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Hybrid numerical and analytical solutions for acoustic boundary problems in thermo-viscous fluids

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Abstract

The present work aims at contributing to the investigation of methods to solve some classes of problems of acoustic propagation in thermo-viscous fluids, in unbounded or bounded media. The focus here is on thermal and vortical diffusion at the boundaries, which have to be considered for an accurate description of the acoustic field in small fluid-filled cavities and ducts. Existing boundary element or finite element acoustic software does not include these phenomena, as they are not compatible with the basic equations involved. A methodology is given to solve such problems when using this software, introducing a hybrid method which combines both numerical solutions and analytical solutions (for the fields inside the boundary layers). A detailed application is presented to validate the process using a boundary elements method.

1. Introduction

Acoustic propagation within a homogeneous thermo-viscous fluid-filled medium at rest, unbounded in all directions, involves reactive and absorbing processes which can be characterized, in the frequency domain, by a complex wavenumber, whose imaginary part is proportional to the shear and bulk viscosity coefficients and the heat conduction coefficient (it can also include dissipation processes due to molecular relaxation using the appropriate complex specific heat ratio [1]).

In a bounded domain (duct or cavity), the reactive and absorbing processes at rigid boundaries arise from interactions between the acoustic movements and both the entropic movement

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(diffusion of heat) and the vortical movement (diffusion of shear waves) which are created on the boundary walls, pumping energy from the acoustic wave. The entropic and vortical perturbations diffuse into the medium in direction normal to the boundary (which is a consequence of the uniform boundary conditions), and die out before reaching the opposite wall (provided that the local curvature and distance between the walls are large enough with respect to the boundary layers thicknesses). In these situations, the absorption of acoustic waves outside the boundary layers can frequently be characterized, in the frequency domain, by the imaginary part of a complex wavenumber which, in most cases, will be proportional to the square root of the shear viscosity coefficient as well as that of the heat conduction coefficient [1].

A somewhat different approach is needed in very small cavities and narrow ducts, where one or two of the dimensions are of similar magnitude to the boundary layers thicknesses. In these situations, which can be found in numerous acoustic devices (more particularly electroacoustic devices), the heat diffusion (entropic movement) and the shear wave diffusion (vortical movement) have amplitudes of the same order of magnitude as the acoustic wave itself (which acts on the wall as a source for the entropic and vortical movement). In these circumstances, the approach must involve a precise description of the particle movement inside the boundary layers. This particle movement can be expressed in terms of a superposition of three kinds of components: first, the original wave (which provides energy), second, thermal and viscous dissipation in the form of correction factors added to the acoustical wave function, and third, diffusion of heat and shear waves also described by means of additive functions [2]. Most of the works devoted to this class of problems take an approach of dividing the cavity into separable shapes, then providing for each an appropriate analytical solution of the basic linear differential equations which govern acoustic, entropic and vortical movements (integral equations not being available yet). Examples of such approaches are given in References [3,4] and in other papers referenced in their bibliographies.

On the basis presented above, in order to secure explicit solutions of problems which need to combine numerical (using boundary element (BEM) or finite element (FEM) commercial software) and analytical solutions (which is the aim of the present paper), three main classes of problems will be considered, which depend on the nature of the domain considered: (i) unbounded domains; (ii) domains of arbitrary shape where the distance between the walls is large compared to the boundary layers thicknesses and (iii) domains where one or two dimensions are of the same order of magnitude as the boundary layers thicknesses. The hybrid method and its numerical implementation presented here is applicable to both unbounded or bounded domains, the only difference being the mesh definition when using BEM (Section 3.1). The application to a large waveguide illustrates an example of acoustic problems of the second class, but the method and the numerical procedure is exactly the same when applied to exterior problems. The third class requires adapted models; the case of capillary slots and tubes is presented in Section 2.2.

The motivation to proceed with the derivation of relevant methods to solve these kinds of problems arises from the growing demand for numerical methods for calculating acoustic fields in a wide range of devices and for new applications. More particularly, there is a strong motivation for introducing into the numerical modelling both the absorption and the reaction resulting from viscous and thermal effects, especially within the boundary layers. When taking into account the reactive and absorbing phenomena, and in the case where the geometry of the bounded domains is not separable, an integral description of the basic equations (including Stokes–Navier equation and conduction of heat equation) would be desirable, especially when existing software can be adapted.

Over the last 10 years, theoretical activities on the subject provide a relevant global formulation which would be able to satisfy the requirements mentioned above for describing the acoustic fields in thermo-viscous fluids. In publications which relate to this subject one finds: (i) the pioneered works of Dokumaci which lead to both a modelling of viscous boundary layers at the boundaries using a boundary elements method [5] and a modelling of heat conduction effects [6]; (ii) the works of Cummings [3] and Beltman *et al.* [4] which uses numerical methods, based on the set of differential equations dedicated to this kind of problem, to model thin layers of viscous fluid trapped between parallel walls and (iii) that of Karra *et al.* which establishes a tridimensional integral formulation for a non-viscous but heat conducting fluid [7,8].

The aim of the present work is to describe a procedure for solving boundary acoustic problems, making use of existing boundary or finite element acoustic software which does not include the viscous and thermal phenomena both in the bulk of the fluid and inside the boundary layers. The domain considered may be either open or closed and may have, or not, dimensions of the order of magnitude of the boundary layers thicknesses.

The basis for this formulation is a hybrid method that combines numerical and analytical solutions. It provides both a global numerical solution and also a local (inside the boundary layers) high-resolution analytical solution which are combined formally by matching, at the boundary layers interfaces with the medium, to construct an accurate combined solution. For this, the length scale for the global problem must be larger than the scale for the local problem in the boundary layers.

The local region associated with the boundary layers is treated separately as a high-resolution local problem to capture the small scale, rapid spatial variations. This scale separation allows the high-resolution analytical problem to be solved with simplified geometry. Outside the boundary layers region both solutions, the large-scale numerical solution and the high-resolution analytical solution, must achieve a satisfactory mathematical overlap so that the transition zone between the local and global solutions is accurate.

The method is applied in Section 3 to calculate the acoustic pressure field in a thermo-viscous fluid-filled large tube, using a classical BEM code (used in industry) and compare with the corresponding analytical solution, other interesting applications being beyond local available computing capacity.

2. Acoustic fields in thermo-viscous fluid-filled bounded media

2.1. Acoustic, entropic and vortical movements

2.1.1. Basic formulation

The variables describing the dynamical and thermodynamical state of the fluid are the pressure variation p, the particle velocity $\mathbf{v}$, the density variation ρ', and the temperature variation τ. The parameters which specify the properties and the nature of the fluid at rest are (i) the ambient values of the same quantities (respectively P_0, $\mathbf{V}_0 = \mathbf{0}$, ρ_0 and T_0) and (ii) the coefficient of shear viscosity μ, the bulk viscosity coefficient η, the coefficient of thermal conductivity λ, the heat coefficients at constant pressure and constant volume per unit of mass C_P and C_V, respectively, the specific heat ratio $\gamma = C_P/C_V$, the increase in pressure per unit increase in temperature at constant density $\hat{\beta} = P_0(\partial P/\partial V)_\rho$, and the adiabatic speed of sound c_0. The characteristic lengths are defined as $\ell_v = (\frac{4}{3}\mu + \eta)/(\rho_0 c_0)$, $\ell'_v = \mu/(\rho_0 c_0)$ and $\ell_h = \lambda/(\rho_0 C_p c_0)$.

The particle velocity $\mathbf{v}$ is written as the sum of the laminar acoustic velocity $\mathbf{v}_a$, which includes (or not) both viscous and thermal effects in the bulk of the domain, the laminar thermal velocity $\mathbf{v}_h$ and the vortical velocity $\mathbf{v}_v$. The thermal and vortical velocities are negligible in comparison with the laminar acoustic velocity in the bulk of the whole domain considered, except inside the boundary layers near the walls.

On these walls, assumed here to be perfectly rigid, the particle velocity and the temperature variation (sum of the acoustic τ_a and the entropic τ_h temperature variations) vanish, namely $\mathbf{v}_a + \mathbf{v}_h + \mathbf{v}_v = \mathbf{0}$ and $\tau_a + \tau_h = 0$, and moreover, because the thermal and vortical velocities created on the boundaries by the acoustic perturbation are directly expressed in terms of diffusion processes along the inward normal to the wall, these velocity components $\mathbf{v}_h$ and $\mathbf{v}_v$ die out over a very short distance from the wall (i.e., the boundary layer thicknesses denoted, respectively, δ_h and δ_v). Hence, assuming that these thicknesses are much lower than the dimensions of the domain considered (unbounded or bounded problems of the first and second classes mentioned in the introduction), the harmonic acoustic pressure field p_{a0} outside the boundary layers, of time dependence $e^{i\omega t}$ (angular frequency ω), is solution of the set of equations

$$(\Delta + k_a^2)p_{a0} = 0 \quad \text{in the domain considered,} \tag{1a}$$

$$\partial_u p_{a0} = 0 \quad \text{on the boundaries.} \tag{1b}$$

The dissipative effect in the bulk of the domain is included in the well-known approximate (discarding the second order term ℓ_{vh}^2) complex wavenumber $k_a = k_0(1 - (i/2)k_0\ell_{vh})$ (Eq. (1a)), where $\ell_{vh} = \ell_v + (\gamma - 1)\ell_h$ and $k_0 = \omega/c_0$.

The vortical and entropic movement being negligible in the bulk of the domain, the temperature variations and the acoustic velocity are given, respectively, by

$$\tau_{a0} = [(\gamma - 1)/(\gamma\hat{\beta})]p_{a0}, \quad \text{(adiabatic relationship),} \tag{2}$$

and

$$\mathbf{v}_{a0} = [i/(\rho_0\omega)]\boldsymbol{\nabla} p_{a0}, \quad \text{(Euler equation).} \tag{3}$$

The acoustic pressure p_{a0}, the temperature variation τ_{a0} and the particle velocity $\mathbf{v}_{a0}$ are considered below as the given external boundary expressions (outside the boundary layers) for the calculation of the velocity $\mathbf{v}$ and of the temperature variations τ near a wall (inside the boundary layers).

2.1.2. Equations of motion near the boundaries

The linear equations which give an accurate description of the small amplitude disturbances inside the viscous and thermal boundary layers must satisfy several assumptions in order to avoid overly intricate formulations, namely: (i) as the pressure variation can be assumed constant over the thermal and viscous boundary layer thicknesses δ_h and δ_v, respectively (because the wavelength is much greater than these thicknesses), the acoustic pressure p is approximated by its zero order expansion p_{a0} (with respect to the very small admittance β equivalent to the boundary layers; Eq. (14b)) ($p \approx p_{a0}$) and the u-component (**u** inwardly directed) v_u of the particle velocity **v** is much lower than its component $\mathbf{v_w}$ parallel to the wall, that is the flow is assumed to be essentially tangential to the wall and then, in a first approximation, the Navier–Stokes equation is substituted by the Hagen–Poiseuille equation for the $\mathbf{v_w}$-components tangents to the wall, the normal component of the velocity v_u being, nevertheless, taken into account in the conservation of mass equation to assume the volume flow conservation; (ii) spatial variations in normal direction **u** of both the velocity **v** and the temperature variation τ are much greater than spatial variations in the tangential directions and hence the spatial variations of these quantities in the tangential directions can be neglected in the Navier–Stokes equation and in the Fourier heat conduction equation. Therefore, the complete set of equations and boundary conditions governing the fluid motion inside the boundary layers, involving these approximations, is straightforwardly obtained, leading to, for a harmonic perturbation [1,9]:

for the **w**-*component of the particle velocity* **v**

$$(1 + (1/k_v^2)\partial^2_{uu})\mathbf{v_w}(u, \mathbf{w}) = (\mathrm{i}/\rho_0\omega)\boldsymbol{\nabla}_\mathbf{w} p_{a0}(\mathbf{w}) \tag{4a}$$

with

$$\mathbf{v_w}(u = 0) = 0 \quad \text{and} \quad \mathbf{v_w}(u > \delta_v) = (\mathrm{i}/\rho_0\omega)\boldsymbol{\nabla}_\mathbf{w}\, p_{a0} \tag{4b}$$

for the temperature variation τ

$$(1 + (1/k_h^2)\partial^2_{uu})\tau = ((\gamma - 1)/\gamma\hat{\beta})p_{a0} \tag{5a}$$

with

$$\tau(u = 0) = 0 \quad \text{and} \quad \tau(u > \delta_h) = ((\gamma - 1)/\gamma\hat{\beta})p_{a0} \tag{5b}$$

for the u-component of the particle velocity (**u** *inwardly directed*)

$$\mathrm{i}\omega\rho' + \rho_0\boldsymbol{\nabla}\mathbf{v} = 0 \tag{6a}$$

with

$$\rho' = (\gamma/c_0^2)(p_{a0} - \hat{\beta}\tau) \quad \text{and} \quad v_u(u = 0) = 0. \tag{6b}$$

In these equations, $k_v = (1 - \mathrm{i})\sqrt{k_0/(2\ell'_v)}$, $k_h = (1 - \mathrm{i})\sqrt{k_0/(2\ell_h)}$, and $u = 0$ means "on the wall considered" (**u** being normal to the wall, inwardly directed), and the time dependence $\mathrm{e}^{\mathrm{i}\omega t}$ is implicitly included in the factor p_a.

2.1.3. Solutions for the acoustic field

The regular solutions of this set of Eqs. (4)–(6) are the appropriate results that are needed to solve boundary problems, making use of post-processing methods to provide these results inside the boundary layers beyond the results for the acoustic pressure p_{a0} obtained from a numerical method. These solutions are straightforwardly obtained from this set of equations, step by step, leading to simple expressions for $\mathbf{v_w}$, τ and $\mathbf{v}_u$, respectively.

The **w**-component $\mathbf{v_w}$ of the particle velocity **v**, solution of Eq. (4a) subject to the boundary conditions (4b), is given by

$$\mathbf{v_w} = (\mathrm{i}/\rho_0\omega)\boldsymbol{\nabla}_\mathbf{w}\, p_{a0}(1 - \mathrm{e}^{-\mathrm{i}k_v u}). \tag{7}$$

In the same manner, the temperature variation τ is solution of Eqs. (5), leading to

$$\tau = ((\gamma - 1)/\gamma\hat{\beta})p_{a0}(1 - e^{-ik_h u}), \tag{8}$$

where the factor $(\gamma - 1)/(\gamma\hat{\beta})p_{a0}$ is the adiabatic temperature variation τ_{a0}.

Finally, the solution for the normal component v_u of the particle velocity can be obtained from the conservation of mass equation (6a) in the following manner. First, invoking Eq. (6b) and solution (8) for τ yields

$$\rho' = (1/c_0^2)p_{a0}[1 + (\gamma - 1)e^{-ik_h u}] \tag{9}$$

Second, the divergence operator is split into two parts $\nabla\mathbf{v} = \partial_u v_u + \nabla_\mathbf{w}\mathbf{v}_\mathbf{w}$, where $\nabla_\mathbf{w}\mathbf{v}_\mathbf{w}$ is expressed as, using solution (7) for $\mathbf{v}_\mathbf{w}$,

$$\nabla_\mathbf{w}\mathbf{v}_\mathbf{w} = (i/\rho_0\omega)\nabla_\mathbf{w}\nabla_\mathbf{w}p_{a0}(1 - e^{-ik_v u}) \tag{10}$$

and then invoking Eq. (1a), the operator $\nabla_\mathbf{w}\nabla_\mathbf{w}$, acting on p_{a0}, is expressed either as $(-k_w^2)$ where k_w is the tangent wavenumber associated to the acoustic field p_{a0} or as $[-(\partial^2_{uu} + k_a^2)p_{a0}]$ which is the propagation equation (1a), depending, respectively, on it is associated or not to the factor $e^{-ik_v u}$ which vanishes for $u > \delta_v = \sqrt{2}/|k_v|$ (such choice permit below to obtain an expression for v_u which is solution both inside and outside the boundary layers).

Then using Eqs. (9), (10) and these last expressions for $\nabla_\mathbf{w}\nabla_\mathbf{w}$, and assuming that $k_a \approx k_0$, the conservation of mass equation (6a) leads to the following differential equation for the last unknown v_u:

$$\partial_u v_u = \frac{i}{\rho_0\omega}\partial^2_{uu}p_{a0} - \left[\frac{k_w^2}{k_0^2}e^{-ik_v u} + (\gamma - 1)e^{-ik_h u}\right]\frac{i\omega}{\rho_0 c_0^2}p_{a0}. \tag{11}$$

Taking into account that $(\partial_u p_{a0}) \approx 0$ for $u < \delta_v$ and $v_u(u = 0) = 0$ (6b), the integration of this last equation from $u = 0$ to every value of u greater or lower than the viscous and thermal boundary layers thicknesses (δ_h and δ_v) yields [2]

$$v_u = \frac{i}{\rho_0\omega}\partial_u p_{a0} - \left[\frac{k_w^2}{k_0^2}\frac{1}{k_v}(1 - e^{-ik_v u}) + \frac{\gamma - 1}{k_h}(1 - e^{-ik_h u})\right]\frac{k_0}{\rho_0 c_0}p_{a0}. \tag{12}$$

The u-component (12) and $\mathbf{w}$-component (7) of the vortical ($\mathbf{v}_v$) and thermal ($\mathbf{v}_h$) particle velocities are given, respectively, by the factors involving $e^{-ik_v u}$ and $e^{-ik_h u}$, respectively. The sum of the other factors represent the acoustic velocity in the thermo-viscous fluid. Especially, the u-component of the acoustic velocity is given by

$$v_{au} = (i/\rho_0\omega)\partial_u p_{a0} - ((k_w^2/k_0^2)(1/k_v) + (\gamma - 1)/k_h)(k_0/\rho_0 c_0)p_{a0}. \tag{13}$$

In the right side of this relationship, the first term is negligible inside the boundary layers and the second one is negligible outside the boundary layers.

2.1.4. Hybrid method to solve the problem

The definition of the pressure variation p_{a0} (1a) implies that its normal derivative vanishes on the boundary ($\partial_u p_{a0} = 0$ for $u = 0$). It follows that Eq. (13) is an admittance-like boundary condition, namely

$$-\rho_0 c_0 v_{au} = \beta p_{a0}, \tag{14a}$$

where

$$\beta = k_0(k_w^2/k_0^2 k_v + (\gamma - 1)/k_h) \tag{14b}$$

is a small admittance which is appropriate to express adequately the viscous and thermal dissipative and reactive processes near the rigid walls (the dissipative process in the bulk of the domain being accounted for in the complex wavenumber k_a). Therefore relevant fundamental equations and solutions which govern the acoustic field (denoted p_a) inside the bulk of the domain considered can be derived from this last result by assuming that the difference $(p_a - p_{a0})$ is one order of magnitude lower than p_a (the function p_{a0} can be considered as a zero order expansion of the function p_a with respect to the small quantity β) as follows:

$$(\Delta + k_a^2)p_a = 0 \quad \text{in the bulk of the domain,} \tag{15a}$$

$$(\partial_u - \mathrm{i}k_0\beta)p_a = 0 \quad \text{on the boundaries,} \tag{15b}$$

where

$$k_a \approx k_0(1 - (\mathrm{i}/2)k_0\ell_{vh}), \tag{15c}$$

$$\beta = [-\rho_0 c_0 v_{au}/p_a]_{u=0} = k_0(k_w^2/k_0^2 k_v + (\gamma - 1)/k_h). \tag{15d}$$

Then, making use of this acoustic pressure field p_a instead of the Neumann acoustic field p_{a0}, the acoustic field inside the boundary layers is always given by the set of Eqs. (7)–(9), where p_{a0} can be replaced by p_a, for the variables $\mathbf{v_w}$, τ and ρ', respectively, the solution for the normal component of the particle velocity $\mathbf{v}_u$ being given now by

$$v_u(u < \delta_v, \delta_h) = v_{au} + \left[\frac{k_w^2}{k_0^2}\frac{1}{k_v}\mathrm{e}^{-\mathrm{i}k_v u} + \frac{\gamma - 1}{k_h}\mathrm{e}^{-\mathrm{i}k_h u}\right]\frac{k_0}{\rho_0 c_0}p_a \tag{16a}$$

with

$$v_{au} = -(\beta/\rho_0 c_0)p_a, \quad \text{(inside the boundary layers).} \tag{16b}$$

The set of equations and solutions (15), (7)–(9) and (16) is the fundamental problem to solve, using numerical method to calculate p_a and a post-processing process to calculate in detail the acoustic field inside the boundary layers, when the dimensions of the domain are everywhere greater than the boundary layer thicknesses.

Nevertheless, it is noteworthy that the small admittance β depends on the factor (k_w^2/k_0^2) (which characterizes the direction of the acoustic velocities on the wall for the non-evanescent modes), so the admittance β is unknown. The best way to obtain approximate values for this factor at each element of the discretized boundary is to solve numerically the problem given by Eqs. (15), substituting to the factor (k_w^2/k_0^2) its mean value (1/2), and then to calculate its real approximate value at each element of the boundary using the numerical solution obtained previously (denoted p_m), that is,

$$(k_w^2/k_0^2) = -(1/k_0^2 p_m)\partial^2_{\mathbf{ww}} p_m. \tag{17}$$

Therefore, the problem considered can be solved by the procedure mentioned above, first using numerical method to calculate p_a and second using a post-processing process to calculate the acoustic field inside the boundary layers.

This method is adapted to express the reactive and absorbing processes, namely the thermal and viscous effects, in the boundary layers near the rigid walls as well as in the bulk of the domain. Especially, when the domain is a rigid walled cavity, the quality factor for the acoustic pressure p_a is realistic, which is important when the frequency is tuned to make the field resonant.

2.2. Capillary domains

When the whole domain or part of the domain has one dimension of the same order of magnitude as the boundary layer thicknesses and much greater than the other dimensions (very thin slots or capillary tubes, third class of problems), the model used to calculate the behavior of the acoustic field inside the boundary layers must be slightly revised as indicated in this subsection, the thicknesses involved being assumed to remain greater than the mean free path (which in air at atmospheric pressure is about 0.1 μm), to ensure that the continuum hypothesis remains valid.

In first approximation, these capillary domains can be modelled either as a very flat rectangular slot or very thin cylindrical tube in every real situation, because the pressure variation can be assumed uniform through the thickness of the fluid film or the fluid column.

Under these circumstances, the theory developed above in Section 2.1.2 remains valid to describe the oscillating flow produced as the fluid is pumped back and forth, either by the forced vibrations of the wall or the forced acoustic field at one or both ends of the capillary slot or tube. Nevertheless, in Eqs. (4)–(6), the normal co-ordinate u must be replaced by the radial co-ordinate r (capillary tube) or the normal co-ordinate z (capillary slot), and the boundary conditions have to be adapted to these co-ordinates; moreover, as the wall is eventually assumed to be subject to external (driving) forced vibrations, the associated volume velocity must be taken into account in the right side of the conservation of mass equation. Therefore, the equations of motion of the fluid in the frequency domain, take respectively the forms, the acoustic pressure field being denoted p,

for the capillary slot, with a thickness ε

$$(1 + (1/k_v^2)\partial_{zz}^2)\mathbf{v}_\mathbf{w}(z, \mathbf{w}) = (\mathrm{i}/\rho_0\omega)\boldsymbol{\nabla}_\mathbf{w} p \tag{18a}$$

with

$$\mathbf{v}_\mathbf{w}(z = 0) = \mathbf{v}_\mathbf{w}(z = \varepsilon) = 0, \tag{18b}$$

$$(1 + (1/k_h^2)\partial_{zz}^2)\tau = ((\gamma - 1)/\gamma\hat{\beta})p \tag{18c}$$

with

$$\tau(z = 0) = \tau(z = \varepsilon) = 0, \tag{18d}$$

$$\mathrm{i}\omega\rho' + \rho_0\boldsymbol{\nabla}_\mathbf{w}\mathbf{v}_\mathbf{w} = -\mathrm{i}\omega\rho_0\xi/\varepsilon, \tag{18e}$$

(ξ being the displacement, outwardly directed, of the wall $z = \varepsilon$) with

$$\rho' = (\gamma/c_0^2)(p - \hat{\beta}\tau) \quad \text{and} \quad v_z(z = 0) = 0; \tag{18f}$$

for the capillary tube, with radius R and axis x

$$(1 + (1/k_v^2)(1/r)\partial_r r \partial_r)v_x = (\mathrm{i}/\rho_0\omega)\partial_x p \tag{19a}$$

with

$$v_x(r = R) = 0, \tag{19b}$$

$$(1 + (1/k_h^2)(1/r)\partial_r r \partial_r)\tau = ((\gamma - 1)/\gamma\hat{\beta})p \tag{19c}$$

with

$$\tau(r = R) = 0, \tag{19d}$$

$$\mathrm{i}\omega\rho' + \rho_0\partial_x v_x = -\mathrm{i}\omega\rho_0\xi/(R/2), \tag{19e}$$

(ξ being the displacement, outwardly directed, of the wall $r = R$) with

$$\rho' = (\gamma/c_0^2)(p - \hat{\beta}\tau). \tag{19f}$$

The solutions of these sets of equations, which assume v_z and v_r to be negligible, are given by

$$\mathbf{v_w} = (\mathrm{i}/\rho_0\omega)\,\boldsymbol{\nabla}_\mathbf{w}\,p[1 - f_v], \tag{20a}$$

$$\tau = ((\gamma - 1)/\gamma\hat{\beta})\,p[1 - f_h] \tag{20b}$$

with

$$f_{v,h} = \cos[k_{v,h}(z - \varepsilon/2)]/\cos k_{v,h}\varepsilon/2, \quad \text{for the capillary slot} \tag{20c}$$

$$f_{v,h} = \mathrm{J}_0(k_{v,h}r)/\mathrm{J}_0(k_{v,h}R), \quad \text{for the capillary tube,} \tag{20d}$$

where J_0 is the zero order cylindrical Bessel function of the first kind, and **w** standing for the transverse co-ordinate(s).

Integrating the solutions across the fluid layer (between the walls $z = 0$ and $z = e$, or $r = 0$ and $r = R$) leads to mean values of $\mathbf{v_w}$ and τ which convey

$$\langle \mathbf{v_w} \rangle = (\mathrm{i}/\rho_0\omega)\boldsymbol{\nabla}_\mathbf{w}\,p[1 - F_v], \tag{21a}$$

$$\langle \tau \rangle = ((\gamma - 1)/\gamma\hat{\beta})\,p[1 - F_h] \tag{21b}$$

with

$$F_{v,h} = \tan[k_{v,h}\varepsilon/2]/k_{v,h}\varepsilon/2, \quad \text{for the capillary slot,} \tag{21c}$$

$$F_{v,h} = (2/k_{v,h}R)\mathrm{J}_1(k_{v,h}R)/\mathrm{J}_0(k_{v,h}R), \quad \text{for the capillary tube,} \tag{21d}$$

where J_1 is the first order cylindrical Bessel function of the first kind.

Then combining Eqs. (21b) and (18f) or Eq. (19f) yields

$$\langle \rho' \rangle = p/\boldsymbol{C}^2 \tag{22a}$$

with

$$(1/\boldsymbol{C}^2) = (\rho_0\chi_T/\gamma)[1 + (\gamma - 1)F_h], \tag{22b}$$

and combining Eqs. (22a) and (18e) or Eq. (19e), along with relation (21a) leads to

$$[\Delta_{\mathbf{w}} + \chi^2]p(\mathbf{w}) = \zeta\xi(\mathbf{w}) \tag{23a}$$

with

$$\chi^2 = (\omega^2/c_0^2)[1 + (\gamma - 1)F_h]/[1 - F_v], \tag{23b}$$

$$\zeta = -(\rho_0\omega^2/\varepsilon)(1/(1 - F_v)), \quad \text{for the capillary slot,} \tag{23c}$$

$$\zeta = -(\rho_0\omega^2/R/2)(1/(1 - F_v)), \quad \text{for the capillary tube.} \tag{23d}$$

When the displacement of the wall is not null, acoustic propagation (Eq. (23a)) is coupled to the propagation equation of the wall, that is

$$(O_{\mathbf{w}} + K^2)\xi(\mathbf{w}) = p_{\text{ext}}(\mathbf{w}) - p(\mathbf{w}), \tag{24}$$

where $p_{\text{ext}}(\mathbf{w})$ represents an external pressure source applied to one of the boundaries of the domain. With a microphone, for instance, p_{ext} represents the (external) acoustic pressure to be measured, and applied to the membrane: the operator $O_{\mathbf{w}}$ meaning then the membrane's mechanical operator $\Delta_{\mathbf{w}}$.

These last results emphasize that the whole acoustic problem for capillary slots or tubes reduces to a complex Helmholtz equation, homogeneous ($\xi = 0$) or not, depending only on the one (tube) or two (slot) tangential co-ordinates $\mathbf{w}$. For a fluid layer, very thin compared to the boundary layer thicknesses δ_v and δ_h, Eq. (21b) does not represent an adiabatic compressibility law, but an isothermal one, and for a layer of fluid whose thickness has the same order of magnitude than the boundary layer thicknesses, $\boldsymbol{C}^2$ is complex, having an imaginary part which contributes to the dissipative process (polytropic law). Anyway, the solutions are given by solving Eq. (23a), assuming usual conditions at the ends of the capillary (which involve acoustic pressure and particle velocity calculated numerically).

3. Numerical implementation of the method in rigid-walled domains

3.1. Introduction

The present analysis allows the possibility of solving numerically several classes of problems of acoustic propagation, including those where (Fig. 1) (i) the domain considered is bounded or not,

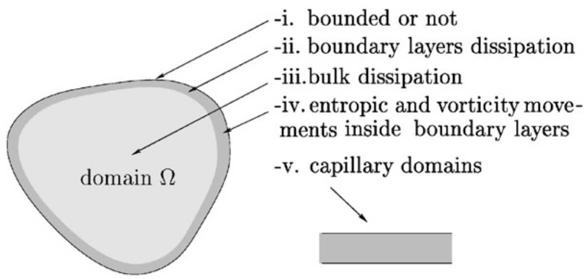

Fig. 1. Phenomena involved in acoustics problems and the sub-domains where they occur.

very small (cavities) or not, very narrow (ducts) or not; (ii) the bounded domain is rigid-walled, the reactive and absorbing processes at rigid boundaries arising from vortical and entropic processes; (iii) the reactive and absorbing processes in the bulk of the domain, characterized, in the frequency domain, by a complex wavenumber involving the shear and bulk viscosity coefficients, the heat conduction coefficient and also the molecular relaxation time if necessary [1], can be included in the calculation of the acoustic field; (iv) the acoustic, the vortical and the entropic movements inside the boundary layer are calculated in detail inside the boundary layers and, when the dimensions of the domain are greater than the boundary layer thicknesses, the fields satisfy continuity conditions with the acoustic field outside the boundary layers; (v) the capillary domains are modelled from an adapted Helmholtz-type equation.

Currently, available boundary elements method packages and finite elements method packages dedicated to acoustics rely on a simple Helmholtz equation for pressure. Therefore, they are not compatible with the basics equations involved and described in Section 1: problems have to be solved according to the procedure described previously, i.e. using those software packages AND dedicated post-processings, making use of each one in turn.

Several aspects that have to be considered, whilst implementing the procedure given here when using boundary element packaging [10] are mentioned below. In fact, most of them are not discussed here either because they imply minor modifications of the software or because they can be basically introduced through post-processing processes which do not require fundamental investigations.

Nevertheless, it is noteworthy that when considering problems involving domains Ω with open boundary or with closed boundary, or involving limited domains whose one (or two) dimensions have the same order of magnitude as the thicknesses of the thermal and viscous boundary layers, most attention has to be paid, respectively, to the shape of the boundaries which must be adequately extended (that is artificially closed to create a small complementary domain (Ω^-, Fig. 2) the solution for the pressure being given finally by the pressure jump), to the irregular frequencies (which leads to solve the problem from the calculation of the pressure jump), and to the rule of meshing [10–12].

Besides, it still remains to present (next Section 3.2) the technique used both to secure an approximate numerical calculation providing, on the boundary layers, the values of the thermo-viscous impedance-like function β which depends on the properties of the unknown acoustic field

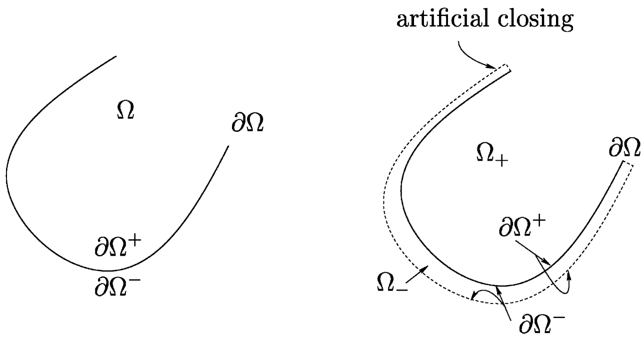

Fig. 2. Idealization (theoretical) of a boundary $\partial\Omega$ and the associated boundaries $\partial\Omega^+$ and $\partial\Omega^-$ for numerical simulation.

on the boundaries (see Section 2.1.4, Eq. (17)) and then to carry out the calculation of the acoustic field in the bulk of the domain which conveys significant improvement because it takes into account the reactive and dissipative effects of the boundary layers.

Even though the final goal of the method is to solve specific engineering complex problems (as mentioned in Section 1), it is not the scope of the present paper to calculate acoustic fields in the subsequent engineering systems. Nevertheless, a numerical solution is given in the last Section 3.3 in order to show improvements that can be reached using the procedure implemented. The studied system is a rigid-walled tube closed at one end by a piston acoustic source and at the other end by a rigid wall, whose well-known analytical solution is used as the reference. Of course, this system is very simple, but because the calculation is performed around its first resonance frequency, it permits emphasise of the main property of the procedure given here that is to take into account the effect, on the acoustic field, in the bulk of the domain, of the reactive and dissipative phenomena inside the boundary layers.

3.2. Numerical procedure to express the thermo-viscous impedance-like boundary condition

When solving a problem in a rigid-walled domain, taking into account the reactive and dissipative processes in the viscous and thermal boundary layers, an admittance-like boundary condition (Eqs. (15b) and (15d)), can be used. Nevertheless, the admittance β involved is unknown because it depends on the unknown factor k_w^2/k_0^2 (Eq. (17), which characterizes the direction of the acoustic velocities on the wall for the non-evanescent modes). In order to express this factor k_w^2/k_0^2 and then to solve numerically the problem under consideration, a three-step procedure is suggested here.

In the first step, the problem (Eq. (15)) is solved in the considered domain Ω, using the main numerical code chosen, absorbing and reactive processes being taken into account by using the admittance-like function (Eq. (14b)) $\beta = k_0(k_w^2/k_0^2 k_v + ((\gamma - 1)/k_h))$ onto the boundaries, where the unknown factor k_w^2/k_0^2 $(0 < |k_w^2/k_0^2| < 1)$ is arbitrarily set to 1/2. For each frequency, one value of the pressure p_a per node is then calculated.

According to the admittance description, the boundary layers thicknesses $\delta_{v,h}$ being negligible, the acoustic pressure calculated on the boundary nodes is nearly equal to pressure values at distance $\delta_{v,h}$ from the boundary (see Section 2.1.4) as illustrated in Fig. 3.

Hence, the pressure field calculated on the boundary nodes is considered as the field at the interface between the boundary layers and the bulk of the domain. This acoustic field is considered as a first approximation of the numerical solution; it provides an accurate spatial acoustic pressure distribution which permits to calculate the unknown factor (k_w^2/k_0^2) in the expression of the admittance β, giving realistic values of this admittance β for the field of interest.

In a second step, from the pressure results calculated in the first step, the unknown factor k_w^2/k_0^2 (Eq. (17)) is estimated on each element, and then the value of the admittance-like function β is calculated with a good accuracy as regards the pressure field to be finally calculated. Depending on the numerical coding, this factor can be calculated on each element using Eq. (17) (which can be reduced to $|\mathbf{k_w}| = |\mathbf{\nabla_w} p / \mathrm{i}k_0 p|$ for simple fields, such as a local plane wave).

This last expression can be used as long as the Shannon spatial criterion is satisfied, that is the wavelength is much greater than the dimensions of each element and the wavenumber $\mathbf{k}_0$ is constant on each element of the boundary.

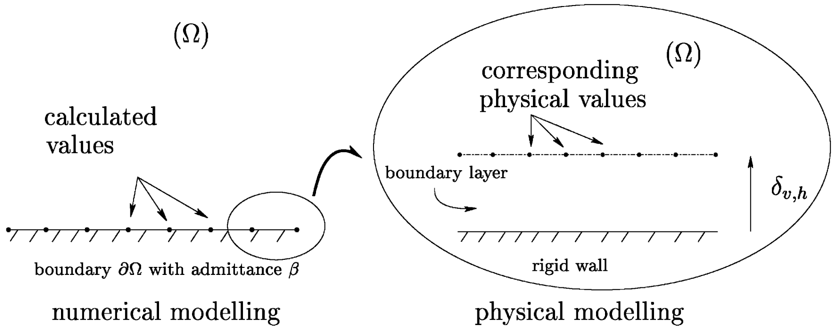

Fig. 3. Admittance-like description and equivalence between numerical pressure results on the boundary and acoustic pressure values at the interface between the boundary layer and the bulk of the domain, the boundary layers thicknesses $\delta_{v,h}$ being negligible compared to the dimensions of the considered domain Ω.

In the present application, an external C-language code (post-processing) is developed to process the input and output files of the chosen numerical packaging. Although linear interpolation functions over triangular elements would be accurate to calculate the tangential derivatives ($\partial_{\mathbf{w}}p$) on each element, the quadratic functions are chosen to ensure a good quality interpolation. Hence, this C-language code needs the numerical model to be based on quadratic interpolation over triangular elements and the processed files need to contain pressure results on each node of the mesh.

In the third step, problem (15) is numerically solved with the adapted values of β on each element of the boundary. Then, if needed, particle velocity $\mathbf{v}$ and temperature variation τ can be calculated in the boundary layers with a post-processing, that is another external C-language code based on Eqs. (7), (8) and (12).

As the method presented in Sections 2.1.4 and 3.2 to calculate the impedance-like β function lies only on the acoustic pressure near the boundaries (calculus of pressure field in the domain Ω is not needed), either BEM or FEM techniques can be used (the numerical packaging chosen here, Rayon, developed by Straco, Compiègne, France [10] makes use of the BEM technique).

3.3. Comparison to analytical results, for a "large" waveguide, and conclusion

Being concerned by the efficiency of the method, we give here a simple application, yet typical of the most important class of problems: those where the boundary layers can be considered as very thin regarding the characteristic dimensions of the acoustic domain. In this class of problems, the admittance-like function β is clearly adequate to express the effect of the boundary layers onto the acoustic field. The chosen application is then a "large", rigid walled waveguide, "large" meaning that the transverse dimensions are much greater than the boundary layer thicknesses. The waveguide is closed at one end ($z = 0$) by a plane piston source (velocity V_0) and at the other end ($z = L_z$) by a rigid wall (Fig. 4).

The frequency range of the study lie over the first axial resonance of the waveguide in order to emphasize the role played by the dissipation process in the boundary layers. For this frequency range, the transverse dimensions are such as the field in the waveguide is a plane wave (under the first cut off frequency). All these requirements are here achieved using a waveguide 170 mm long and 5 mm large (square cross-section $L_x = L_y$) in the frequency range 900–1100 Hz.

The amplitude of the acoustic pressure in the waveguide at the end opposite to the source is shown in Fig. 5. Curves (1) and (2) are, respectively, the analytic result and the numerical one, taking into account only the losses which occur in the thermal boundary layers [7], and curves (3)

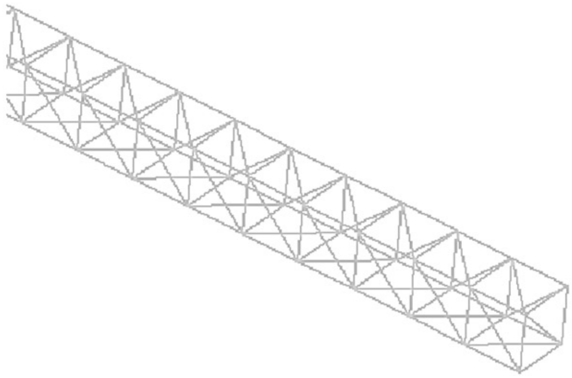

Fig. 4. Part (one end) of the "large" waveguide considered as an example and view of the meshing used.

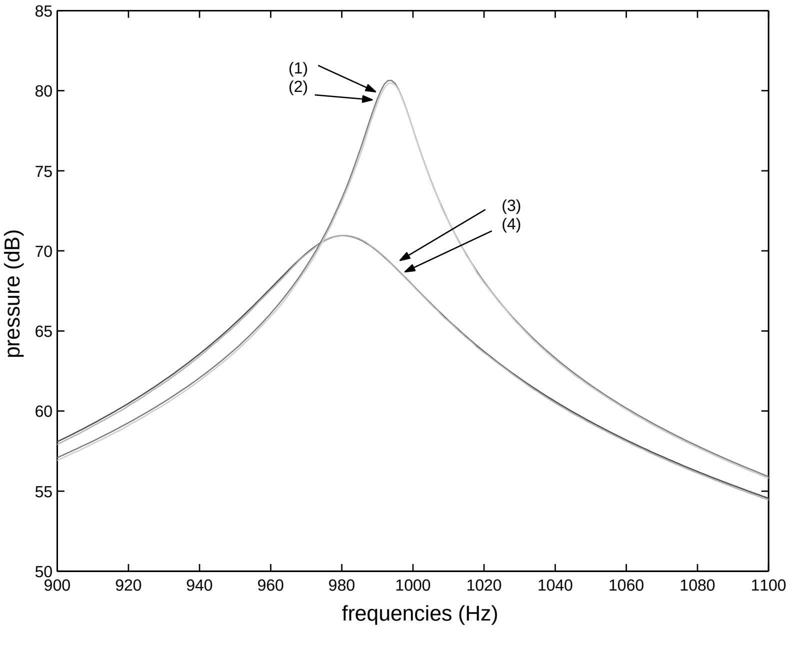

Fig. 5. Amplitude (dB) of pressure calculated on a numerical node at the end opposite to the source of the waveguided considered, as function of the frequency. Analytical result (curve 1) and computed result (curve 2) when taking into account only the thermal effects [7,8]; exact analytical result (curve 3) and computed result (curve 4) when taking into account both thermal and viscous effects using the procedure presented Section 3.2.

and (4) are, respectively, the exact analytic result, namely [1]

$$p(L_z) = 2(k_0/k_{z00})\rho_0 c_0 (1 + V_0 \cos(\mathrm{i}k_{z00}L_z))/(2\mathrm{i}\sin k_{z00}L_z),$$

where

$$k_{z00}^2 = k_0^2[1 + (1-\mathrm{i})\eta_0]$$

and

$$\eta_0 = (1/\sqrt{2})\, 2(L_x + L_y)/(L_x L_y)(1/\sqrt{k_0})[\sqrt{\ell'_v} + (\gamma - 1)\sqrt{\ell_h}]$$

and the numerical one obtained in the present work (see a view of the meshing used in Fig. 4), that is when considering both thermal and viscous losses inside the boundary layers.

To conclude, it can be emphasized that this simple example has highlighted the advantage of the method in having given results when a resonance occurs in a waveguide, that is when the thermal and viscous losses play an important and very sensitive role. Given the specificity of this example, there is seen to be very close agreement between computational and precise analytical result, thereby supporting the method presented in this paper for engineering problems involving complex shape domains.

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