Maximum principle for parabolic SPDE's : first approach

Laurent DENIS,* Anis MATOUSSI† and Lucretiu STOICA‡

April 15, 2010

1 Introduction

In the theory of Partial Differential Equations, the maximum principle plays an important role and there is a huge literature on this subject. It permits one to study the local behavior of solutions of PDE since it gives a relation between the bound of the solution on the boundary and a bound on the whole domain. The maximum principle for quasilinear parabolic equations was proved by Aronson -Serrin (see Theorem 1 of [1]). In two recent papers [7], [8] we have proved a maximum principle for quasilinear stochastic PDE's. The main step is to get an estimate of the uniform norm of the solution with null boundary Dirichlet condition. In these papers, we have considered a very general case and used sophisticated spaces and tools, so that the links between the deterministic proof given by Aronson-Serrin and our proof in the stochastic case is not clear. The main objective of this paper is to explain how one can estimate the uniform norm of the solution in a simpler case in order to point out the interest of Moser's iteration method.
We are concerned with the following stochastic partial differential equation (in short SPDE):

$$du_t(x) + Au_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \partial_i\big[g_i(t,x,u_t(x),\nabla u_t(x))\big] = h_j(t,x,u_t(x),\nabla u_t(x))\, dB^j_t,$$

where A is an elliptic second order symmetric differential operator defined on some bounded domain $\mathcal{O} \subset \mathbb{R}^d$, with null Dirichlet condition. We are interested in studying the behavior of the weak solution. More precisely, we prove that under suitable conditions (Lipschitz continuity of the coefficients and boundness of the initial conditions), the solution u belongs to all the L^p spaces and in the case where h does not depend on ∇u, we obtain a maximum principle, namely we prove that for all $T > 0$,

$$E\|u\|_{L^\infty([0,T]\times\mathcal{O})} < +\infty,$$

which implies that P-almost surely, $u(t,x)$ is uniformly bounded in t and x.
To prove this, we first show an Ito's formula for the p-integral over $\mathcal{O}$ of the solution and then, we adapt the iterative technics introduced first by Moser [17] and developed by Aronson-Serrin [1] for parabolic non-linear PDE. The main idea is to iterate some inequalities on the $L^{p,q}$-spaces and to use an interpolation theorem between those spaces.
There is a huge literature about SPDE's but, to our knowledge, not so many authors consider the L^p-theory for solutions of SPDE's. One has to mention the works of Pardoux and Peng [21] which treat the case where $g = 0$ and prove the smoothness of the solutions using the theory of Backward Stochastic Differential Equations, see also [2] and [15]. Gyöngy and Rovira [10] prove L^p-estimates in the case where f and g do not depend on ∇u (but g may have polynomial growth) by using L^p-estimates for the Green's kernel. Let us also mention that an L^p-theory for solutions of parabolic

*Département de Mathématiques, Equipe "Analyse et Probabilités, Université d'Evry-Val-d'Essonne, Boulevard F. Mitterrand, 91 025 EVRY Cedex, FRANCE. e-mail: ldenis@univ-evry.fr
†Université du Maine, Département de Mathématiques, Equipe Statistique et Processus, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France. e-mail: anis.matoussi@univ-lemans.fr
‡University of Bucharest, Faculty of Mathematics, Str. Academiei 14, Bucharest RO -70109, ROMANIA. e-mail: lstoica@fmi.unibuc.ro

quasilinear SPDE's driven by an infinite dimensional noise (an analytic approach) is developped by Krylov [13] and also by Mikulevicius and Rozovskii [16]. Moreover, Kuksin et al [11] obtained Hölder norm estimates of the solutions for linear SPDE by using L^p-estimates and Kolmogorov type criteria. Our approach is different and is based on technics developed by Aronson-Serrin and Moser for non-linear PDE's in the late 60's.
This paper is divided as follows: in the next section, we set hypotheses and notations and we recall results of existence and uniqueness proved in [6], then in the third section we establish an Ito's formula for the solution which permits to obtain L^p-estimates and finally a maximum principle. The last section is an appendix in which we recall some (well-known) lemmas that we need.

2 Preliminaries

2.1 Hypotheses and definitions

Let $\{B_t := (B_t^j)_{j\in\{1,\cdots,d_1\}}\}_{t\geqslant 0}$ be a d_1-dimentional Brownian motion defined on a standard filtered probability space $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},P\big)$.
Let $\mathcal{O}\subset\mathbb{R}^d$ be an open domain with finite Lebesgue measure on $\mathbb{R}^d$ and $L^2(\mathcal{O})$ the set of square integrable functions with respect to the Lebesgue measure on $\mathcal{O}$. Let A be a symmetric second order differential operator given by $A := -\sum_{i,j=1}^d \partial_i(a^{i,j}\,\partial_j)$. We assume that $a=(a^{i,j})_{i,j}$ is a measurable symmetric matrix defined on $\mathcal{O}$ which satisfies the uniform ellipticity condition

$$\lambda|\xi|^2 \leqslant \sum_{i,j=1}^d a^{i,j}(x)\xi^i\xi^j \leqslant \Lambda|\xi|^2,\quad \forall x\in\mathcal{O},\ \xi\in\mathbb{R}^d\,,$$

where λ and Λ are positive constants.
Let $(F,\mathcal{E})$ be the associated Dirichlet form given by $F:=\mathcal{D}(A^{1/2})=H_0^1(\mathcal{O})$ and

$$\mathcal{E}(u,v):=\big(A^{1/2}u,\ A^{1/2}v\big)\quad\text{and}\quad \mathcal{E}(u,u)=\|\sqrt{A}u\|^2,\ \forall(u,v)\in F$$

where $(\cdot,\cdot)$ and $\|\,.\,\|$ are respectively the inner product and the norm on $L^2(\mathcal{O})$. $H_0^1(\mathcal{O})$ is the first order Sobolev space of functions vanishing at the boundary . For the notion and definition of Dirichlet forms, we follow [9] or [3].

We consider the quasilinear stochastic partial differential equation (in short SPDE) for the real-valued random field $u_t(x)$

$$\begin{aligned}(1)\qquad du_t(x)+\big[Au_t(x)+f(t,x,u_t(x),\nabla u_t(x))+\sum_{i=1}^d\partial_i\big[g_i(t,x,u_t(x),\nabla u_t(x))\big]dt\\ =\sum_{j=1}^{d_1}h_j(t,x,u_t(x),\nabla u_t(x))\,dB_t^j.\end{aligned}$$

with initial condition $u(0,.)=\xi(.)$ and Dirichlet boundary condition

$$u_t(x)=0,\quad\text{for all }(t,x)\in\mathbb{R}_+\times\partial\mathcal{O}\,.$$

We assume that we have predictable random functions

$$\begin{aligned} f\quad &:\quad \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}\ ,\\ g\quad &=\quad (g_1,...,g_d)\ :\ \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d\\ h\quad &:\quad \mathbb{R}_+\times\Omega\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^{d_1}\ ,\end{aligned}$$

that satisfy the Lipschitz conditions with respect to the last two variables

$$
\begin{aligned}
&|f(t,\omega,x,y,z) - f(t,\omega,x,y^{'},z^{'})| \leqslant C\big(|y-y^{'}| + |z-z^{'}|\big)\,, \\
(2)\qquad &\Big(\sum_{i=1}^{d} |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y^{'},z^{'})|^2\Big)^{\frac{1}{2}} \leqslant C\,|y-y^{'}| + \alpha\,|z-z^{'}|\,, \\
&\Big(\sum_{j=1}^{d_1} |h_j(t,\omega,x,y,z) - h_j(t,\omega,x,y^{'},z^{'})|^2\Big)^{\frac{1}{2}} \leqslant C\,|y-y^{'}| + \beta\,|z-z^{'}|\,,
\end{aligned}
$$

where C, α, β are non negative constants.
These conditions imply the following inequalities, $\forall t \geqslant 0$,

$$
\begin{aligned}
\left\|f\left(u,\nabla u\right) - f\left(v,\nabla v\right)\right\| &\leq C\left\|u-v\right\| + C\lambda^{-\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}\left(u-v\right), \\
\left\|g\left(u,\nabla u\right) - g\left(v,\nabla v\right)\right\|_{L^2(\mathcal{O};\mathbb{R}^d)} &\leq C\left\|u-v\right\| + \alpha\lambda^{-\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}\left(u-v\right), \\
\left\|h\left(u,\nabla u\right) - h\left(v,\nabla v\right)\right\|_{L^2\left(\mathcal{O};\mathbb{R}^{d_1}\right)} &\leq C\left\|u-v\right\| + \beta\lambda^{-\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}\left(u-v\right),
\end{aligned}
$$

which have the same form as the conditions (H1) in [6]. In order to check that the condition (H2) from that paper is satisfied we should also estimate the norm of the following operator $R: L^2\left(\mathcal{O};\mathbb{R}^d\right) \to F'$, defined by

$$
R\left(v_1,...,v_d\right) = \sum_{i=1}^{d} \partial_i v_i,
$$

in the sense of distributions (with F' the topological dual of F). Since one obviously has

$$
\left\langle \varphi, R\left(v_1,...,v_d\right)\right\rangle \leq \lambda^{-\frac{1}{2}}\mathcal{E}^{\frac{1}{2}}\left(\varphi\right)\left\|v\right\|_{L^2(\mathcal{O};\mathbb{R}^d)}
$$

for any $\varphi \in \mathcal{C}_c^{\infty}\left(\mathcal{O}\right)$ and $v = \left(v_1,...,v_d\right) \in L^2\left(\mathcal{O};\mathbb{R}^d\right)$, it follows that $\|R\| \leq \lambda^{-\frac{1}{2}}$. Now we can see that the condition (H2) of [6] is satisfied provided that the constants intervening in our assumption (2) satisfy the inequality

$$
2\alpha + \beta^2 < 2\lambda\,. \tag{3}
$$

We will assume throughout this paper that this condition is satisfied so that our equation has a unique solution by Theorem 9 of [6]. This last condition means that the sizes of the second order perturbation introduced by g and of the first order perturbation blended through h with the Brownian motion should be relatively small.
Moreover we assume that for any $T > 0$

$$
\begin{aligned}
&\xi \in L^2(\Omega \times \mathcal{O}) \text{ is an } \mathcal{F}_0\text{-measurable random variable} \\
(4)\qquad &f(\cdot,\cdot,\cdot,0,0) := f^0 \in L^2\big([0,T]\times\Omega\times\mathcal{O};\mathbb{R}\big) \\
&h(\cdot,\cdot,\cdot,0,0) := h^0 \in L^2\big([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^{d_1}\big) \\
&g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0,...,g_d^0) \in L^2\big([0,T]\times\Omega\times\mathcal{O};\mathbb{R}^d\big).
\end{aligned}
$$

2.2 Weak and mild solutions

Let $L^2_{loc}\left(\mathbb{R}_+; H^1_0(\mathcal{O})\right)$ be the space of all measurable functions $u: \mathbb{R}_+ \to H^1_0(\mathcal{O})$ such that

$$
\|u\|_T := \left(\int_0^T \left(\|u_t\|^2 + \mathcal{E}\left(u_t\right)\right)dt\right)^{1/2} < \infty, \qquad \text{for any } T > 0.
$$

$\mathcal{H}$ is the space of $H_0^1(\mathcal{O})$-valued predictable processes $(u_t)_{t\geqslant 0}$ such that

$$\|u\|_{E,T} := \left(\sup_{0\leqslant t\leqslant T} E\|u_t\|^2 + \int_0^T E\,\mathcal{E}(u_t)\,dt \right)^{1/2} < \infty\,, \quad \text{for each}\,, T>0\,.$$

The space of test functions in our study will be $\mathcal{D} = \mathcal{C}_c^\infty \otimes \mathcal{D}(A)$, where $\mathcal{C}_c^\infty$ denotes the space of C^∞ real valued functions with compact support. Since $\mathcal{C}_c^\infty$ is dense in $L^2_{loc}(\mathbb{R}_+)$ and $\mathcal{D}(A)$ is dense in $H_0^1(\mathcal{O})$ and in $L^2(\mathcal{O})$ respectively, it follows that $\mathcal{D}$ is dense bothly in $L^2_{loc}\left(R_+; H_0^1(\mathcal{O})\right)$ and in $L^2_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right)$.

Definition 1. *We say that $u \in \mathcal{H}$ is a weak solution of equation* (2.2) *with initial condition $\xi \in L^2(\Omega\times\mathcal{O})$, if the following relation holds almost surely, for each $\varphi\in\mathcal{D}$,*

$$\begin{aligned}\int_0^\infty [(u_s,\partial_s\varphi) - \mathcal{E}(u_s,\varphi_s) - (f(s,u_s,\nabla u_s),\varphi_s) + \sum_{i=1}^d (g_i(s,u_s,\nabla u_s),\partial_i\varphi_s)]ds \\ + \int_0^\infty (h(s,u_s,\nabla u_s),\varphi_s)\,dB_s + (\xi,\varphi_0) = 0.\end{aligned} \tag{5}$$

The proof of the next Theorem may be found first in the pioneering work of Pardoux [19] or in [6].

Theorem 2. *Under previous hypotheses (2)-(4), the SPDE (2.2) admits a unique solution $u\in\mathcal{H}$ and this solution has $L^2(\mathcal{O})$-continuous trajectories.*

Existence and uniqueness have been proved by Krylov and Rozovskii [12] for quasilinear SPDE under weaker hypotheses. Indeed, they considered the case of a monotone operator A. In our context, we could write our equation (2.2) in the following form:

$$du_t(x) + \widetilde{A}u_t(x)dt + \left[f(t,x,u_t(x),\nabla u_t(x)) + \sum_{i=1}^d \partial_i\left[g_i(t,x,u_t(x),0)\right]dt = \sum_{j=1}^{d_1} h_j(t,x,u_t(x),\nabla u_t(x))\,dB_t^j.$$

where $\widetilde{A} = \sum_{i,j=1}^d \partial_i\left(\widetilde{a}^{i,j}\partial_j\right)$ with $\widetilde{a}^{i,j} = a^{i,j}(x) + \widetilde{g}_{ij}(t,x,u_t(x),\nabla u_t(x))$. Therefore, $\widetilde{A}$ is uniformly elliptic but it becomes non-symmetric and fully nonlinear operator.

3 L^p-estimates of the solution

3.1 Ito's formula for the L^p-norm

All along this section, we assume that hypotheses (2)-(4) hold. We point out that we always denote by $c>0$ a constant whose value may change from line to line and that, for any $\epsilon>0$, we denote by c_ϵ a constant which depend on ϵ like the one appearing in the following typical inequality

$$ab \leqslant \epsilon a^2 + c_\epsilon b^2, \quad a,b\in\mathbb{R}\,.$$

We will denote by $u := \mathcal{U}(\xi,f,g,h)$ the solution of the equation (2.2) with initial condition ξ and coefficients f, g, h. In order to prove an Ito type formula with respect to the p -integral over $\mathcal{O}$ (see Proposition 6 below) we first study solutions of the equation (2.2) with ξ, f, g, h of a particular type. In the next lemma we consider the linear case, that is we assume that f, g, h do not depend on the last two variables.

Lemma 3. *1) If f, $h_1,\cdots,h_{d_1}$ belong to $\mathcal{C}_c^\infty\otimes L^2(\Omega)\otimes\mathcal{D}(A)$, $g_1,\cdots,g_d$ belong to $\mathcal{C}_c^\infty\otimes L^2(\Omega)\otimes\mathcal{D}(A^{3/2})$ and if ξ belongs to $L^2(\Omega)\otimes\mathcal{D}(A)$ then $u:=\mathcal{U}(\xi,f,g,h)$ is an $L^2(\mathcal{O})$-valued square integrable semimartingale.*
2) If f, $h_1,\cdots,h_{d_1}$, $g_1,\cdots,g_d$ belong to $L^2_{loc}\left(\mathbb{R}^+; L^2(\Omega\times\mathcal{O})\right)$ and $\xi\in L^2\left(\Omega\times\mathcal{O}\right)$, then there exists a sequence $(u^k)_{k\in\mathbb{N}}$ of $L^2(\mathcal{O})$-valued square integrable semimartingales which approximates $u:=\mathcal{U}(\xi,f,g,h)$ in the sense that $\lim_{k\to\infty} E\|u^k-u\|_T^2 = 0$ for all $T>0$.

The proof of this last lemma was given in Denis et al (Lemma 6 in [7]). Now, we give the following preliminary result

Lemma 4. *Assume that* f, $h_1 \cdots h_{d_1}$, $g_1, \cdots, g_d$ *belong to* $L^2_{loc}\big(\mathbb{R}^+; L^2(\Omega\times\mathcal{O})\big)$ *and* $\xi \in L^2\big(\Omega\times\mathcal{O}\big)$ *and consider* $u = \mathcal{U}(\xi, f, g, h)$. *Let* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ *be a function of class* C^2 *and assume that* φ'' *is bounded and* $\varphi'(0) = 0$. *Then* P*-a.s. for all* $t \in [0, T]$

$$
\begin{aligned}
&\int_{\mathcal{O}} \varphi(u_t(x))\, dx \;+\; \int_0^t \mathcal{E}\big(\varphi'(u_s),\, u_s\big)\, ds = \int_{\mathcal{O}} \varphi(\xi)\, dx - \int_0^t (\varphi'(u_s), f_s)\, ds \\
&+ \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x))\partial_i u_s(x)\, g_i(s,x)\, dx\, ds + \sum_{j=1}^{d_1} \int_0^t (\varphi'(u_s),\, h_j(s))\, dB^j_s \\
&+ \frac{1}{2}\sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) h_j^2(s,x)\, dx\, ds\,,
\end{aligned}
\tag{6}
$$

where the term $t \to \sum_{j=1}^{d_1} \int_0^t (\varphi'(u_s),\, h_j(s))\, dB^j_s$ *is well-defined as an integrable martingale.*

Proof. Assume first that $f, h_1, \cdots, h_{d_1} \in \mathcal{C}^\infty_c \otimes L^2(\Omega) \otimes \mathcal{D}(A)$ $g_1, \cdots, g_d$ belong to $\mathcal{C}^\infty_c \otimes L^2(\Omega) \otimes \mathcal{D}(A^{3/2})$ and $\xi \in L^2(\Omega) \otimes \mathcal{D}(L)$, then u is a semimartingale and one has

$$
\forall t \geqslant 0,\; u_t = \xi + \int_0^t A u_s ds - \int_0^t f(s)\, ds - \sum_{i=1}^d \int_0^t \partial_i g_i(s)\, ds + \sum_{j=1}^{d_1} \int_0^t h_j(s)\, dB^j_s\,.
$$

Due to the regularity assumptions that we put on the coefficients, it's clear that all the terms in the above equation are well defined and Ito's formula for Hilbert-valued semimartingales (see [4] for example) yields

$$
\begin{aligned}
&\int_{\mathcal{O}} \varphi(u_t(x))\, dx = \int_{\mathcal{O}} \varphi(\xi(x))\, dx + \int_0^t (\varphi'(u_s), A u_s)\; ds - \int_0^t (\varphi'(u_s), f_s)\, ds \\
&- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi'(u_s(x))\partial_i\, g_i(s,x)\, dx\, ds + \sum_{j=1}^{d_1} \int_0^t (\varphi'(u_s),\, h_j(s))\, dB^j_s \\
&+ \frac{1}{2}\sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi''(u_s(x)) h_j(s,x)^2\, dx\, ds\,.
\end{aligned}
\tag{7}
$$

Then, as

$$
\forall s \geqslant 0,\; (\varphi'(u_s), L u_s) = -\mathcal{E}\left(\varphi'(u_s), u_s\right),
$$

and

$$
\sum_{i=1}^d \int_{\mathcal{O}} \varphi'(u_s(x))\partial_i\, g_i(s,x)\, dx = -\sum_{j=1}^{d_1} \int_{\mathcal{O}} \varphi''(u_s(x))\partial_i u_s(x)\, g_i(s,x)\, dx,
$$

we get the desired equality.

For the martingale part, let us remark that the square root of its brackets is dominated as follows

$$
\begin{aligned}
\Big(\sum_{j=1}^{d_1} \int_0^T (\varphi'(u_s),\, h_j(s))^2\, ds\Big)^{\frac{1}{2}} \quad &\leqslant \quad \Big(\sum_{j=1}^{d_1} \int_0^T \parallel \varphi'(u_s) \parallel^2 \parallel h_j(s) \parallel^2\, ds\Big)^{\frac{1}{2}} \\
&\leqslant \quad \sup_{s\in[0,T]} \parallel \varphi'(u_s) \parallel^2 + \sum_{j=1}^{d_1} \int_0^T \parallel h_j(s) \parallel^2 ds\,.
\end{aligned}
$$

Since φ'' is bounded the first derivative φ' has at most linear growth, and since $u \in \mathcal{H}$ it follows that $\sup_{s\in[0,T]} \|\varphi'(u_s)\|^2$ belongs to $L^1(\Omega)$. Therefore the square root of the bracket belongs to $L^1(\Omega)$, so that, by the BDG inequality we deduce that $t \to \sum_{j=1}^{d_1} \int_0^t (\varphi'(u_s),\, h_j(s))\, dB_s^j$ is an integrable martingale.
The general case is obtained by approximation thanks to previous lemma. □

Lemma 5. *Assume that the conditions (2), (3) are fulfilled and that the condition (4) is replaced by the stronger one that is*

$$\xi \in L^\infty\big(\Omega \times \mathcal{O}\big), \quad f^0,\, g^0,\, h^0 \in L^\infty\big(\mathbb{R}_+ \times \Omega \times \mathcal{O}\big)\,.$$

and denote by

$$K = \|\xi\|_{L^\infty(\Omega\times\mathcal{O})} \vee \|f^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \vee \|h^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \vee \|g^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O};\mathbb{R}^d)}\,.$$

Then for each $T > 0$, $u \in \bigcap_{p\geqslant 2} L^p\big([0,T]\times\mathcal{O}\times\Omega\big)$. Moreover there exist constants $c0$ and $c'0$ which only depend on K, C, α, β such that for all real $l \geqslant 2$, we have

$$E\int_{\mathcal{O}} |u_t(x)|^l\, dx \leqslant cl(l-1)e^{c\,l(l-1)t} \tag{8}$$

and

$$E\int_0^t\int_{\mathcal{O}} |u_t(x)|^{l-2}|\nabla u_t(x)|^2\, dx \leqslant c'l(l-1)e^{c\,l(l-1)t}\,. \tag{9}$$

Proof. We fix a real $l \geqslant 2$ and introduce the sequence $(\varphi_n)_{n\in\mathbb{N}^*}$ of functions such that for all $n \in \mathbb{N}^*$:

$$\forall x\in\mathbb{R},\ \varphi_n(x) = \begin{cases} |\, x\,|^l & \text{if } |\,x\,| \leqslant n \\ n^{l-2}\left[\frac{l(l-1)}{2}(|x|-n)^2 + l\,n(|x|-n) + n^2\right] & \text{if } |\,x\,| > n \end{cases}$$

One can easily verify that for fixed n, φ_n is twice differentiable with bounded second derivative, $\varphi_n''(x) \geqslant 0$, and as $n\to\infty$ one has $\varphi_n(x) \longrightarrow |x|^l$, $\varphi_n'(x) \longrightarrow l sgn(x)|x|^{l-1}$, $\varphi_n''(x) \longrightarrow l(l-1)|x|^{l-2}$. Moreover, the following relations hold, for all $x\in\mathbb{R}$:

1. $|\, x\varphi_n'(x)\,| \leqslant l\varphi_n(x)$.
2. $|\, \varphi_n'(x)\,| \leqslant |\, x\varphi_n''(x)\,|$.
3. $|\, x^2\varphi_n''(x)\,| \leqslant l(l-1)\varphi_n(x)$.
4. $|\varphi_n'(x)| \leqslant l(\varphi_n(x)+1)$.
5. $|\varphi_n''(x)| \leqslant l(l-1)(\varphi_n(x)+1)$.

From Lemma 4 we have P-a.s. for all $t\in[0,T]$

(10)

$$\begin{aligned}
&\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx + \int_0^t \mathcal{E}\big(\varphi_n'(u_s),\, u_s\big)\, ds = \int_{\mathcal{O}} \varphi_n(\xi)\, dx - \int_0^t\int_{\mathcal{O}} \varphi_n'(u_s) f(s,x,u_s,\nabla u_s)\, dxds \\
&+ \sum_{i=1}^d \int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))\partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds + \sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}} \varphi_n'(u_s)\, h_j(s,x,u_s,\nabla u_s)\, dxdB_s^j \\
&+ \frac12 \sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,.
\end{aligned}$$

By the uniform ellipticity of the operator A we get

$$\mathcal{E}\big(\varphi_n'(u_s),\, u_s\big) \;\geqslant\; \lambda \int_{\mathcal{O}} \varphi_n''(u_s)|\nabla u_s|^2\, dx.$$

Let $\epsilon > 0$ be fixed. Using the Lipschitz condition on f and the properties of the functions $(\varphi_n)_n$ we get

$$\begin{aligned}
|\varphi_n'(u_s)|\, |f(s,x,u_s,\nabla u_s)| \;&\leqslant\; |\varphi_n'(u_s)|\,\big(|f^0(s,x)| + C\,(|u_s| + |\nabla u_s|)\,\big)\\
&\leqslant\; |\varphi_n'(u_s)||f^0(s,x)| + |u_s||\varphi_n''(u_s)|\,(C|u_s| + C|\nabla u_s|)\,)\\
&\leqslant l(\varphi_n(u_s)+1)\,|f^0(s,x)| + C|u_s|^2|\varphi_n''(u_s)| + C|u_s||\nabla u_s||\varphi_n''(u_s)|\\
&\leqslant l(\varphi_n(u_s)+1)\,|f^0(s,x)| + (C+c_\epsilon)\,|u_s|^2\varphi_n''(u_s) + \,\epsilon\varphi_n''(u_s)|\nabla u_s|^2.
\end{aligned}$$

Now using Cauchy-Schwarz inequality and the Lipschitz condition on g we get

$$\begin{aligned}
&\sum_{i=1}^{d} \varphi_n''(u_s)\partial_i u_s\, g_i(s,x,u_s,\nabla u_s) \;\leqslant\; \varphi_n''(u_s)\,|\nabla u_s|\,\big(|g^0(s,x)| + C|u_s| + \alpha|\nabla u_s|\,\big)\\
&\leqslant\; \epsilon\,\varphi_n''(u_s)|\nabla u_s|^2 + 2c_\epsilon\varphi_n''(u_s)\,\big(K^2 + C^2|u_s|^2\,\big) + \alpha\,\varphi_n''(u_s)|\nabla u_s|^2\\
&\leqslant l(l-1)c_\epsilon K^2 + 2c_\epsilon(K^2+C^2)l(l-1)|\varphi_n(u_s)| + (\alpha+\epsilon)\,\varphi_n''(u_s)|\nabla u_s|^2
\end{aligned}$$

In the same way as before we get

$$\begin{aligned}
\sum_{j=1}^{d_1} \varphi_n''(u_s)h_j^2(s,u_s,\nabla u_s) \;&\leqslant\; \varphi_n''(u_s)\,\big(\,c_\epsilon'(|h^0(s,x)| + C|u_s|)^2 \;+\; (1+\epsilon)\beta^2\,|\nabla u_s|^2\,\big)\\
&\leqslant \varphi_n''(u_s)\,\big(2c_\epsilon' K^2 \;+\; 2c_\epsilon' C^2|u_s|^2 +\; (1+\epsilon)\beta^2\,|\nabla u_s|^2\,\big)\\
&\leqslant\; 2c_\epsilon' l(l-1)K^2 + 2c_\epsilon'(K^2+C^2)l(l-1)\varphi_n(u_s) + (1+\epsilon)\,\beta^2\,\varphi_n''(u_s)|\nabla u_s|^2
\end{aligned}$$

Thus taking the expectation (one has to remember that, as a consequence of the previous lemma, the expectation of the martingale part is null), we deduce

(11)

$$\begin{aligned}
E\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx &+ (\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha+\epsilon)\,)\, E\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))\,|\nabla u_s|^2\, dx\, ds\\
&\leqslant\; l(l-1)c_\epsilon'' K^2 \;+\; c_\epsilon'' l(l-1)\big(K^2 + C^2 + C + c_\epsilon\big)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x))\, dx\, ds
\end{aligned}$$

On account of the condition (3), one can choose $\epsilon > 0$ small enough such that

$$(\lambda - \frac{1}{2}(1+\epsilon)\beta^2 - (\alpha+\epsilon) > 0$$

and then

$$E\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx \leqslant\; cl(l-1) \;+\; cl(l-1)E\int_0^t\int_{\mathcal{O}} \varphi_n(u_s(x))\, dx\, ds\,.$$

We obtain by Gronwall's Lemma, that

$$E\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx \leqslant\; c\,l(l-1)\,\exp\big(c\,l(l-1)\,t\big)$$

and so it is now easy from (11) to get

$$E\int_0^t\int_{\mathcal{O}} \varphi_n''(u_s(x))\,|\nabla u_s|^2\, dx\, ds \;\leqslant\; c'\,l\,(l-1)\,\exp\big(cl(l-1)\,t\big)$$

Finally, letting $n \to \infty$ by Fatou's lemma we deduce (8) and (9). □

Proposition 6. *Let $u = u(t,x)$ be the solution of the SPDE (2.2) and assume the hypotheses of the previous lemma. Then for $l \geqslant 2$, we get the following Itô's formula, P-almost surely, for all $t \geqslant 0$*

$$\begin{aligned}\int_{\mathcal{O}} |u_t(x)|^l \, dx + \int_0^t \mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) ds = \int_{\mathcal{O}} |\xi(x)|^l \, dx \\ - \; l \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_s(x)|^{l-1} f(s,x,u_s,\nabla u_s)\, dxds \\ + \; l(l-1) \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} |u_s(x)|^{l-2} \partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds \\ + l \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} sgn(u_s)|u_t(x)|^{l-1} h_j(s,x,u_s,\nabla u_s)\, dxdB_s^j \\ + \frac{l(l-1)}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} |u_t(x)|^{l-2} h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,.\end{aligned} \tag{12}$$

where $\mathcal{E}\left(l\,(u_s)^{l-1}\, sgn(u_s),\, u_s\right) = l(l-1) \sum_{i,j=1}^d \int_{\mathcal{O}} |u_s(x)|^{l-2} a^{ij}(x)\, \partial_i u_s(x)\, \partial_j u_s(x)\, dx.$

Proof. From Lemma 4 with the same notations, we have P-almost surely, and for all $t \geqslant 0$ and $n \in \mathbb{N}$

$$\begin{aligned}&\int_{\mathcal{O}} \varphi_n(u_t(x))\, dx \;+\; \int_0^t \mathcal{E}\big(\varphi_n'(u_s),\, u_s\big)\, ds = \int_{\mathcal{O}} \varphi_n(\xi(x))\, dx - \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s(x)) f(s,x,u_s,\nabla u_s)\, dxds \\ &+ \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x)) \partial_i u_s(x)\, g_i(s,x,u_s,\nabla u_s)\, dx\, ds + \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n'(u_s(x))\, h_j(s,x,u_s,\nabla u_s)\, dxdB_s^j \\ &+ \frac{1}{2} \sum_{j=1}^{d_1} \int_0^t \int_{\mathcal{O}} \varphi_n''(u_s(x)) h_j^2(s,x,u_s,\nabla u_s)\, dx\, ds\,.\end{aligned}$$

Passing to the limit as $n \to \infty$ (thanks to the previous lemma) clearly we obtain by the dominated convergence theorem the desired result. □

3.2 A Maximum Principle for SPDE

For fixed $T > 0$, and any real numbers p, $q \geqslant 1$, we denote by $L^{p,q}([0,T] \times \mathcal{O})$ the space of measurable functions $v : \mathbb{R}_+ \times \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$\|v\|_{p,q} := \Big(\int_0^T \Big(\int_{\mathcal{O}} |v(t,x)|^p \, dx \Big)^{q/p} dt \Big)^{1/q}$$

is finite. Of course one can use the L^∞-norms (or the essential sup-norm) in the case when p or q take the value ∞. See the appendix for more properties of these spaces.
We shall restrict our study to the case where the coefficient h in the equation (2.2) does not depend on the gradient of the solution u of the SPDE. This means that we take the constant β equal to zero in (2) and (3) will be replaced by

$$\alpha < \lambda\,. \tag{13}$$

Theorem 7 (Maximum principle)**.** *Assume the condition (2) holds with $\beta = 0$, that (13) is satisfied and that*

$$\xi \in L^\infty\big(\Omega\times\mathcal{O}\big), \quad f^0,\, g^0,\, h^0 \,\in L^\infty\big(\mathbb{R}_+\times\Omega\times\mathcal{O}\big)\,.$$

and denote by

$$K = \|\xi\|_{L^\infty(\Omega\times\mathcal{O})} \vee \|f^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \vee \|h^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O})} \vee \|g^0\|_{L^\infty(\mathbb{R}^+\times\Omega\times\mathcal{O};\mathbb{R}^d)}\,.$$

Let u be a solution of the SPDE (2.2). Then there exists a constant Θ which depends only on λ, α, C, $|\mathcal{O}|$, d, T and K such that

$$E\,\|u\|_{L^\infty\big([0,\,T]\times\mathcal{O}\big)} \,\leqslant\, \Theta\,. \tag{14}$$

Proof. We assume for simplicity that the Lebesgue measure of the set $\mathcal{O}$ is equal to 1 (i.e. $|\mathcal{O}| = 1$). Let $l \geqslant 2$. By Proposition 6, P-almost surely, and for each $t\in[0,\,T]$ we have

$$\begin{aligned}\int_{\mathcal{O}}|u_t|^l\,dx + l(l-1)\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}\big(\sum_{i,j=1}^d a^{ij}\,\partial_i u_s\,\partial_j u_s\,\big)\,dx\,ds = \int_{\mathcal{O}}|\xi(x)|^l\,dx\\ -\,l\int_0^t\int_{\mathcal{O}} sgn(u_s)\,|u_s|^{l-1}\,f(s,x,u_s,\nabla u_s)\,dx\,ds\\ +\,l(l-1)\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}\big(\sum_{i=1}^d \partial_i u_s\,g_i(s,x,u_s,\nabla u_s)\big)\,dx\,ds\\ +\,l\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}} sgn(u_s)\,|u_s|^{l-1}\,h_j(s,x,u_s)\,dx\,dB_s^j\\ +\,\frac{l(l-1)}{2}\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}h_j^2(s,x,u_s)\,dx\,ds\,.\end{aligned} \tag{15}$$

Step 1 : We use the uniform ellipticity of the matrix $(a^{i,j})_{1\,\leqslant\,i,\,j\,\leqslant\,d}$ in the second term of the left hand side of the last equation to get

$$\begin{aligned}\int_{\mathcal{O}}|u_t|^l\,dx + \lambda l(l-1)\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}\,|\nabla u_s|^2\,dx\,ds \leqslant \int_{\mathcal{O}}|\xi|^l\,dx\\ -\,l\int_0^t\int_{\mathcal{O}} sgn(u_s)\,|u_s|^{l-1}\,f(s,x,u_s,\nabla u_s)\,dx\,ds\\ +\,l(l-1)\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}\big(\sum_{i=1}^d \partial_i u_s\,g_i(s,x,u_s,\nabla u_s)\big)\,dx\,ds\\ +\,\frac{l(l-1)}{2}\sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}}|u_s|^{l-2}h_j^2(s,x,u_s)\,dx\,ds\\ +\,l\,M_t\end{aligned} \tag{16}$$

where $M_t := \sum_{j=1}^{d_1}\int_0^t\int_{\mathcal{O}} sgn(u_s)\,|u_s|^{l-1}\,h_j(s,x,u_s,\nabla u_s)\,dx\,dB_s^j$ is an integrable martingale.

Step 2 : Now, using the Lipschitz assumption on f and the algebra inequality : $2ab \leqslant a^2/\epsilon + \epsilon\, b^2$ with $\epsilon > 0$, we get

$$\begin{aligned}|u_s|^{l-1}\,|f(s,x,u_s,\nabla u_s)| \,&\leqslant\, |u_s|^{l-1}\,\big(|f_s^0| + C\,(|u_s| + |\nabla u_s|)\,\big)\\ &\leqslant\, |u_s|^{\frac{l}{2}}\,|u_s|^{\frac{l}{2}-1}\,\big(|f_s^0| + C\,(|u_s| + |\nabla u_s|)\,\big)\\ &\leqslant\, C\epsilon\,|u_s|^{l-2}\,|\nabla u_s|^2 \,+\, \big(C \,+\, \frac{C}{4\epsilon} + \frac{1}{2}\big)\,|u_s|^l \,+\, \frac{1}{2}|u_s|^{l-2}|f_s^0|^2.\end{aligned} \tag{17}$$

We use Schwarz inequality and the same argument as before to get

$$
(18)\qquad
\begin{aligned}
|u_s|^{l-2}\Big|\sum_{i=1}^{d}\partial_i u_s\, g_i(s,x,u_s,\nabla u_s)\Big| &\leqslant |u_s|^{l-2}\,|\nabla u_s|\,\big(\sum_{i=1}^{d}|g_i(s,x,u_s,\nabla u_s)|^2\big)^{1/2}\\
&\leqslant |u_s|^{l-2}\,|\nabla u_s|\,\Big(|g_s^0|+C|u_s|+\alpha|\nabla u_s|\Big)\\
&\leqslant \alpha|u_s|^{l-2}|\nabla u_s|^2+\epsilon|u_s|^{l-2}|\nabla u_s|^2+\frac{1}{4\epsilon}|u_s|^{l-2}|g_s^0|^2+C\epsilon|u_s|^{l-2}|\nabla u_s|^2+\frac{C}{4\epsilon}|u_s|^l\\
&\leqslant (\alpha+\epsilon+C\epsilon)\,|u_s|^{l-2}\,|\nabla u_s|^2+\frac{C}{4\epsilon}|u_s|^l+\frac{1}{4\epsilon}|u_s|^{l-2}|g_s^0|^2,
\end{aligned}
$$

and in the same way

$$
(19)\qquad
\begin{aligned}
|u_s|^{l-2}\big(\sum_{j=1}^{d_1}|h_j(s,x,u_s)|^2\big) &\leqslant |u_s|^{l-2}\big(|h_s^0|+C|u_s|\,\big)^2\\
&\leqslant |u_s|^{l-2}\,\big(|h_s^0|^2+C^2|u_s|^2+2C|h_s^0||u_s|\big)\\
&\leqslant \big(C^2+C\big)\,|u_s|^l\;+C\,|u_s|^{l-2}|h_s^0|^2\,.
\end{aligned}
$$

As a consequence we get

$$
\begin{aligned}
\int_{\mathcal O}|u_t|^l\,dx+l(l-1)\Big[\,\lambda-\big(\frac{\epsilon}{l-1}+\alpha+2\epsilon C\big)\,\Big]\int_0^t\int_{\mathcal O}|u_s|^{l-2}\,|\nabla u_s|^2dx\,ds&\\
\leqslant \int_{\mathcal O}|\xi|^l\,dx\;+l^2\,\big(\frac{3}{2}\,C+\frac{C}{2\epsilon}+\frac{C^2}{2}+\frac{1}{2}\big)\int_0^t\int_{\mathcal O}|u_s|^l\,dx\,ds&\\
+\,l^2\,\big(\frac{C}{2}+\frac{1}{4\epsilon}+\frac{1}{2}\big)\int_0^t\int_{\mathcal O}|u_s|^{l-2}\,\big[|f_s^0|^2+|g_s^0|^2+|h_s^0|^2\,\big]\,dx\,ds&\\
+\,l\,M_t\,.&
\end{aligned}
$$

Set $k_\epsilon:=\lambda-(\alpha+\epsilon+2\epsilon C)$ and choose $\epsilon 0$ small enough so that

$$
0<k_\epsilon\;\leqslant\;\lambda-\big(\frac{\epsilon}{l-1}+\alpha+2\epsilon C\big)\,.
$$

Then for $l\geqslant 2$, $t\in[0,\,T]$ and thanks to the Hölder's inequality we get

$$
(20)\qquad
\begin{aligned}
\int_{\mathcal O}|u_t|^l\,dx\;+\;l(l-1)k_\epsilon\int_0^t\int_{\mathcal O}|u_s|^{l-2}\,|\nabla u_s|^2\,dx\,ds\leqslant\;\leqslant\;K^l\;+\;l^2\,c'_\epsilon\int_0^T\int_{\mathcal O}|u_s|^l dx\,ds&\\
+\;l^2\,c'_\epsilon\,K^2\int_0^T\big(\int_{\mathcal O}|u_s|^l\,dx\big)^{\frac{l-2}{l}}\,ds+\;l\,M_T^*\,.&
\end{aligned}
$$

where $M_T^*=\sup_{0\,\leqslant\,t\,\leqslant\,T}|M_t|$.

Step 3 : We set now $v:=|u|^{l/2}$, then $\partial_i v=(l/2)\partial_i u\, sign(u)\,|u|^{(l-2)/2}$ and so $|\nabla v|^2=(l/2)^2|u|^{l-2}|\nabla u|^2$. Making the change of variables in the second term of the left hand side of the last equation and using the Sobolev's inequality (see Appendix) we get

$$
\begin{aligned}
l(l-1)k_\epsilon\int_0^t\int_{\mathcal O}|u_s|^{l-2}\,|\nabla u_s|^2\,dxds\;&=\;\frac{4k_\epsilon(l-1)}{l}\int_0^t\int_{\mathcal O}|\nabla v_s|^2\,dxds\\
&\geqslant\frac{4k'_\epsilon(l-1)}{l}\int_0^t\Big(\int_{\mathcal O}|v_s|^{2^*}\,dx\Big)^{2/2^*}\,ds\\
&=\;\frac{4k'_\epsilon(l-1)}{l}\,\|v\|^2_{2^*,2}
\end{aligned}
$$

where $2^* := \dfrac{2d}{d-2}$ if $d>2$, while 2^* may be any number in $]2,\infty[$ if $d=2$ and $2^*=\infty$ for $d=1$. k'_ϵ is a constant which depends only on ϵ, λ, C, α, $|\mathcal{O}|$ and the dimension d. Moreover, one can choose $\delta 0$ such that for all $\delta \geqslant \frac{4k'_\epsilon(l-1)}{l}$. Taking the supremum over $t\in[0,T]$ in the left side of the estimation (20), this yields

(21)
$$\|v\|^2_{2,\infty} \vee \left(\delta\, \|v\|^2_{2^*,2}\right) \leqslant K^l \;+ l^2\, c'_\epsilon \int_0^T \int_{\mathcal{O}} |u_s|^l dx\, ds + l^2\, c'_\epsilon\, K^2 \int_0^T \big(\int_{\mathcal{O}} |u_s|^l\, dx\big)^{\frac{l-2}{l}} ds + lM^*_T$$

Step 4 : We apply Lemma 8 from the Appendix with the couples $(2,\,\infty)$ and $(2^*,2)$ and obtain the interpolating inequality

$$\|v\|_{\varrho,2\varrho} \leqslant \|v\|^{\nu}_{2,\infty}\|v\|^{1-\nu}_{2^*,2} \leqslant \|v\|_{2,\infty} \vee \|v\|_{2^*,2}$$

where $\varrho := 2(d+1)/d$ for $d>2$ and $\rho=5/2$ for $d=1,\,2$ and $\nu = 1-1/\varrho$. Moreover, without loss of generality we may assume that $\delta \le 1$, so that we have

$$\delta\left(\|v\|^2_{2,\infty} \vee \|v\|^2_{2^*,2}\right) \le \|v\|^2_{2,\infty} \vee \left(\delta\, \|v\|^2_{2^*,2}\right),$$

and hence the estimate (21) becomes

$$\delta\, \|v\|^2_{\varrho,2\varrho} \leqslant K^l \;+ l^2\, c'_\epsilon \int_0^T \int_{\mathcal{O}} |u_s|^l dx\, ds + l^2\, c'_\epsilon\, K^2 \int_0^T \big(\int_{\mathcal{O}} |u_s|^l\, dx\big)^{\frac{l-2}{l}} ds + lM^*_T$$

which is equivalent, using the equality

(22)
$$\forall\, \sigma>0, \quad \forall\, p,\; q \geqslant 1, \qquad \|u^\sigma\|^{1/\sigma}_{p,q} = \|u\|_{\sigma p,\sigma q}$$

to

$$\delta \|u^l\|_{\frac{\varrho}{2},\rho} = \delta\|v^2\|_{\frac{\varrho}{2},\rho} \leqslant K^l \;+ l^2\, c'_\epsilon \int_0^T \int_{\mathcal{O}} |u_s|^l dx\, ds$$
$$+\, l^2\, c'_\epsilon\, K^2 \int_0^T \big(\int_{\mathcal{O}} |u_s|^l\, dx\big)^{\frac{l-2}{l}} ds + lM^*_T\,.$$

Set $\sigma := \dfrac{\rho}{2} > 1$. We apply the previous inequality with $l := \sigma^m$, for all $m \geqslant m_0$, where m_0 is the least integer such that $\sigma^{m_0} \geqslant 2$. Hence

$$\delta \|u^{\sigma^m}\|_{\sigma,2\sigma} \leqslant K^{\sigma^m} + \sigma^{2m}\, c'_\epsilon \int_0^T \int_{\mathcal{O}} |u_s|^{\sigma^m} dx\, ds + \sigma^{2m}\, c'_\epsilon\, K^2 \int_0^T \big(\int_{\mathcal{O}} |u_s|^{\sigma^m}\, dx\big)^{\frac{\sigma^m-2}{\sigma^m}} ds$$
$$+\, \sigma^m M^*_T\,,$$

or equivalently using the definition of the $L^{p,q}$-norm

$$\delta\, \|u^{\sigma^m}\|_{\sigma,2\sigma} \leqslant K^{\sigma^m} \;+ \sigma^{2m}\, c'_\epsilon \|u^{\sigma^m}\|_{1,1} + \sigma^{2m}\, c'_\epsilon\, K^2 \int_0^T \big(\int_{\mathcal{O}} |u_s|^{\sigma^m}\, dx\big)^{\frac{\sigma^m-2}{\sigma^m}} ds$$
$$+\, \sigma^m M^*_T\,.$$

By the Hölder's inequality

$$\delta\, \|u^{\sigma^m}\|_{\sigma,2\sigma} \leqslant K^{\sigma^m} \;+ \sigma^{2m}\, c'_\epsilon T^{\frac12} \|u^{\sigma^m}\|_{1,2} + \sigma^{2m}\, c'_\epsilon\, T^{\frac12+\frac{1}{\sigma^m}}\, K^2 \Big(\int_0^T \big(\int_{\mathcal{O}} |u_s|^{\sigma^m}\, dx\big)^{\frac{2\sigma^m}{\sigma^m}} ds\Big)^{\frac{\sigma^m-2}{2\sigma^m}}$$
$$+\, \sigma^m M^*_T\,.$$

Moreover by (22)

$$\delta\,\|u\|_{\sigma^{m+1},2\sigma^{m+1}}^{\sigma^m} \leqslant K^{\sigma^m} + \sigma^{2m}\,c'_\epsilon\,T^{\frac12}\|u\|_{\sigma^m,2\sigma^m}^{\sigma^m} + \sigma^{2m}\,c'_\epsilon\,T^{\frac12+\frac{1}{\sigma^m}}\,K^2\,\|u\|_{\sigma^m,2\sigma^m}^{\sigma^m-2} + \sigma^m M_T^*\,.$$

Define the sequence $e_m := K \vee \|u\|_{\sigma^m,2\sigma^m}$ for all $m \geqslant m_0$. Then the last estimate becomes

$$\delta\,e_{m+1}^{\sigma^m} \leqslant \Big(1 + \sigma^{2m}c'_\epsilon T^{\frac12}\left(1+T^{\frac{1}{\sigma^m}}\right)\Big)\,e_m^{\sigma^m} + \sigma^m M_T^*\,. \tag{23}$$

Hence

$$e_{m+1}^{\sigma^m} \leqslant \frac{1}{\delta}\Big(1+\sigma^m + \sigma^{2m}c'_\epsilon T^{\frac12}\left(1+T^{\frac{1}{\sigma^m}}\right)\Big)\,(e_m^{\sigma^m}\vee M_T^*),$$

or equivalently

$$e_{m+1} \leqslant \Big[\frac{1}{\delta}\Big(1+\sigma^m+\sigma^{2m}c'_\epsilon T^{\frac12}\left(1+T^{\frac{1}{\sigma^m}}\right)\Big)\Big]^{1/\sigma^m}\left(e_m \vee (M_T^*)^{1/\sigma^m}\right).$$

Define $b_m := \Big[\frac{1}{\delta}\Big(1+\sigma^{2m}c'_\epsilon T^{\frac12}\left(1+T^{\frac{1}{\sigma^m}}\right)\Big)\Big]^{1/\sigma^m}$ and take the expectation in the last inequality

$$E\,e_{m+1} \leqslant b_m\,E\left[\,e_m \vee (M_T^*)^{1/\sigma^m}\,\right],\qquad \forall\, m \geqslant m_0.$$

By Lemma 10 in the Appendix this is further majorised by

$$E\,e_{m+1} \leq b_m 2^{1/\sigma^{\frac{m}{2}}}\,E\left[e_m \vee \langle M\rangle_T^{1/2\sigma^m}\right], \tag{24}$$

with $m \geq n_0 = m_0 \vee m_0'$, where m_0' is the least integer such that $\frac{1}{\sigma^m}$ belongs to the interval $(0,p_0)$ given by that lemma .

Step 5 : We need now an estimate for the quadratic variation of $(M)_{t\,\geqslant\,0}$ namely

$$\forall\, t\geqslant 0,\quad < M >_t = \sum_{j=1}^{d_1}\int_0^t\Big(\int_{\mathcal O}|u_s|^{l-1}\,h_j(s,x,u_s)\,dx\Big)^2 ds\,.$$

We use the Lipschitz property of h and Hölder inequality to get, for any $l \geqslant 2$,

$$\begin{aligned}
|< M >_T| &\leqslant \int_0^T d_1\Big(\int_{\mathcal O}\left[\,|u_s|^{l-1}|h^0| + C\,|u_s|^l\,\right]dx\Big)^2 ds\\
&\leqslant 2d_1K^2\int_0^T\Big(\int_{\mathcal O}|u_s|^l\,dx\Big)^{2\frac{l-1}{l}} ds + 2d_1\,C^2\int_0^T\Big(\int_{\mathcal O}|u_s|^l\,dx\Big)^2 ds\\
&\leqslant 2\,d_1\,K^2T^{\frac1l}\,\|u^l\|_{1,2}^{2\frac{l-1}{l}} + 2d_1C^2\|u^l\|_{1,2}^2\\
&\leqslant \left(2d_1T^{\frac1l}+2d_1C^2\right)\left(K\vee\|u\|_{l,2l}\right)^{2l}\\
&\leqslant 2\,d_1\left(\left(T+1\right)^{\frac12}+C^2\right)e_m^{2\sigma^m}\,.
\end{aligned} \tag{25}$$

Using this estimate to dominate the right hand side of relation (24) we get

$$E\,e_{m+1} \leqslant b_m\,2^{1/\sigma^{\frac{m}{2}}}\,E\left[\,e_m \vee \left[\,2\,d_1\left(\left(T+1\right)^{\frac12}+C^2\right)\right]^{1/2\sigma^m} e_m\right],\quad \forall\, m\geqslant n_0\,.$$

Denoting $b'_m := \left[\,2\,d_1\left(\left(T+1\right)^{\frac12}+C^2\right)\right]^{1/2\sigma^m}$ we may further write

$$E\,e_{m+1} \leqslant b_m\,b'_m\,2^{1/\sigma^{\frac{m}{2}}}\,E\,e_m\,.$$

Denote $A' := \prod_{m=n_0}^{\infty} b_m\, b'_m\, 2^{1/\sigma^{\frac{m}{2}}}$ which converges because $\sigma > 1$. Thus

$$E\, e_m \leqslant A'\, E\, e_{n_0}, \quad \forall\, m \geqslant n_0$$

Taking the limit as $m \to \infty$ and using the monotone convergence theorem we get

$$\begin{aligned} E\, \|u\|_{L^\infty\big([0,T]\times\mathcal{O}\big)} &\leqslant A'\, E\, e_{n_0} \\ &\leqslant A' \left(K + E\, \|u\|_{\sigma^{n_0}, 2\sigma^{n_0}} \right) \\ &\leqslant A' \Big(K + \Big(\int_0^T E\, \big[\int_O |u_s(x)|^{\sigma^{n_0}}\, dx \big]^2\, ds \Big)^{\frac{1}{2\sigma^{n_0}}} \Big) \\ &\leqslant A' \Big(K + \Big(\int_0^T E\, \big[\int_O |u_s(x)|^{2\sigma^{n_0}}\, dx \big]\, ds \Big)^{\frac{1}{2\sigma^{n_0}}} \Big). \end{aligned}$$

Applying Lemma 5 it is now easy to conclude. □

4 Appendix

4.1 $L^{p,q}$-spaces

For fixed $T > 0$, and for each real numbers p, $q \geqslant 1$, we denote by $L^{p,q}([0,T] \times \mathcal{O})$ the space of measurable functions $v : \mathbb{R}_+ \times \mathcal{O} \longrightarrow \mathbb{R}$ such that

$$\|v\|_{p,q} := \Big(\int_0^T \Big(\int_{\mathcal{O}} |v(t,x)|^p\, dx \Big)^{q/p} dt \Big)^{1/q}$$

is finite. We need the following results concerning the spaces $L^{p,q}$, which were also used by Aronson-Serrin [1]

Lemma 8. *If $u \in L^{p,p_1} \bigcap L^{q,q_1}$, then $u \in L^{r,r_1}$, where*

$$\frac{1}{r} = \frac{\gamma}{p} + \frac{(1-\gamma)}{q}, \qquad \frac{1}{r_1} = \frac{\gamma}{p_1} + \frac{(1-\gamma)}{q_1}, \qquad 0 \leqslant \gamma \leqslant 1.$$

Moreover we have

$$\|u\|_{r,r_1} \leqslant \|u\|_{p,p_1}^{\gamma}\, \|u\|_{q,q_1}^{1-\gamma} \tag{26}$$

Lemma 9 (Sobolev's inequality)**.** *Assume $d > 2$. Let $u \in H_0^1(\mathcal{O})$, then $u \in L^{2^*}(\mathcal{O})$ where $2^* := \dfrac{2d}{d-2}$ and there exists a constant $c > 0$ which depends only on the dimension d such that*

$$\|u\|_{2^*} \leqslant c\, \|\nabla u\|_2\, . \tag{27}$$

If $d = 2$, 2^* may be any number in $]2, \infty[$ and if $d = 1$, $2^* = \infty$, while the constant c depends also on $|\mathcal{O}|$ (i.e. the Lebesgue measure of $\mathcal{O}$).

A consequence of the Sobolev's inequality which will be used in our context is the following

$$\|u\|_{2^*,2} \leqslant c\, \|\nabla u\|_{2,2}, \qquad \forall\, u \in L^2\big([0, T];\; H_0^1(\mathcal{O}) \big)\, . \tag{28}$$

4.2 An application of the good λ-inequality

We use in the proof of the main theorem the following lemma.

Lemma 10. *There exists $p_0 > 0$ such that for any continuous local martingale M, any nonnegative random variable Z, and any $T > 0$, the following inequality holds*

$$E\left(M_T^* \vee Z\right)^p \leq 2^{p^{\frac{1}{2}}} E\left(\langle M\rangle_T^{\frac{1}{2}} \vee Z\right)^p, \qquad \forall p \in]0, p_0[\,.$$

Proof. We first recall the notion of "good λ -inequalities" which was used by Burkholder to prove the BDG estimates. The reader is refered to Revuz-Yor [22] or Rogers-Williams [23] for more details. Suppose that ϕ is a positive real function defined on $]0, a]$ such that $\phi(\delta) \downarrow 0$ as $\delta \downarrow 0$ and $\beta > 1$ a real number.

A pair (X, Y) of nonnegative random variables is said to satisfy the good λ -inequalities $I(\phi, \beta)$ provided that

$$P(X > \beta\lambda,\ Y \leq \delta\lambda) \leq \phi(\delta)\, P(X > \lambda)\,,$$

for any $\lambda > 0$ and $\delta \in]0, a]$.
The main result related to this notion that interests us is the following : there exists a constant b depending only on ϕ, β and $p > 0$ such that

$$EX^p \leq bEY^p,$$

(see Revuz-Yor [22], chap. IV, Lemma 4.9 and Rogers-Williams [23], Lemma 42.3).
More precisely, here we are interested in finding an estimate of b for a particular function ϕ and p near zero.
Let W be the standard brownian motion and P^0 the Wiener measure. Let us define

$$\varphi(\delta) = P^0\left(W_1^* \geq \frac{\beta - 1}{\delta}\right), \delta > 0\,.$$

It is known that M_T^* and $\langle M\rangle_T^{\frac{1}{2}}$ satisfy the $I(\varphi, \beta)$ inequalities with any $\beta 1$ and φ defined above (see the proof of Theorem IV.4.10 in [Revuz Yor]). On the other hand, it is easy to check that, in general, if X and Y satisfy $I(\phi, \beta)$ and Z is a nonnegative random variable, then $X \vee Z$ and $Y \vee Z$ satisfy $I(\phi, \beta)$ too. Moreover, by the classical arguments, the function φ is dominated by the simpler function $\psi(\delta) = \frac{\delta}{\beta - 1}$.
Therefore the estimate of the lemma follows from the next lemma applied with $\gamma = 2$ and $\alpha = \frac{1}{2}$. $\square$

Lemma 11. *Let X, Y satisfy the $I(\psi, \beta)$ inequalities with $\beta > 1$ and $\psi(\delta) = c\delta, c > 0$. For each $\gamma 1$ and $\alpha \in (0, 1)$ there exists $p_0 > 0$ such that*

$$E\, X^p \,\leq\, \gamma^{p^\alpha}\, E\, Y^p,$$

for any $p \in (0, p_0)$.

Proof. The standard argument of good λ -inequalities implies

$$\left(\frac{1}{\beta^p} - \psi(\delta)\right) EX^p \leq \frac{1}{\delta^p} EY^p.$$

In order to prove the estimate from the statement it is enough to show that

$$\gamma^{-p^\alpha} \leq \frac{\delta^p}{\beta^p}\left(1 - \beta^p \psi(\delta)\right),$$

for p in a neighbourhood of 0 and a suitable choosen number $\delta(p)$. In fact we will show that $\delta(p) = \nu^{p^{\alpha-1}}$, with a constant $\nu \in \left(\gamma^{-1}, 1\right)$, does the job. With this δ the last relation takes the form

$$\gamma^{-1} \leq \frac{\nu}{\beta^{p^{1-\alpha}}} \left(1 - \beta^p c \nu^{p^{\alpha-1}}\right)^{p^{-\alpha}} .$$

Since obviously $\lim_{p \to 0} \beta^{p^{1-\alpha}} = 1$, and since $\gamma^{-1} < \nu$, to find out an interval $(0, p_0)$ where the preceding inequality holds it suffices to note that

$$\lim_{p \to 0} \left(1 - \beta^p c \nu^{p^{\alpha-1}}\right)^{p^{-\alpha}} = 1 .$$

□

References

[1] Aronson, D.G. and Serrin J. (1967) : Local behavior of solutions of quasi-linear parabolic equations., in : *Archive for Rational Mechanics and Analysis*, vol. **25** (1967), pp. 81-122. .

[2] Bally V. and Matoussi A. (2001) : Weak solutions for SPDE's and Backward Doubly SDE's, in : *J. of Theoret.Probab.* **14**, 125-164.

[3] Bouleau N. and Hirsch F. (1993) : *Dirichlet forms and analysis on Wiener space*, Kluwer.

[4] Da Prato G. and Zabczyk J. (1992) : *Stochastic equations in Infinite Dimensions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press.

[5] Denis L. (2004) : Solutions of SPDE considered as Dirichlet Processes, in : *Bernoulli Journal of Probability*, **10** (5).

[6] Denis L. and Stoica, I. L. (2001) : A general analytical result for non-linear s.p.d.e.'s and applications, in : *Electronic Journal of Probability*, **9**, p. 674-709.

[7] Denis L., Matoussi A. and Stoica, I. L. (2005) : L^p estimates for the uniform norm of solutions of quasilinear SPDE's, in : *Probab. Theory Relat. Fileds* ***133****, 437-463.*

[8] Denis L., Matoussi A. and Stoica, I. L. (2008) : *Maximum principle and Comparison Theorem for Quasilinear Stochastic PDE's. Submitted paper, preprint 2007.*

[9] Fukushima M., Oshima Y. and Takeda M. (1994) : *Dirichlet Forms and Symmetric Markov Processes, de Gruyter studies in Math.*

[10] Gyöngy I. and Rovira C. (2000) : *On L^P-solutions of semilinear stochastic partial differental equations, in : Stochastic Processes and their Applications,* **90** , 83-108.

[11] Kuksin, S. B., Nadirashvili, N. S. adn Piatnitski, A. L. (2004) : Hölder estimates for solutions of parabolic SPDES. *Theory Probab. Appl.*, **47** , 157-64.

[12] Krylov N. V., Rozovskii, N.V. (1981) : Stochastic evolution equations. *Journal of Soviet Mathematics*, Vol. **14** , 1233-1277.

[13] Krylov N. V. (1999) : An analytic approach to SPDEs. *Stochastic Partial Differential Equations* : Six Perspectives, AMS Mathematical surveys an Monographs, **64**, 185-242.

[14] Kunita H. (1994) : Generalized Solutions of a Stochastic Partial Differential Equation, in : *J. of Theoret. Probab*, **7**, 279-308.

[15] Matoussi A. and Scheutzow, M. (2002) : Semilinear Stochastic PDE's with nonlinear noise and Backward Doubly SDE's, in : *J. of Theoret. Probab.* **15**, 1-39.

[16] Mikulevicius R. and Rozovskii B.L. (2001) : A Note on Krylov's Lp -Theory for Systems of SPDES, in : *Electronic Journal of Probability*, 6, paper No 12, pp. 1-35.

[17] Moser J. (1961) : On Harnack's theorem for elliptic differential equation, in *Communications on Pures and applied Mathematics,* **14**, 577-591.

[18] Nualart D. and Pardoux E. (1992) : White noise driven quasilinear SPDEs with reflectio,, in : *Probab. Theory Related Fields* **93** , 77-89.

[19] Pardoux, E. (1975) : Equations aux dérivées partielles stochastiques montones. Etude de solutions fortes de type Itô. *Thèse de Doctorat Université Paris Sud.*

[20] Pardoux E. (1979) : Stochastic partial differential equations and filtering of diffusion process. *Stochastics* **3**, 127-167.

[21] Pardoux E. and Peng, S. (1994) : Backward doubly SDE's and systems of quasilinear SPDEs. *Probab. Theory Relat. Field* **98**, 209-227.

[22] Revuz, D. and Yor, M.(1991) : *Continuous Martingales and Brownian Motion.* Springer, third edition.

[23] Rogers, L. C. G. and Williams, D. (1987) : *Diffusions, Markov Processes and Martingales*, Vol. II, Itô Calculus, Wiley.

[24] Rozovskii B.L. (1990) : *Stochastic Evolution Systems*, Kluver, Dordrecht- Boston- London.

[25] Walsh, J.B. (1986) : An introduction to stochastic partial differential equations. *Ecole d'Eté de St-Flour XIV, 1984, Lect. Notes in Math*, **1180**, Springer Verlag.