) \varphi^*(u, X_{u-}, C_u) (K_T^u)^* \left(u \wedge t - \frac{ut}{T} \right) dN_t dN_u$.

Proof. On the set $\{N_T = i\}$, we have, for every $0 \leq j \leq i$,

$$\begin{aligned} X_T^{T_j}(x) &= \Phi_{T_i,T}(., Z_i) \circ \Psi(T_i, ., Z_i) \circ \cdots \circ \Psi(T_{j+1}, ., Z_{j+1}) \circ \Phi_{T_j,T_{j+1}}(x, Z_j) \\ &= F(\Phi_{T_j,T_{j+1}}(x, Z_j)), \end{aligned}$$

where $F = \Phi_{T_i,T}(., Z_i) \circ \Psi(T_i, ., Z_i) \circ \cdots \circ \Psi(T_{j+1}, ., Z_{j+1})$. We will prove that

$$\partial_{T_j} X_T^{T_j}(x) = -\nabla_x X_T^{T_j}(x) f(T_j, x, Z_j). \tag{6.22}$$

Indeed, the left hand side equals to

$$-F'(\Phi_{T_j,T_{j+1}}(x, Z_j)) \partial_t \Phi_{T_j,T_{j+1}}(x, Z_j),$$

and the right hand side equals to

$$-F'(\Phi_{T_j,T_{j+1}}(x, Z_j)) \nabla_x \Phi_{T_j,T_{j+1}}(x, Z_j) f(T_j, x, Z_j).$$

The equation (6.22) is obtained by using the result

$$\partial_t \Phi_{T_j,T_{j+1}}(x, Z_j) = \nabla_x \Phi_{T_j,T_{j+1}}(x, Z_j) f(T_j, x, Z_j).$$

We have

$$X_{T_j-}(x) = \Phi_{T_{j-1},T_j}(., Z_{j-1}) \circ \Psi(T_{j-1}, ., Z_{j-1}) \circ \cdots \circ \Psi(T_1, ., Z_1) \circ \Phi_{0,T_1}(x, Z_0),$$

which deduces

$$\begin{aligned} \partial_{T_j} X_{T_j-}(x) &= \partial_t \Phi_{T_{j-1},T_j}(., Z_{j-1}) \left(\Psi(T_{j-1}, ., Z_{j-1}) \circ \cdots \circ \Psi(T_1, ., Z_1) \circ \Phi_{0,T_1}(x, Z_0) \right) \\ &= f(T_j, X_{T_j-}(x), Z_j). \end{aligned}$$

Moreover, $X_{T_j}(x) = \Psi(T_j, X_{T_j-}(x), Z_j)$, so

$$\begin{aligned}
\partial_{T_j} X_{T_j}(x) &= \partial_t \Psi(T_j, X_{T_j-}(x), Z_j) + \nabla_x \Psi(T_j, X_{T_j-}(x), Z_j)\partial_{T_j} X_{T_j-}(x) \\
&= \partial_t \Psi(T_j, X_{T_j-}(x), Z_j) + \nabla_x \Psi(T_j, X_{T_j-}(x), Z_j) f(T_j, X_{T_j-}(x), Z_j).
\end{aligned}$$

As $X_T(x) = X_T^{T_j}(X_{T_j}(x))$, by using Equation (6.22) we deduce

$$\begin{aligned}
\partial_{T_j} X_T(x) &= \partial_{T_j} X_T^{T_j}(X_{T_j}(x)) + \nabla_x X_T^{T_j}(X_{T_j}(x))\partial_{T_j} X_{T_j}(x) \\
&= -K_T^{T_j}(x)\varphi(T_j, X_{T_j-}(x), Z_j).
\end{aligned}$$

Hence,

$$\begin{aligned}
D_s X_T &= \sum_{j=1}^{i} \frac{\partial X_T}{\partial T_j} D_s T_j = -\sum_{j=1}^{i} K_T^{T_j}\varphi(T_j, X_{T_j-}, Z_j) D_s T_j \\
&= -\sum_{j=1}^{i} K_T^{T_j}\varphi(T_j, X_{T_j-}, Z_j)\left(\frac{T_j}{T} - \mathbf{1}_{[0,T_j]}(s)\right) \\
&= -\int_0^T K_T^t \varphi(t, X_{t-}, C_t)\left(\frac{t}{T} - \mathbf{1}_{[0,t]}(s)\right) dN_t.
\end{aligned}$$

And the second statement follows as

$$\begin{aligned}
&\Gamma[X_T] = \int_0^T D_s X_T (D_s X_T)^* ds \\
&= \int_0^T ds\Big(\int_0^T K_T^t \varphi(t, X_{t-}, C_t)\left(\frac{t}{T} - \mathbf{1}_{[0,t]}(s)\right) dN_t \\
&\quad \times \int_0^T \varphi^*(u, X_{u-}, C_u)(K_T^u)^*\left(\frac{u}{T} - \mathbf{1}_{[0,u]}(s)\right) dN_u\Big) \\
&= \int_0^T \int_0^T K_T^t \varphi(t, X_{t-}, C_t)\varphi^*(u, X_{u-}, C_u)(K_T^u)^* \\
&\quad \times \left(\int_0^T \left(\frac{t}{T} - \mathbf{1}_{[0,t]}(s)\right)\left(\frac{u}{T} - \mathbf{1}_{[0,u]}(s)\right) ds\right) dN_t dN_u \\
&= \int_0^T \int_0^T K_T^t \varphi(t, X_{t-}, C_t)\varphi^*(u, X_{u-}, C_u)(K_T^u)^*\left(u \wedge t - \frac{ut}{T}\right) dN_t dN_u.
\end{aligned}$$

□

Now we can use the criterion of density in $\mathbb{D}^{1,2}$. We define

$$\mathcal{C} = \{det\left(\int_0^T \int_0^T K_T^t \varphi(t, X_{t-}, C_t)\varphi^*(u, X_{u-}, C_u)(K_T^u)^*\left(u \wedge t - \frac{ut}{T}\right) dN_t dN_u\right) > 0\}.$$

As a consequence of Theorem 5.1, we have

Proposition 6.1. *(Existence of density of X_T) If $\mathbb{P}(\mathcal{C}) > 0$ then the conditional law of $X_T(x)$ given $\mathcal{C}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Now we consider the case $d = 1$. For each subset A of $\mathcal{K}$, we denote by N_t^A the number of times that the Markov chain C passes through A up to t.

Proposition 6.1. *Assume that there exists a subset A of $\mathcal{K}$ such that for every $k \in A$ and every $t \in \mathbb{R}^+$, $x \in \mathbb{R}^d$, $\varphi(t, x, k) \neq 0$. Then the conditional law of $X_T(x)$ given $\{N_T^A \geq 1\}$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$.*

Proof. The results is obtained by using Proposition 6.1 and

$$\Gamma[X_T] = \int_0^T \left(\int_0^T K_T^t \varphi(t, X_{t-}, C_t) \left(\frac{t}{T} - \mathbf{1}_{[0,t]}(s) \right) dN_t \right)^2 ds \geq 0.$$

By using the same argument as Proposition 5.0, $\Gamma[X_T] = 0$ deduces that

$$K_T^t \varphi(t, X_{t-}, C_t) = 0, \text{ a.s. } \forall t \in [0, T].$$

This can not happen since $K_T^t \neq 0\, a.s. \forall t \in [0, T]$ and $\varphi(t, X_{t-}, C_t) \neq 0\, a.s.$ when $C_t \in A$. $\square$

In case the functions f and g do not depend on t, we can see that the function φ vanishes when $g(x, k) = -x$ and $f(0, k) = 0$. In this case, the solution jumps to 0 at the first jump and then stays there. In the next proposition, we will give a sufficient condition for which the condition in Proposition 6.1 will be satisfied.

Theorem 6.2. *A sufficient condition for the measure $(\mathbf{1}_{\{N_T \geq 1\}} \mathbb{P}) X_T^{-1}$ to be absolutely continuous with respect to one dimensional Lebesgue measure is that:*

$$|W(g, f)(x, k)| > \frac{1}{2} ||f''(., k)||_\infty ||g(., k)||_\infty^2, \ \forall x \in \mathbb{R}, k \in \mathcal{K},$$

where $W(g, f)(x, k) = g'(x, k) f(x, k) - f'(x, k) g(x, k)$ is the Wronskian of f and g, and all the derivatives are with respect to x.

Proof. By applying Taylor expansion for $f(x, k)$ on x, we have

$$\begin{aligned} \varphi(x, k) &= f(x + g(x, k), k) - f(x, k) - g'(x, k) f(x, k) \\ &= g(x, k) f'(x, k) + \frac{1}{2} f''(\xi_x, k) g^2(x, k) - g'(x, k) f(x, k) \\ &= \frac{1}{2} f''(\xi_x, k) g^2(x, k) - W(g, f)(x, k), \end{aligned}$$

for some ξ_x between x and $x + g(x, k)$. $\square$

7 Computation of Greeks

In this section, we would like to use the same technique as the classical Malliavin calculus for the case of Markov chain to compute greeks by using the integration by parts formula as done for the case of Poisson process in [11]. As usual for $n \in \mathbb{N}\bigcup\{+\infty\}$, we shall denote by $C_b^n(\mathbb{R})$ the set of bounded real functions which are n times differentiable with bounded derivatives.

Lemma 7.2. *Let (a,b) be an open interval of $\mathbb{R}$. Let $(F^x)_{x\in(a,b)}$ and $(G^x)_{x\in(a,b)}$ be two families of random variables such that the maps $x \in (a,b) \mapsto F^x \in \mathbb{D}^{1,2}$ and $x \in (a,b) \mapsto G^x \in \mathbb{D}^{1,2}$ are continuously differentiable. Let $m \in \mathcal{H}$ satisfy*

$$D_m F^x \neq 0,\ a.s\ on\ \{\partial_x F^x \neq 0\},\ x \in (a,b),$$

and such that $mG^x \partial_x F^x / D_m F^x$ is continuous in x in $Dom\delta$. We have

$$\frac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)] = \mathbb{E}\left[f(F^x)\delta\left(G^x m \frac{\partial_x F^x}{D_m F^x}\right)\right] + \mathbb{E}[\partial_x G^x f(F^x)] \quad (7.23)$$

for any $f \in C_b^1(\mathbb{R})$.

Proof. We have:

$$\begin{aligned}
\frac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)] &= \mathbb{E}[G^x f'(F^x)\partial_x F^x] + \mathbb{E}[\partial_x G^x f(F^x)]. \\
&= \mathbb{E}\left[G^x \frac{D_m f(F^x)}{D_m F^x}\partial_x F^x\right] + \mathbb{E}[\partial_x G^x f(F^x)] \\
&= \mathbb{E}\left[G^x \frac{\partial_x F^x}{D_m F^x} D_m f(F^x)\right] + \mathbb{E}[\partial_x G^x f(F^x)] \\
&= \mathbb{E}\left[f(F^x)\delta\left(G^x m \frac{\partial_x F^x}{D_m F^x}\right)\right] + \mathbb{E}[\partial_x G^x f(F^x)].
\end{aligned}$$

The last equation follows from Remark .4. □

In practice, it is interesting to consider the case where the function f is only piecewise continuous. To this end, we introduce the set $\mathbb{T}_0 \subset \mathbb{R}$ of points $y \in \mathbb{R}$ such that

$$\lim_{n\to+\infty} \sup_{x\in(a,b)} \mathbb{P}(F^x \in (y - \frac{1}{n}, y + \frac{1}{n})) = 0,$$

and we put $\mathbb{T} = \mathbb{T}_0 \bigcup\{-\infty, +\infty\}$.

Let us remark that if for any $x \in (a,b)$, F^x admits a density which is locally bounded uniformly w.r.t. x then $\mathbb{T} = \mathbb{R}$, see for example [14] for such an example.

Finally we consider the set $\mathcal{L}$ of real functions f of the form

$$\forall y \in \mathbb{R},\ f(y) = \sum_1^n \Phi_i(y)\mathbf{1}_{A_i}(y),$$

where $n \in \mathbb{N}$, for all i, A_i is an interval with endpoints in $\mathbb{T}$ and Φ_i is continuous and bounded.

Proposition 7.2. *Assume that families* $(F^x)_{x\in(a,b)}$ *and* $(G^x)_{x\in(a,b)}$ *satisfy hypotheses of Lemma 7.2. Let* f *be in* $\mathcal{L}$. *Then the map* $x \mapsto \mathbb{E}[G^x f(F^x)]$ *is continuously differentiable and relation* (7.23) *still holds.*

Proof. We consider a compact interval $I \subset (a,b)$ and prove the result on I. Consider first the case where f is continuous and bounded. Then, for any $n \in \mathbb{N}^*$ we can approximate f by a function $f_n \in C_b^1(\mathbb{R})$ such that

- $|f(y) - f_n(y)| \leq \dfrac{1}{n} \ \forall y \in [-n, n]$;
- $|f(y) - f_n(y)| \leq \|f\|_\infty \ \forall y,\ |y| \geq n$;

As a consequence of the previous Lemma and the hypotheses of continuity we made, we know that the map $x \mapsto \mathbb{E}[G^x f_n(F^x)]$ is continuously differentiable on I and we have

$$\frac{\partial}{\partial x}\mathbb{E}[G^x f_n(F^x)] = \mathbb{E}\left[f_n(F^x)\delta\left(G^x m \frac{\partial_x F^x}{D_m F^x}\right)\right] + \mathbb{E}[\partial_x G^x f_n(F^x)].$$

It remains to prove that $f_n(F^x)$ converges to $f(F^x)$ in L^2 uniformly in x on I. Indeed, thanks to the Cauchy-Schwarz inequality, this clearly implies the uniform convergence of $\dfrac{\partial}{\partial x}\mathbb{E}[G^x f_n(F^x)]$ to $\dfrac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)]$ on I. We have for all $x \in I$

$$\begin{aligned}\mathbb{E}[|f_n(F^x) - f(F^x)|^2] &\leq \frac{1}{n^2} + \|f\|_\infty^2 \mathbb{P}(|F^x| \geq n) \\ &\leq \frac{1}{n^2} + \|f\|_\infty^2 \frac{\sup_{x\in I}\mathbb{E}[|F^x|^2]}{n^2},\end{aligned}$$

hence $\lim_{n\to+\infty} \sup_{x\in I} \mathbb{E}[|f_n(F^x) - f(F^x)|^2] = 0$, which yields the result in this case.

Consider now the general case. We just need to consider the case where

$$f = \Phi \mathbf{1}_{[c,+\infty[} \text{ or } f = \Phi \mathbf{1}_{(c,+\infty[},$$

with Φ is bounded and continuous and $c \in \mathbb{T}_0$. Then, we can approximate f by a sequence of function $(f_n)_n$ in $C_b^0(\mathbb{R})$ converging to f everywhere and such that for any $n \in \mathbb{N}^*$

- $f_n(y) = f(y) \ \forall y \in [c + \frac{1}{n}, +\infty[$;
- $|f(y) - f_n(y)| \leq \|f\|_\infty \ \forall y \in [c - \frac{1}{n}, c + \frac{1}{n}]$;
- $f(y) = 0$ if $y \leq c - \frac{1}{n}$.

Since f_n is bounded and continuous, we are in the previous case and here again we just have to prove that $f_n(F^x)$ converges to $f(F^x)$ in L^2 uniformly in x on I. We have for all $x \in I$

$$\mathbb{E}[|f_n(F^x) - f(F^x)|^2] \quad \leq \quad \|f\|_\infty^2 \mathbb{P}(F^x \in (c - \frac{1}{n}, c + \frac{1}{n})).$$

Since c belongs to $\mathbb{T}_0$ we get

$$\lim_{n \to +\infty} \sup_{x \in I} \mathbb{E}[|f_n(F^x) - f(F^x)|^2] = 0,$$

and the proof is complete since I is arbitrary. $\square$

Thanks to Lemma 7.2 and Proposition 7.2, we can compute the weights for first and second derivatives with respect to x as done in [11]. The results are slightly different due to the additional term in $\delta(m)$.

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