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Laure Lagny; Michel Bruneau; François Gautier

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Pedagogically treating topics as having several ways to be addressed: Harmonically varying states of free-edge circular plates

Laure Lagny, Michel Bruneau,[a] and François Gautier
Laboratoire d'Acoustique de l'Université du Mans (LAUM), UMR 6613, Institut d'Acoustique-Graduate School (IA-GS), CNRS, Le Mans Université, Le Mans, 72085 Cedex 9, France

ABSTRACT:
The paper is mainly concerned with the training of students by treating exercises as having several ways to be addressed. The topic considered herein focuses on the vibrations of a free edge axisymmetric homogeneous circular thin plate excited by a time-periodic source. This topic is prepared using the three analytic methods available (but these are not fully used analytically in the literature) to show the different aspects of the problem: modal expansion, integral formulation, and exact general solution, against which the other models are tested. Several results are provided when the source is localized at the center of the plate to validate the methods with each other, and these are discussed before concluding. https://doi.org/10.1121/10.0019635

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I. INTRODUCTION

The intent of the paper is to show, through an example, how to deepen the training of students by treating exercises as having several ways to be addressed, and how solutions of a given well-posed problem with apparently very different analytical forms lead to a unique result (that should be shown to students). The topic considered herein focuses on the vibrations of a free edge axisymmetric homogeneous circular thin plate excited by a time-periodic source, where the results are presented when the source is localized at the center of the plate (formulas are also displayed for a source uniformly distributed over the plate surface). Many textbooks and papers contain modelling, analytical results, and gray scale plots on the behavior of the displacement fields of such a device, including the well-known Chladni's patterns.[1–4] Nevertheless, a systematic framework which treats this topic of using the three analytic methods available to show the different aspects of the problem and compare the results obtained (to validate the methods with each other) is not available. This approach presented should allow the students that need to deepen their knowledge or just want to practice through exercises to gain a deeper understanding of the problem considered. Moreover, starting from the well-posed problem (Sec. II), they can practice in a complete manner when solving the details of this problem analytically in an exact or approximate manner (Sec. III) or even numerically (but from fully analytical expression herein; Sec. IV).

In Sec. II, given a pressure source acting on the plate, the fundamental equation for the displacement field is presented in the framework of Kirchhoff theory of thin plates.[1–3] The associate Green's function for the infinite plate is given.[1,3,5] The expressions of the source functions considered and boundary conditions retained are detailed.[6,7] In Sec. III, the first model (Sec. III A), against which the other models are tested, starts from the exact general solution of the circular plate, the solution of the problem considered is obtained from applying inner edge (line source distributed on a circle whose radius vanishes) and outer edge conditions.[4] This exact solution is presented below with details which clarify some issues. The second model (Sec. III B) involves a sum over the natural modes of the simple finite homogeneous free plate,[2] where the individual terms involve the known eigenfunctions and eigenvalues for the plate. Note that the weakness of this model arises from the fact that the required sum is still difficult to evaluate. Section III C presents the third model, which relies on an integral formulation,[8,9] giving a prominent role to the infinite homogeneous plate Green's function in the frequency domain, and the results are obtained analytically, not numerically (boundary-element method as usual).[9] Several results are finally discussed in Sec. IV before concluding.

II. ANALYTIC APPROACHES FOR POTOTYPE PROBLEM: HOMOGENEOUS PLANE THIN ELASTIC PLATE

A. The well-posed problem

As indicated in the Introduction, we consider analytic approaches to express the displacement field of a homogeneous plane thin elastic circular plate, which can be (or not) considered as an annular plate with an inner radius, $a \rightarrow 0$, and outer radius, denoted by b. The origin of the coordinates is set at $r = 0$. The boundary conditions on the contour

[a]Electronic mail: michel.bruneau@univ-lemans.fr

denoted below C (a and b) are assumed to be of any kind in this subsection (Fig. 1). The plate is subjected to a mechanical excitation localized at $r_s = a \to 0$, which is given by its sinusoidal normal force (N).

Consider an area surface enclosed in a contour C and let $\mathbf{r}$ designate an arbitrary interior point. Assume that a homogeneous, isotropic, and finite plate of mass per unit area, M_S, and bending moment, $D_h = Eh^3/12(1-\nu^2)$, extend throughout this surface area (E denotes the Young modulus, h is the thickness, and ν is Poisson's ratio of the plate). If a pressure source, $p_s(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ (unit Pa), acts on the plate, the equation for the subsequent displacement field, $w(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t)$ (harmonically time varying state), can be written classically as[1]

$$D_h[\Delta^2 - k^4]w(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t) = -p_s(\mathbf{r},\omega)\exp(-\mathrm{i}\omega t), \quad (1)$$

where to the lower order of the small quantity, $R_a/(\omega M_S)$,

$$k = \left[\left(\frac{M_S}{D_h}\right)\omega^2\left(1 - \frac{1}{i\omega}\frac{R_a}{M_S}\right)\right]^{1/4} \cong \left(\frac{M_S}{D_h}\omega^2\right)^{1/4}\left[1 - \frac{1}{i\,\omega}\frac{R_a}{4M_S}\right] \quad (2)$$

is the complex wavenumber which accounts for the structural damping that is assumed to behave as viscous damping (given by $R_a \partial w/\partial t$ in the time domain).

The partial differential equation is solved and subjected to any kind of classical boundary conditions, namely, simple edge conditions (free, clamped, and simply supported). Below, the free conditions are chosen by way of example because this choice suffices to validate the methods presented in Secs. II B and II C.

B. The Green's function

Assuming the axisymmetric behavior of the problem, the normal force (unit N) of the source localized at $r_s = 0$ can be expressed as (frequency domain)

$$\begin{aligned} T_S^{(\omega)} &= \lim_{r_s\to 0} 2\pi \int_0^{r_s} p_s(r,\omega) r\,\mathrm{d}r \\ &\cong \lim_{r_s\to 0} 2\pi p_s(0,\omega)\int_0^{r_s} r\,\mathrm{d}r \\ &= \lim_{r_s\to 0} \pi r_s^2 p_s(r_s,\omega), \end{aligned} \quad (3)$$

which leads to

$$\begin{aligned} p_s(r,\omega) &= \left\{ \begin{array}{ll} \lim_{r_s\to 0} T_S^{(\omega)}/(\pi r_s^2) & \text{if } r \le r_s \\ 0 & \text{if } r > r_s \end{array} \right\} \\ &= T_S^{(\omega)} \frac{\delta(r)}{2\pi r}. \end{aligned} \quad (4)$$

The associated infinite plate free-field Green's functions ($b \to \infty$) denoted by $G(\mathbf{r},\mathbf{r}_0;\omega)$, which satisfy Eqs. (1) and (2) with Eq. (4), take the form[1]

$$G(\mathbf{r},\mathbf{r}_0;\omega) = \frac{1/(\mathrm{i}\omega)}{8\sqrt{M_S D_h}}\left[H_0^{(1)}(k\rho) - H_0^{(1)}(\mathrm{i}k\rho)\right], \quad (5)$$

where

$$\begin{aligned} &H_0^{(1)}(k\rho) = J_0(k\rho) + \mathrm{i}N_0(k\rho), \\ &H_0^{(1)}(\mathrm{i}\,k\rho) = J_0(\mathrm{i}\,k\rho) + \mathrm{i}\,N_0(\mathrm{i}\,k\rho), \\ &\rho = |\mathbf{r} - \mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2\,rr_0\cos\theta_0}, \end{aligned} \quad (6)$$

where $J_0(z)$ and $N_0(z)$ are the zero-order Bessel and Neumann functions, respectively, $H_0^{(1)}(z)$ is the zero-order Hankel function of the first kind, and the wavenumber k is given by Eq. (2). Note that because of the choice of the time dependence, $e^{-i\omega t}$, the Sommerfeld radiation condition at infinity is satisfied by this Green's function.

The solutions of Eqs. (1) and (4), $T_S^{(\omega)}\, G(\mathbf{r},\mathbf{r}_0;\omega)$, are measured in meters because $T_S^{(\omega)}$ is measured in Newtons (N), and the Green's function $G(\mathbf{r},\mathbf{r}_0;\omega)$ represents a displacement field per unit amplitude of the shear force $T_S^{(\omega)}$. Note that the function $((M_S/D_h)\omega^2)^{1/4}$ in Eq. (2) shows clearly the effects of the dispersion phenomenon due to the fact that the speed of the flexural waves depends on the frequency,

$$c_f = (D_h/M_S)^{1/4}\sqrt{\omega} = \sqrt{\zeta}\sqrt{h}\sqrt{\omega}, \quad (7)$$

where

$$\zeta = \sqrt{\frac{E/\rho_0}{12(1-\nu^2)}},$$

and ρ_0 is the density of the plate (E and ν are defined above).

III. SOLUTIONS

A. The direct (exact) solution

This solution relies on the basic method used to obtain the general result of the posed problem in the frequency domain, which is frequently observed in mathematical textbook discussions, but authors, in practice, often assume a solution that can be used only when dealing with an eigenvalue problem, as we will see below. Considering the circular ($a \to 0$) problem, the solution of Eq. (1) takes the form (involving the quartet of zero-order cylindrical Bessel functions of the first and second kinds)[4]

$$w(kr) = AJ_0(\xi) + BI_0(\xi) + C[N_0(\xi) + DK_0(\xi)], \quad (8)$$

where $\xi = kr$ and satisfies four boundary conditions (two at $r = a$ and two at $r = b$), which gives the four integration constants. It is noteworthy that one of the boundary conditions in the problems considered herein involves the line source term set at $r = a$ and is expressed by its prescribed normal force, $T_S^{(\omega)}$, given in Eq. (4) such that

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$$p_s(r,\omega) = T_S^{(\omega)}\delta(r-a)/(2\pi r), \tag{9}$$

which can be taken into account either on the right-hand side of Eq. (1) or as the following boundary condition:

$$2\pi a D_h\, \partial_r \Delta w = -T_S^{(\omega)}. \tag{10}$$

When dealing with the circular plate ($a \to 0$), Eq. (10) holds. The other conditions at the center of the plate are indifferently given by any of the following requirements:

(a) the displacement w remains finite at the origin, leading to $D = \lim_{\xi\to 0}[-N_0(\xi)/K_0(\xi)] = 2/\pi$, functions $N_0(\xi)$ and $K_0(\xi)$ have a logarithmic infinity at $\xi = 0$, but these can be cancelled against each other;

(b) the slope

$$\partial w(\xi)/\partial\xi = -AJ_1(\xi) + BI_1(\xi) - C\left[N_1(\xi) + (2/\pi)K_1(\xi)\right], \tag{11}$$

vanishes;

(c) the resulting bending moment $\lim_{r=a\to 0} 2\pi r D_h[\partial_r + (\nu/r)]\partial_r w$, which is proportional to

$$\begin{aligned}\lim_{\xi\to 0} \xi\left[\partial_\xi + (\nu/\xi)\right]\partial_\xi w &= \xi\{-AJ_0(\xi) + BI_0(\xi) - C\left[N_0(\xi) + (2/\pi)K_0(\xi)\right] \\ &\quad + (\nu-1)(-AJ_1(\xi) + BI_1(\xi) \\ &\quad - C\left[N_1(\xi) + (2/\pi)K_1(\xi)\right])\},\end{aligned} \tag{12}$$

vanishes, according to the behavior of each Bessel function at the origin; and

(d) then, accounting for

$$\partial_\xi \Delta w(\xi) = AJ_1(\xi) + BI_1(\xi) + C\left[N_1(\xi) - (2/\pi)K_1(\xi)\right]$$

and

$$\lim_{\xi\to 0}\left[N_1(\xi) + (2/\pi)K_1(\xi)\right] = -4/(\pi\xi), \tag{13}$$

Equation (10) gives the scale factor

$$C = -T_S^{(\omega)}/\left(8D_h k^2\right). \tag{14}$$

The advantage of using such a so-called "direct method" to solve the problem considered in the frequency domain is that it gives the exact solution, and the integration constants are given by the boundary conditions at the outer edge of the disk. Equation (14) implies that the third derivative of the solution (8) has a jump of size one at the origin because it is an odd function. Then, it is a Green's function in the frequency domain for a circular plate,

$$\begin{aligned} w(kr) &= G(kr) \\ &= AJ_0(kr) + BI_0(kr) \\ &\quad - \frac{1}{8D_h k^2}\left[N_0(kr) + \frac{2}{\pi}K_0(kr)\right]. \end{aligned} \tag{15}$$

The integration constants A and B are obtained by writing the boundary conditions at $r = b$, leading to the algebraic equations,

$$\begin{pmatrix} O_M J_0(kb) & O_M I_0(kb) \\ O_T J_0(kb) & O_T I_0(kb) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{8D_h k^2}\begin{pmatrix} O_M\left[N_0(kb) + (2/\pi)K_0(kb)\right] \\ O_T\left[N_0(kb) + (2/\pi)K_0(kb)\right] \end{pmatrix}, \tag{16}$$

with

$$O_M = \left(\frac{\partial}{\partial r_0} + \frac{\nu}{r_0}\right)\frac{\partial}{\partial r_0}$$

and

$$O_T = \frac{\partial}{\partial r_0}\left(\frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial}{\partial r_0}\right).$$

Note that the functions in Eq. (16) involve only zero- and one-order Bessel functions.

It is worth noting that the eigenfunctions of the circular plate mentioned in Sec. III B satisfies the homogeneous Eq. (1) with $p_s = 0$, namely, with $T_S^{(\omega)} = 0$. Then $C = 0$, and it follows that the eigenfunctions imply only the Bessel functions J_0 and I_0, which are regular at the origin.

B. The modal expansion method

The construction of the solution in terms of axisymmetric eigenmodes of the circular homogeneous plates makes use of the modal wave functions $\psi_j(r) = \alpha_j J_0(k_j r) + \beta_j I_0(k_j r)$ [or $\psi_j(r)$] and their associated eigenvalues, k_j,[2,4,8] which are solutions of the homogeneous equation associated with Eq. (1) ($p_s = 0$) and subjected to the free boundary conditions chosen here,[2,4]

$$w(r,\omega) = \sum_j \eta_j(\omega)\,\psi_j(k_j r), \tag{17}$$

where the coefficients of expansion η_j are given by (d$\mathbf{r}$ meaning $r\,dr\,d\theta$)

$$\eta_j(\omega) = \int\!\!\int_S w(r,\omega)\psi_j(r)\mathrm{d}\mathbf{r} \Big/ \int\!\!\int_S \psi_j^2(r)\mathrm{d}\mathbf{r}.$$

The constants α_j and β_j are expressed in terms of each other by solving the system of two homogeneous algebraic equations given by the two edge conditions,

$$\begin{pmatrix} O_M J_0(k_j b) & O_M I_0(k_j b) \\ O_T J_0(k_j b) & O_T I_0(k_j b) \end{pmatrix}\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{18}$$

where one of the integration constants is chosen by writing that the eigenfunctions have the norm equal to one (for example) and the eigenvalues k_j result from writing that the determinant of this system should be null.

The solution of Eq. (1) for an axisymmetric source function $p_s(r,\omega)$ on the right-hand side of the equation, which makes use of these separable wave functions, can be derived by using the following procedure: multiply Eq. (1) by $\psi_\ell(r)$ and integrate over the surface S of the plate, and one readily obtains (invoking orthogonality)

$$\left[-\omega^2 - \mathrm{i}\omega\frac{R_a}{M_S} + \omega_j^2\right]\eta_j(\omega) = \frac{-1}{N_j M_S}\int\int_{\mathsf{S}}\psi_j(r)p_s(\mathbf{r},\omega)\mathrm{d}\mathbf{r}, \tag{19}$$

where

$$\nabla^4\psi_j = k_j^4\psi_j, \quad k_j^4 = (M_s/D)\omega_j^2$$

and

$$N_j = \int\int_{\mathsf{S}}\psi_j^2(r)\mathrm{d}\mathbf{r}, \tag{20}$$

which readily leads to

$$\eta_j(\omega) = \frac{\exp(-\mathrm{i}\omega t)}{M_S N_j\left(\omega^2 + 2\mathrm{i}\omega\omega_j/Q_j - \omega_j^2\right)}\int\int_{\mathsf{S}}\psi_j(r)p_s(r,\omega)\mathrm{d}\mathbf{r}, \tag{21}$$

where $Q_j = 2\omega_j M_S/R_a$ represents the quality factor of the plate at the angular frequencies that are significant only at the frequencies close to the resonant frequencies $\omega \cong \omega_j$, and $\int\int_{\mathsf{S}}\psi_j(r)p_s(r,\omega)\mathrm{d}\mathbf{r} = \psi_j(r_s)T_s(\omega)$ when exciting the plate with a localized source set at $r = r_s$.

The relevant point here is that the weakness of this model arises from the fact that the required sum is still difficult to evaluate.

C. The integral method

Given the free-field Green's functions [Eq. (5)], solutions of the originally posed problem [Eq. (1)] can be subsequently achieved with the aid of Green's integral theorem. This subsection is dedicated to the derivation of such a form of the solution.

Applying Green's theorem within the bounded (or unbounded) surface S of the plate, interior to the contour C, it follows that[9]

$$\begin{aligned}\int\int_{\mathsf{S}}&\{w(\mathbf{r}_0,\omega)D_h\Delta_0\Delta_0 G(\mathbf{r},\mathbf{r}_0;\omega)\\ &- G(\mathbf{r},\mathbf{r}_0;\omega)D_h\Delta_0\Delta_0 w(\mathbf{r}_0,\omega)\}\mathrm{d}\mathbf{r}_0\\ =\oint_{\mathsf{C}}&\mathbf{n}\cdot\{-\Delta_0 G(\mathbf{r},\mathbf{r}_0;\omega)D_h\nabla_0 w(\mathbf{r}_0,\omega)\\ &+ D_h\Delta_0 w(\mathbf{r}_0,\omega)\nabla_0 G(\mathbf{r},\mathbf{r}_0;\omega)\\ &+w(\mathbf{r}_0,\omega)\nabla_0 D_h\Delta_0 G(\mathbf{r},\mathbf{r}_0;\omega)\\ &- G(\mathbf{r},\mathbf{r}_0;\omega)\nabla_0 D_h\Delta_0 w(\mathbf{r}_0,\omega)\}\mathrm{d}\ell_0,\end{aligned} \tag{22}$$

where C is traversed counterclockwise and $\mathbf{n}$ is the unit normal to C, which is directed to the exterior of the surface S. On the other hand, invoking Eq. (1) on S and the reciprocity of the Green's function, the left-hand side of Eq. (22) becomes

$$\int\int_{\mathsf{S}}G(\mathbf{r},\mathbf{r}_0;\omega)p_s(\mathbf{r}_0,\omega)\mathrm{d}\mathbf{r}_0 - w(\mathbf{r},\omega). \tag{23}$$

On substituting Eq. (23) into Eq. (22), it follows that

$$\begin{aligned}w(\mathbf{r},\omega) = &\int\int_{\mathsf{S}}G(\mathbf{r},\mathbf{r}_0;\omega)p_s(\mathbf{r}_0,\omega)\mathrm{d}\mathbf{r}_0\\ &+\oint_{\mathsf{C}}\mathbf{n}\cdot\{\Delta_0 G(\mathbf{r},\mathbf{r}_0;\omega)D_h\nabla_0 w(\mathbf{r}_0,\omega)\\ &- D_h\Delta_0 w(\mathbf{r}_0,\omega)\nabla_0 G(\mathbf{r},\mathbf{r}_0;\omega)\\ &-w(\mathbf{r}_0,\omega)\nabla_0 D_h\Delta_0 G(\mathbf{r},\mathbf{r}_0;\omega)\\ &+ G(\mathbf{r},\mathbf{r}_0;\omega)\nabla_0 D_h\Delta_0 w(\mathbf{r}_0,\omega)\}\mathrm{d}\ell_0.\end{aligned} \tag{24}$$

When the contour C is circular (radii a and b), the operator $\mathbf{n}\cdot\nabla_0$ reduces to $\pm\partial/\partial r$ (the sign depends on the circle considered). On the other hand, it is convenient to express the factors which involve the Green's function as functions of the associated functions:

- the outward normal derivative of displacement,

$$\theta_G(\mathbf{r},\mathbf{r}_0;\omega) = \frac{\partial}{\partial r_0}G(\mathbf{r},\mathbf{r}_0;\omega); \tag{25}$$

- the bending moment,

$$M_G(\mathbf{r},\mathbf{r}_0;\omega) = D_h\left(\frac{\partial}{\partial r_0} + \frac{\nu}{r_0}\right)\frac{\partial}{\partial r_0}G(\mathbf{r},\mathbf{r}_0;\omega); \tag{26}$$

- the shear force,

$$T_G(\mathbf{r},\omega) = D_h\frac{\partial}{\partial r_0}\left(\frac{\partial^2}{\partial r_0^2} + \frac{1}{r_0}\frac{\partial}{\partial r_0}\right)G(\mathbf{r},\mathbf{r}_0;\omega); \tag{27}$$

and equivalently for the displacement w.

The result that emerges is

$$\begin{aligned}w(\mathbf{r},\omega) = &\int\int_{\mathsf{S}}G(\mathbf{r},\mathbf{r}_0;\omega)p_s(\mathbf{r}_0,\omega)\mathrm{d}\mathbf{r}_0\\ &-2b\int_0^\pi\{w(b,\omega)T_G(\mathbf{r},\mathbf{b};\omega) - \theta(b,\omega)M_G(\mathbf{r},\mathbf{b};\omega)\\ &+M_w(b,\omega)\theta_G(\mathbf{r},\mathbf{b};\omega) - T_w(b,\omega)G(\mathbf{r},\mathbf{b};\omega)\}\mathrm{d}\theta_0\\ &+2a\int_0^\pi\{w(a,\omega)T_G(\mathbf{r},\mathbf{a};\omega) - \theta(a,\omega)M_G(\mathbf{r},\mathbf{a};\omega)\\ &+M_w(a,\omega)\theta_G(\mathbf{r},\mathbf{a};\omega) - T_w(a,\omega)G(\mathbf{r},\mathbf{a};\omega)\}\mathrm{d}\theta_0.\end{aligned} \tag{28}$$

Note that the first term on the right-hand side of Eq. (28) represents any source term set in the domain between concentric circular lines of radii a and b. When the source term is concentrated on the internal circular boundary layer of radius a [Eqs. (9) and (10)], the last term on the right-hand side of Eq. (28) can be substituted for the first term because

$$\begin{aligned}-2a&\int_0^\pi T_w(a,\omega)G(\mathbf{r},\mathbf{a};\omega)\mathrm{d}\theta_0\\ &= \int\int_{\mathsf{S}}G(\mathbf{r},\mathbf{r}_0;\omega)p_s(\mathbf{r}_0,\omega)\mathrm{d}\mathbf{r}_0\\ &= T_S^{(\omega)}G(\mathbf{r},\mathbf{a};\omega).\end{aligned} \tag{29}$$

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The integral over the interval $(0, 2\pi)$ can be expressed using the following results:[10]

$$2\int_0^{\pi} J_0\left(k\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}\right)\mathrm{d}\theta_0 = 2\pi J_0(kr)J_0(kr_0), \tag{30}$$

$$2\int_0^{\pi} N_0\left(k\sqrt{r^2+r_0^2-2rr_0\cos\theta_0}\right)\mathrm{d}\theta_0 = 2\pi J_0(kr)N_0(kr_0), \quad r<r_0. \tag{31}$$

Note that knowing that the latter functions are continuous, the field can be obtained on the edge $r_1 = r_2$ because it can be calculated as close as one wants to the edge. Note also that due to the axisymmetric properties of the posed problems emphasized by Eqs. (30) and (31), it is readily verifiable that the integrands are always spatially regular at the boundaries (including the center of the plate). Then, they can be evaluated in the standard sense.

It is noteworthy that by differentiating Eq. (28) with respect to r and using Eqs. (25)–(27) and Eqs. (30) and (31), we obtain a set of four integral equations for the displacement $w(\mathbf{r}, \omega)$, the rotation $\theta(r;\omega)$, bending moment $M(r;\omega)$, and shear stress $T(r;\omega)$, which are needed to solve the problem. These equations are valid for any point within the domain, but to formulate the final solution, one must first take the point to the boundary: this classical procedure gives a set of algebraic equations satisfied by a set of unknown parameters involving the values of w, θ, M, and T on the boundaries. As an example, for the annular plate, there are eight algebraic equations (four at $r = a$ and four at $r = b$) which give the eight unknown parameters, namely, w, θ, M, and T at $r = a$ and at $r = b$. Substituting these values into Eq. (28) gives the sought solution for w and then for each element of the state vector, $W = (w,\ \theta,\ M,\ T)^T$.

Taking into account both of the Green's functions displayed above [Eqs. (5) and (6)] and Eqs. (30) and (31), the set of integral equations just mentioned takes the following form for an external normal force on the inner edge of the plate denoted by T_w and when assuming, for example, free outer edge, namely, assuming that $M(b;\omega)$ and $T(b;\omega)$ vanish:

$$\begin{aligned} w(r;\omega) = {} & b\left[\theta(b,\omega)\tilde{M}_G(r,b;\omega) - w(b,\omega)\tilde{T}_G(r,b;\omega)\right] \\ & -a\left[\theta(a,\omega)\tilde{M}_G(r,a;\omega) \right. \\ & \left. - w(a,\omega)\tilde{T}_G(r,a;\omega) + T_w\tilde{G}(r,a;\omega)\right], \end{aligned} \tag{32}$$

$$\begin{aligned} \theta(r;\omega) = {} & b\left[\theta(b,\omega)\tilde{M}'_G(r,b;\omega) - w(b,\omega)\tilde{T}'_G(r,b;\omega)\right] \\ & -a\left[\theta(a,\omega)\tilde{M}'_G(r,a;\omega) \right. \\ & \left. - w(a,\omega)\tilde{T}'_G(r,a;\omega) + T_w\tilde{G}'(r,a;\omega)\right], \end{aligned} \tag{33}$$

and so on for $M(r,R;\omega)$ and $T(r,R;\omega)$, where $(')$ denotes the derivative with respect to r and with the notation

$$\tilde{\Phi} = 2\int_0^{\pi}\Phi(\theta_0)\mathrm{d}\theta_0. \tag{34}$$

Taking into account the expressions (5) and (6) of $G(\mathbf{r},\mathbf{r}_0;\omega)$ and the expressions (26) and (27) of $M_G(\mathbf{r},\mathbf{r}_0;\omega)$ and $T_G(\mathbf{r},\mathbf{r}_0;\omega)$, respectively, and using the well-known properties of the Bessel functions, one readily obtains

$$\begin{aligned} \tilde{M}_G(r,R;\omega) &= 2\int_0^{\pi} M_G(\mathbf{r},\mathbf{R};\omega)\mathrm{d}\theta_0 \\ &= \mathrm{i}\frac{\pi}{4}\left[J_0(kr)H_0^{(1)}(kR) + J_0(ikr)H_0^{(1)}(\mathrm{i}kR)\right] \\ &\quad -\frac{\pi}{4k}\frac{1-\nu}{R}\left[\mathrm{i}J_0(kr)H_1^{(1)}(kR) \right. \\ &\quad \left. + J_0(\mathrm{i}kr)H_1^{(1)}(\mathrm{i}kR)\right], \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{T}_G(r,R;\omega) &= 2\int_0^{\pi} T_G(\mathbf{r},\mathbf{R};\omega)\mathrm{d}\theta_0 \\ &= \frac{\pi}{4}k\left[-\mathrm{i}J_0(kr)H_1^{(1)}(kR) + J_0(\mathrm{i}kr)H_1^{(1)}(\mathrm{i}kR)\right], \end{aligned} \tag{36}$$

and the expressions for their derivative with respect to r follows directly in using the equation $(\partial/\partial r)J_0(kr) = -kJ_1(kr)$. Note that when solving the integral equation for the displacement field w, the related variables θ, M, and T can be expressed from simply differentiating w with respect to the spatial coordinate instead of using the derivatives of the integral equation for w [Eq. (33) and others].

If one allows the radius a to shrink to zero, on one hand, the limiting values $\lim_{a\to 0} a\tilde{M}_G(r,a;\omega)$, $\lim_{a\to 0} a\tilde{T}_G(r,a;\omega)$, and those of their derivatives vanish, and on the other hand, one can write $\lim_{a\to 0} aT_w\tilde{G}(r,a;\omega) = T_S G(r,0;\omega)$, and in doing so, Eqs. (32) and (33) yield to the integral equations for the circular plate,

$$\begin{aligned} w(r;\omega) = {} & T_S^{(\omega)}G(r,0;\omega) + b\left[\theta(b,\omega)\tilde{M}_G(r,b;\omega) \right. \\ & \left. - w(b,\omega)\tilde{T}_G(r,b;\omega)\right], \end{aligned} \tag{37}$$

$$\begin{aligned} \theta(r;\omega) = {} & T_S^{(\omega)}G'(r,0;\omega) + b\left[\theta(b,\omega)\tilde{M}'_G(r,b;\omega) \right. \\ & \left. - w(b,\omega)\tilde{T}'_G(r,b;\omega)\right]. \end{aligned} \tag{38}$$

Therefore, following the procedure described above, we are left with a set of two algebraic equations at $r = b$, namely,

$$\begin{aligned} w(b;\omega) = {} & T_S^{(\omega)}G(b,0;\omega) + b\left[\theta(b,\omega)\tilde{M}_G(b,b;\omega) \right. \\ & \left. - w(b,\omega)\tilde{T}_G(b,b;\omega)\right], \end{aligned} \tag{39}$$

$$\begin{aligned} \theta(b;\omega) = {} & T_S^{(\omega)}G'(b,0;\omega) + b\left[\theta(b,\omega)\tilde{M}'_G(b,b;\omega) \right. \\ & \left. - w(b,\omega)\tilde{T}'_G(b,b;\omega)\right], \end{aligned} \tag{40}$$

whose solutions $w(b,\omega)$ and $\theta(b,\omega)$ give the unknown parameters included in Eqs. (37) and (38), then to the sought solution $w(r;\omega)$ of the problem itself for any point r on the surface of the circular plate.

When considering the specific problem, frequently encountered in small electroacoustic components (especially Micro Electro Mechanical Systems components), where a uniform harmonic pressure source $p_s(\mathbf{r},\omega)$ (independent of $\mathbf{r}$) acts on the whole surface of a circular plate,[11] the previous results hold, subject to writing

- the exact solution as the sum of the general solution of the homogeneous equation associated to Eq. (1) and a particular solution of Eq. (1) itself,

$$w(r,\omega) = AJ_0(kr) + B\,I_0(kr) + C\left[N_0(kr) + \frac{2}{\pi}K_0(kr)\right] + \frac{p_s}{k^4 D_h}; \tag{41}$$

- the surface integral in the expansion coefficients [Eq. (19)], involved in the sum over the natural modes [Eq. (16)], as

$$\iint_S \psi_j(r) p_s(r,\omega)\mathrm{d}\,\mathbf{r} = p_s \iint_S \psi_j(r)\mathrm{d}\mathbf{r}; \tag{42}$$

- and the source term in the integral solution [Eq. (28)] as

$$\iint_S G(\mathbf{r},\mathbf{r}_0;\omega)p_s(\mathbf{r}_0,\omega)\mathrm{d}\mathbf{r}_0 = p_s \iint_S G(\mathbf{r},\mathbf{r}_0;\omega)\mathrm{d}\mathbf{r}_0. \tag{43}$$

This problem (among others) is not considered here.

IV. SOME RESULTS AND CONCLUSION

The values of the parameters of the plate chosen are the following: radius, $b = 0.5$ m; thickness, $h = 5\cdot 10^{-3}$ m; Young modulus, $E = 7\times 10^{10}$ Pa; Poisson's ratio, $\nu = 0.33$; density, $\rho = 2700\ kg/m^3$; and quality factor, $Q_j = 500$. The results presented below start from the expressions of the solutions for the point source obtained through the three methods presented above, which are given by Eqs. (15), (17), (19), and (37). First and foremost, it is convenient to verify that even if these three expressions are very different from each other, they lead to the same results because the solution of a well-posed problem must be unique. Figure 2 (among others not presented here) shows the mobility $i\omega w(r,\omega)/T_S^{(\omega)}$, relating the amplitude of the velocity $i\omega w(r,\omega)$ at a field point, r, to a time-harmonic force, $T_S^{(\omega)} = 1\,N$, which is applied normally to the structure at the driving point $r = 0$ (logarithmic scale) as a function of the frequency (up to 10 kHz) for the receiving radius $r = 0$: direct method and integral method (perfectly overlapped thin solid curves) and modal method (dashed curve, 31 modes, eigenfrequencies up to 50 kHz; dashed-dotted curve, ten modes, eigenfrequencies up to 5 kHz). The crosses show the values of the eigenfrequencies given in the literature.[2] The discrepancies, due to the numerical errors, are almost negligible everywhere, except those between the modal method and the other methods in the higher frequency range as a result of the truncation of the modal sum: in the higher frequency range, the limitation of the modal method appears

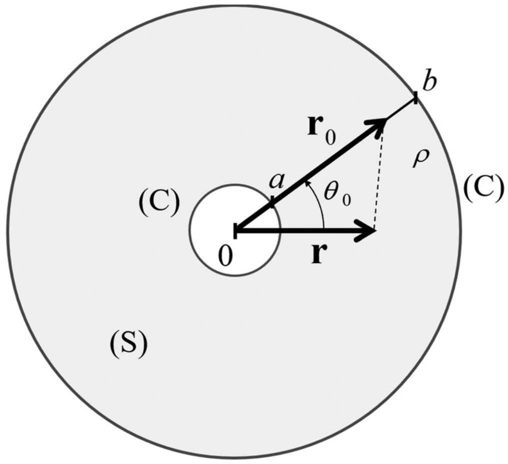

FIG. 1. Circular homogeneous plate, showing surface S and contour C (internal radius, $r_s = a \to 0$; external radius, $r = b$). Notations: $\rho = |\mathbf{r}-\mathbf{r}_0| = \sqrt{r^2 + r_0^2 - 2rr_0\cos\theta_0}$.

at frequencies near the eigenfrequency of the higher mode considered.

The curves or map presented in Figs. 2–4 have been calculated using the direct method and the integral method (curves perfectly overlapped). Figure 3 shows the amplitude of the velocity (logarithmic scale, $T_S^{(\omega)} = 1$, mobility) as a function of the frequency (up to 10 kHz) for several values of the coordinate $r \in (O, b = 0.5\text{ m})$ of the receiving point such that (a) $r = b$, (b) $r = 3b/4$, (c) $r = 3b/5$, (d) $r = b/2$, (e) $r = 2b/5$, and (f) $r = b/4$ (see $r = 0$ in Fig. 2). The differences between the shapes of the curves appear clearly. Nevertheless, each curve always exhibits the fundamental resonances and antiresonances of the circular plate. The gray scale plot presented in Fig. 4 represents the amplitude of the velocity as function of the frequency (horizontal axis) and the coordinate $r \in (O, b = 0.5\text{ m})$ of the receiving point (vertical axis): this map permits one to interpret clearly the shapes shown in Fig. 3. For example, at the distance $r = 2b/5$ [Fig. 3(e)], the resonance and antiresonance in the vicinity of the frequency $f = 7$ kHz, which are very close

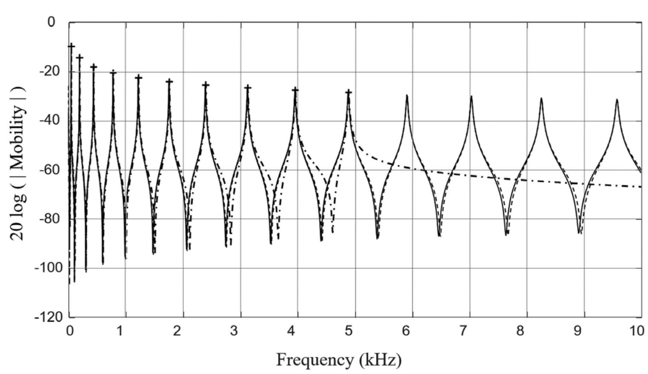

FIG. 2. Amplitude of the velocity (logarithmic scale, with $T_S^{(\omega)} = 1$, mobility) at the center of the plate as a function of the frequency (up to 10 kHz): direct method and integral method (perfectly overlapped thin solid curves) and modal method (dashed curve, 31 modes, eigenfrequencies up to 50 kHz; dashed-dotted curve, 10 modes, eigenfrequencies up to 5 kHz).

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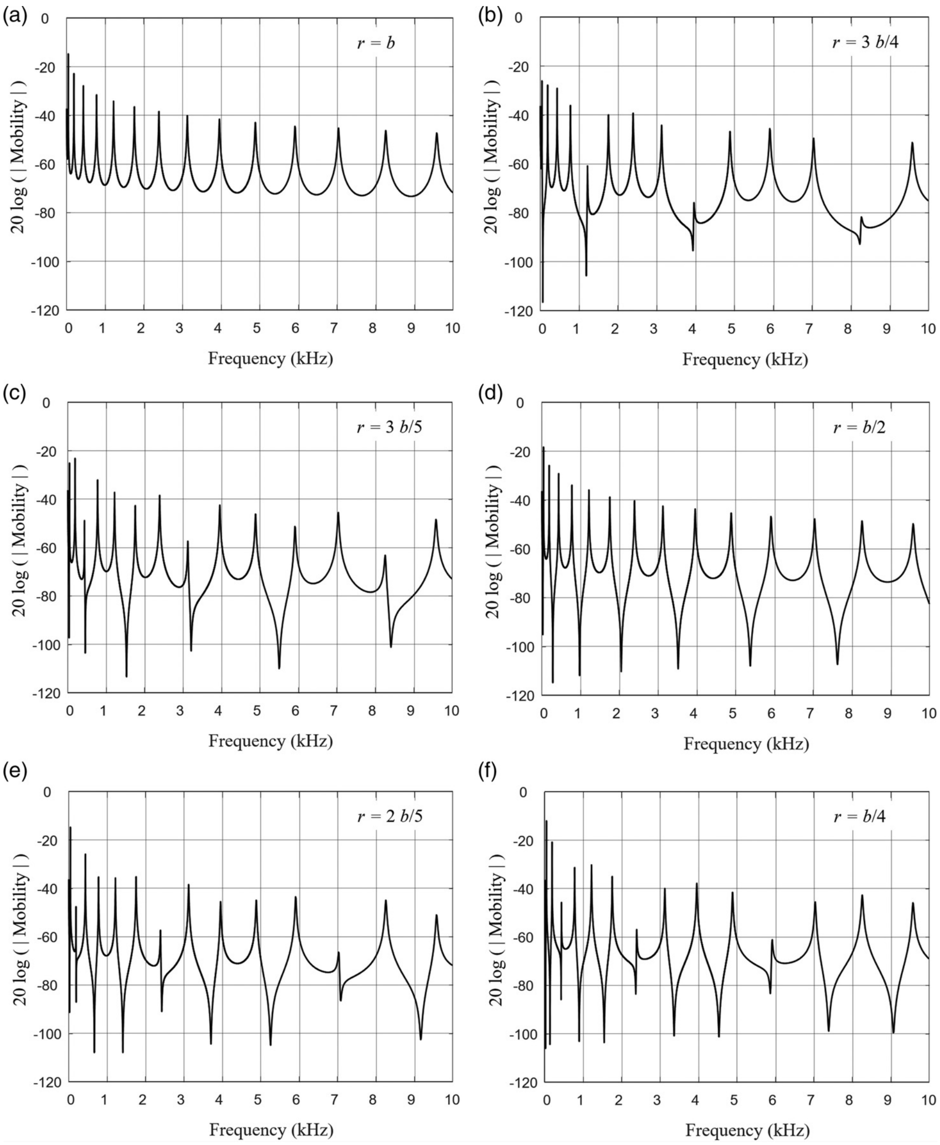

FIG. 3. Amplitude of the velocity (logarithmic scale, $T_S^{(\omega)} = 1$, mobility) as a function of the frequency (up to 10 kHz) for several values of the coordinate $r \in (O, b = 0.5\text{ m})$ of the receiving point, where (a) $r = b$, (b) $r = 3b/4$, (c) $r = 3b/5$, (d) $r = b/2$, (e) $r = 2b/5$, and (f) $r = b/4$ (see $r = 0$ in Fig. 1).

together, correspond in the map (Fig. 4) to the intersection between a white and black trace.

In conclusion, the purpose of this paper is not to present a parametric investigation of the solutions considered (including the modal behavior) because it exists in the literature for each method separately (only numerically for the third method). However, it presents an analytic investigation into several methods that are potentially of use in mode-matching problems involving the behavior of circular plates excited with a harmonically varying source set at the center of the plate. The comparison between the direct method, the integral method, and the modal method is particularly significant in that it justifies the use of this set of solutions as a means of representing such problems. Actually, it shows that the direct formulation and the integral formulation incorporate the characteristic modal quantities which involve eigenfunctions and eigenvalues for the circular plate. This is all the more important from a pedagogical point of view as other problems, such as the nonuniform plate (r-dependent properties), cannot be solved by a modal method because of the absence of orthogonal eigenmodes, but it can be solved by the other methods (eventually through a discretization of the profile). Moreover, it is worth noting that the map displayed in Fig. 4 permits one to interpret clearly the shape of the velocity of the plate as a function of the frequency and coordinate r.

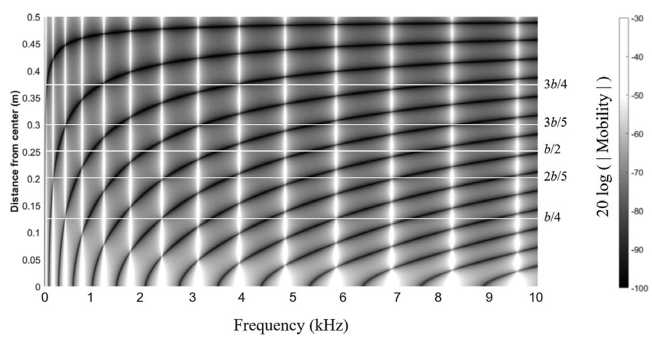

FIG. 4. Amplitude of the velocity (for $T_S^{(\omega)} = 1$, mobility) as function of the frequency (horizontal axis, up to 10 kHz) and the coordinate r of the receiving point [vertical axis, $r \in (O, b = 0.5\text{ m})$].

Also notice that the methods presented here in the frame of the vibrations of a circular homogeneous plate can be applied to many other problems, including those involving the simple Laplacian in the equation of motion instead of the bi-Laplacian involved here.

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