

Acoustic fields in binary gas mixtures: Mutual diffusion effects throughout and beyond the boundary layers

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The acoustic behavior in thermo-viscous gas mixtures, both in proximity of walls and far from them (outside the boundary layers), involves deviations from the adiabatic and laminar movements in pure gases, which result from the influence of several diffusive fields, namely, shear, entropic, and concentration variation fields (their energy being provided by the acoustic field itself). Owing to the boundary conditions, that are slip condition, isothermal condition and concentration flux vanishing on the walls, a strong coupling between these fields occurs inside the boundary layers while their effects appear to be simple additive processes in the bulk of the medium. Although recent literature on this subject leads to interesting results, opening the way to several new issues [R. Raspet *et al.*, *J. Acoust. Soc. Am.* **105**, 65–73 (1999); R. Raspet *et al.*, *J. Acoust. Soc. Am.* **112**, 1414–1422 (2002); G. W. Swift and P. S. Spoor, *J. Acoust. Soc. Am.* **106**, 1794–1800 (1999); D. A. Geller and G. W. Swift, *J. Acoust. Soc. Am.* **111**, 1675–1684 (2002)], the results available still have limitations because they do not provide complete solutions for the propagative and diffusive fields throughout and beyond the boundary layers. The present work aims at providing these solutions in the whole domains considered. The results allow interpreting analytically the behavior of the fields above mentioned in closed cavities and ducts, and particularly in spherical cavities which are best suited to develop metrological applications. © 2012 Acoustical Society of America.

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NOMENCLATURE

a_0	Speed of sound
b	$cM^2/(M_1M_2)$
c	Mass concentration variation of the component 1
C, C_0	Total and static mass concentrations of the component 1
C_P, C_V	Isobaric and isochoric specific heat capacities
D	Diffusion coefficient
g, g_i	Chemical potential per unit of mass of the gas and of the component i
$J_n(z)$	n order Bessel function
\vec{i}	Concentration flux density (diffusion transfer)
$k_0 = \omega/a_0$	Adiabatic wavenumber
k_P, k_T	Baro- and thermal diffusion ratios
k_a, k_v, k_h, k_D	Wavenumbers associated to the acoustic, vortical, entropic, and diffusive movements, respectively
$\ell_v, \ell'_v, \ell_h, \ell_D$	Viscous (with and without bulk viscosity), thermal and diffusive characteristic lengths
ℓ_{vhd}	$\ell_v + (\gamma - 1)\ell_h + \gamma x(1 - x)\alpha^2\ell_D$

M, M_i	Molar mass of gas and of component i
p	Pressure variation
P, P_0	Total and static pressures
\vec{q}	Thermal flux density
R	Molar gas constant
S	Entropy per unit of mass
T, T_0	Total and static temperatures
\vec{v}	Particle velocity
$x = C_0M/M_1$	Mole fraction of component 1
α	$\alpha_T(\gamma - 1)/\gamma + \alpha_P$
α_P	$k_P/[x(1 - x)]$
α_T	$k_T/[x(1 - x)]$
β	Increase in pressure per unit increase in temperature at constant density
$\hat{\beta}$	$P\beta \approx P_0\beta \approx P_0/T_0$
Δ, Δ_w	Respectively, Laplacian and Laplacian with respect to the w_1 and w_2 coordinates
$\gamma = C_P/C_V$	Specific heat ratio
ϵ	$k_T\alpha_T(\gamma - 1)/\gamma$
η	Bulk viscosity
λ	Thermal conductivity
μ	Shear viscosity
ω	Angular frequency
ρ, ρ_0	Total and static densities (with $\rho_0 a_0^2 = \gamma P_0 \approx \gamma \hat{\beta} T_0$)
ρ'	Density variation
τ	Temperature variation

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I. INTRODUCTION

The general behavior of binary gas mixtures was described by Landau and Lifshitz¹ starting from hydrodynamical and thermodynamical fundamental equations, and by Chapman and Cowling² from the point of view of kinetic theory. Both approaches emphasize that the thermophysical properties of a gas mixture are not reduced to a simple weighted mean of the individual properties of each component, but that they result from the cross-interactions between the different components of the mixture. Particularly, the phenomena involved are mutual diffusions of the components depending on the main parameters involved in an acoustic field, including concentration variations actually due to pressure variations (barodiffusion) and temperature variations (thermal diffusion).

In a theoretical approach to the description of sound absorption in air, Rocard calculated the thermal diffusion effects in the bulk of an air-filled domain. In this case, these effects are very small with respect to the viscous and thermal effects, and the interactions between the mutual diffusion and the viscous effects (in the Navier–Stokes equation) are negligible.³ In fact, the mutual diffusion effects on the acoustic behavior of gas mixtures are more significant in disparate mass mixtures. They are thus very small in air, and usually neglected in several aerial acoustic problems. Later, several works based on hydrodynamical and thermodynamical fundamental equations, or based on kinetic theory, provided more complete expressions of mutual diffusion effects (including thermal diffusion and barodiffusion).^{4–6}

Recent works,^{7–10} which enabled a description of the mutual diffusion effects coupled to the viscous and thermal effects in the boundary layers, were specifically dedicated to applications using gas mixtures or humid air. But, despite these advances in the modeling of acoustic fields in gas mixtures, today a general analytical model of acoustic behaviors of a binary gas mixture, near and far from the boundaries, is not yet available.

It is the aim of this paper to investigate the analytical approach more closely to provide such a model for binary gas mixtures, starting from the hydrodynamical and thermodynamical fundamental equations (Sec. II), leading to solutions in the bulk (Sec. III) and also close to boundaries (Sec. IV). The theoretical results obtained from this approach are used to calculate the effects of mutual diffusion on the acoustic fields in binary gas mixtures for several practical applications (Sec. V).

II. LINEARIZED FUNDAMENTAL EQUATIONS OF ACOUSTICS IN BINARY GAS MIXTURES

Most of the equations presented in this section are adapted from the formalism of Landau and Lifshitz.¹

In the Navier–Stokes equation

$$\rho_0 \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} p + \left(\eta + \frac{4}{3} \mu \right) \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) - \mu \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{v}, \quad (1)$$

the variables p and \vec{v} represent the particle velocity and the pressure variation, respectively; and ρ_0 , η , and μ are, respectively, the mass density, the bulk, and shear viscosity coefficients of the mixture.

The following differential expression for the density ρ , involves the pressure P , the temperature T , and the molar mass of each component M_i ($i = 1, 2$),

$$d\rho = \rho \left[\frac{1}{P} dP - \frac{1}{T} dT - \frac{M_2 - M_1}{M_1 M_2} M dC \right],$$

where C_0 and x are, respectively, the mass concentration and the mole fraction of component “1,”

$$M = \frac{M_1 M_2}{M_1 + (M_2 - M_1) C_0} = x M_1 + (1 - x) M_2. \quad (2)$$

Making use of this differential expression to remove the density variation ρ' from the mass conservation equation for the gas mixture

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v} = 0$$

leads to

$$\frac{\partial \tau}{\partial t} - \frac{T_0}{P_0} \frac{\partial p}{\partial t} + T_0 \frac{M_2 - M_1}{M_1 M_2} M \frac{\partial c}{\partial t} - T_0 \vec{\nabla} \cdot \vec{v} = 0. \quad (3)$$

The mass conservation equation for the component “1” of the gas mixture

$$\frac{\partial (C\rho)}{\partial t} + \vec{\nabla} \cdot (C\rho\vec{v}) + \vec{\nabla} \cdot \vec{i} = 0,$$

where the concentration flux density is written as

$$\vec{i} = -\rho D \frac{M_1 M_2}{M^2} \left[\frac{M^2}{M_1 M_2} \vec{\nabla} C + \frac{k_P}{P} \vec{\nabla} P + \frac{k_T}{T} \vec{\nabla} T \right], \quad (4)$$

D , k_P , and k_T being the diffusion coefficient, and the baro and thermal diffusion ratios respectively, can be linearized as follows, taking account of Eq. (3),

$$\left(\frac{\partial}{\partial t} - D\Delta \right) b = D \left[\frac{k_P}{P_0} \Delta p + \frac{k_T}{T_0} \Delta \tau \right], \quad (5)$$

where

$$b = \frac{M^2}{M_1 M_2} c. \quad (6)$$

Note that the barodiffusion ratio is exactly determined by the following expression:

$$k_P = P_0 \frac{(\partial V / \partial C_0)_{P,T}}{(\partial g / \partial C_0)_{P,T}} = x(1-x)\alpha_P, \quad (7)$$

where $\alpha_P = (M_2 - M_1)/M$.

Inserting the following differential expression for the entropy per unit mass S

$$dS = \frac{C_P}{T} dT - \frac{1}{\rho T} dP - \left(\frac{\partial g}{\partial T} \right)_{P,C} dC,$$

where $g = g_1/M_1 - g_2/M_2$ is the chemical potential of the mixture per unit of mass and g_i the chemical potential of component i , in the Fourier's equation

$$\rho_0 T_0 \frac{\partial S}{\partial t} = -\vec{\nabla} \cdot (\vec{q} - g\vec{i}),$$

the heat flux density being given by

$$\vec{q} = \left[\frac{k_T}{x(1-x)} \frac{M^2 RT}{M_1 M_2} - T \left(\frac{\partial g}{\partial T} \right)_{P,C} + g \right] \vec{i} - \lambda \vec{\nabla} T, \quad (8)$$

leads to the following linearized diffusion equation:

$$\left(\frac{\partial}{\partial t} - \frac{\lambda}{\rho_0 C_P} \Delta \right) \tau = \frac{1}{\rho_0 C_P} \frac{\partial p}{\partial t} + \frac{P_0}{\rho_0 C_P} \alpha_T \frac{\partial b}{\partial t}, \quad (9)$$

where $\alpha_T = k_T/[x(1-x)]$.

Then, using the following identities (perfect gas)

$$T_0/P_0 = 1/\hat{\beta}, \quad \rho_0 C_P = \frac{\gamma \hat{\beta}}{\gamma - 1}, \quad (10a)$$

and using the following notations:

$$\begin{aligned} \ell_D &= \frac{D}{a_0}, \quad \ell_h = \frac{\lambda}{\rho_0 a_0 C_P}, \quad \ell'_v = \frac{\mu}{\rho_0 a_0}, \\ \ell_v &= \frac{1}{\rho_0 a_0} \left(\eta + \frac{4}{3} \mu \right), \end{aligned} \quad (10b)$$

the linearized fundamental set of acoustic equations in binary gas mixtures is given by

$$\frac{1}{a_0} \frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_0 a_0} \vec{\nabla} p + \ell_v \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) - \ell'_v \vec{\nabla} \wedge \vec{\nabla} \wedge \vec{v}, \quad (11)$$

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{T_0} \frac{\partial \tau}{\partial t} - \frac{1}{P_0} \frac{\partial p}{\partial t} + \alpha_P \frac{\partial b}{\partial t}, \quad (12)$$

$$\left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_h \Delta \right) \tau = \frac{\gamma - 1}{\gamma \hat{\beta}} \frac{1}{a_0} \frac{\partial}{\partial t} (p + P_0 \alpha_T b), \quad (13)$$

$$\left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_D \Delta \right) b = \ell_D \left[\frac{k_P}{P_0} \Delta p + \frac{k_T}{T_0} \Delta \tau \right]. \quad (14)$$

In Eqs. (13) and (14), the pressure variation p can be expressed as the sum of a perturbed acoustic pressure p_a and an excess of pressure variation $p_h + p_b$ associated to the heat (index h) and the concentration (index b) diffusion movements. It is the same for the temperature variation $\tau = \tau_a + \tau_h + \tau_b$.

On the other hand, in these fundamental equations (12), (13), and (14), consistent with previous works,⁷ it is clear that, as described by Landau and Lifshitz,¹ three different kinds of diffusion effects occur in the gas mixture, namely the diffusion flows due to (i) concentration gradients, whose effects are expressed by the diffusion coefficient D inversely proportional to the static pressure P_0 and weakly depending on x , (ii) temperature gradients, whose effects are expressed by the coefficient of thermal diffusion $k_T D$, (iii) pressure gradients, whose effects are expressed by the coefficient of barodiffusion $k_P D$.

The thermal diffusion and barodiffusion ratios both vanish when $x = 0$ or $x = 1$ (pure gases).

For coherence with the previous approximations (linear acoustics), terms of order higher than one of the characteristic lengths (i.e., order two of the penetration depths) are neglected in the following.

III. ACOUSTIC FIELD IN INFINITE DOMAIN

Substituting Eq. (13) into the equation obtained by applying the operator $[(1/a_0)(\partial/\partial t) - \ell_h \Delta]$ in Eq. (12), then inserting the resulting equation into the equation obtained by applying the operator $[(1/a_0)(\partial/\partial t) - \ell_h \Delta]$ in Eq. (11), and neglecting the shear viscosity effects, leads to a couple of equations that can be identified to remove the variable \vec{v} , giving then

$$\left[\frac{1}{a_0^3} \frac{\partial^3}{\partial t^3} - \left(1 + (\ell_v + \gamma \ell_h) \frac{1}{a_0} \frac{\partial}{\partial t} \right) \frac{1}{a_0} \frac{\partial}{\partial t} \Delta + \ell_h \left(1 + \gamma \ell_v \frac{1}{a_0} \frac{\partial}{\partial t} \right) \Delta \right] p = \rho_0 a_0^2 \left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_v \Delta \right) \left(\alpha \frac{1}{a_0} \frac{\partial}{\partial t} - \ell_h \alpha_P \Delta \right) \frac{1}{a_0} \frac{\partial}{\partial t} b.$$

After employing the factorization and approximations suggested in Ref. 11 (pp. 79–81) (which can be verified straightforwardly), making use of Eq. (14) and assuming that $p = p_a + p_h + p_b \approx p_a$ and $\tau = \tau_a + \tau_h + \tau_b \approx \tau_a \approx (\gamma - 1)/(\hat{\beta} \gamma) p_a$, the operator $[(1/a_0)(\partial/\partial t) - \ell_D \Delta]$ acting on the equation obtained leads to

$$\begin{aligned} & \left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_D \Delta \right) \left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_h \Delta \right) \left[\frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} - \left(1 + (\ell_v + (\gamma - 1) \ell_h) \frac{1}{a_0} \frac{\partial}{\partial t} \right) \Delta \right] (p_a + p_h + p_b) \\ & = \gamma \ell_D \left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_v \Delta \right) \left(\alpha \frac{1}{a_0} \frac{\partial}{\partial t} - \ell_h \alpha_P \Delta \right) \frac{1}{a_0} \frac{\partial}{\partial t} \left[k_P \Delta p_a + \hat{\beta} k_T \Delta \tau_a \right]. \end{aligned}$$

Finally, assuming that $\ell_{D,h,v}\Delta p_a \ll 1/a_0\partial p_a/\partial t$ leads straightforwardly to

$$\frac{1}{a_0^2} \frac{\partial^2}{\partial t^2} p_a - \left[1 + \ell_{vhd} \frac{1}{a_0} \frac{\partial}{\partial t} \right] \Delta p_a \approx 0, \quad (15a)$$

$$\left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_h \Delta \right) p_h \approx 0, \quad (15b)$$

$$\left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell_D \Delta \right) p_h \approx 0, \quad (15c)$$

where $\ell_{vhd} = \ell_v + (\gamma - 1)\ell_h + \gamma x(1-x)\alpha^2 \ell_D$.

Considering an harmonic motion of angular frequency ω , Eq. (15a) can be written as

$$(\Delta + k_a^2) p_a \approx 0, \quad (16a)$$

the acoustic wavenumber k_a being given by

$$k_a^2 = k_0^2(1 - j k_0 \ell_{vhd}). \quad (16b)$$

It is easy to verify that the term $\ell_D \gamma x(1-x)\alpha^2$ related to the mutual diffusion phenomena vanishes in a pure gas, leading then to the well-known results for the wavenumber k_a in a thermo-viscous gas.

This result, which accounts for the mutual diffusion effects, coupled to the viscous and thermal effects in the bulk of the fluid, is in agreement with those of previous works.⁴⁻⁶

IV. ACOUSTIC FIELD CLOSE TO THE BOUNDARIES

The domain considered here is close to a rigid wall. The coordinate normal to the wall inwardly directed is denoted u , the origin (denoted s) being on the wall, and the coordinates tangent to the wall are denoted (w_1, w_2) or \vec{w} for the both of them. In order to avoid overly intricate formulations, the following appropriate approximations are assumed on the fundamental linear equations (11)–(14), giving an accurate description of the small amplitude disturbances inside the boundary layers.

- (1) Mostly through the boundary layers, the component normal to the wall of the acoustic velocity v_u is much lower than the tangential components v_{w_i} .
- (2) The spatial variation of the acoustic velocity is much higher in the normal direction u than in the tangential directions w_i .

According to these assumptions, the normal and tangential components of the Navier–Stokes equation (11) lead to the following relations, using expression (12) for $\vec{\nabla} \cdot \vec{v}$, and assuming that $\tau \approx (\gamma - 1)/(\gamma\beta)p$ in the lower order terms

$$\rho_0 \frac{\partial}{\partial t} v_u \approx \frac{\partial}{\partial u} \left[- \left(1 + \ell_v \frac{1}{a_0} \frac{\partial}{\partial t} \right) p + \rho_0 a_0 \ell_v \alpha_P \frac{\partial}{\partial t} b \right], \quad (17a)$$

$$\left(\frac{1}{a_0} \frac{\partial}{\partial t} - \ell'_v \frac{\partial^2}{\partial u^2} \right) \vec{v}_w \approx - \frac{1}{\rho_0 a_0} \left(1 + \ell_v \frac{1}{a_0} \frac{\partial}{\partial t} \right) \vec{\nabla}_w p + \ell_v \alpha_P \frac{\partial}{\partial t} \vec{\nabla}_w b. \quad (17b)$$

It is noteworthy that, inside the boundary layers (see below), the order of magnitude of each term of Eq. (17a) is lower than the corresponding term of Eq. (17b). Therefore, in the applications presented below, Eq. (17a) remains unused.

To sum up, for an harmonic motion (angular frequency ω), the particle velocity and the pressure, temperature and concentration variations are described by the following set of equations

$$\left(1 + \frac{1}{k_v^2} \frac{\partial^2}{\partial u^2} \right) \vec{v}_w \approx - \frac{1}{j\omega\rho_0} (1 + jk_0\ell_v) \vec{\nabla}_w p + a_0 \ell_v \alpha_P \vec{\nabla}_w b, \quad (18)$$

$$\frac{\partial}{\partial u} v_u + \vec{\nabla}_w \cdot \vec{v}_w = -j\omega \frac{\gamma}{\rho_0 a_0^2} (p - \hat{\beta}\tau) + j\omega \alpha_P b, \quad (19)$$

$$\left(1 + \frac{1}{k_h^2} \Delta \right) \tau \approx \frac{\gamma - 1}{\gamma\hat{\beta}} (p + P_0 \alpha_T b), \quad (20)$$

$$\left(1 + \frac{1}{k_D^2} \Delta \right) b \approx - \frac{\gamma}{\rho_0 a_0^2 k_D^2} [k_P \Delta p + \hat{\beta} k_T \Delta \tau], \quad (21)$$

where

$$k_v^2 = - \frac{jk_0}{\ell'_v}, \quad k_h^2 = - \frac{jk_0}{\ell_h}, \quad k_D^2 = - \frac{jk_0}{\ell_D}, \quad (22)$$

and are subject to the following boundary conditions:

$$v_u(u, \vec{w}) = 0, \quad u = s, \quad \forall \vec{w}, \quad (23a)$$

$$\vec{v}_w(u, \vec{w}) = 0, \quad u = s, \quad \forall \vec{w}, \quad (23b)$$

$$\tau(u, \vec{w}) = 0, \quad u = s, \quad \forall \vec{w}, \quad (24)$$

$$i_u(u, \vec{w}) = 0, \quad u = s, \quad \forall \vec{w}, \quad (25)$$

where the expression of the concentration flux density (4), assuming that the gradient of the acoustic pressure with respect to the coordinate u in the direction normal to the wall is negligible, gives

$$\frac{\partial b}{\partial u} + \frac{\hat{\beta}}{P_0} k_T \frac{\partial \tau}{\partial u} \approx 0, \quad (26a)$$

which, eliminating b in accordance with Eq. (21) [$\epsilon = k_T \alpha_T (\gamma - 1)/\gamma$], leads to

$$(1 + \epsilon) \frac{\partial \tau}{\partial u} \Big|_{u=s} + \frac{1}{k_h^2} \frac{\partial^3 \tau}{\partial u^3} \Big|_{u=s} \approx 0. \quad (26b)$$

It should be noticed that the boundary conditions on the particle velocity (23a) is not satisfied only when the mixture does not involve phase-change process (namely, here condensation-evaporation). Indeed, when such process occurs, the walls that

do not behave as perfectly rigid walls anymore, leading to a non-zero normal particle velocity v_u and to several other boundary conditions on the vapor pressure and normal particle velocity.⁷

However, the acoustic model presented here should be usable for the study of humid air under physical conditions (humidity ratio, temperature, pressure) away from saturation.

A. Solutions for the temperature variation

Applying the operator $(1 + (1/k_D^2)\Delta)$ on the equation of diffusion of temperature (20) and making use of Eq. (21) to remove the variable b leads to the following equation for τ (consistent at the lowest order with previous works⁹):

$$\left[1 + \left(\frac{1}{k_h^2} + \frac{1+\epsilon}{k_D^2}\right)\Delta + \frac{1}{k_h^2 k_D^2}\Delta\Delta\right]\tau \approx \frac{\gamma-1}{\gamma\hat{\beta}} \left[1 + \frac{1}{k_D^2}(1 - \alpha_T k_P)\Delta\right]p. \quad (27)$$

To emphasize the main properties of the phenomena (strong coupling between temperature and concentration variations inside the boundary layers) the appropriate form of the operator on the left hand side is a “product” of two spatial second order operators,⁹ leading to (expression of the complex wavenumbers k_{hD} and k_{Dh} are given in Appendix A)

$$\left(1 + \frac{1}{k_{hD}^2}\Delta\right)\left(1 + \frac{1}{k_{Dh}^2}\Delta\right)\tau \approx \frac{\gamma-1}{\gamma\hat{\beta}} \left[1 + \frac{1}{k_D^2}(1 - \alpha_T k_P)\Delta\right]p. \quad (28)$$

Expressing the Laplacian Δ as the sum $\Delta = (\partial^2/\partial u^2) + \Delta_w$ and assuming that the movement tangent to the wall is almost the acoustic movement, which implies $\Delta_w \tau \approx [(\gamma-1)/\hat{\beta}\gamma]\Delta_w p$, this last equation takes the following form:

$$\left(1 + \frac{1}{k_{hD}^2}\frac{\partial^2}{\partial u^2}\right)\left(1 + \frac{1}{k_{Dh}^2}\frac{\partial^2}{\partial u^2}\right)\tau \approx \frac{\gamma-1}{\gamma\hat{\beta}} \left[1 + \frac{1}{k_D^2}\left(1 - \alpha_T k_P - \frac{\Delta_w}{k_h^2}\right)\Delta - \left(\frac{1}{k_{hD}^2} + \frac{1}{k_{Dh}^2}\right)\Delta_w\right]p;$$

and inasmuch as the terms of order greater than one of the characteristic lengths ℓ , namely, $\ell_{hD} = -jk_0/k_{hD}^2$ and $\ell_{Dh} = -jk_0/k_{Dh}^2$, can be neglected, this equation acquires the following form:

$$\left(1 + \frac{1}{k_{hD}^2}\frac{\partial^2}{\partial u^2}\right)\left(1 + \frac{1}{k_{Dh}^2}\frac{\partial^2}{\partial u^2}\right)\tau \approx \frac{\gamma-1}{\gamma\hat{\beta}}p_\tau, \quad (29a)$$

with $(\Delta$ and Δ_w being here identified to $-k_a^2)$

$$p_\tau(\vec{w}) \approx \left[1 + \frac{k_a^2}{k_D^2}x(1-x)\alpha_T\alpha + \frac{k_a^2}{k_h^2}\right]p. \quad (29b)$$

The form of this equation reflects the nature of the temperature variation field τ : it is a superposition of two diffusion

processes, labeled hD and Dh , where the concentration and thermal fields of the gas mixture are strongly intricate, pointing out the important coupling of the two phenomena. Note that the thicknesses of the boundary layers associated to these diffusion processes are $\delta_{hD} = \sqrt{2}/|k_{hD}|$ and $\delta_{Dh} = \sqrt{2}/|k_{Dh}|$.

It is noteworthy that the analogous procedure used below to express the equation which governs the behavior of the concentration variation b inside the boundary layers conveys similar interpretation.

The following relations can readily be deduced from Eqs. (A1) and (A2):

$$\frac{1}{k_{hD}^2} + \frac{1}{k_{Dh}^2} = \frac{1}{k_h^2} + \frac{1+\epsilon}{k_D^2}, \quad k_{hD}^2 k_{Dh}^2 = k_h^2 k_D^2. \quad (30)$$

Assuming that the term p_τ of Eq. (29a) is quasi-uniform in the u direction inside the boundary layers, the solution of this equation for the temperature variation τ , subjected to the boundary conditions (24) and (26b), is expressed as the sum of two functions as follows:

$$\begin{aligned} \tau(u, \vec{w}) &= \frac{\gamma-1}{\hat{\beta}\gamma} p_\tau(\vec{w}) [1 - A_\tau \varphi_{hD}(u) - (1 - A_\tau) \varphi_{Dh}(u)] \\ &= \frac{\gamma-1}{\hat{\beta}\gamma} p_\tau(\vec{w}) [1 - \psi_\tau(u)], \end{aligned} \quad (31)$$

the constant parameter A_τ being derived from the boundary condition (26b) and given in Appendix B for several geometries.

In this expression, the first term $[(\gamma-1)/\hat{\beta}\gamma]p_\tau(\vec{w})$ is the solution of the equation when the movement is forced by the pressure $p_\tau(\vec{w})$ and the second one involves the general solutions $\varphi_{hD}(u) = \phi_{hD}(u)/\phi_{hD}(s)$ and $\varphi_{Dh}(u) = \phi_{Dh}(u)/\phi_{Dh}(s)$ of the homogeneous diffusion equations previously identified.

Finally, just notice that in pure gases, according to the empirical models available for k_T ,^{2,12} ϵ vanishes as expected, and expression (31) should reduce to

$$\tau(u, \vec{w}) = \frac{\gamma-1}{\hat{\beta}\gamma} \left(1 + \frac{k_a^2}{k_h^2}\right) p(\vec{w}) [1 - \varphi_h(u)].$$

B. Expression of the concentration variation

The expression of the concentration variation b is derived from Eq. (20) making use of Eq. (31), leading to

$$\begin{aligned} b(u, \vec{w}) &= \frac{\gamma}{\rho_0 a_0^2} x(1-x)p \\ &\times \left[\frac{k_a^2}{k_D^2} \alpha(1 - \psi_\tau(u)) - \frac{1}{k_T} \psi_b(u) \right], \end{aligned} \quad (32)$$

with

$$\begin{aligned} \psi_b(u) &= A_\tau \frac{\ell_D - \ell_{Dh}}{\ell_D} \varphi_{hD}(u) \\ &+ (1 - A_\tau) \frac{\ell_D - \ell_{hD}}{\ell_D} \varphi_{Dh}(u). \end{aligned} \quad (33)$$

In pure gases, b vanishes as expected.

C. Solutions for the tangential particle velocity

The solution for \vec{v}_w of Eq. (18), in which the term $a_0\alpha_P\ell_v\vec{\nabla}_w b$ is neglected because it leads to a second order term of the characteristic lengths [the variations of the function $\psi_b(u)$ being small], is written as, accounting for the boundary condition (23b),

$$\vec{v}_w(u, \vec{w}) \approx -\frac{1}{j\omega\rho_0} (1 + jk_0\ell_v)\vec{\nabla}_w p(\vec{w}) \left[1 - \frac{\phi_v(u)}{\phi_v(s)} \right], \quad (34)$$

where $\psi_v = \phi_v(u)/\phi_v(s)$, the solution of the homogeneous equation (18), depends on the geometry considered.

This solution for the tangential velocity shows that the mutual diffusion in the gas mixture has no effect on the particle velocity at the order one of the characteristic lengths.

D. Relation between the particle velocity normal to the wall and the pressure variation

Making use of the solutions (31), (32), (34) for the temperature and concentration variation, and for the tangential particle velocity, in Eq. (19) leads straightforwardly to the final relation between the normal component v_u of the particle velocity \vec{v} and the pressure variation p , at the order one of the characteristic lengths,

$$j\omega\rho_0 \frac{\partial}{\partial u} v_u = (1 - \psi_v(u))\Delta_w p + k_0^2 [1 - jk_0\ell_{vhd} + \gamma \frac{\alpha_P}{\alpha_T} \psi_b(u) + (\gamma - 1)\psi_\tau(u) + jk_0(\ell_v\psi_v(u) + ((\gamma - 1)\ell_h + \ell_D)\gamma x(1 - x)\alpha^2)\psi_\tau(u)] p. \quad (35)$$

This relation can be simplified with respect to the geometry and the accuracy required (see examples below).

E. Particular cases

In this section, there are two major concerns regarding the effects of the phenomena which occur in the boundary layers presented above: the reflection on a quasi-plane rigid wall and the acoustic propagation in waveguides.

1. Reflection on a quasi-plane wall

The acoustic pressure p being quasi-uniform over the depth of the boundary layers, and the wall being characterized by its specific admittance y_s (which vanishes when the wall is perfectly rigid), the mean values of the terms of Eq. (35) over a distance from the wall lower than or equal to these depths can be expressed easily as given below (note that the same notation is used for the variable of integration and the upper limit of the integral):

$$\int_s^u \frac{\partial}{\partial u} v_u du = v_u(u) - v_u(s) = v_u(u) + \frac{y_s}{\rho_0 a_0} p, \quad (36a)$$

$$\int_s^u \Delta_w (1 - \psi_v) p du \approx -k_w^2 p \left(\delta - \int_s^u \psi_u du \right), \quad (36b)$$

$\delta = (u - s)$ being of the same order of magnitude as the boundary layer thicknesses, the variations of the functions ψ_X in the boundary layers being much higher than those of p (subscript X in the following stands for subscripts v , τ , b , hD , and Dh indifferently). Considering the reflection on a wall in a semi-infinite domain, functions $\phi_X(u)$ are

$$\phi_X(u) = e^{-jk_X u} \quad \text{and} \quad \phi_X(s) = e^{-jk_X s},$$

leading then to

$$\int_s^u \frac{\phi_X(u)}{\phi_X(s)} du \approx \frac{1 - j}{\sqrt{2}} \sqrt{\frac{\ell_X}{k_0}},$$

the exponential function $e^{-jk_X(u-s)}$ vanishing rapidly when $\delta \approx \delta_X$ (evanescent waves associated to diffusion processes from the wall).

Then, it turns out, by integrating Eq. (35) from s to u , and using expression (B2) of the coefficient A_τ , that

$$-\rho_0 a_0 \frac{v_u}{p} \approx \frac{1 + j}{\sqrt{2}} \sqrt{k_0} \times \left[\frac{k_w^2}{k_0^2} \sqrt{\ell'_v} + (\gamma - 1) \sqrt{\ell_h} \frac{\sqrt{\ell_{hD}} + \sqrt{\ell_{Dh}}}{\sqrt{\ell_h} + \sqrt{\ell_D}} \right] + y_s. \quad (37)$$

In pure gases, $\ell_{hD} = \ell_h$ and $\ell_{Dh} = \ell_D$, leading then to the well-known expression of the viscous and thermal effects on the reflection of an acoustic wave on a quasi-plane rigid wall.¹¹

2. Propagation in waveguides

Considering a waveguide with perfectly rigid walls, the mean value of Eq. (35) over the section S of the guide allows to derive a wave equation which governs the pressure variation of quasi-plane waves, the normal velocity v_u being removed because the mean value of the left hand side of Eq. (35) vanishes:

$$\langle \partial v_u / \partial u \rangle_S = 0. \quad (38)$$

Then, denoting K_X the mean value of the functions $\phi_X(u) = \varphi_X(u)/\varphi_X(s)$, viz.,

$$K_X = \langle \varphi_X(u) \rangle_S, \quad (39)$$

which are known functions involving Bessel or trigonometric functions depending on the shape of the section [Eqs. (B3), (B2)], the mean value of Eq. (35) leads readily to the following propagation equation:

$$(\Delta_w + k_w^2)p = 0, \quad (40)$$

where, neglecting terms which are of order greater than one and using expressions (B4) or (B6) of the coefficients A_τ , the wavenumber k_w takes the following form:

$$k_w^2 \approx k_0^2 \left[\frac{1 + (\gamma - 1)K_{hD}K_{Dh}F_{hD}}{1 - K_v} - jk_0\ell_{vhd} \right], \quad (41)$$

with

$$F_{hD} = \frac{\ell_h/\ell_{hD} - \ell_h/\ell_{Dh}}{K_{Dh}(1 - \ell_h/\ell_{Dh}) - K_{hD}(1 - \ell_h/\ell_{hD})}. \quad (42)$$

Inasmuch as the terms of order one of the characteristic lengths ℓ_v , ℓ_h , and ℓ_D are accounted for in the formalism, the bulk effects [Eq. (16b)] are included in the model, in the last term of the right hand side of Eq. (41) (this result is unusual when modeling the phenomena inside the boundary layers). Note that, in pure gases, $\ell_{hD} = \ell_h$ and $\ell_{Dh} = \ell_D$, leading then to the well-known expression of the viscous and thermal effects in an acoustic waveguide¹¹ [first term in the right hand side of Eq. (41)], depending on the shape of the guide section.

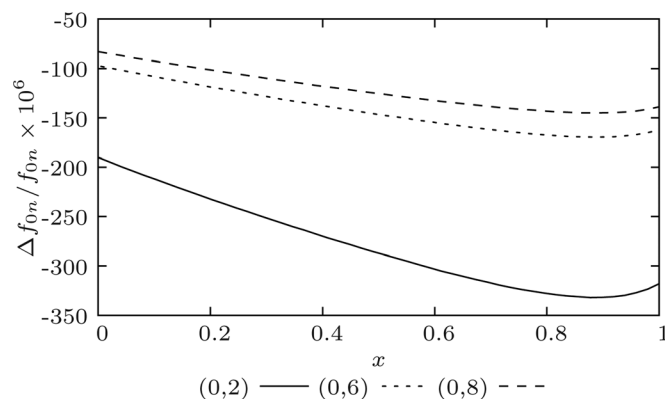
a. Higher frequency range ("large waveguide"). When the transverse dimensions of the waveguide are much greater than the boundary layers depths (higher frequency range), an estimate of the coefficients K_X at the lowest order of the characteristic lengths is usually suitable:

$$K_X \approx \frac{1-j}{\sqrt{2}} \sqrt{\frac{\ell_X A}{k_0 V}},$$

where A/V is the surface/volume ratio of a given length of the waveguide. Then, the wavenumber k_w at the lowest order of the characteristic lengths takes the following form:

$$k_w^2 \approx k_0^2 \left[1 + \frac{1-j}{\sqrt{2}} \sqrt{\frac{1}{k_0 V} A} \times \left(\sqrt{\ell'_v} + (\gamma - 1) \sqrt{\ell_h} \frac{\sqrt{\ell_{hD}} + \sqrt{\ell_{Dh}}}{\sqrt{\ell_h} + \sqrt{\ell_D}} \right) \right]. \quad (43)$$

b. Lower frequency range ("capillary duct"). When the transverse dimensions of the waveguide are much smaller than the boundary layers depths (lower frequency range), the asymptotic limit yields



$$K_X \approx 1 - \frac{1}{4/3 + 2(A/V)^2 j \ell_X / k_0},$$

$$\frac{1}{1 - K_X} \approx \frac{4}{3} + 2 \left(\frac{A}{V} \right)^2 \frac{j \ell_X}{k_0} \approx 2 \left(\frac{A}{V} \right)^2 \frac{j \ell_X}{k_0},$$

leading then to

$$k_w^2 \approx 2jk_0^2 \gamma \frac{\ell'_v}{k_0} \left(\frac{A}{V} \right)^2 \quad (44)$$

for the wavenumber k_w at the order one of the characteristic lengths. The propagation in capillary ducts should not then be modified by the mutual diffusion phenomena.

V. APPLICATIONS

Several applications, already considered in the literature and involving acoustic propagation in gas mixtures helium-argon and helium-xenon, are reconsidered below.

The thermophysical properties of pure gases are accurately determined from *ab initio* calculations of the atomic interactions potential or from dedicated equations of state.^{13–15} The virial coefficients and speed of sound of the gas mixtures considered are expressed as functions of the temperature and of the static pressure from extrapolation of previous experimental results or from empirical models.^{15,16} The transport properties of the mixtures are calculated making use of a mole fraction weighing kinetic method having a good agreement with experimental results.^{17,18}

Finally, the diffusion coefficient D , which depends on the gas composition and the temperature and inversely proportional to the static pressure, is determined by interpolating experimental results.^{2,19} The thermal diffusion ratio k_T , as suggested in recent works,^{9,12} is expressed by different functions of x for each gas mixture. For the gas mixtures considered (He-Ar and He-Xe), the component 1 is chosen to be helium. This choice of the lightest gas for the component 1 imposes the thermal diffusion ratio k_T to be negative.²

A. Resonance properties of gas-filled spherical cavity

The methods for speed of sound measurement based on the use of gas-filled cavities are well-known and have been

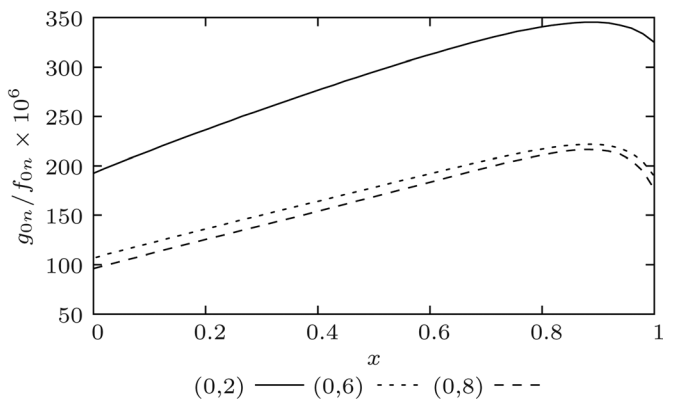


FIG. 1. Viscous thermal and diffusive effects on the resonance properties of radial modes of a spherical cavity filled with helium-argon mixture at 100 kPa and 273 K.

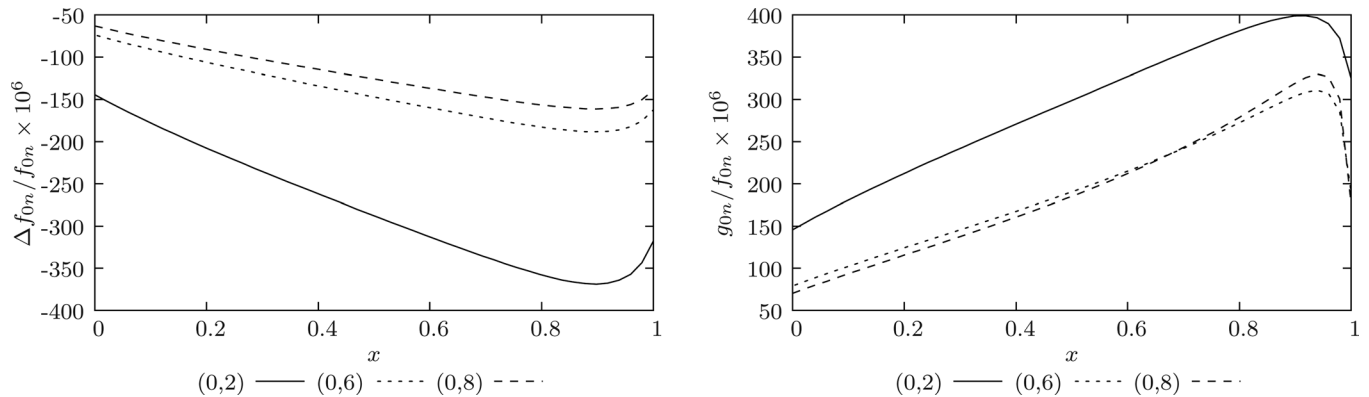


FIG. 2. Viscous thermal and diffusive effects on the resonance properties of radial modes of a spherical cavity filled with helium-xenon mixture at 100 kPa and 273 K.

constantly improved for decades. In particular, they are used for primary acoustic thermometry^{20,21} and for the accurate determination of the Boltzmann constant,^{22,23} where the speed of sound is derived from the accurate measurement of the acoustic resonance frequencies of a pure gas-filled spherical or quasi-spherical cavity.

Indeed, the thermophysical properties of the gas (speed of sound, virial coefficients, transport properties, diffusion coefficients, ...) determine the resonance frequencies and quality factors of the acoustic field within the cavity. Thus, the extremely high accuracy achievable on these experiments indicates that the method may be suitable for an accurate determination of these properties from acoustic measurements.

For mixtures, this acoustic approach looks promising for several metrological and fundamental purposes such as validation of kinetic theory based on gas models¹⁷ or a new approach to carrying out an alternative method for determining the collision diameters and intermolecular force-law from the diffusion coefficients,² or even for industrial applications like thermoacoustic engines.⁹

The acoustic pressure p in a spherical cavity satisfies the propagation equation (16a), in which the dissipative effects in the bulk are expressed in the wavenumber k_a [Eq. (16b)], associated to the following boundary condition (out of sound source):

$$\frac{\partial p}{\partial u} + jk_0 y_{vthd} p = 0, \quad (45a)$$

y_{vthd} being the specific admittance expressing the viscous, thermal and diffusive effects in the boundary layers [Eq. (37)] written as follows:

$$y_{vthd} = \frac{1+j}{\sqrt{2}} \sqrt{k_0(\gamma-1)} \sqrt{\ell_h} \frac{\sqrt{\ell_{hD}} + \sqrt{\ell_{Dh}}}{\sqrt{\ell_h} + \sqrt{\ell_D}}, \quad (45b)$$

considering here a radial resonant acoustic mode $(0, n)$ generated in the cavity. Indeed, in this case the term k_w^2/k_0^2 of Eq. (37), where k_w is the wavenumber normal to the wall, is zero.

The perturbing viscous, thermal and diffusive effects on the resonance properties of the mode (m, n) of a spherical cavity^{4,22-24} are given by

$$\frac{\Delta f_{0n} + jg_{0n}}{f_{0n}} = \frac{jy_s}{z_{0n}} + j \frac{k_0}{2} \ell_{vhd}, \quad (46)$$

z_{0n} being the eigenvalue, f_{0n} the resonance frequency without perturbing effects, Δf_{0n} the deviation from f_{0n} due to the viscous, thermal and diffusive effects, and g_{0n} the corresponding halfwidth of the radial mode $(0, n)$.

For a cavity of 80 mm radius filled with helium-argon and helium-xenon mixtures of helium molar concentration x

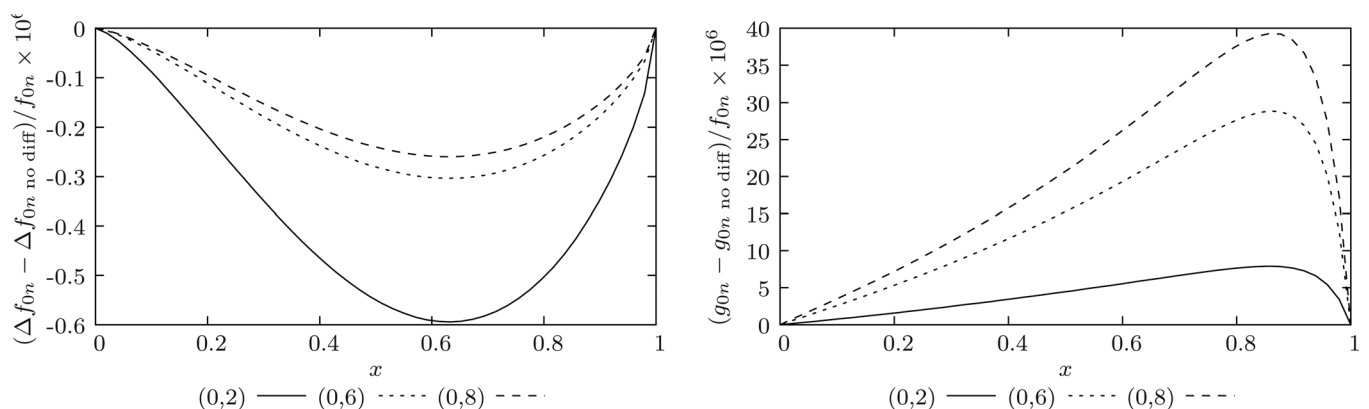


FIG. 3. Effect of the mutual diffusion on the resonance properties of radial modes of a spherical cavity filled with helium-argon mixture at 100 kPa and 273 K.

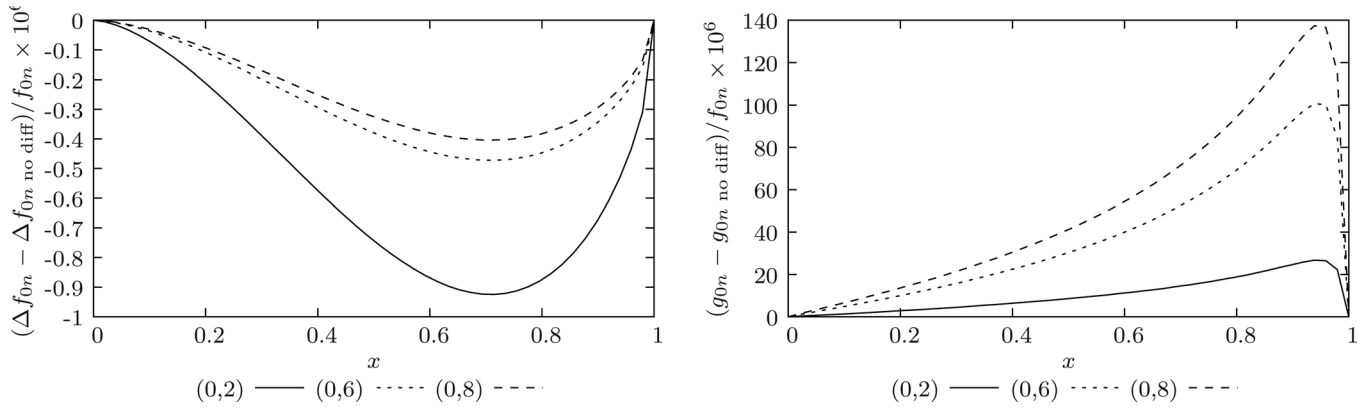


FIG. 4. Effect of the mutual diffusion on the resonance properties of radial modes of a spherical cavity filled with helium-xenon mixture at 100 kPa and 273 K.

varying from 0 to 1 for $P_0 = 100$ kPa and $T_0 = 273.16$ K, the effects ($\Delta f_{0n}/f_{0n}$ and g_{0n}/f_{0n}) on several radial modes, $m = 0$ and $n = 2, 6, 8$, are represented in Figs. 1 and 2, respectively. The isolated contribution of the mutual diffusion effects [$(\Delta f_{0n} - \Delta f_{0n, no diff})/f_{0n}$ and $(g_{0n} - g_{0n, no diff})/f_{0n}$] for both mixtures is showed on Figs. 3 and 4.

The mutual diffusion effects in the bulk being dissipative only, the frequency deviations Δf_{0n} on Figs. 3 and 4 are only due to the boundary layers effects, and, here the contributions to the halfwidth are essentially due to the effects in the bulk of the gas mixture.

Relative variations of few parts in 10^{-2} in several gas thermophysical properties (speed of sound, thermal conductivity, diffusion coefficients, virial coefficients, ...) lead to variations of few parts in 10^{-6} in the resonance properties of the cavity, which is small but significant regarding accurate gas metrology. Thus, acoustic resonators of optimized dimensions and shape, might be used to determine the thermal conductivity λ and the first acoustic virial coefficient of a gas mixture from radial modes resonance frequencies measurements, and the coefficient of mutual diffusion D from the halfwidths measurements.

B. Thermoacoustic engines: Dissipation in a duct

Helium-argon and helium-xenon gas mixtures are commonly used in thermoacoustic engines for their low Prandtl number that allows to increase the thermal effects without increasing the viscous effects in the stack. The effects of mutual diffusion inside thermoacoustic engines stacks have already been investigated⁹ to express the acoustic power dissipated in a duct filled with a gas mixture, in order to estimate if the mutual diffusion effects have a significant part in the total dissipation, and then if they could be experimentally observed in thermoacoustic engines.

The time-averaged acoustic power dissipated in a length dw of the duct is given by

$$\frac{\partial \mathcal{P}}{\partial w} = -\frac{1}{2} \text{SRe} \left[\langle \hat{v}_w \rangle \frac{\partial p}{\partial w} + \hat{p} \frac{\partial \langle v_w \rangle}{\partial w} \right], \quad (47)$$

leading to, making use of expression (34) of v_w ,

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial w} = & \frac{S\omega}{2} \left[\rho_0 |\langle v_w \rangle|^2 [k_0 \ell_v - \text{Im}(K_v)] \right. \\ & + \frac{|p|^2}{\rho_0 a_0^2} [k_0 ((\gamma - 1)\ell_h + x(1-x)\gamma\alpha^2 \ell_D) \\ & \left. - (\gamma - 1)\text{Im}(K_{hD} F_{hD}) \right], \quad (48) \end{aligned}$$

the factor F_{hD} being given by the expression (42).

At the lowest order, when the transverse dimensions of the duct are much larger than the boundary layers depth, the previous expression leads to the approximate form

$$\begin{aligned} \frac{\partial \mathcal{P}}{\partial w} = & \frac{\omega S A}{2} \frac{1}{V \sqrt{2k_0}} \left[\rho_0 |\langle v_w \rangle|^2 \sqrt{\ell'_v} + \frac{|p|^2}{\rho_0 a_0^2} \right. \\ & \left. \times (\gamma - 1) \sqrt{\ell_h} \frac{\sqrt{\ell_{hD}} + \sqrt{\ell_{Dh}}}{\sqrt{\ell_h} + \sqrt{\ell_D}} \right], \quad (49) \end{aligned}$$

which is consistent with previous results,⁹ when neglecting the bulk dissipative effects.

The ratio of the second term in the right hand side of Eq. (48), for helium-argon and helium-xenon mixture, and the same term when the mutual diffusion effects are neglected, viz., $A_{diff} = k_0 [(\gamma - 1)\ell_h + x(1-x)\gamma\alpha^2 \ell_D] - (\gamma - 1)\text{Im}(K_{hD} F_{hD}) / (\gamma - 1)[k_0 \ell_h - \text{Im}(K_h)]$, is displayed on Fig. 5 for atmospheric pressure and temperature.

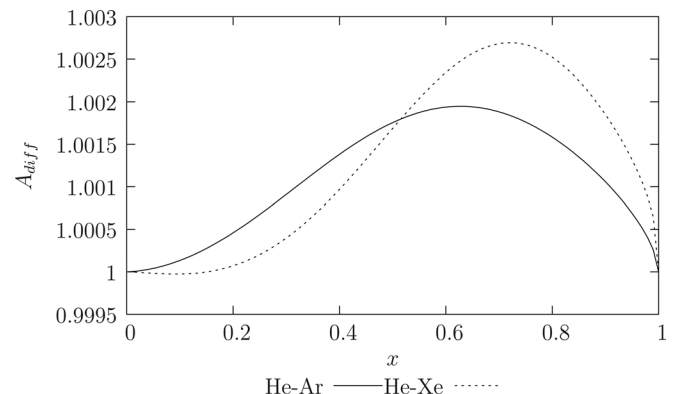


FIG. 5. Factor by which mutual diffusion multiplies the dissipation of acoustic power along a duct filled with helium-argon and helium-xenon gas mixtures.

These results are nearly the same as those showed by Swift,⁹ and confirm then that the mutual diffusion effects would hardly be experimentally observed in thermoacoustic engines. However, these dissipative effects in waveguides would be significant for other applications having higher precision requirements.

VI. CONCLUSION

The mutual diffusion phenomena have a small effect on the acoustic propagation that would be negligible in most practical cases, but significant in several others, in particular for the use of acoustics in several metrological applications such as the determination of gas mixtures thermophysical properties by acoustical methods.

Considering humid air as a binary gas mixture of air and water vapor, the modeling presented here would apply as long as this mixture is not water saturated. But when it is water saturated, condensation/evaporation occurs on the boundaries; the modeling could be extended with the appropriate boundary conditions.⁷

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APPENDIX A: EXPRESSION OF THE WAVENUMBERS k_{hD} AND k_{Dh}

The solutions for the wavenumbers k_{hD} and k_{Dh} are readily obtained from Eqs. (27), (28):

$$\frac{1}{k_{hD}^2} = -\frac{A + \sqrt{\Gamma}}{2jk_0} \quad \text{and} \quad \frac{1}{k_{Dh}^2} = -\frac{A - \sqrt{\Gamma}}{2jk_0},$$

where

$$A = \ell_h + \ell_D(1 + \epsilon),$$

$$\Gamma = (\ell_h - \ell_D)^2 + 2\epsilon\ell_D(\ell_h + \ell_D(1 + \epsilon/2)).$$

When dealing with a pure gas, the temperature/concentration coupling factor ϵ vanishes, leading to

$$\frac{1}{k_{hD}^2} \Big|_{\epsilon=0} = \frac{1}{2jk_0} [\ell_h + \ell_D + |\ell_h - \ell_D|],$$

$$\frac{1}{k_{Dh}^2} \Big|_{\epsilon=0} = \frac{1}{2jk_0} [\ell_h + \ell_D - |\ell_h - \ell_D|],$$

that is to say, $k_{hD} = k_h$ and $k_{Dh} = k_D$ if $\ell_h \geq \ell_D$ and $k_{hD} = k_D$ and $k_{Dh} = k_h$ if $\ell_h \leq \ell_D$.

The convention chosen for the expressions of k_{hD} and k_{Dh} leads to the following writing:

$$k_{hD}^2 = -\frac{jk_0}{\ell_{hD}}, \quad k_{Dh}^2 = -\frac{jk_0}{\ell_{Dh}}, \quad (\text{A1})$$

where

$$\ell_{hD} = \frac{1}{2} [\ell_h + \ell_D(1 + \epsilon) + \sqrt{\Gamma}],$$

$$\ell_{Dh} = \frac{1}{2} [\ell_h + \ell_D(1 + \epsilon) - \sqrt{\Gamma}]. \quad (\text{A2})$$

APPENDIX B: CALCULATION OF THE COEFFICIENTS A_τ

The coefficients A_τ are derived from the boundary condition (26b), making use of the expression (31) of τ .

1. Reflection on a quasi-plane wall in semi-infinite space

In this case, the expression of the functions $\phi_X(u)/\phi_X(s)$ is given in Sec. IVE 1

$$\frac{\phi_X(u)}{\phi_X(s)} = e^{-jk_X(u-s)},$$

and the first and third derivatives on the wall are $-jk_X$ and jk_X^3 , respectively. Making use of the following expressions:

$$\frac{\ell_{hD} + \ell_{Dh}}{\ell_h} = 1 + \frac{\ell_D}{\ell_h}(1 + \epsilon), \quad \ell_{hD}\ell_{Dh} = \ell_h\ell_D, \quad (\text{B1})$$

gives then for the coefficients A_τ :

$$A_\tau = \frac{\sqrt{\ell_D\ell_{Dh}} - \sqrt{\ell_h\ell_{hD}}}{(\sqrt{\ell_{Dh}} - \sqrt{\ell_{hD}})(\sqrt{\ell_h} + \sqrt{\ell_D})}. \quad (\text{B2})$$

As expected, in a pure gas ($\epsilon = 0$) A_τ is equal to 1 when $\ell_{hD} = \ell_h$, $\ell_{Dh} = \ell_D$, and equal to 0 when $\ell_{hD} = \ell_D$, $\ell_{Dh} = \ell_h$.

2. Cylindrical duct

In a cylindrical duct of radius R , the functions $\phi_X(u)/\phi_X(s)$ and coefficients K_X are given by the Bessel functions of the first kind

$$\frac{\phi_X(u)}{\phi_X(s)} = \frac{J_0(k_X r)}{J_0(k_X R)}, \quad K_X = \frac{2}{k_X R} \frac{J_1(k_X R)}{J_0(k_X R)}, \quad (\text{B3})$$

giving, respectively, for the first and third derivatives on the lateral wall of the duct, $k_X^2 K_X R/2$ and $k_X^4 K_X R/2$.

Making use of the relations (B1) then leads to

$$A_\tau = \frac{K_{Dh}(1 - \ell_h/\ell_{Dh})}{K_{Dh}(1 - \ell_h/\ell_{Dh}) - K_{hD}(1 - \ell_h/\ell_{hD})}. \quad (\text{B4})$$

As expected, in a pure gas ($\epsilon = 0$) A_τ is equal to 1 when $\ell_{hD} = \ell_h$, $\ell_{Dh} = \ell_D$, and equal to 0 when $\ell_{hD} = \ell_D$, $\ell_{Dh} = \ell_h$.

3. Rectangular slit

In a rectangular slit of depth d , the functions $\phi_X(u)/\phi_X(s)$ and coefficients K_X are

$$\psi_X(u) = \frac{\cos k_X(z - d/2)}{\cos k_X d/2}, \quad K_X = \frac{2}{k_X d} \tan k_X d/2, \quad (\text{B5})$$

giving, respectively, for the first and third derivatives on the lateral wall of the duct, $k_X^2 K_X d/2$ and $k_X^4 K_X d/2$, leading then to

$$A_\tau = \frac{K_{Dh}(1 - \ell_h/\ell_{Dh})}{K_{Dh}(1 - \ell_h/\ell_{Dh}) - K_{hD}(1 - \ell_h/\ell_{hD})}. \quad (\text{B6})$$

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