Improved Formulation of the Acoustic Transfer Admittance of Cylindrical Cavities

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Summary
The reciprocity calibration method using cylindrical cavities (called plane wave couplers) described in standards assumes that only plane wave-motion exists inside the cavity, whereas radial wave-motion also exists. A previous investigation, which accounts for the radial wave-motion, analysed the acoustic field in cylindrical cavities, providing a correction to the plane wave theory quoted in the current IEC standard 61094-2 (1992) and pointing out its relevance in the calibration uncertainties. The author expresses the acoustic field excited by the transmitting microphone (set at one end of the cavity) as a sum over the eigenmodes of the cavity which satisfy Neumann boundary conditions and gives a prominent role to the shape of the diaphragm of this transmitting microphone. But in this analytical formulation, the influence of the finite acoustic impedance of the receiving microphone (closing the end of the cavity) on the shape of the acoustic pressure field inside the cavity is neglected. This limitation is pointed out in the present paper: the cavity boundaries need not be so highly idealised, in order to obtain a more realistic representation of the acoustic field inside the cylindrical cavity and thus a deeper accuracy in microphone calibration results.

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1. Introduction

The nature and consequences of the acoustic fields within the closed cylindrical cavities used in the pressure reciprocity calibration of microphones, is of importance for determining the complex pressure sensitivity of Laboratory Standard Microphones. This reciprocity calibration method is described in the IEC standard 61094-2 (1992) [1]. In this current standard, the analytical model quoted for describing the acoustic field inside the cavity (the coupler) assumes a plane wave solution. In a paper published later on (1993) [2], Rasmussen, considering that the influence of the radial wave-motion in the calibration process is not negligible in comparison with the calibration uncertainties, pointed out that the plane wave model is not appropriate for this specific application (especially in the highest frequency range). The formulation presented by the author gives a prominent role to the shape of the membrane of the transmitting microphone, which closes one end of the cylindrical cavity (Figure 1): the construction of the solution in terms of eigenmodes of the cavity makes use of modal wavefunctions that are everywhere regular solutions of the homogeneous Helmholtz equation subject to the Neumann boundary conditions at the cavity walls (then both the eigenfunctions and eigenvalues are real). This formulation governs the spatial variation of the acoustic response due to a given time-periodic source distribution (shape of the vibrating membrane of the transmitting microphone) inside the cavity (with such an inhomogeneous source it is obvious that the plane wave solution is not appropriate).

Nevertheless the solution outlined by the author need to be reinforced. In fact, it has a shortcoming in terms of analytical convenience because it implies that the impedance of the receiving microphone is infinite, that is to say the receiving diaphragm is perfectly rigid. This boundary of the cavity needs not be so highly idealized because it acts on the acoustic field in the same way as the other end of the cavity (the transmitting diaphragm). On the presumption that results relying on such simplified model (perfectly rigid receiving membrane) may not be wholly correct regarding the calibration uncertainties obtainable today, an improved solution for the pressure field inside the cavity which takes into account the velocity field of the receiving membrane is suggested here. Moreover, the viscous and thermal boundary layer effects are included in the formalism because they are easy to express, the relevant mathematics being unchanged. The resulting expression for the acoustic transfer admittance of the coupler is

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given and calculated for several combinations of couplers and microphones. The results obtained are compared to both those given in the IEC standard and those given by Rasmussen [2]. These comparisons show in our opinion that it should be obvious to make use of the formulation proposed here, which in some respect reinforces the former idea suggested by Rasmussen.

2. The fundamental problem in the closed coupler and its solution

The heart of the pressure reciprocity calibration of microphones comprises a cylindrical cavity filled with gas (Figure 1). The length ℓ of the cavity has the same order of magnitude as its radius a, both being much greater than boundary layer thicknesses. The cavity is closed at its ends by the diaphragms of two flush mounted transducers, one used as a harmonic sound source (transmitter, radius a_t) set at $z = 0$, and the other one as a receiver (radius a_r) set at $z = \ell$. The radii a_t and a_r are smaller than the radius a of the cavity but close to it. The diaphragms of the transmitter and the receiver vibrate respectively at the velocity v_t (the velocity field of the diaphragm of the driver) and the velocity v_R (created by the acoustic pressure field on its membrane).

As the system has an axial symmetry, the acoustic field is assumed to be independent of the azimuthal component θ. Nevertheless, for being plainly clear, we proceed at the beginning of the paper with the derivation of the relevant equations which include this coordinate, before giving the closed form of the solutions which does not depend on the θ-coordinate (taking into account the azimuthal in the solutions would overshadow the purpose of this paper, mentioned above). Besides the general assumption of linear acoustics, the dissipation in the bulk of the cavity is neglected in the formalism presented in the following sections, but the viscous and thermal dissipation and reaction in the boundary layers are taken into account.

2.1. Fundamental equations and lateral boundary conditions: quasi plane wave approximation

The variables describing the dynamic and thermodynamic states of the fluid are the pressure variation p, the particle velocity $\vec{v}$, the density variation ρ', the entropy variation σ, and the temperature variation τ. The parameters which specify the properties and the nature of the fluid are the ambient values of the density ρ_0, the static pressure P_0, the shear viscosity coefficient μ, the coefficient of thermal conductivity λ, the specific heat coefficient at constant pressure and constant volume per unit of mass C_P and C_V respectively, the specific heat ratio γ, and the increase in pressure per unit increase in temperature at constant density β.

A complete set of linearized homogeneous equations governing small amplitude disturbances of the fluid includes the Navier-Stokes equation, the Fourier equation for heat conduction and the conservation of mass equation. This set of linear equations, which gives an accurate description of the acoustic field, must satisfy several assumptions in order to avoid overly intricate formulation. It assumes quasi plane wave approximation, namely: i- the r- and θ-components of the particle velocity are much lower than their z-component $\underline{v}_z$ (the reason why $\underline{v}_z$ is here underlined appears hereafter), that is the flow is assumed to be essentially tangential to the wall $r = a$, and then, in a first approximation, the Navier-Stokes equation is substituted by the Hagen-Poiseuille equation for the $\underline{v}_z$-component tangent to the lateral wall, the normal component of the velocity v_r being, nevertheless, taken into account in the conservation of mass equation to assume the volume flow conservation; ii- spatial variations in normal direction r of both the particle velocity $\underline{v}_z$ and the temperature variation $\underline{\tau}$ are much greater than their spatial variations in the z-direction and hence the spatial variations of these quantities in the z-direction can be neglected in the Navier-Stokes equation and in the Fourier heat conduction equation.

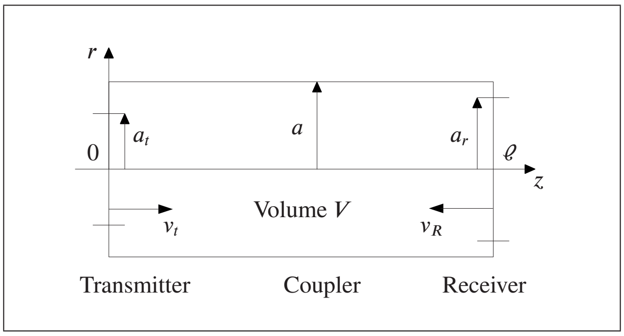

Figure 1. Geometry of the coupler and the transmitting and receiving microphones.

In addition, we will assume three associated boundary conditions on the lateral wall $r = a$ (the boundary conditions on both the transmitting membrane ($z = 0$) and the receiving one ($z = \ell$) will be considered later, section 2.3): the tangential z-component of the particle velocity must vanish (non slip condition) that is $\underline{v}_z(a, \theta, z) = 0$, and the temperature variation is required to vanish, so $\underline{\tau}(a, \theta, z) = 0$.

Therefore, the complete set of equations and boundary conditions governing the fluid motion in the quasi plane wave approximation, involving the abovementioned approximations, is straightforwardly obtained, leading to, for a harmonic perturbation (angular frequency ω):

- for the z-component of the particle velocity (Hagen-Poiseuille equation)

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} - \ell'_v \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\right)\underline{v}_z(r, \theta, z) = -\frac{1}{\rho_0 c_0}\frac{\partial}{\partial z} p(r, \theta, z), \quad (1)$$

where c_0 is the adiabatic speed of sound and where the characteristic length ℓ'_v is defined as

$$\ell'_v = \frac{\mu}{\rho_0 c_0}, \quad \text{with} \quad \underline{v}_z(a, \theta, z) = 0 \quad \forall z, \theta, \quad (2)$$

and with the condition that the field remains finite at $r = 0$, i.e.

$$\underline{v}_z(0, \theta, z) \quad \text{finite} \quad \forall z, \theta,$$

- for the temperature variation (Fourier equation)

$$\left(\frac{1}{c_0} \frac{\partial}{\partial t} - \ell_h \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \right) \underline{\tau}(r, \theta, z) = \frac{\gamma - 1}{\beta\gamma} \frac{1}{c_0} \frac{\partial}{\partial t} p(r, \theta, z), \tag{3}$$

where the characteristic length ℓ_h is defined as

$$\ell_h = \frac{\lambda}{\rho_0 c_0 C_P}, \quad \text{with} \quad \underline{\tau}(a, \theta, z) = 0 \quad \forall z, \theta, \tag{4}$$

and with the condition that the temperature field remains finite at $r = 0$, i.e.

$$\underline{\tau}(0, \theta, z) \quad \text{finite} \quad \forall z, \theta,$$

- for the r-component and the θ-component (negligible) of the particle velocity (conservation of mass equation)

$$\frac{\gamma}{\rho_0 c_0^2} \frac{\partial}{\partial t} (p - \beta \underline{\tau}) + \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} v_\theta + \frac{\partial}{\partial z} \underline{v}_z = 0. \tag{5}$$

The regular solutions of this set of equations (1 to 5) are the appropriate results that are needed to obtain the equation of propagation for the quasi plane wave in the z-direction, when assuming that all the quantities are replaced by their mean values across the section of the cavity, leading to a boundary problem which depends only on the z-coordinate. These results are presented in the next section.

2.2. The equation of propagation for the quasi plane wave approximation

The quasi plane wave approximation assumes that the acoustic pressure field does not vary significantly with the variable r in the viscous and thermal boundary layers, near the lateral wall. On the other hand, due to the viscous boundary layer, the z-component $\underline{v}_z$ of the particle velocity depends strongly on the coordinate r inside this boundary. Then the right hand side of equation (1) can be assumed independent of the r-coordinate leading to the approximate solution of this equation, subject to the boundary condition (2), for a harmonic motion (the factor $e^{j\omega t}$ is omitted):

$$\underline{v}_z(r, \theta, z) \approx \frac{j}{k_0 \rho_0 c_0} \frac{\partial}{\partial z} p(r, \theta, z) \left[1 - \frac{J_0(k_v r)}{J_0(k_v a)} \right], \tag{6a}$$

$k_0 = \omega / c_0$ being the "adiabatic" wavenumber and the wavenumber k_v, associated with the vortical movement due to viscosity effects, being given by

$$k_v = \frac{1 - j}{\sqrt{2}} \sqrt{k_0 / \ell'_v}. \tag{6b}$$

Moreover, when considering a quasi plane wave approximation, the acoustic pressure is quasi independent of the position across a section of the cylindrical cavity. Therefore, the effect of the viscous boundary layer given by this solution can be approximated by its mean value across the section of the cavity, i.e. it is appropriate to integrate equation (6) on a section of the cavity, neglecting that p depends on r- and θ-coordinates when integrating. The approximate solution for the z-component of the particle velocity denoted v_z, then reduces to

$$\begin{aligned} v_z &\approx \frac{j}{k_0 \rho_0 c_0} \frac{\partial}{\partial z} p(r, \theta, z) \cdot \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \left[1 - \frac{J_0(k_v r)}{J_0(k_v a)} \right] r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \frac{j}{k_0 \rho_0 c_0} \frac{\partial}{\partial z} p(r, \theta, z) \left[1 - \frac{2}{k_v a} \frac{J_1(k_v a)}{J_0(k_v a)} \right]. \end{aligned} \tag{7}$$

It must be noticed here that the r-component of the particle velocity, which does not depend on the viscous effect inside the viscous boundary layer because it is normal to the surface $r = a$ of the cavity, is given by the usual Euler equation (harmonic motion)

$$v_r(r, \theta, z) = \frac{j}{k_0 \rho_0 c_0} \frac{\partial}{\partial r} p(r, \theta, z). \tag{8}$$

The θ-component which is nearly negligible here, can be also approximated by using Euler equation:

$$v_\theta(r, \theta, z) = \frac{j}{k_0 \rho_0 c_0} \frac{1}{r} \frac{\partial}{\partial \theta} p(r, \theta, z). \tag{9}$$

In the same manner as for obtaining the solution $\underline{v}_z$ (6), the solution $\underline{\tau}$ of equation (3), subjects to the boundary conditions (4) is given by

$$\underline{\tau}(r, \theta, z) = \frac{\gamma - 1}{\beta\gamma} p(r, \theta, z) \left[1 - \frac{J_0(k_h r)}{J_0(k_h a)} \right], \tag{10a}$$

where the expression of the wavenumber k_h (associated with the entropy diffusion due to heat conduction) is given by

$$k_h = \frac{1 - j}{\sqrt{2}} \sqrt{k_0 / \ell_h}, \tag{10b}$$

and the solution (10a) can be approximated by its mean value across the section of the cylindical cavity (admitting the same approximations as those on $\underline{v}_z$, eq. 8):

$$\tau(r, \theta, z) = \frac{\gamma - 1}{\beta\gamma} p(r, \theta, z) \left[1 - \frac{2}{k_h a} \frac{J_1(k_h a)}{J_0(k_h a)} \right]. \tag{11}$$

Then, mean value across a section of equation (5), making use of equations (8), (8), (9) and (11), leads to the following propagation equation in the z-direction:

$$\left[k_0^2 \frac{1 + (\gamma - 1) K_h}{1 - K_v} + \frac{1}{1 - K_v} \left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) + \frac{\partial^2}{\partial z^2} \right] p(r, \theta, z) = 0, \tag{12}$$

the expressions for $K_{h,v}$ (i.e. respectively K_h and K_v) being given by

$$K_{h,v} = \frac{2}{k_{h,v}a}\frac{J_1(k_{h,v}a)}{J_0(k_{h,v}a)}. \tag{13}$$

By assuming that the spatial derivatives of the acoustic pressure p with respect to the r-and θ-coordinates are much lower than the derivative with respect to the z-coordinate, the complex factor K_v can be neglected in these terms (because it is much lower than one) and then this last equation (12) simplifies to

$$\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} + k^2\right]p(r,\theta,z) = 0, \tag{14}$$

the wavenumber k being given by

$$\begin{aligned} k^2 &= k_0^2\frac{1+(\gamma-1)K_h}{1-K_v} \\ &\approx k_0^2\left[1+\frac{1-\mathrm{j}}{\sqrt{2}}\frac{2}{a\sqrt{k_0}}\left(\sqrt{\ell'_v}+(\gamma-1)\sqrt{\ell_h}\right)\right]. \end{aligned} \tag{15}$$

Then, the motion of the fluid is governed by this three-dimensional Helmholtz wave equation, the solution being subjected to the boundary conditions on the walls $z = 0$ and $z = \ell$ which take into account the effects of both the velocity profile of the diaphragms of the microphones and the viscous and thermal boundary layers. The remainder of the paper is devoted to the construction of such a solution, using the Green theorem, and to the interpretation of the results obtained by comparison with those available in the literature.

2.3. The boundary problem within the framework of quasi plane wave approximation, integral formulation

The problem addressed above is given by the propagation equation (14) associated to the following boundary conditions :

$$v_r = 0, \quad r = a,\ \theta \in (0, 2\pi),\ z \in (0,\ell), \tag{16a}$$

(because the viscous and thermal boundary layers which imply the r-component of the particle velocity are accounted for in the complex wavenumber k given in the previous section 2.2 in the framework of the quasi plane wave approximation), and

$$\rho_0c_0v_z = \rho_0c_0v_t - \zeta_{z0}p, \quad r \in (0,a_t),\ \theta\in(0,2\pi),\ z = 0, \tag{16b}$$

$$\rho_0c_0v_z = -\rho_0c_0v_R + \zeta_{z\ell}p, \quad r \in (0,a_r),\ \theta\in(0,2\pi),\ z = \ell, \tag{16c}$$

$$\rho_0c_0v_z = -\zeta_{z0}p, \quad r \in (a_t,a),\ \theta\in(0,2\pi),\ z = 0, \tag{16d}$$

$$\rho_0c_0v_z = \zeta_{z\ell}p, \quad r \in (a_r,a),\ \theta\in(0,2\pi),\ z = \ell, \tag{16e}$$

where v_t and v_R represent the velocity fields of the transmitting and the receiving diaphragms respectively, and where the specific admittances ζ_{z0} and $\zeta_{z\ell}$ on the walls at $z = 0$ and $z = \ell$, which express the viscous and the thermal boundary layer effects on the acoustic field inside the cavity [3], are detailed in section 2.4 equation (25d) and in appendix.

Upon using the notation (13) and invoking respectively equations (8) and (8), equations (16a) to (16e) lead to, to the first order of the small quantities ζ_{z0}, $\zeta_{z\ell}$ and K_v (and for any value of $\theta \in (0, 2\pi)$),

$$\frac{\partial p}{\partial n} = \frac{\partial p}{\partial r} = 0, \quad r = a,\ z\in(0,\ell), \tag{17a}$$

$$\begin{aligned}\frac{\partial p}{\partial n} &= -\frac{\partial p}{\partial z} = \frac{\mathrm{j}\,k_0\rho_0c_0}{1-K_v}v_z \\ &\approx \frac{\mathrm{j}\,k_0\rho_0c_0}{1-K_v}v_t - \mathrm{j}\,k_0\zeta_{z0}\,p, \quad r\in(0,a_t),\ z=0,\end{aligned} \tag{17b}$$

$$\begin{aligned}\frac{\partial p}{\partial n} &= \frac{\partial p}{\partial z} = -\frac{\mathrm{j}\,k_0\rho_0c_0}{1-K_v}v_z \\ &\approx \frac{\mathrm{j}\,k_0\rho_0c_0}{1-K_v}v_R - \mathrm{j}\,k_0\zeta_{z\ell}\,p, \quad r\in(0,a_r),\ z=\ell,\end{aligned} \tag{17c}$$

$$\frac{\partial p}{\partial n} = -\frac{\partial p}{\partial z} = -\mathrm{j}\,k_0\zeta_{z0}\,p, \quad r\in(a_t,a),\ z=0, \tag{17d}$$

$$\frac{\partial p}{\partial n} = \frac{\partial p}{\partial z} = -\mathrm{j}\,k_0\zeta_{z\ell}\,p, \quad r\in(a_r,a),\ z=\ell. \tag{17e}$$

The set of equations (14) and (17a) to (17e), while appropriate to addressing the acoustic pressure field in the coupler, is in the following expressed in using the integral formulation (Green theorem, [4, pp. 319–321]). With a Green function satisfying at the cavity walls the same admittance boundary conditions as the acoustic pressure p (17a to e) but homogeneous, namely

$$\frac{\partial}{\partial n}G(\vec{r},\vec{r}_0) = -\mathrm{j}\,k_0\zeta_{z0}G(\vec{r},\vec{r}_0), \quad r\in(0,a),\ \theta\in(0,2\pi),\ z=0, \tag{18a}$$

$$\frac{\partial}{\partial n}G(\vec{r},\vec{r}_0) = -\mathrm{j}\,k_0\zeta_{z\ell}G(\vec{r},\vec{r}_0), \quad r\in(0,a),\ \theta\in(0,2\pi),\ z=\ell, \tag{18b}$$

$$\frac{\partial}{\partial n}G(\vec{r},\vec{r}_0) = 0, \quad r=a,\ \theta\in(0,2\pi),\ z\in(0,\ell), \tag{18c}$$

the solution for p, subjected to the boundary conditions (17a) to (17e), is given by

$$p(\vec{r}) = \frac{\mathrm{j}\,k_0\rho_0c_0}{1-K_v}\left[\iint_{S_t} v_t(\vec{r}_0)G(\vec{r},\vec{r}_0)\,\mathrm{d}S_0 + \iint_{S_r} v_R(\vec{r}_0)G(\vec{r},\vec{r}_0)\,\mathrm{d}S_0\right], \tag{19}$$

where S_t and S_r are respectively the surfaces of the transmitter and receiver diaphragms.

This modelling provides improvements (see ref. [2]) in taking account for both the effects of the thermo-viscous boundary layers in a realistic manner and the symmetrical role of the diaphragms velocity fields in the pressure field expression (this last result is important because the reciprocity method relies on this property of symmetry). Moreover, it allows more advanced modelling which would account for the velocity profile of the diaphragms in a whole formulation (including an appropriate definition of the efficiency of the microphones).

2.4. The modal solution

The construction of solutions in terms of eigenmodes of the cavity involving only radial and azimutal coordinates (r,θ) makes use of modal wave functions that are everywhere regular solutions of the homogeneous Helmholtz equation

$$\left(\Delta_{\vec{w}} + k^2_{w\mu\nu}\right)\psi^d_{\mu\nu}(r,\theta) = 0, \quad \forall r \in (0,a),\ \forall\theta \in (0,2\pi), \tag{20a}$$

with $$\Delta_{\vec{w}} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

and subject to the Neumann boundary condition

$$\frac{\partial}{\partial z}\psi^d_{\mu\nu}(r,\theta) = 0, \quad r = a,\ \forall\theta \in (0,2\pi). \tag{20b}$$

These modal eigenfunctions are (the subscript d standing for either s or c)

$$\psi^c_{\mu\nu}(r,\theta) = \frac{1}{a_\mu b_{\mu\nu}}\cos\mu\theta J_\mu(k_{w\mu\nu}r), \tag{21a}$$

$$\psi^s_{\mu\nu}(r,\theta) = \frac{1}{a_\mu b_{\mu\nu}}\sin\mu\theta J_\mu(k_{w\mu\nu}r), \tag{21b}$$

where the functions J_μ are the cylindrical Bessel functions of the first kind, and the corresponding eigenvalues are

$$k_{w\mu\nu} = \gamma_{\mu\nu}/a, \tag{22}$$

where $\gamma_{\mu\nu}$ $(\nu = 0,1,2,3...)$ is the $(\nu+1)$th zero of the first derivative with respect to r of the Bessel function $J_\mu(k_{w\mu\nu}r)$. In the following, these eigenfunctions conform to the orthonormality condition (see appendix)

$$\int_0^a r\,\mathrm{d}r \int_0^{2\pi}\mathrm{d}\theta\,\left[\psi^d_{\mu\nu}(r,\theta)\right]^2 = 1. \tag{23}$$

The Green's function, which satisfies the boundary conditions (18a) to (18c), is taken as

$$G(\vec{r},\vec{r}_0) = \sum_{\substack{\mu,\nu=0,1...\\ d=c,s}} g_{\mu\nu}(z,z_0)\psi^d_{\mu\nu}(r_0,\theta_0)\psi^d_{\mu\nu}(r,\theta), \tag{24}$$

the coefficients $g_{\mu\nu}(z,z_0)$ being given by [5, pp. 272–275] (see appendix)

$$g_{\mu\nu}(z,z_0) = -\frac{\cos\left(k_{z\mu\nu}z+\varphi_{0\mu\nu}\right)\cos\left(k_{z\mu\nu}(z_0-\ell)-\varphi_{\ell\mu\nu}\right)}{k_{z\mu\nu}\sin\left(k_{z\mu\nu}\ell+\varphi_{0\mu\nu}+\varphi_{\ell\mu\nu}\right)}, \quad z < z_0, \tag{25a}$$

$$g_{\mu\nu}(z,z_0) = -\frac{\cos\left(k_{z\mu\nu}z_0+\varphi_{0\mu\nu}\right)\cos\left(k_{z\mu\nu}(z-\ell)-\varphi_{\ell\mu\nu}\right)}{k_{z\mu\nu}\sin\left(k_{z\mu\nu}\ell+\varphi_{0\mu\nu}+\varphi_{\ell\mu\nu}\right)}, \quad z > z_0, \tag{25b}$$

where $$k^2_{z\mu\nu} = k^2 - k^2_{w\mu\nu}, \tag{25c}$$

and $$\tan\varphi_{0,\ell\mu\nu} = -\frac{\mathrm{j}\,k_0\zeta_{z0,\ell\mu\nu}}{k_{z\mu\nu}}, \tag{25d}$$

the specific admittance $\zeta_{z0,\ell\mu\nu}$, equivalent to the effects of viscous and thermal boundary layers on the walls $z = 0$ and $z = \ell$ for the mode $(\mu\nu)$ being given by equation (A7, appendix 3).

Given the Green's function displayed above, equation (19) leads readily to the acoustic pressure

$$p(r,\theta,z) = \sum_{\mu,\nu,d} p^d_{\mu\nu}(z)\,\psi^d_{\mu\nu}(r,\theta), \tag{26}$$

with

$$p^d_{\mu\nu}(z) = \frac{\mathrm{j}\,k_0\rho_0 c_0}{1-K_v}\Big[g_{\mu\nu}(z,0)\left\langle v_t|\psi^d_{\mu\nu}\right\rangle_t + g_{\mu\nu}(z,\ell)\left\langle v_R|\psi^d_{\mu\nu}\right\rangle_r\Big], \tag{27}$$

where $\langle v_t|\psi^d_{\mu\nu}\rangle_t$ is the integral over the surface S_t of the function $v_t(r,\theta)\psi^d_{\mu\nu}(r,\theta)$ and $\langle v_R|\psi^d_{\mu\nu}\rangle_r$ is the integrale over the surface S_r of the function $v_R(r,\theta)\psi^d_{\mu\nu}(r,\theta)$.

With the assumption that the velocity field of the transmitter diaphragm v_t is independant of the azimutal coordinate θ, the acoustic pressure does not depend on the θ-coordinate (because the cavity and the receiver diaphragm are axisymmetric), neither does the velocity field of the receiver diaphragm v_R.

These results are consistent with those which can be obtained from the surface integrals $\langle v_{t,R}|\psi^d_{\mu\nu}\rangle_{t,r}$ ($v_{t,R}$ denoting respectively v_t and v_R), namely

$$\left\langle v_{t,R}|\psi^d_{\mu\nu}\right\rangle_{t,r} = \int_0^{a_{t,r}} r_0\,\mathrm{d}r_0 \int_0^{2\pi}\mathrm{d}\theta_0\, v_{t,R}(r_0,\theta_0)\,\psi^d_{\mu\nu}(r_0,\theta_0). \tag{28}$$

In point of fact, using the expression (21aa) and (21ab) of the eigenfunctions and assuming that the velocity $v_{t,R}$ does not depend on the θ-coordinate, these integrals vanish when $\mu \neq 0$, and also when $d = s$ and $\mu = 0$ simultaneously. Then, what results is

$$p(r,z) = \sum_\nu p^c_{0\nu}(z)\,\psi^c_{0\nu}(r). \tag{29}$$

The source function $\langle v_t|\psi^c_{0\nu}\rangle_t$ (eq. 27) remains unknown so long as the coupling between the movement of the diaphragm and the field in the rear cavity of the emitting microphone is not modelled precisely, the electrical force acting on the diaphragm being expressed accurately. Such precise modelling, which would permit to deduced the velocity profile theoretically (this may be the subject for further study), is currently missing. In the current modelling, in practice, the velocity profile of the emitting diaphragm is approximated using the surface motion of an unloaded homogeneous membrane

$$v_t(r) = V_t\frac{J_0(k_t r) - J_0(k_t a_t)}{J_2(k_t a_t)}, \tag{30}$$

where V_t is the mean value of the velocity (which would represent an equivalent uniform profile proportional to the electrical excitation) and where $k_t = \omega\sqrt{M_S/T}$ (M_S being the surface density and T the tension of the transmitter diaphragm), or using other profile such as

$$v_t(r) = A\psi^c_{00} + B\psi^c_{01}(r), \tag{31}$$

where the constants A and B are given by the conditions,

$$v_t(a_t) = 0 \quad \text{and} \quad \langle v_t\rangle_t = V_t. \tag{32}$$

Finally, to determine the integral $< v_R|\psi^c_{0\nu} >_r$ (eq. 27), the velocity field v_R of the receiving diaphragm is expressed as

$$S_r v_R(r) = -\underline{Y_r}(r)\, p(r, \ell), \tag{33}$$

where $\underline{Y_r}(r)$ is the local admittance of the receiving microphone (depending on the r-coordinate) which can be written as a function of its mean value across the surface S_r of the diaphragm $Y_r = \langle\underline{Y_r}\rangle_r$ in the following manner:

$$\underline{Y_r}(r) = \left\langle \underline{Y_r} \right\rangle_r + \delta Y_r(r), \tag{34}$$

with $\langle\delta Y_r\rangle_r = 0$.

Then, in accordance with equation (29)

$$-S_r\langle v_R|\psi^c_{0\nu}\rangle_r = \sum_{\nu'} p^c_{0\nu'}(\ell)\left[Y_r\left\langle \psi^c_{0\nu'}\,\middle|\,\psi^c_{0\nu}\right\rangle_r + \left\langle \delta Y_r\psi^c_{0\nu'}\,\middle|\,\psi^c_{0\nu}\right\rangle_r\right], \tag{35}$$

and, disregarding the correction term linked to the higher order modes $\nu' \neq 0$ in the last term (in the right hand side of equation 35) which leads to

$$\left\langle \delta Y_r\psi^c_{0\nu'}\,\middle|\,\psi^c_{0\nu}\right\rangle_r \approx \left\langle \delta Y_r\psi^c_{00}\,\middle|\,\psi^c_{00}\right\rangle_r = \left\langle \delta Y_r\right\rangle_r\left\langle \psi^c_{00}\,\middle|\,\psi^c_{00}\right\rangle_r = 0,$$

one obtains

$$-S_r\left\langle v_R\,\middle|\,\psi^c_{0\nu}\right\rangle_r = \sum_{\nu'} p^c_{0\nu'}(\ell)\, Y_r\left\langle \psi^c_{0\nu'}\,\middle|\,\psi^c_{0\nu}\right\rangle_r. \tag{36}$$

Substituting this result in the expressions (26) and (27) of the pressure field p (with $d = c$ and $\mu = 0$) and neglecting the terms expressing the coupling between modes having different quantum numbers ($\nu \neq \nu'$) gives

$$p(r, z) = \frac{\mathrm{j}\,k_0}{1-K_\nu}\frac{\rho_0 c_0}{S}\sum_\nu\left[g_{0\nu}(z,0)S\langle v_t|\psi^c_{0\nu}\rangle_t - Y_{r0\nu}g_{0\nu}(z,\ell)p^c_{0\nu}(\ell)\right]\psi^c_{0\nu}(r), \tag{37}$$

where $Y_{r0\nu} = Y_r\frac{a^2}{a_r^2}\langle\psi^c_{0\nu}|\psi^c_{0\nu}\rangle_r$, and $\langle\psi^c_{0\nu}|\psi^c_{0\nu}\rangle_r$ is the integral over the surface S_r of the product $\psi^c_{0\nu}\psi^c_{0\nu}$.

The coefficients of this expansion on the eigenfunctions $\psi^c_{0\nu}(r)$

$$p^c_{0\nu}(z) = \frac{\mathrm{j}\,k_0}{1-K_\nu}\frac{\rho_0 c_0}{S}\left[g_{0\nu}(z,0)S\left\langle v_t|\psi^c_{0\nu}\right\rangle_t - Y_{r0\nu}g_{0\nu}(z,\ell)p^c_{0\nu}(\ell)\right], \tag{38}$$

which depends on the z-coordinate, give, for $z = \ell$,

$$p^c_{0\nu}(\ell) = \frac{\dfrac{\mathrm{j}\,k_0}{1-K_\nu}\dfrac{\rho_0 c_0}{S}g_{0\nu}(\ell,0)S\langle v_t|\psi^c_{0\nu}\rangle_t}{1+\dfrac{\mathrm{j}\,k_0}{1-K_\nu}\dfrac{\rho_0 c_0}{S}Y_{r0\nu}g_{0\nu}(\ell,\ell)}, \tag{39}$$

and then they take the form

$$p^c_{0\nu}(z) = \frac{\mathrm{j}\,k_0}{1-K_\nu}\frac{\rho_0 c_0}{S}\left[g_{0\nu}(z,0)S\langle v_t|\psi^c_{0\nu}\rangle_t - Y_{r0\nu}g_{0\nu}(z,\ell)\frac{\dfrac{\mathrm{j}\,k_0\rho_0 c_0}{1-K_\nu}g_{0\nu}(\ell,0)\langle v_t|\psi^c_{0\nu}\rangle_t}{1+\dfrac{\mathrm{j}\,k_0}{1-K_\nu}\dfrac{\rho_0 c_0}{S}Y_{r0\nu}g_{0\nu}(\ell,\ell)}\right], \tag{40}$$

the acoustic pressure field being given by (eq. 29).

3. Mean acoustic transfer admittance of the coupler

The current quantity of interest for the pressure reciprocity calibration method is the acoustic transfer admittance Y_T, the "quotient of the short-circuit volume velocity produced by the microphone used as a transmitter by the sound pressure acting on the diaphragm of the microphone used as a receiver" [1]. In order to allow to verify that the results available in the literature can be considered as approximations of those obtained here, the descriptor of the acoustic transfer admittance Y_T must be taken as the same quotient but replacing the parameters which depend on the coordinate r by their mean values across the surface of the concerned transducer, yielding:

$$Y_T = \frac{S_t\left\langle v_t(r)\right\rangle_t + \left\langle \underline{Y_t}(r)p(r,0)\right\rangle_t}{\left\langle p(r,\ell)\right\rangle_r}, \tag{41}$$

where $\underline{Y_t}(r)$ is the local admittance of the transmitting microphone (depending on the r-coordinate). Expressing this admittance $\underline{Y_t}(r)$ in the same manner as $\underline{Y_r}(r)$ (eq. 34)

$$\underline{Y_t}(r) = \left\langle \underline{Y_t}\right\rangle_t + \delta Y_t(r), \tag{42}$$

and the acoustic pressure at $z = 0$ as

$$p(r,0) = \left\langle p(0)\right\rangle_t + \delta p(r,0), \tag{43}$$

leads to

$$\left\langle \underline{Y_t}(r)p(r,0)\right\rangle_t \approx \left\langle \underline{Y_t}\right\rangle_t\left\langle p(0)\right\rangle_t + \left\langle \underline{Y_t}\right\rangle_t\left\langle \delta p(0)\right\rangle_t + \left\langle p(0)\right\rangle_t\left\langle \delta Y_t\right\rangle_t, = \left\langle \underline{Y_t}\right\rangle_t\left\langle p(0)\right\rangle_t. \tag{44}$$

Then, the acoustic transfer admittance Y_T is given by

$$Y_T = \frac{S_t V_t + Y_t\left\langle p(0)\right\rangle_t}{\left\langle p(\ell)\right\rangle_r}, \tag{45}$$

where

$$V_t = \left\langle v_t \right\rangle_t = \frac{2}{a_t^2}\int_0^{a_t} v_t(r_0) r_0 \, \mathrm{d}r_0,$$

and $$Y_t = \langle \underline{Y_t} \rangle_t = \frac{2}{a_t^2}\int_0^{a_t} \underline{Y_t}(r_0) r_0 \, \mathrm{d}r_0,$$

the factors $\langle p(0)\rangle_t$ and $\langle p(\ell)\rangle_r$ being given respectively by

$$\left\langle p(0)\right\rangle_t = \sum_\nu p^c_{0\nu}(0)\left\langle \psi^c_{0\nu}\right\rangle_t,$$

and $$\left\langle p(\ell)\right\rangle_r = \sum_\nu p^c_{0\nu}(\ell)\left\langle \psi^c_{0\nu}\right\rangle_r.$$

A straightforward but lengthy calculation leads to

$$\begin{aligned} Y_T = & \Bigg[1 + Y_t \sum_\nu \frac{SV_{0\nu}}{a_0 b_{0\nu}}\left\langle \psi^c_{0\nu}\right\rangle_t \\ & \cdot \left[\cos k_{z0\nu}\ell + \frac{Y_{p0\nu}+Y_{r0\nu}}{Y_{i0\nu}} \mathrm{j}\sin k_{z0\nu}\ell\right] \\ & \cdot \left[Y_{T00\nu} + Y_{r0\nu}\left(\frac{Y_{p0\nu}}{Y_{i0\nu}}\mathrm{j}\sin k_{z0\nu}\ell + \cos k_{z0\nu}\ell\right)\right]^{-1} \\ & \cdot \Bigg[\sum_\nu \frac{SV_{0\nu}}{a_0 b_{0\nu}}\left\langle \psi^c_{0\nu}\right\rangle_r \qquad (46) \\ & \cdot \left[Y_{T00\nu} + Y_{r0\nu}\left(\frac{Y_{p0\nu}}{Y_{i0\nu}}\mathrm{j}\sin k_{z0\nu}\ell + \cos k_{z0\nu}\ell\right)\right]^{-1}\Bigg]^{-1}, \end{aligned}$$

where $k_{z0\nu}$, $Y_{r0\nu}$, a_0 and $b_{0\nu}$ are given respectively by equations (25c), (37), (A1) and (A2), and where

$$Y_{i0\nu} = \frac{S}{\rho_0 c_0}\frac{k_{z0\nu}}{k_0}(1-K_\nu), \qquad (47)$$

$$Y_{p0\nu} = \frac{S}{\rho_0 c_0}\zeta_{z0,\ell\nu} \qquad (48)$$

the specific admittance $\zeta_{z0,\ell\nu}$ being given by equation (A8a), and

$$Y_{T00\nu} = \left[Y_{i0\nu} + \frac{Y^2_{p0\nu}}{Y_{i0\nu}}\right]\mathrm{j}\sin k_{z0\nu}\ell + 2Y_{p0\nu}\cos k_{z0\nu}\ell. \qquad (49)$$

The coefficients $V_{0\nu}$ depend on the expression choosen for the velocity field of the transmitter $v_t(r)$ (eq. 30 or 31) and are given by

$$V_{0\nu} = \frac{1}{V_t}\frac{2}{a_t^2}\int_0^{a_t} v_t(r) J_0(k_{w0\nu}r) r \, \mathrm{d}r. \qquad (50)$$

Note that when assuming both the exact plane wave approximation ($\nu = 0$) and the same value for the diameter of the diaphragms and the acoustic coupler ($a_r = a_t = a$), the expression (46) reduces to

$$\begin{aligned} Y_T = & \, \mathrm{j}\sin k_{z00}\ell\left[Y_{i00} + \frac{(Y_t+Y_{p00})(Y_r+Y_{p00})}{Y_{i00}}\right] \\ & + \cos k_{z00}\ell(Y_t + Y_r + 2Y_{p00}), \qquad (51) \end{aligned}$$

Table I. Parameters of LS2P microphones used to obtain the results shown on Figures 2 and 3.

	Transmitter	Receiver
Equivalent volume (mm^3)	8.07	8.03
Resonance frequency (Hz)	23000	20500
Loss factor	1.22	1.22

where the wavenumber k^2_{z00}, the admittance Y_{p00} which account for the boundary layers, and iterative admittance Y_{i00} are given respectively by

$$\begin{aligned} k^2_{z00} &= k^2 = k_0^2\left[1 + \frac{1-\mathrm{j}}{\sqrt{2}}\frac{1}{\sqrt{k_0}}\frac{2}{a}\left(\sqrt{\ell'_\nu} + (\gamma-1)\sqrt{\ell_h}\right)\right], \\ Y_{p00} &= \frac{S}{\rho_0 c_0}\zeta_{z00} = \frac{S}{\rho_0 c_0}\frac{1+\mathrm{j}}{\sqrt{2}}\sqrt{k_0}(\gamma-1)\sqrt{\ell_h}, \\ Y_{i00} &= \frac{S}{\rho_0 c_0}\frac{k_{z00}}{k_0}(1-K_\nu) \qquad (52) \\ &\approx \frac{S}{\rho_0 c_0}\left[1 + \frac{1-\mathrm{j}}{\sqrt{2}}\frac{1}{a\sqrt{k_0}}\left(-\sqrt{\ell'_\nu} + (\gamma-1)\sqrt{\ell_h}\right)\right]. \end{aligned}$$

This last result is exactly the expression obtained in a previous paper [6] which in turn represents a corrected version of the expression given in IEC publication 61094-2 [1].

4. Results and discussion

The differences (dB) between the acoustic transfer admittances calculated with the expression (51), which assumes plane wave approximation, and those calculated with the expression (46) in the present paper show significant deviations between both models in the upper frequency range, even the mean values across the surface of the transducers does not allow to emphasize the relevance of the model presented here. These main results can be seen in Figures 2 and 3 for LS2P microphones, which parameters (diameter, equivalent volume, resonance frequency, loss factor) are given in Table I, with cylindrical cavities 9.3 mm in diameter and having a total length from 4.04 to 7.08 mm. These differences can reach 0.1 dB, which is non negligible compared to the reproducibility of the measurements, nearly equal to 0.01–0.02 dB [7].

Curves on Figure 2 show these differences when the velocity profile is approximated using surface motion of an unloaded homogeneous membrane (eq. 30) and curves on Figure 3 show these differences when the velocity profile is approximated as given in equation (31). We notice that the results are quite insensitive to the velocity profiles used for the diaphragms. On the other hand, we have verified that the contribution of both the higher order modes ($\nu\rangle 1$) and the terms expressing the coupling between modes having different quantum numbers (section 2.4) in the modal expansions do not contribute significantly to the results presented here.

On one hand, Figure 4 shows the differences (dB) between the acoustic transfer admittances calculated with

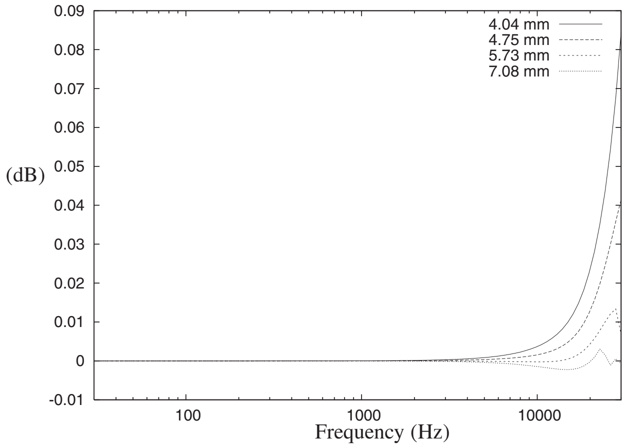

Figure 2. Differences between the acoustic transfer admittances calculated with the expressions (51) and (46), for LS2P microphones when $v_t(r)$ is given by (eq. 30).

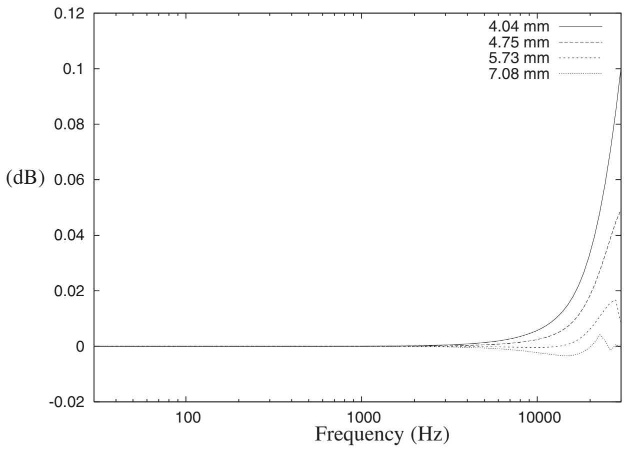

Figure 3. Differences between the acoustic transfer admittances calculated with the expressions (51) and (46), for LS2P microphones when $v_t(r)$ is given by (eq. 31).

the expression (51), which assumes plane wave approximation, and those calculated with the expression (46) in the present paper, and on the other hand Figure 5 shows the differences (dB) between the acoustic transfer admittances calculated with the expression (51) and those calculated with the expression (46) the mean value of the acoustic pressure $\langle p(\ell)\rangle_r$ across the surface of the receiving diaphragm being replaced by a mean value weighted with the normalised velocity distribution of the diaphragm (like suggested by Rasmussen [2]), both curves been obtained for LS1P microphones (parameters given in Table II), using cavities 18.6 mm in diameter and between 7.5 and 13.5 mm for their total length. It can be seen that the discrepancies between these results reach one order of magnitude, emphasizing that the sensitivity of the transfer admittance to the weighted mean value is significant. Moreover, the discrepancies between the results shown on Figure 5 and the results given by Rasmussen [2, Figure 4] (using the same weihgted mean value), with the same microphones and the same cavities, are for example the order of 0.2 dB at 12.5 kHz for an acoustic cavity 7.5 mm long,

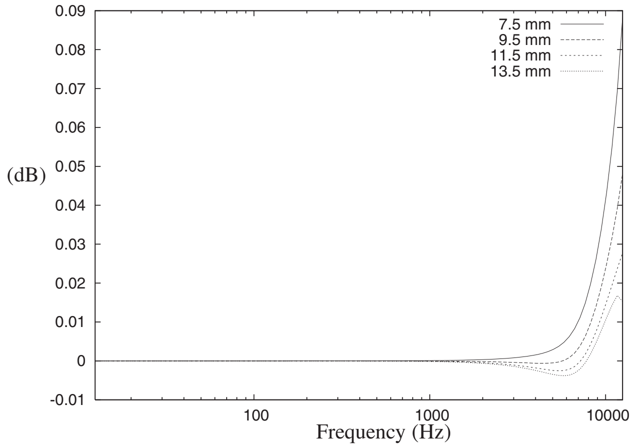

Figure 4. Differences between the acoustic transfer admittances calculated with the expressions (51) and (46), for LS1P microphones when $v_t(r)$ is given by (eq. 30).

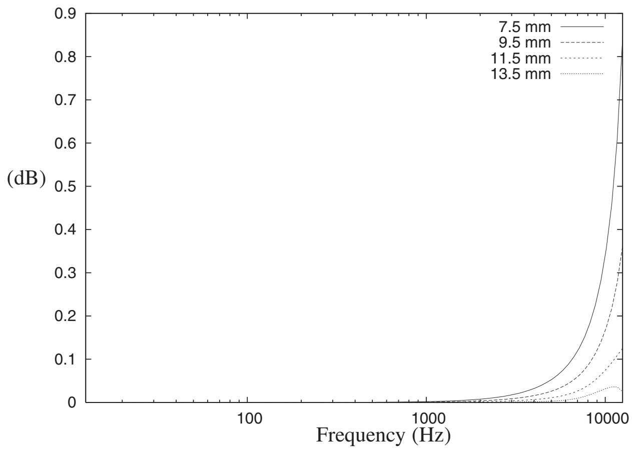

Figure 5. Differences between the acoustic transfer admittances calculated with the expressions (51) and (46), for LS1P microphones when $v_t(r)$ is given by (eq. 30) and replacing $\langle p(\ell)\rangle_r$ by a mean value weighted with the normalised velocity distribution of the diaphragm.

Table II. Parameters of LS1P microphones used to obtain the results shown on Figures 4 and 5.

	Transmitter	Receiver
Equivalent volume (mm^3)	147.72	131.36
Resonance frequency (Hz)	8500	8500
Loss factor	1.05	1.05

which is not negligible compared to the reproducibility mentioned above (note that the quantity represented in [2, Figure 4] is the correction factor to the microphone sensitivity ; therefore, the correction to the acoustic admittance proposed by Rasmussen reaches about 1.0 dB at 12.5 kHz for a 7.5 mm cavity).

5. Conclusion

At the end of the present paper, all the quantities of interest at $z = 0$ and $z = \ell$ are approximated by replacing the parameters depending on the coordinate r with their mean values across the surface of the concerned transducers, allowing to verify that the results available in the literature can be considered as approximations of those obtained here. But making use of these mean values (at $z = 0$ and $z = \ell$) instead of the functions which depend on the coordinate r, one can anticipate that the interest of the modelling presented in the present paper would be reduced drastically in essence. Therefore, as suggested by Rasmussen [2], this inconsistency could be here partially circumvent in replacing the acoustic pressure field on the receiving diaphragm (which depends on the r-coordinate) by a mean value weighted with the normalised velocity distribution of the diaphragm. On the other hand, as assumed in the current modelling, the velocity profile of the emitting diaphragm could be approximated using the surface motion of an unloaded homogeneous membrane or using other profile such as an expansion on the main acoustic modes of the cavity (the results for the acoustic pressure field being quite insensitive to the profile used). Doing this, the results obtained here compared to those given by Rasmussen show discrepancies which are probably due to the difference between the construction of the models describing the acoustic field inside the coupler.

In conclusion, the modelling presented in the paper provides improvements in taking account for both the effects of the thermo-viscous boundary layers in a realistic manner and the symmetrical role of the diaphragms velocity fields in the pressure field expression (this last result is important because the reciprocity method relies on this property of symmetry). Moreover, it allows more advanced modelling which would account for the velocity profile of the diaphragms in the formulation, that is to say in the expression of the efficiency of the microphones, in the behaviour of the rear cavity of the microphones, and finally, through the transfer admittance, in the reciprocity calibration system as a whole where the different parts (electrical, acoustical, and mechanical parts) are strongly coupled together. This fine modelling may be the subject for further researches.

Appendix

Normalization coefficients

The normalization coefficients a_μ and $b_{\mu\nu}$ (21aa and b), are given by the orthonormality condition (23), namely

$$a_\mu^2 = \int_0^{2\pi} \cos^2 \mu\theta \, \mathrm{d}\theta = \int_0^{2\pi} \sin^2 \mu\theta \, \mathrm{d}\theta = (1 + \delta_{\mu 0})\pi, \quad \text{(A1)}$$

and

$$b_{\mu\nu}^2 = \int_0^a J_\mu^2(k_{w\mu\nu} r) r \, \mathrm{d}r,$$

yielding

$$\begin{cases} b_{00}^2 = \frac{a^2}{2} \\ b_{\mu\nu}^2 = \frac{a^2}{2}\left(1 - \frac{\mu^2}{\gamma_{\mu\nu}^2}\right) J_\mu^2(\gamma_{\mu\nu}), \quad (\mu, \nu) \neq (0, 0). \end{cases} \quad \text{(A2)}$$

Coefficients $g_{\mu\nu}(z, z_0)$

Upon using the modal expansion (24) as the solution of the Green equation

$$\left[\Delta_{\vec{w}} + \frac{\partial^2}{\partial z^2} + k^2\right] G(\vec{w}, \vec{w}_0; z, z_0) = -\delta(\vec{w} - \vec{w}_0)\delta(z - z_0), \quad \text{(A3)}$$

yield straightforwardly (see standard textbook)

$$\left[\frac{\partial^2}{\partial z^2} + k_{z\mu\nu}^2\right] g_{\mu\nu}(z, z_0) = -\delta(z - z_0), \quad \forall z \in (0, \ell), \quad \text{(A4)}$$

with $k_{z\mu\nu}^2 = k^2 - k_{w\mu\nu}^2$, the boundary conditions on the diaphragms being (18a,b)

$$\left(\frac{\partial}{\partial z} - \mathrm{j}\, k_0 \zeta_{z0\mu\nu}\right) g_{\mu\nu}(z, z_0) = 0, \quad z = 0, \quad \text{(A5a)}$$

$$\left(\frac{\partial}{\partial z} + \mathrm{j}\, k_0 \zeta_{z\ell\mu\nu}\right) g_{\mu\nu}(z, z_0) = 0, \quad z = \ell. \quad \text{(A5b)}$$

The solution takes the form [4]

$$g_{\mu\nu}(z, z_0) = -g_2(z) \int_0^z \frac{g_1(x)\delta(x - z_0)}{W_g} \mathrm{d}x - g_1(z) \int_z^\ell \frac{g_2(x)\delta(x - z_0)}{W_g} \mathrm{d}x. \quad \text{(A6)}$$

where g_1 and g_2 are solutions of the homogeneous form of equation (A4)

$$g_1(z) = \cos(k_{z\mu\nu} z + \varphi_{0\mu\nu})$$

and

$$g_2(z) = \cos(k_{z\mu\nu}(z - \ell) - \varphi_{\ell\mu\nu}),$$

the function W_g being the wronskian $W_g = g_1 g_2' - g_2 g_1'$. The expression (A6) gives straigthforwardly the solutions (25a) and (25b). Then, equation (25d) results directly from the boundary conditions (A5a) and (A5b).

Admittances equivalent to the effect of the viscous and thermal boundary layers

The admittances, equivalent to the effects of the viscous and thermal boundary layers, on the walls $z = 0$ and $z = \ell$, has the form [3]

$$\zeta_{z0,\ell\mu\nu} \approx \frac{1 + \mathrm{j}}{\sqrt{2}} \sqrt{k_0} \left[\left(1 - \frac{k_{z\mu\nu}^2}{k_0^2}\right) \sqrt{\ell'_v} + (\gamma - 1)\sqrt{\ell_h}\right]. \quad \text{(A7)}$$

leading to, respectively for the propagative and evanescent modes,

$$\zeta_{z0,\ell\mu\nu} \approx \frac{1+\mathrm{j}}{\sqrt{2}} \sqrt{k_0} \left[\frac{\gamma_{\mu\nu}^2 / a^2}{k_0^2} \sqrt{\ell'_v} + (\gamma - 1)\sqrt{\ell_h}\right] \quad \text{(A8a)}$$

$$\zeta_{z0,\ell\mu\nu} \approx \frac{1+\mathrm{j}}{\sqrt{2}} \sqrt{k_0} \left[\sqrt{\ell'_v} + (\gamma - 1)\sqrt{\ell_h}\right]. \quad \text{(A8b)}$$

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